Path Integration in Statistical Field Theory: from QM to Interacting Fermion Systems

Andreas Wipf

Theoretisch-Physikalisches Institut
Friedrich-Schiller-University Jena

Methods of Path Integration in Modern Physics
25.-31. August 2019

Andreas Wipf (TPI Jena)
Introduction

2. Path Integral Approach to Systems in Equilibrium: Finite Number of DOF
   - Canonical approach
   - Path integral formulation

3. Quantized Scalar Field at Finite Temperature
   - Lattice regularization of quantized scalar field theories
   - Äquivalenz to classical spin systems

4. Fermionic Systems at Finite Temperature and Density
   - Path Integral for Fermionic systems
   - Thermodynamic potentials of relativistic particles

5. Interacting Fermions
   - Interacting fermions in condensed matter systems
   - Massless GN-model at Finite Density in Two Dimensions
   - Interacting fermions at finite density in $d = 1 + 1$
why do we discretize quantum (field) theories?

- weakly coupled subsystems: perturbation theory
- if not: strongly coupled system

Properties can only be explained by strong correlations of subsystems

Example of strongly coupled systems:
- ultra-cold atoms in optical lattices
- high-temperature superconductors
- statistical systems near phase transitions
- strong interaction at low energies
- exactly soluble models (large symmetry, QFT, TFT)
- approximations
  - mean field, strong coupling expansion, . . .
- restriction to effective degrees of freedom
  - Born-Oppenheimer approximation, Landau-theory, . . .
- functional methods
  - Schwinger-Dyson equations
  - functional renormalization group equation
- numerical simulations
  - lattice field theories = particular classical spin systems
  \[ \Rightarrow \text{powerful methods of statistical physics and stochastics} \]
Quantum mechanical system in thermal equilibrium

- Hamiltonian $\hat{H} : \mathcal{H} \mapsto \mathcal{H}$
- System in thermal equilibrium with heat bath

\[ \hat{\varrho}_\beta = \frac{1}{Z_\beta} \hat{K}(\beta), \quad \hat{K}(\beta) = e^{-\beta \hat{H}}, \quad \beta = \frac{1}{kT} \]

- Canonical

- Normalizing partition function

\[ Z_\beta = \text{tr} \hat{K}(\beta) \]

- Expectation value of observable $\hat{O}$ in ensemble

\[ \langle \hat{O} \rangle_\beta = \text{tr} (\hat{\varrho}_\beta \hat{O}) \]
inner and free energy

\[ U = \langle \hat{H} \rangle_\beta = -\frac{\partial}{\partial \beta} \log Z_\beta, \quad F_\beta = -kT \log Z_\beta \]

⇒ all thermodynamic potentials, entropy \( S = -\partial_T F, \ldots \)

specific heat

\[ C_V = \langle \hat{H}^2 \rangle_\beta - \langle \hat{H} \rangle_\beta^2 = -\frac{\partial U}{\partial \beta} > 0 \]

system of particles: specify Hilbert space and \( \hat{H} \)

identical bosons: symmetric states

identical fermions: antisymmetric states

traces on different Hilbert spaces
Path integral for partition function in quantum mechanics

- Euclidean evolution operator $\hat{K}$ satisfies diffusion type equation

$$\hat{K}(\beta) = e^{-\beta \hat{H}} \Rightarrow \frac{d}{d\beta} \hat{K}(\beta) = -\hat{H}\hat{K}(\beta)$$

- Compare with time-evolution operator and Schrödinger equation

$$\hat{U}(t) = e^{-i\hat{H}/\hbar} \Rightarrow i\hbar \frac{d}{dt} \hat{U}(t) = \hat{H}\hat{U}(t)$$

- Formally: $\hat{U}(t = -i\hbar\beta) = \hat{K}(\beta)$, imaginary time

- $\hbar$ quantum fluctuations, $kT$ thermal fluctuations
evaluate trace in position space

\[ \langle q | e^{-\beta \hat{H}} | q' \rangle = K(\beta, q, q') \implies Z_{\beta} = \int dq \ K(\beta, q, q) \]

“initial condition” for kernel: \( \lim_{\beta \to 0} K(\beta, q, q') = \delta(q, q') \)

free particle in \( d \) dimensions (Brownian motion)

\[ \hat{H}_0 = -\frac{\hbar^2}{2m} \Delta \implies K_0(\beta, q, q') = \left( \frac{m}{2\pi \hbar^2 \beta} \right)^{d/2} e^{-\frac{m}{2\hbar^2 \beta} (q' - q)^2} \]

Hamiltonian \( \hat{H} = \hat{H}_0 + \hat{V} \) bounded from below \( \Rightarrow \)

\[ e^{-\beta (\hat{H}_0 + \hat{V})} = s - \lim_{n \to \infty} \left( e^{-\beta \hat{H}_0 / n} e^{-\beta \hat{V} / n} \right)^n, \quad \hat{V} = V(\hat{q}) \]
insert for every identity $1$ in

$$( e^{-\frac{\beta}{n} \hat{H}_0} e^{-\frac{\beta}{n} \hat{V}} ) 1 ( e^{-\frac{\beta}{n} \hat{H}_0} e^{-\frac{\beta}{n} \hat{V}} ) 1 \ldots 1 ( e^{-\frac{\beta}{n} \hat{H}_0} e^{-\frac{\beta}{n} \hat{V}} )$$

the resolution $1 = \int dq \, |q\rangle \langle q|$ $\Rightarrow$

$$K(\beta, q', q) = \lim_{n \to \infty} \langle q' | ( e^{-\frac{\beta}{n} \hat{H}_0} e^{-\frac{\beta}{n} \hat{V}} )^n | q \rangle$$

$$= \lim_{n \to \infty} \int dq_1 \ldots dq_{n-1} \prod_{j=0}^{j=n-1} \langle q_{j+1} | e^{-\frac{\beta}{n} \hat{H}_0} e^{-\frac{\beta}{n} \hat{V}} | q_j \rangle,$$

initial and final positions $q_0 = q$ and $q_n = q'$

define small $\varepsilon = \hbar \beta / n$ and finally use

$$e^{-\frac{\beta}{n} V(\hat{q})} | q_j \rangle = | q_j \rangle e^{-\frac{\beta}{n} V(q_j)}$$

$$\langle q_{j+1} | e^{-\frac{\beta}{n} \hat{H}_0} | q_j \rangle = \left( \frac{m}{2\pi \hbar \varepsilon} \right)^{d/2} e^{-\frac{m}{2\hbar \varepsilon} (q_{j+1} - q_j)^2}$$
discretized “path integral”

\[ K(\beta, q', q) = \lim_{n \to \infty} \int dq_1 \cdots dq_{n-1} \left( \frac{m}{2\pi \hbar \varepsilon} \right)^{n/2} \cdot \exp \left\{ -\frac{\varepsilon}{\hbar} \sum_{j=0}^{j=n-1} \left[ \frac{m}{2} \left( \frac{q_{j+1} - q_j}{\varepsilon} \right)^2 + V(q_j) \right] \right\} \]

divide interval \([0, \hbar \beta]\) into \(n\) sub-intervals of length \(\varepsilon = \hbar \beta / n\)

consider path \(q(\tau)\) with sampling points \(q(\tau = k\varepsilon) = q_k\)
• Riemann sum in exponent approximates Riemann integral

\[
S_E[q] = \int_0^\beta d\tau \left( \frac{m}{2} \dot{q}^2(\tau) + V(q(\tau)) \right)
\]

• \(S_E\) is Euclidean action (\(\propto\) action for imaginary time)

• integration over all sampling points \(n \rightarrow \infty\) formal path integral \(\mathcal{D}q\)

• path integral with real and positive density

\[
K(\beta, q', q) = C \int_{q(0)=q}^{q(\hbar \beta)=q'} \mathcal{D}q \ e^{-S_E[q]/\hbar}
\]
on diagonal = integration over all path $q \rightarrow q$

$$K(\beta, q, q) = \langle q | e^{-\beta \hat{H}} | q \rangle = C \int_{q(0)=q}^{q(\hbar \beta)=q} \mathcal{D}q \ e^{-S_E[q]/\hbar}$$

trace

$$\text{tr} \ e^{-\beta \hat{H}} = \int dq \langle q | e^{-\beta \hat{H}} | q \rangle = C \int_{q(0)=q(\hbar \beta)}^{q(\hbar \beta)=q} \mathcal{D}q \ e^{-S_E[q]/\hbar}$$

partition function $Z(\beta)$ integral over all periodic paths with period $\hbar \beta$.

