# Chapter 3

# The Harmonic Oscillator

To get acquainted with path integrals we consider the harmonic oscillator for which the path integral can be calculated in closed form. We allow for an arbitrary time-dependent oscillator strength and later include a time dependent external force. We begin with the discretized path integral (2.29) and then turn to the continuum path integral (2.32).

## 3.1 Solution by discretization

The action of a one-dimensional harmonic oscillator with mass $m$ is

$$S = \frac{m}{2}\int_{t'}^{t} ds\,\Big(\dot{w}^2(s) - \omega^2(s)w^2(s)\Big)\,, \tag{3.1}$$

where $\omega(s)$ is a *time-dependent* circular frequency. To calculate the propagator from $q'$ at initial time $t'$ to $q$ at final time $t$ we divide the time interval in $n$ intervals of equal length $\epsilon = (t-t')/n$. Our starting point is (2.35) with the following classical action for a broken line path

$$S^{(n)}(w) = \frac{m}{2}\sum_{j=0}^{n-1}\Big[\frac{1}{\epsilon}(w_{j+1}-w_j)^2 - \epsilon\,\omega_j^2 w_j^2\Big] \quad \text{with} \quad \omega_j = \omega(t'+j\epsilon). \tag{3.2}$$

For the following manipulations is it convenient to introduce two $n-1$-tupels, one with the integration variables as entries and the other with the positions of the endpoints,

$$\xi = (w_1, w_2, \dots, w_{n-1}) \quad \text{and} \quad \eta = (q', \underbrace{0,\dots,0}_{\text{n}-3\text{ times}}, q). \tag{3.3}$$

Then the action can be rewritten as

$$S^{(n)}(w) = S^{(n)}(\eta,\xi) = \frac{m}{2}\left(\frac{1}{\epsilon}\,(\eta,\eta) + \frac{1}{\epsilon}(\xi, C\xi) - \frac{2}{\epsilon}(\xi,\eta) - \epsilon\,\omega_0^2 q'^{\,2}\right), \tag{3.4}$$

where the $n-1$-dimensional matrix $C$ is

$$C = \begin{pmatrix} \mu_1 & -1 & 0 & \cdots & 0 \\ -1 & \mu_2 & -1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & -1 & \mu_{n-1} \end{pmatrix}, \quad \mu_j = 2 - \epsilon^2 \omega_j^2. \tag{3.5}$$

For vanishing $\omega_j$ the square matrix $C$ is proportional to the discretized second derivative (one-dimensional lattice Laplacian) on the discrete time lattice. We are left with calculating the *Gaussian integral*

$$K_\omega(t,q,t',q') = \lim_{n\to\infty} A_\epsilon^n \int d^{n-1}\xi \; e^{iS^{(n)}(\xi,\eta)/\hbar}, \quad \text{where} \quad A_\epsilon = \left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2}, \tag{3.6}$$

and the lattice action (3.4) is a quadratic function of the integration variables $\xi$. As a function of these variables it is extremal at $\xi_{\rm cl}$, given by

$$\frac{\delta S^{(n)}}{\delta \xi_i}(\xi = \xi_{\rm cl}) = 0 \quad \text{or} \quad C\xi_{\rm cl} = \eta. \tag{3.7}$$

$\xi_{\rm cl}$ is the classical solution of the discretized equation of motion. Expanding the action about this solution yields

$$S^{(n)}(\xi_{\rm cl} + \xi) = S^{(n)}(\xi_{\rm cl}) + \frac{m}{2\epsilon}(\xi, C\xi) \tag{3.8}$$

with the following action of the classical solution

$$S^{(n)}(\xi_{\rm cl}) = \frac{m}{2\epsilon}\left[\eta^2 - (\eta, C^{-1}\eta)\right] - \frac{1}{2}m\omega_0^2\epsilon q'^2. \tag{3.9}$$

Terms linear in $\xi$ are absent since $\xi_{\rm cl}$ is an extremum of $S$. Inserting (3.9) into (3.6) leads to

$$K_\omega(t,q,t',q') = \lim_{n\to\infty} A_\epsilon^n \, e^{iS^{(n)}(\xi_{\rm cl})/\hbar} \int d^{n-1}\xi \; e^{\,im/2\epsilon\hbar\,(\xi,C\xi)}. \tag{3.10}$$