- can construct well-defined Wiener-measure
  - measure(differentiable paths) = 0
  - measure(continuous paths) = 1
not only classical paths contribute
exercise (Mehler formula)

show that the harmonic oscillator with Hamiltonian

\[
\hat{H}_\omega = -\frac{\hbar^2}{2m} \frac{d^2}{dq^2} + \frac{m\omega^2}{2} q^2
\]

has heat kernel

\[
K_\omega(\beta, q', q) = \sqrt{\frac{m\omega}{2\pi\hbar \sinh(\hbar\omega\beta)}} \exp \left\{ - \frac{m\omega}{2\hbar} \left[ (q^2 + q'^2) \coth(\hbar\omega\beta) - \frac{2qq'}{\sinh(\hbar\omega\beta)} \right] \right\}
\]

• equation of euclidean motion \( \ddot{q} = \omega^2 q \) has for given \( q, q' \) the solution

\[
q(\tau) = q \cosh(\omega\tau) + (q' - q \cosh(\omega\beta)) \frac{\sinh(\omega\tau)}{\sinh(\omega\beta)}
\]

• action

\[
S = \frac{m}{2} \int_0^\beta \left( \dot{q}^2 + \omega^2 q^2 \right) = \frac{m\omega}{2 \sinh \omega\beta} \left( (q^2 + q'^2) \cosh \omega\beta - 2qq' \right)
\]
second derivative

\[
\frac{\partial^2 S}{\partial q \partial q'} = -\frac{m\omega}{\sinh(\omega \beta)}
\]

semiclassical formula exact for harmonic oscillator

\[
K(\beta, q', q) = \sqrt{-\frac{1}{2\pi} \frac{\partial^2 S}{\partial q \partial q'}} e^{-S}
\]

yields above results for heat kernel

diagonal elements

\[
K_\omega(\beta, q, q) = \sqrt{\frac{m\omega}{2\pi \hbar \sinh(\hbar \omega \beta)}} \exp \left\{ -\frac{2m\omega q^2}{\hbar} \frac{\sinh^2(\hbar \omega \beta/2)}{\sin(\hbar \omega \beta)} \right\}
\]
• partition function

\[ Z_\beta = \frac{1}{2 \sinh(\frac{\hbar \omega \beta}{2})} = \frac{e^{-\frac{\hbar \omega \beta}{2}}}{1 - e^{-\hbar \omega \beta}} = e^{-\frac{\hbar \omega \beta}{2}} \sum_{n=0}^{\infty} e^{-n\hbar \omega \beta} \]

• evaluate trace with energy eigenbasis of \( \hat{H} \) ⇒

\[ Z(\beta) = \text{tr} e^{-\beta \hat{H}} = \langle n | e^{-\beta \hat{H}} | n \rangle = \sum_n e^{-\beta E_n} \]

• comparison of two sums ⇒

\[ E_n = \hbar \omega \left( n + \frac{1}{2} \right), \quad n \in \mathbb{N} \]
- position operator

\[ \hat{q}(t) = e^{it\hat{H}/\hbar}\hat{q} e^{-it\hat{H}/\hbar}, \quad \hat{q}(0) = \hat{q} \]

- imaginary time \( t = -i\tau \Rightarrow \) euclidean operator

\[ \hat{q}_E(\tau) = e^{\tau\hat{H}/\hbar}\hat{q} e^{-\tau\hat{H}/\hbar}, \quad \hat{q}_E(0) = \hat{q} \]

- correlations at different \( 0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq \beta \) in ensemble

\[ \langle \hat{q}_E(\tau_n) \cdots \hat{q}_E(\tau_1) \rangle_\beta \equiv \frac{1}{Z(\beta)} \text{tr} \left( e^{-\beta\hat{H}} \hat{q}_E(\tau_n) \cdots \hat{q}_E(\tau_1) \right) \]

- consider thermal two-point function (now we set \( \hbar = 1 \))

\[ \langle \hat{q}_E(\tau_2)\hat{q}_E(\tau_1) \rangle_\beta = \frac{1}{Z(\beta)} \text{tr} \left( e^{-(\beta-\tau_2)\hat{H}} \hat{q} e^{-(\tau_2-\tau_1)\hat{H}} \hat{q} e^{-\tau_1\hat{H}} \right) \]
spectral decomposition: $|n\rangle$ orthonormal eigenstates of $\hat{H} \Rightarrow$

$$\langle \ldots \rangle_\beta = \frac{1}{Z(\beta)} \sum_n e^{-(\beta - \tau_2)E_n} \langle n| \hat{q} e^{-(\tau_2 - \tau_1)}\hat{H} \hat{q}|n\rangle e^{-\tau_1 E_n}$$

insert $1 = \sum |m\rangle \langle m| \Rightarrow$

$$\langle \ldots \rangle_\beta = \frac{1}{Z(\beta)} \sum_{n,m} e^{-(\beta - \tau_2 + \tau_1)E_n} e^{-(\tau_2 - \tau_1)E_m} \langle n| \hat{q} |m\rangle \langle m| \hat{q}|n\rangle$$

low temperature $\beta \to \infty$: contribution of excited states to $\sum_n (\ldots)$ exponentially suppressed, $Z(\beta) \to \exp(-\beta E_0) \Rightarrow$

$$\langle \hat{q}_E(\tau_2) \hat{q}_E(\tau_1) \rangle_\beta \xrightarrow{\beta \to \infty} \sum_{m \geq 0} e^{-(\tau_2 - \tau_1)(E_m - E_0)} |\langle 0| \hat{q} |m\rangle|^2$$

likewise

$$\langle \hat{q}_E(\tau) \rangle_\beta \longrightarrow \langle 0| \hat{q}|0\rangle$$
connected two-point function

\[ \langle \hat{q}_E(\tau_2) \hat{q}_E(\tau_1) \rangle_{c,\beta} \equiv \langle \hat{q}_E(\tau_2) \hat{q}_E(\tau_1) \rangle_{\beta} - \langle \hat{q}_E(\tau_2) \rangle_{\beta} \langle \hat{q}_E(\tau_1) \rangle_{\beta} \]

term with \( m = 0 \) in \( \sum_m (\ldots) \) cancels \( \Rightarrow \) exponential decay with \( \tau_1 - \tau_2 \):

\[ \lim_{\beta \to \infty} \langle \hat{q}_E(\tau_2) \hat{q}_E(\tau_1) \rangle_{c,\beta} = \sum_{m \geq 1} e^{-(\tau_2 - \tau_1)(E_m - E_0)} |\langle 0 | \hat{q} | m \rangle|^2 \]

energy gap \( E_1 - E_0 \) and matrix element \( |\langle 0 | \hat{q} | 1 \rangle|^2 \) from

\[ \langle \hat{q}_E(\tau_2) \hat{q}_E(\tau_1) \rangle_{c,\beta} \to \infty \quad \to e^{-(E_1 - E_0)(\tau_2 - \tau_1)} |\langle 0 | \hat{q} | 1 \rangle|^2, \quad \tau_2 - \tau_1 \to \infty \]
for path-integral representation consider matrix elements

\[ \langle q' | e^{-\beta \hat{H}} e^{\tau_2 \hat{H}} \hat{q} e^{-\tau_2 \hat{H}} e^{\tau_1 \hat{H}} \hat{q} e^{-\tau_1 \hat{H}} | q \rangle \]

resolution of the identity and \( \hat{q} | u \rangle = u | u \rangle \):

\[ \langle \ldots \rangle = \int d\nu d\sigma \langle q' | e^{-(\beta - \tau_2) \hat{H}} | \nu \rangle \nu \langle \nu | e^{-(\tau_2 - \tau_1) \hat{H}} | u \rangle u \langle u | e^{-\tau_1 \hat{H}} | q \rangle \]

path integral representations each propagator \((\beta > \tau_2 > \tau_1)\):

- sum over paths with \( q(0) = q \) and \( q(\tau_1) = u \)
- sum over paths with \( q(\tau_1) = u \) and \( q(\tau_2) = \nu \)
- sum over paths with \( q(\tau_2) = \nu \) and \( q(\beta) = q' \)
- multiply with intermediate positions \( q(\tau_1) \) and \( q(\tau_2) \)

\[ \int d\nu d\sigma : \text{path integral over all paths with } q(0) = q \text{ and } q(\beta) = q' \]
insertion of \(q(\tau_2)q(\tau_1)\) in path integral

\[
\langle \hat{q}_E(\tau_2) \hat{q}_E(\tau_1) \rangle_\beta = \frac{1}{Z(\beta)} \oint Dq e^{-S_E[q]} q(\tau_2)q(\tau_1)
\]

similarly: thermal \(n\)-point correlation functions

\[
\langle \hat{q}_E(\tau_n) \cdots \hat{q}_E(\tau_1) \rangle_\beta = \frac{1}{Z(\beta)} \oint Dq e^{-S_E[q]} q(\tau_n) \cdots q(\tau_1)
\]

**conclusion**

there exist a path integral representation for all equilibrium quantities, e.g.

- thermodynamic potentials, equation of state, correlation functions
real time: quantum mechanics

- **action from mechanics**
  \[ S = \int dt \left( \frac{m}{2} \dot{q}^2 - V(q) \right) \]

- **real time path integral**
  \[ \langle q' | e^{-it\hat{H}/\hbar} | q \rangle = C \int_{q(0)=q}^{q(t)=q'} Dq e^{iS[q]/\hbar} \]

- **correlation functions**
  \[ \langle 0 | T \hat{q}(t_1) \hat{q}(t_2) | 0 \rangle = C \int Dq e^{iS[q]/\hbar} q(t_1)q(t_2) \]

- **oscillatory integrals**

imaginary time: quantum statistics

- **euclidean action**
  \[ S_E = \int d\tau \left( \frac{m}{2} \dot{q}^2 + V(q) \right) \]

- **imaginary time path integral**
  \[ \langle q' | e^{-\beta\hat{H}/\hbar} | q \rangle = C \int_{q(0)=q}^{q(h\beta)=q'} Dq e^{-S_E[q]/\hbar} \]

- **correlation functions**
  \[ \langle \hat{q}_E(\tau_1) \hat{q}_E(\tau_2) \rangle_{\beta} = C \int Dq e^{-S_E[q]/\hbar} q(\tau_1)q(\tau_2) \]

- **exponentially damped integrals**
stochastic methods are required

- numerical simulations: discrete (euclidean) time
- system on time lattice = classical spin system

\[ Z_\beta = \lim_{n \to \infty} \int \prod_{i=1}^{n} dq_i \left( \frac{m}{2\pi \hbar \epsilon} \right)^{n/2} e^{-S_E(q_1, \ldots, q_n)/\hbar} \]

- expectation values of observables

\[ \int \prod_{i=1}^{n} dq_i \prod_{j=1}^{n} dq_j F(q_1, \ldots, q_n) \]

high-dimensional integral (sometimes \( n = 10^6 \) required)

- curse of dimension: analytical and numerical approaches do not work
- stochastic methods, e.g. Monte-Carlo important sampling

what can be determined?

- energies, transitions amplitudes and wave functions in QM
- potentials, phase transitions, condensates and critical exponents
- bound states, masses and structure functions in particle physics . . .
Monte-Carlo simulation (Metropolis algorithm)
square of the ground state wave function
parameters in units of lattice constant $\varepsilon$

A. Wipf, Lecture Notes Physics 864 (2013)
exercise: harmonic chain

find free energy for periodic chain of coupled harmonic oscillators

\[ H = \frac{1}{2m} \sum_{i=1}^{N} p_i^2 + \frac{m \omega^2}{2} \sum_i (q_{i+1} - q_i)^2, \quad q_i = q_{i+N} \]

- periodic \( q(\tau) \Rightarrow \) may integrate by parts in

\[ L_E = \frac{m}{2} \int d\tau \left( \dot{q}^2 + \omega^2 (q_{i+1} - q_i)^2 \right) \]

- matrix notation

\[ L_E = \frac{m}{2} \int d\tau q^T \left( - \frac{d^2}{d\tau^2} + A \right) q, \quad A = \omega^2 (2\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}) \]

- hint: non-negative eigenvalues and orthonormal eigenvectors of \( A \):

\[ \omega_k = 2\omega \sin \frac{\pi k}{N} \quad \text{and} \quad e_k \]
expand \( q(\tau) = \sum c_k(\tau) e_k \)

\[
L_E = \sum_k \frac{m}{2} \int d\tau \left( \dot{c}_k^2 + \omega_k^2 c_k^2 \right)
\]

\( N \) decoupled oscillators with frequencies \( \omega_k \) ⇒

\[
\langle q \mid e^{-\beta H} \mid q \rangle = \prod_k K_{\omega_k}(\beta, q_k, q_k)
\]

results for one-dimensional oscillator ⇒

\[
Z_\beta = \prod_k \frac{e^{\beta \omega_k/2}}{e^{\beta \omega_k} - 1} = \prod_k \frac{e^{-\beta \omega_k/2}}{1 - e^{-\beta \omega_k}}, \quad \omega_k = 2 \omega \sin \frac{\pi k}{N}
\]

free energy contains zero-point energy

\[
F_\beta = \frac{1}{2} \sum_k \hbar \omega_k + kT \sum_k \log \left( 1 - e^{-\hbar \omega_k/kT} \right)
\]
• spin 0: scalar field (Higgs particle, inflaton, . . . )
• spin $\frac{1}{2}$: spinor field (electron, neutrinos, quarks, . . . )
• spin 1: vector field (photon, W-bosons, Z-boson, gluons, . . . )

A quick way from quantum mechanics to quantized scalar field theory:

• scalar field $\phi(t, x)$ satisfies Klein-Gordon type equation ($\hbar = c = 1$)

$$\Box \phi + V'(\phi) = 0$$

• Lagrangian = integral of Lagrangian density over space

$$L[\phi] = \int_{\text{space}} d\mathbf{x} \ L(\phi, \partial_{\mu} \phi), \quad L = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi)$$
- momentum field, Legendre transform $\Rightarrow$ Hamiltonian (fixed time $t$)

$$\pi(x) = \frac{\delta L}{\delta \dot{\phi}(x)} = \frac{\partial L}{\partial \dot{\phi}(x)} = \dot{\phi}(x)$$

$$H = \int dx (\pi \dot{\phi} - L) = \int dx \mathcal{H}, \quad \mathcal{H} = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi)$$