Here we encounter for the first time a *Gaussian integral*. Such integrals appear frequently in path integral calculations. The one-dimensional Gaussian integral is

$$\int d\xi \, e^{-\alpha\xi^2/2} = \sqrt{\frac{2\pi}{\alpha}}. \tag{3.11}$$

The generalization to multi-dimensional Gaussian integrals follows after a diagonalization of the matrix defining the quadratic form in the exponent and is given by

$$\int d^p\xi \; \exp\left(-\frac{1}{2}(\xi, B\xi)\right) = \frac{(2\pi)^{p/2}}{\sqrt{\det B}}. \tag{3.12}$$

Here $B$ is a $p$-dimensional symmetric matrix with non-negative real part. For a non-symmetric $B$ the antisymmetric part does not contribute to the integral and $B$ is replaced by $(B+B^\dagger)/2$ on the right hand side. For an imaginary $B$ the result (3.12) holds in the distributional sense. Using this useful formula in (3.10) and performing the continuum limit yields

$$K_\omega(t,q,t',q') = \lim_{\epsilon\to 0}\sqrt{\frac{m}{2\pi i\hbar}}\,\frac{1}{\sqrt{\epsilon\det C}}\;e^{iS^{(n)}(\xi_{\rm cl})/\hbar}. \tag{3.13}$$

It remains to calculate the determinant of the matrix $C$ and the matrix element $(\eta, C^{-1}\eta)$ entering the classical action in (3.9).

To find the determinant of the $n-1$-dimensional matrix $C$ in (3.6) we consider the $p$-dimensional matrix

$$C_p = \begin{pmatrix} \mu_1 & -1 & 0 & \cdots & 0 \\ -1 & \mu_2 & -1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & & \cdots & -1 & \mu_p \end{pmatrix}, \qquad \mu_j = 2-\epsilon^2\omega_j^2, \tag{3.14}$$

and denote its determinant by $d_p$. Expanding the determinant in the last row yields the recursion relation $d_p = \mu_p d_{p-1} - d_{p-2}$ with the initial conditions $d_1 = \mu_1$ and $d_0 = 1$. To solve this recursion relation we write it in the form

$$d_p - 2d_{p-1} + d_{p-1} = -\epsilon^2\omega_p d_{p-1} \tag{3.15}$$

and divide by $\epsilon^2$. Furthermore we set $d_p = d(s_p)$, where $s_p = t' + p\epsilon$ denotes the time after $p$ time-steps have passed since the initial time $t'$. For $\epsilon \to 0$ we may approximate differences by differentials such that the recursion relation turns into the differential equation,

$$\ddot{d}(s) = -\omega^2(s)d(s). \tag{3.16}$$

The initial slope of $d$ diverges in the continuum limit since $d_2 - d_1 = 1 + O(\epsilon^2)$. Hence we rescale $d(s) \to D(s) = \epsilon\, d(s)$ in order to get a non-singular function. At initial time $t'$ the rescaled function vanishes and has unit slope. Hence in the continuum limit we have

$$\epsilon\det C = \epsilon d_{n-1} \overset{\epsilon\to 0}{\longrightarrow} D(t,t'), \tag{3.17}$$

where the $D$-function solves the *Gelfand-Yaglom initial value problem* [5]

$$\frac{d^2D(s,t')}{ds^2} = -\omega^2(s)D(t,t'), \qquad D(t',t') = 0, \quad \frac{\partial D(s,t')}{\partial s}\Big|_{s=t'} = 1. \tag{3.18}$$

Note that the D-function depends on the initial time $t'$ since it solves the initial values problem. The determinant is the values of $D$ at the final time $t$. The factor $\epsilon$ in $\epsilon\det C = D(t)$ chancels against $\epsilon$ in (3.15) and in the continuum limit we obtain a finite evolution kernel.