- free particle: $V \propto \phi^2 \Rightarrow$ Klein-Gordan $\Box \phi + m^2 \phi = 0$
- infinitely many dof: one at each space point
- one of many possible regularizations: discretize space
- field theory on space lattice: $x = \varepsilon n$ with $n \in \mathbb{Z}^{d-1}$

$$\phi(t, x) \longrightarrow \phi_{x=\varepsilon n}(t), \quad \int dx \longrightarrow \varepsilon^{d-1} \sum_n$$
finite hypercubic lattice in space

\[ x = \varepsilon n \quad \text{with} \quad n_i \in \{1, 2, \ldots, N_i\} \]

continuum field \( \phi(x) \rightarrow \) lattice field \( \phi_x \)

integral \( \rightarrow \) Riemann sum

\[ \int \, dx \longrightarrow \varepsilon^{d-1} \sum_n \]

derivative \( \rightarrow \) difference quotient

\[ \frac{\partial \phi(x)}{\partial x_i} \longrightarrow (\partial_i \phi)_x \]

example: symmetric “lattice derivative”

\[ (\partial_i \phi)_x = \frac{\phi_{x+\varepsilon e_i} - \phi_{x-\varepsilon e_i}}{2\varepsilon} \]
finite lattice $\rightarrow$ mechanical system with finite number of dof

$$H = \varepsilon^{d-1} \sum_{x \in \text{lattice}} \left( \frac{1}{2} \pi_x^2 + \frac{1}{2} (\partial \phi)_x^2 + V(\phi_x) \right)$$

path integral quantization known

$$\langle \{ \phi'_x \} | e^{-i\hat{H}/\hbar} | \{ \phi_x \} \rangle = C \int \prod_x D\phi_x \ e^{iS[\{ \phi_x \}]/\hbar}$$

(formal) path integral over paths $\{ \phi_x(t) \}$ in configuration space

$$\phi_x(0) = \phi_x \quad \text{and} \quad \phi_x(t) = \phi'_x, \quad \forall \ x = \varepsilon n$$

high-dimensional quantum mechanical system with action

$$S[\{ \phi_x \}] = \int dt \varepsilon^{d-1} \sum_x \left( \frac{1}{2} \dot{\phi}_x^2 - \frac{1}{2} (\partial \phi)_x^2 - V(\phi_x) \right)$$
• canonical partition function

\[ Z_\beta = C \int \mathcal{D}\phi_x \ e^{-S_E[\{\phi_x\}]/\hbar}, \quad \phi_x(\tau) = \phi_x(\tau + \hbar \beta) \]

• real euclidean action

\[ S_E[\{\phi_x\}] = \int d\tau \ \varepsilon^{d-1} \sum_x \left( \frac{1}{2} \dot{\phi}_x^2 + \frac{1}{2} (\partial \phi)^2_x + V(\phi_x) \right) \]

• path-integral well-defined after discretization of “time”

• convenient: same lattice constant \( \varepsilon \) in time and spatial directions

• replace \( \tau \in [0, \hbar \beta] \mapsto \tau \in \{\varepsilon, 2\varepsilon, \ldots, N_0 \varepsilon\} \) with \( N_0 \varepsilon = \hbar \beta \)

• lattice sites \( (x^\mu) = (\tau, x) = (\varepsilon n^\mu) \) with \( n_\mu \in \{1, 2, \ldots, N_\mu\} \)

\( \Rightarrow d \)-dimensional hypercubic space-time lattice
lattice field $\phi_x$ defined on sites of space-time lattice $\Lambda$
- $d$-dimensional Euclid'sche space-time $\rightarrow$ lattice $\Lambda$, sites $x \in \Lambda$
- continuous field $\phi(x) \rightarrow$ lattice field $\phi_x, x \in \Lambda$
  - finite lattice: extend in direction $\mu$: $L_\mu = \varepsilon N_\mu$
  - finite temperature: $L_1 = \cdots = L_{d-1} \gg L_0 \equiv \beta = 1/(kT)$
  - scalar field periodic in imaginary time direction
    \[
    \phi_{x=(x^0+\varepsilon N_0, \varepsilon)} = \phi_{x=(x^0, \varepsilon)} \implies \text{temperature-dependence}
    \]
- typically: also periodic in spatial directions
  $\Rightarrow$ identification $x^\mu \sim x^\mu + L_\mu$ (torus)
- space-time volume $V = \varepsilon^d N_1 N_2 \cdots N_d$
- some freedom in choice of lattice derivative (use symmetries)
dimensionless fields and couplings ($\hbar = c = 1$)

- natural units $\hbar = c = 1 \Rightarrow$ all units in powers of length $L$
- dimensionless action (unit $L^0$)

$$S_E = \int d^d x \left( \frac{1}{2} (\partial \phi)^2 + \sum_a \lambda_{ph}^a \phi^a \right)$$

- $\int d^d x (\partial \phi)^2$ dimensionless $\Rightarrow [\partial \phi] = L^{-d/2} \Rightarrow [\phi] = L^{1-d/2}$
- $\lambda_{ph}^a \int d^d x \phi^a$ dimensionless $\Rightarrow [\lambda_{ph}^a] = L^{-d-a+ad/2}$
- in particular $\lambda_{ph}^2 \propto m^2 \Rightarrow [m] = L^{-1}$
- 4 space-time dimensions $\Rightarrow \lambda_{ph}^4$ dimensionless
- dimensionless lattice field and lattice constants ($x = \epsilon n$)

$$\phi_x = \epsilon^{1-d/2} \phi_n, \quad \lambda_{ph}^a = \epsilon^{-d-a+ad/2} \lambda_a$$
lattice action with dimensionful quantities

\[
S^\text{ph}_L = \varepsilon^d \sum_x \left( \frac{1}{2} \left( \frac{\phi_{x+\varepsilon e_\mu} - \phi_{x-\varepsilon e_\mu}}{2\varepsilon} \right)^2 + \sum_a \lambda^\text{ph}_a \phi^a_x \right)
\]

⇒ lattice action with dimensionless quantities

\[
S_L = \sum_n \left( \frac{1}{2} \left( \phi_{n+e_\mu} - \phi_{x-e_\mu} \right)^2 + \sum_a \lambda_a \phi^a_n \right)
\]

partition function

\[
Z_\beta = C \int \prod_{n=1}^{N_0 N_1 \ldots} d\phi_n \ e^{-S_L[\{\phi_n\}]}
\]

finite-dimensional well-defined integral (lattice regularization)
lattice formulation without any dimensionful quantity
processor knows numbers, not units!
merely letting $\varepsilon \to 0$: no meaningful continuum limit

$\lambda_a$ must be changed as $\varepsilon \to 0$

condition: dimensionful observables approach well-defined finite limits

existence of such continuum limit not guaranteed

example: consider correlation length in

$$\langle \phi(n)\phi(m) \rangle_c \propto e^{-(n-m)/\xi}, \quad \frac{1}{\xi} = m = \text{dimensionless mass}$$

$\xi$ depends on dimensionless couplings $\xi = \xi(\lambda_a)$

relates to (given) dimensionful mass $m^{ph} = 1/(\varepsilon \xi) \Rightarrow \varepsilon$

$m^{ph}$ from experiment, $\xi(\lambda_a)$ measured on lattice

renormalization: keep $m^{ph}$ (and further observables) fixed $\Rightarrow \lambda_a$
• extend of physical objects $\gg$ separation of lattice points
• extend of physical objects $\ll$ box size
• conditions (scaling window)
  • small discretization effects $\xi \gg 1$
  • small finite size effects $\xi \ll N_\mu$
  • strict continuum limit: $\xi \to \infty$
• 2’nd order phase transition required in system with $N_{\text{spatial}} \to \infty$
• theory renormalizable: only a small number of $\lambda_a$ must be tuned
• relevant renormalizable field theories
  • non-Abelian gauge theories in $d \leq 4$
  • scalar field theories in $d < 4$
  • four-Fermi theories in $d \leq 3$
  • non-linear sigma-models in $d \leq 3$
  • Einstein-gravity in $d \leq 4$ (???)
- **input in simulations:** only a few observables (masses)
- simulate with stochastic algorithms in scaling window
- repeat simulations with same observables but decreasing $\varepsilon$
- **output:** many (dimensionful) observables
- extrapolate to $\varepsilon \to 0$
- if theory renormalizable: converge to a continuum limit as $\varepsilon \to 0$
- finite temperature: $N_0$ given, $\varepsilon$ from matching to observable $\Rightarrow \beta = \varepsilon N_0$.
  $\Rightarrow$ temperature dependence of
  - free energy
  - condensates
  - pressure, densities
  - free energy of two static charges (confinement)
  - phase diagram
  - screening effects
  - correlations in heat bath, ...
path integral for finite temperature QFT = classical spin model
no non-commutative operators, instead: path or functional integration over fields
scalar field: assign $\phi_n \in \mathbb{R}$ to each lattice site
sigma models: $\phi_n \in \text{Sphere}$
discrete spin models: $\phi_n \in \text{discrete group}$
example: Potts-model: $\phi_n \in \mathbb{Z}_q$
figures: 3–state Potts-type model
electron, muon, quarks, ... are described by 4-component spinor field $\psi_\alpha(x)$

- metric tensor in Minkowski space-time
  
  $$ (\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1) $$

- 4 × 4 gamma-matrices
  
  $$ \gamma^0, \ldots, \gamma^3, \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} 1 $$

- covariant Dirac equation for free massive fermions
  
  $$ (i\partial - m) \psi(x) = 0, \quad \partial = \gamma^\mu \partial_\mu $$

- Euler-Lagrange equation of invariant action
  
  $$ S = \int d^4x \bar{\psi}(i\partial - m)\psi, \quad \bar{\psi} = \psi^\dagger \gamma^0 \implies \pi_\psi = -i\psi^\dagger $$
• quantization: \( \psi(x) \rightarrow \hat{\psi}(x) \)
• satisfies anti-commutation relation

\[
\{ \hat{\psi}_\alpha(t, x), \hat{\psi}_\beta^\dagger(t, y) \} = \delta_{\alpha\beta} \delta(x - y)
\]

• Hamilton operator: \( \beta = \gamma^0 \), \( \alpha = \gamma^0 \gamma \):

\[
\hat{H} = \int dx \ \hat{\psi}^\dagger(x)(\hat{h}\hat{\psi})(x), \quad \hat{h} = i \alpha \cdot \nabla + m \beta
\]

• derive path integral representation of partition function

\[
Z_\beta = \text{tr} e^{-\beta \hat{H}}
\]

• leads to imaginary time path integral
• replace \( t \rightarrow -i\tau \) and

\[
\gamma^0_E = \gamma^0 \quad \text{and} \quad \gamma^i_E = i\gamma^i
\]
ACR with euclidean metric

\[ \{ \gamma^\mu_E, \gamma^\nu_E \} = 2\delta^{\mu\nu} \mathbb{1}, \quad \gamma^\mu_E \text{ hermitean} \]

lattice regularization (drop index \( E \))

space-time \( \mathbb{R}^4 \rightarrow \) finite (hypercubic) lattice \( \Lambda \)
continuum field \( \psi(x) \) on \( \mathbb{R}^4 \rightarrow \) lattice field \( \psi_x \)

expected path integral

\[
Z_\beta = \text{tr}_{\text{reg}} e^{-\beta \hat{H}} = C \oint \prod_{\alpha, x \in \Lambda} d\psi^\dagger_{\alpha, x} d\psi_{\alpha, x} \ e^{-S_L[\psi, \psi^\dagger]} 
\]

integration over anti-periodic fields (ACR for \( \psi \), see below)

\[ \psi_x(\tau + \beta) = -\psi_x(\tau), \quad \text{also on time lattice} \]

\( S_L \) some lattice regularization of

\[
S_E = \int d^d x \ \psi^\dagger (i\partial + im) \psi
\]
quantized scalar field obey equal-time CR

\[
\left[ \hat{\phi}(t, x), \hat{\phi}(t, y) \right] = 0, \quad x \neq y
\]

⇒ commuting fields in path integral

\[
[\phi(x), \phi(y)] = 0, \quad \forall x, y
\]

quantized fermion field obey equal-time ACR

\[
\{ \hat{\psi}_\alpha(t, x), \hat{\psi}_\beta(t, y) \} = 0, \quad x \neq y,
\]

⇒ anti-commuting fields in path integral

\[
\{ \psi_\alpha(x), \psi_\beta^\dagger(y) \} = 0, \quad \forall x, y
\]

variables \( \{ \psi_{\alpha, n}, \psi_{\alpha, n}^\dagger \} \) in fermion path integral: Grassmann variables
free theories have quadratic action

Gaussian integrals with $A = A^T$ positive matrix; exercise $\Rightarrow$

$$\int \prod_{n=1}^{N} d\phi_n \exp \left( -\frac{1}{2} \sum_{n,m} \phi_n A_{nm} \phi_m \right) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

what do we get for fermions?

simplify notation: $\psi_{\alpha,n} \equiv \eta_i$ and $\psi^\dagger_{\alpha,n} \equiv \bar{\eta}_i$ with $i = 1, \ldots, m$

objects $\{\eta_i, \bar{\eta}_i\}$ form complex Grassmann algebra:

$$\{\eta_i, \eta_j\} = \{\bar{\eta}_i, \bar{\eta}_j\} = \{\eta_i, \bar{\eta}_j\} = 0 \implies \eta_i^2 = \bar{\eta}_i^2 = 0$$

Grassmann integration defined by $(a, b \in \mathbb{C})$

$$\int \text{linear}, \quad \int d\eta_i (a + b \eta_i) = b, \quad \int d\bar{\eta}_i (a + b \bar{\eta}_i) = b$$
Grassmann integrals with

\[ \mathcal{D} \bar{\eta} \mathcal{D} \eta \equiv \prod_{i=1}^{m} d\bar{\eta}_i d\eta_i \]

free fermions \( \Rightarrow \) Gaussian Grassmann integral

\[ Z = \int \mathcal{D} \bar{\eta} \mathcal{D} \eta \ e^{-\bar{\eta} A \eta}, \quad \bar{\eta} A \eta = \sum_{i,j} \bar{\eta}_i A_{ij} \eta_j \]

expand exponential function: \( \int \mathcal{D} \bar{\eta} \mathcal{D} \eta \ (\bar{\eta} A \eta)^k = 0 \) for \( k \neq m \)

remaining contribution (use \( \bar{\eta}_i^2 = 0 \))

\[
\frac{1}{n!} \int \mathcal{D} \bar{\eta} \mathcal{D} \eta \ (\bar{\eta} A \eta)^m = \int \mathcal{D} \bar{\eta} \mathcal{D} \eta \ \sum_{i_1,\ldots,i_m} (\bar{\eta}_1 A_{i_1 i_1} \eta_{i_1}) \cdots (\bar{\eta}_m A_{i_m i_m} \eta_{i_m}) \\
= \int \mathcal{D} \bar{\eta} \mathcal{D} \eta \ \prod_i (\bar{\eta}_i \eta_i) \ \sum_{i_1,\ldots,i_m} \varepsilon_{i_1\ldots,i_m} A_{i_1 i_1} \cdots A_{i_m i_m} \\
= (-1)^m \int \prod_i (d\bar{\eta}_i \eta_i) \ \det A = (-1)^m \ \det A
\]
simple formula

\[ \int \mathcal{D}\bar{\eta}\mathcal{D}\eta \ e^{-\bar{\eta}A\eta} = \det A \]

generalization: generating function

\[ Z(\bar{\alpha}, \alpha) = \int \mathcal{D}\bar{\eta}\mathcal{D}\eta \ e^{-\bar{\eta}A\eta + \bar{\alpha}\eta + \bar{\eta}\alpha} = (e^{-\bar{\alpha}A^{-1}\alpha}) \det A \]

expand in powers of \(\bar{\alpha}, \alpha\) \Rightarrow

\[ \langle \bar{\eta}_i\eta_j \rangle \equiv \frac{1}{Z} \int \mathcal{D}\bar{\eta}\mathcal{D}\eta \ e^{-\bar{\eta}A\eta} \bar{\eta}_i\eta_j = (A^{-1})_{ij} \]

application to Dirac fields: above partition function

\[ Z_\beta = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \ e^{-S_L}, \quad \mathcal{D}\bar{\psi}\mathcal{D}\psi = \prod_{\alpha, n} d\psi^{\dagger}_{\alpha,n} \ d\psi_{\alpha,n} \]
dimensionless field and couplings

\[ S_L = \sum_{n \in \Lambda} \psi^\dagger_n (i\phi_{nm} + im\delta_{nm})\psi_n = \sum_n \bar{\psi}_n D_{nm} \psi_m \]

lattice partition function

\[ Z_\beta = C \det D \]

expectation value in canonical ensemble

\[ \langle \hat{A} \rangle_\beta = \frac{1}{Z_\beta} \oint D\bar{\psi} D\psi \ A(\bar{\psi}, \psi) \ e^{-S_L(\psi, \bar{\psi})} \]

formula for complex scalar field

\[ Z_\beta = \oint \mathcal{D}\phi \mathcal{D}\bar{\phi} \ \exp \left(-\sum \bar{\phi}_m C_{mn} \phi_n \right) \propto \frac{1}{\det C} \]

boson fields: periodic in imaginary time
fermion fields: anti-periodic in imaginary time
neutral scalars (+: periodic bc)

\[ S_E = \frac{1}{2} \int \phi(-\Delta + m^2) \phi \implies F_\beta = \frac{kT}{2} \log \det (+(-\Delta + m^2)) + \ldots \]

Dirac fermions (−: anti-periodic bc)

\[ S_E = \int \psi^\dagger (i\partial + im) \psi \implies F_\beta = -2kT \log \det (-(-\Delta + m^2)) + \ldots \]

exercise

Try to prove the results for fermions (including sign and overall factor)

zeta-function for second order operator \( A > 0 \)

\[ \zeta_A(s) = \sum_n \lambda_n^{-s}, \quad \text{eigenvalues } \lambda_n \]
absolute convergent series in half-plane $\Re(s) > d/2$
meromorphic analytic continuation, analytic in neighborhood of $s = 0$
defines $\zeta -$function regularized determinant

$$\log \det A = \text{tr} \log A = \sum \log \lambda_n = - \frac{d \zeta_A(s)}{ds} \bigg|_{s=0}$$
correct for matrices
Mellin transformations

$$\int_0^\infty dt \, t^{s-1} e^{-t\lambda} = \Gamma(s) \lambda^{-s}$$

⇒ relation to heat kernel

$$\zeta_A(s) = \sum_n \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} e^{-t\lambda_n} = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \text{tr} \left( e^{-tA} \right)$$
coordinate representation

\[ \zeta_A(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t \, t^{s-1} \int \mathrm{d}x \, K(t; x, x), \quad K(t) = e^{-tA} \]

heat kernel of \( A = -\Delta + m^2 \) on cylinder \([0, \beta] \times \mathbb{R}^{d-1}\)

\[ K^\pm(t; x, x') = \frac{e^{-m^2 t}}{(4\pi t)^{d/2}} \sum_{n \in \mathbb{Z}} (\pm 1)^n e^{-\left\{ (\tau - \tau' + n\beta)^2 + (x - x')^2 \right\}/4t} \]

integrate over diagonal elements

\[ \zeta_A^\pm(s) = \frac{\beta V}{(4\pi)^{d/2} \Gamma(s)} \int \mathrm{d}t \, t^{s-1-d/2} e^{-m^2 t} \sum_{n=-\infty}^{\infty} (\pm)^n e^{-n^2 \beta^2 / 4t} \]

Jacobi theta function
integral representation of Kelvin functions

\[ \int_0^\infty dt \, t^a e^{-bt-c/t} = 2 \left( \frac{c}{b} \right)^{(a+1)/2} K_{a+1} \left( 2\sqrt{bc} \right) \]

\Rightarrow \text{series representation; in } d = 4

\[ \zeta_A^\pm(s) = \frac{\beta V}{16\pi^2} \frac{m^{4-2s}}{\Gamma(s)} \left( \Gamma(s-2) + 4 \sum_{n=1}^{\infty} (-1)^n \left( \frac{nm\beta}{2} \right)^{s-2} K_{2-s}(nm\beta) \right) \]

identities

\[ \frac{\Gamma(s-2)}{\Gamma(s)} = \frac{1}{(s-1)(s-2)} \quad \text{and} \quad \frac{1}{\Gamma(s)} = s + O(s^2) \]
derivative at $s = 0 \Rightarrow$

$$F_{\beta}^\pm = -\frac{m^4 V C_\pm}{128\pi^2} \left( 3 - 2 \log \frac{m^2}{\mu^2} + 64 \sum_{n=1,2\ldots} (\pm)^n \frac{K_2(nm\beta)}{(nm\beta)^2} \right)$$

real scalars $C_+ = 1$, complex fermions $C_- = -4$

well-known results for massless particles $K_2(x) \sim 2/x^2$

$$\lim_{m \to 0} f^+(\beta) = -\frac{\pi^2}{90} T^4 \quad , \quad \lim_{m \to 0} f^-(\beta) = -\frac{2}{45\pi^2} T^4$$

**questions**

Why is there a relative factor of 4? What is the free energies of complex scalars, Majorana fermions and photons. What is free energy of complex fermions in $d$ dimensions?
condensed matter systems in $d = 2 + 1$

- tight binding approximation for small excitation energies
- honeycomb lattice for graphen (GN): 2 atoms in every cell, 2 Dirac points $\Rightarrow$ 4-component spinor field
- interaction-driven transition metal $\leftrightarrow$ insulator
- long rang order: AF, CDW, . . .
- interacting fermions (symmetries!)

condensed matter systems in $d = 1 + 1$

- conducting polymers (Trans- and Cis-polyacetylen) Su, Schrieffer, Heeger
- quasi-one-dimensional inhomogeneous superconductor Mertsching, Fischbeck

![Relativistic dispersion relations for electronic excitations on honeycomb lattice](image)

from Castro Neto et al.
interacting fermions in $1 + 1$ and $2 + 1$ dimensions

- irreducible spinor in two and three dimensions has 2 components
- $N_f$ species (flavours) of spinors, $\Psi = (\psi_1, \ldots, \psi_{N_f})$
- relativistic fermions

\[
\mathcal{L}_{\text{GN}} = \bar{\Psi} i \phi \Psi + i m \bar{\Psi} \Psi + \mathcal{L}_{\text{Int}}(\Psi, \bar{\Psi}), \quad \text{e.g. } \bar{\Psi} \Psi = \sum \bar{\psi}_i \psi_i
\]

- parity invariant models

\[
\mathcal{L}_{\text{Int}} = \frac{g_{\text{GN}}^2}{2N_f} (\bar{\Psi} \Psi)(\bar{\Psi} \Psi) \quad \text{scalar-scalar, Gross-Neveu}
\]

\[
\mathcal{L}_{\text{Int}} = -\frac{g_{\text{Th}}^2}{2N_f} (\bar{\Psi} \gamma^\mu \Psi)(\bar{\Psi} \gamma_\mu \Psi) \quad \text{vector-vector, Thirring}
\]

\[
\mathcal{L}_{\text{Int}} = \frac{g_{\text{PS}}^2}{2N_f} (\bar{\Psi} \gamma^* \Psi)(\bar{\Psi} \gamma^* \Psi) \quad \text{pseudoscalar-pseudoscalar}
\]

- in even dimensions $\gamma^* \propto \prod \gamma_\mu$
- Hubbard-Stratonovich trick with scalar, vector and pseudoscalar field
combinations thereof in $d = 4$

- non-renormalizable Fermi theory of weak interaction
- effective models for chiral phase transition in QCD (Jona Lasino)

2 spacetime dimensions: $[g] = L^0$

- massless ThM: soluble
- massless ThM in curved space with $\mu$: soluble
- GNM: asymptotically free, integrable

3 spacetime dimensions: $[g] = L^s$

- not renormalizable in PT
- renormalizable in large-$N$ expansion
- interacting UV fixed point $\rightarrow$ asymptotically safe
- can exhibit parity breaking at low $T$

lattice theories:

- generically: sign problem even for $\mu = 0$
- partial solution of sign problem
GN shows breaking of discrete chiral symmetry

order parameter \( i \Sigma = \langle \bar{\Psi} \Psi \rangle \)

\[
\psi_a \rightarrow i \gamma^* \psi_a, \quad \bar{\psi}_a \rightarrow i \bar{\psi}_a \gamma^* \quad \Rightarrow \quad i \Sigma = \langle \bar{\Psi} \Psi \rangle \rightarrow -\langle \bar{\Psi} \Psi \rangle
\]

equivalent formulation with auxiliary scalar field Hubbard-Stratonovich transformation

\[
L_{GN} = L_{\sigma} = \bar{\Psi} (i D \otimes \mathbb{1}_{N_f}) \psi + \frac{N_f}{2g} (\bar{\Psi} \Psi)^2
\]

\[
L_{\sigma} = \bar{\Psi} (i D \otimes \mathbb{1}_{N_f}) \psi + \lambda N_f \sigma^2, \quad D = \vec{\phi} - \sigma \neq D^\dagger
\]
conserved fermion charge

\[ Q = \int_{\text{space}} dx \, j^0 = \int_{\text{space}} dx \, \psi^\dagger \psi \]

partition function of grand canonical ensemble

\[ Z_{\beta, \mu} = \text{tr} \, e^{-\beta (\hat{H} - \mu \hat{Q})} \]

functional integral with above \( L_\sigma \) wherein

\[ D = \hat{\mathcal{D}} + \sigma + \mu \gamma^0 \]

expectation values

\[ \langle \mathcal{O} \rangle = \frac{1}{Z_{\beta, \mu}} \int D\bar{\psi} D\psi D\sigma \, e^{-S_\sigma \mathcal{O}} \]
fermion integral in

\[ Z_{\beta, \mu} = \int D\bar{\psi} D\psi D\sigma \ e^{-S_{\sigma}[\sigma, \psi, \bar{\psi}]} = \int D\sigma \ e^{-N_f S_{\text{eff}}[\sigma]} \]

- \( N_f \) fermion species couple identically to auxiliary field \( \Rightarrow \)

\[ \det (iD \otimes 1) = (\det iD)^{N_f} \]

- \( \psi \) anti-periodic in imaginary time, \( \sigma \) periodic

- effective action after fermion integral

\[ S_{\text{eff}} = \lambda \int d^2 x \ \sigma^2 - \log(\det iD) \]

- Ward identity (lattice regularization)

\[ \frac{1}{Z_{\beta, \mu}} \int D\psi D\bar{\psi} D\sigma \frac{d}{d\sigma(x)} \left( e^{-S[\sigma, \psi, \psi^\dagger]} \right) = -\left\langle \frac{dS}{d\sigma(x)} \right\rangle = 0 \]
**exact relation**

\[ \Sigma \equiv -i \langle \bar{\psi}(x) \psi(x) \rangle = \frac{N_f}{g^2} \langle \sigma(x) \rangle \]

**for \( N_f \to \infty \) saddle point (steepest descend) approximation**

\[ Z_{\beta,\mu} = \int D\sigma \, e^{-N_f S_{\text{eff}}[\sigma]} \xrightarrow{N_f \to \infty} e^{-N_f \min S_{\text{eff}}[\sigma]} \]

**translation invariance \( \Rightarrow \) minimizing \( \sigma \) constant:**

\[ S_{\text{eff}} = (N_f \beta L) U_{\text{eff}} \]

\[
U_{\text{eff}} = \frac{\sigma^2}{4\pi} \left( \log \frac{\sigma^2}{\sigma_0^2} - 1 \right) - \frac{1}{\pi} \int_0^\infty dp \, \frac{p^2}{\varepsilon_p} \left( \frac{1}{1 + e^{\beta(\varepsilon_p + \mu)}} + \frac{1}{1 + e^{\beta(\varepsilon_p - \mu)}} \right)
\]

**one-particle energies**

\[ \varepsilon_p = \sqrt{p^2 + \sigma^2} \]

**IR-scale**

\[ \sigma_0 = \langle \sigma \rangle_{T=\mu=0} \]
condensate in the $(\mu, T)$-plane

- symmetric phase for large $T, \mu$
- homogeneously broken phase for small $T, \mu$
- special points: $(T_c, \mu) = (e^\gamma / \pi, 0), (T, \mu_c) = (0, 1 / \sqrt{2})$
- Lifschitz-Punkt bei $(T, \mu_0) \approx (0.608, 0.318)$

(Wolff, Barducci)
is homogeneity assumption really justified?

possible QCD-phase diagram

- crystalline LOFF phase (color superconductive phase)?
- problem: $\mu \neq 0 \Rightarrow$ complex fermion determinant 😞
- large $\mu$ beyond reach in simulations
- are there inhomogeneous crystalline phases in model systems?
• discrete $\varepsilon_n$ energies of Dirac Hamiltonian on $[0, L]$

$$h_\sigma = \gamma^0 \gamma^1 \partial_x + \gamma^0 \sigma(x)$$

• hidden supersymmetry

$$h^2_\sigma = -\frac{d^2}{dx^2} + \sigma^2(x) - \gamma^1 \sigma'(x) = \begin{pmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{pmatrix}, \quad A = -\frac{d}{dx} + \sigma$$

• renormalization: fix (constant) condensate $\sigma_0$ at $\mu = T = 0$
• introduce constant companion field

$$\bar{\sigma}^2 = \frac{1}{L} \int dx \sigma^2(x)$$

• constant $\sigma \Rightarrow \bar{\sigma} = \sigma$
renormalized effective action for $\sigma = \sigma(x)$

$$S_{\text{eff}}[\sigma] = \frac{\beta L}{4\pi} \bar{\sigma}^2 \left( \log \frac{\bar{\sigma}^2}{\sigma_0^2} - 1 \right) + \beta \left( \sum_{n: \varepsilon_n < 0} \varepsilon_n - \sum_{n: \bar{\varepsilon}_n < 0} \bar{\varepsilon}_n \right) - \sum_{n: \varepsilon_n > 0} \left( \log \left( 1 + \exp^{-\beta (\varepsilon_n + \mu)} \right) + \log \left( 1 + \exp^{-\beta (\varepsilon_n - \mu)} \right) \right)$$

derive gap equation for inhomogeneous field

$$\frac{\delta S_{\text{eff}}}{\delta \sigma(x)} = \frac{1}{2\pi} \sigma(x) \log \frac{\bar{\sigma}^2}{\sigma_0^2} + \sum_{n: \varepsilon_n < 0} \psi_n^{\dagger} \gamma^0 \psi_n - \frac{\sigma(x)}{\bar{\sigma}} \sum_{n: \bar{\varepsilon}_n < 0} \bar{\psi}_n^{\dagger} \gamma^0 \bar{\psi}_n + \sum_{n: \varepsilon_n > 0} \left( \frac{1}{1 + \exp^{-\beta (\varepsilon_n + \mu)}} + \frac{1}{1 + \exp^{-\beta (\varepsilon_n - \mu)}} \right) \psi_n^{\dagger} \gamma^0 \psi_n = 0$$

solution in terms of elliptic functions $\Rightarrow$ crystal of baryons at large $\mu$, low $T$
inhomogeneous condensate for small $T$, large $\mu$

$\Rightarrow$ breaking of translation invariance ($N_f \to \infty$)

wave-length of condensate $\leftrightarrow \mu$

all phase transitions are second order

cp. Peierls-Fröhlich model, ferromagnetic superconductors
no-go theorems and $N_f \to \infty$

- inhomogeneous $\langle \bar{\psi} \psi \rangle$ breaks translation invariance $\rightarrow$ massless Goldstone-excitations $\rightarrow$ should not exist in $d = 1 + 1$
- no-go theorems not valid for $N_f \to \infty$
- phase diagram $=$ artifact of $N_f \to \infty$?
- is there a inhomogeneous condensate for $N_f < \infty$?
- number of massless Goldstone excitations:
  - $n_k$ number of type $k$ Goldstone modes
  - type 1: $\omega \sim |k|^{2n+1}$, e.g. relativistic dispersion relation
  - type 2: $\omega \sim |k|^{2n}$, e.g. non-relativistic dispersion relation
- inner symmetries $n_1 + 2n_2 =$ number of broken directions
- spacetime symmetries $n_1 + 2n_2 \leq$ number of broken directions
- large $\mu$: dispersion relation need not be relativistic
update with (nonlocal) determinant of huge matrix $D$

potential sign-problem for finite $\mu$

can prove: fermion determinant is indeed real

$\Rightarrow$ no sign problem for even $N_f$

hybrid MC algorithm, pseudo fermions

rational approximation of inverse fermion matrix

simulations with chiral fermions only

naive fermions for $N_f = 8, 16$ ($\rightarrow$ doublers)

simulations with SLAC fermions for $N_f = 2, 8, 16$

action of pseudo-fermion field with parallized Fourier transformation

scale setting: condensate $\sigma_0$ at $T = \mu = 0$

simulations on large lattices $N_s \leq 1024$
- low temperature $T = 0.038 \sigma_0$, medium density $\mu = 0.5 \sigma_0$
- typical configuration for $N_f = 8$ and $L = 64$
strong evidences for inhomogeneous condensate

spatial correlation function of chiral condensate

\[ C(x) = \frac{1}{L} \sum_y \langle \sigma(y, t)\sigma(y + x, t) \rangle \]

- \( N_f = 8, L = 64 \) naive fermions
- top: homogeneous phase
  \( \mu = 0 \)
  \( T/\sigma_0 \in \{0.082, 0.988\} \)
- bottom: inhomogeneous phase
  \( T = 0.082\sigma_0 \)
  \( \mu/\sigma_0 \in \{0.5, 0.7, 1.0\} \)
Fourier transform of the spatial correlation function

\[ \tilde{C}(k) \propto \sum_x e^{ikx} C(x) \]

- \( N_f = 8, \ L = 64 \)
  - naive fermions
- top: homogeneous phases
  - \( \mu = 0 \)
  - \( T/\sigma_0 \in \{0.082, 0.988\} \)
- bottom: inhomogeneous phase
  - \( T = 0.082 \sigma_0 \)
  - \( \mu/\sigma_0 \in \{0.5, 0.7, 1.0\} \)
inhomogeneous phase: $\mu$-dependence

Spatial correlation function and Fourier-transform

- $N_f = 8, L = 64$
- SLAC-fermions
- low temperature $T = 0.038\sigma_0$
- different chemical potentials $\mu/\sigma_0 \in \{0, 0.4, 0.5, 0.7\}$
- violet: symmetric phase $\mu = 0, T = 0.61\sigma_0$

Andreas Wipf (TPI Jena)  Path Integration in Statistical Field Theory
\( L = 64, \, \sigma_0 \approx 0.41, \, \mu \approx 0.7, \, N_t = 64 \)

- \( N_f = 8 \)
- crystalline phase
- spatial correlation function for naive and SLAC fermions
- Fourier transform
The diagram shows the dependence of \( \bar{n}_B \cdot L \) on \( \mu \) for different values of \( \sigma_0 \). For \( L = 63 \), \( \sigma_0 = 0.4100 \), and \( N_t = 64 \), the simulation results are plotted. The x-axis represents \( \mu \) ranging from 0.0 to 0.8, and the y-axis represents \( \bar{n}_B \cdot L \) ranging from 0.000 to 1.000.
phase diagram: homogeneous phases from $\sigma$

$\sigma, L = 64, \sigma_0 \approx 0.20$
phase diagram: all three phases from $C_{\min}$

$C_{\min}, L = 64, \sigma_0 \approx 0.20$

Andreas Wipf (TPI Jena)  Path Integration in Statistical Field Theory
correlations function for $N_f \gg 1$

$$C(x, y) \sim \frac{1}{|x - y|^{1/N_f}}$$

may look like SSB for large $N_f$ on small lattices

dependence on system size

smallest available $N_f = 2$

check algorithmic aspects (e.g. thermalization)
$N_f = 2$, smaller lattice $N_s = 125$, chiral SLAC fermions

Andreas Wipf (TPI Jena)
Path Integration in Statistical Field Theory
$N_f = 2$, large lattice, $N_s = 525$, chiral SLAC fermions

Andreas Wipf (TPI Jena)  Path Integration in Statistical Field Theory
$N_f = 2$, comparison

Path Integration in Statistical Field Theory
• first simulation for GN model at finite $\mu, T, N_f$
• no sign problem for even $N_f$
• comparable results for $N_f = 8$ and $N_f = 16$
  naive and chiral SLAC fermions
• phase diagrams are similar as for $N_f \to \infty$
  wave length and amplitude of condensate
• simulations for $N_f = 2$ on sizable lattices
• Goldstone-theorem, . . .
• situation in higher dimensions
• domain walls, vortices, . . . ???
Some remarks concerning interacting fermions in $d = 2 + 1$ dimensions

- asymptotically safe ($1/N_f$ expansion, FRG)
- GN model show 2nd order phase transition for all $N_f$
- $N_f$ odd: parity breaking
- Thirring models:
  - even $N_f$: no phase transition
  - odd $N_f$: phase transition for $N_f \leq N_f^{\text{crit}}$
- critical $N_f^{\text{crit}}$ determined
- spectrum of light (would be Goldstone) particles
- average spectral density of Dirac operator
- full phase diagram in $(\lambda, N_f)$-plane