Besides the determinant we need the matrix element $(\eta, C^{-1}\eta)$ in the classical action. It only depends on the elements in the corners of the matrix $C^{-1}$. These are given by

$$C^{-1} = \frac{1}{d_{n-1}} \begin{pmatrix} c_{n-2} & \cdots & 1 \\ . & \cdots & . \\ 1 & \cdots & d_{n-2} \end{pmatrix}, \tag{3.19}$$

The elements on the diagonal are $c_{n-2} = d(t, t' + \epsilon)$ and $d_{n-2} = d(t - \epsilon, t')$. Expanding in $\epsilon$ the classical action is now seen to depend only on the function $D$ and its time derivatives as the initial and final time as follows,

$$S^{(n)}(w_{\rm cl}) \stackrel{\epsilon\to 0}{\longrightarrow} S[w_{\rm cl}] = \frac{m}{2D(t,t')}\left(q^2 \frac{dD(t,t')}{dt} - {q'}^2 \frac{dD(t,t')}{dt'} - 2qq'\right). \tag{3.20}$$

Since the solution $D$ of the initial value problem (3.18) determines both the classical action and the determinantal factor, see (3.17), it determines the exact time evolution kernel

$$K_\omega(t, q, t', q') = \sqrt{\frac{m}{2\pi\hbar i}} \frac{1}{\sqrt{D(t,t')}}\, e^{iS[w_{\rm cl}]/\hbar}. \tag{3.21}$$

Differentiating the action of the classical path $S[w_{\rm cl}]$ with respect to the initial and final position we recover the $D$-function,

$$\frac{1}{m}\,\partial_q \partial_{q'} S[w_{\rm cl}] = -\frac{1}{D(t,t')}. \tag{3.22}$$

We see that the classical action determines both the phase factor and the determinantel factor infront of the phase factor. The evolution kernel of the time-dependent oscillator is completely determined by the classical action,

$$K_\omega(t, q, t', q') = \sqrt{\frac{1}{2\pi\hbar i}} \left(-\frac{\partial^2 S[w_{\rm cl}]}{\partial q \partial q'}\right)^{1/2} e^{iS[w_{\rm cl}]/\hbar}. \tag{3.23}$$

For the oscillator with *constant frequency* $\omega$ the $D$-function reads

$$D(t,t') = \frac{1}{\omega} \sin\omega(t - t'). \tag{3.24}$$

Setting $t' = 0$ we find the following explicit formula for the evolution kernel

$$K_\omega(t, q, q') = \sqrt{\frac{m\omega}{2\pi i\hbar \sin(\omega t)}} \exp\left\{\frac{im\omega}{2\hbar}\left[({q'}^2 + q^2)\cot(\omega t) - \frac{2qq'}{\sin(\omega t)}\right]\right\}. \tag{3.25}$$

It is not difficult to see that this kernel satisfies the Schrödinger equation and for $t \to 0$ it reduces to the free evolution kernel (2.22) and thus to the delta function. Hence it obeys the initial condition (2.20). The kernel $K_\omega(t,q,q')$ is singular for $\omega t = n\pi$. This apparent problem can be dealt with by integrating the kernel against wave packets. The Feynman path integral for

$$\psi(t,q) = \int dq' \, \langle q|e^{-itH/\hbar}|q'\rangle \, \psi_0(0,q'), \tag{3.26}$$

has no singularities.

After this rather involved manipulation let us recapitulate the crucial steps in deriving the evolution kernels. First we replaced the integration variables $\xi$ by $\xi_{\rm cl}+\xi$, where $\xi_{\rm cl}$ has been an extremum of the classical 'action'. This shift eliminates the linear in $\xi$ terms in the classical action. Without mentioning, we also assumed the measure to be translational invariant, $d^{n-1}(\xi_{\rm cl}+\xi) = d^{n-1}\xi$, which is of course correct for a finite product of Lebesgue measures. The resulting Gaussian integral can be calculated and is given in (3.13).

## 3.2 Oscillator with external source

One may wonder whether the *formal continuum path integral* is of any practical use for realistic quantum systems. Fortunately the answer is yes and we shall see how to use the continuum path integral if one allows for certain formal manipulations.

Here we derive the path integral for an oscillator with time-dependent frequency and driven by a time-dependent and position-independent *external force*. The Hamiltonian function reads

$$H = \frac{1}{2m}p^2 + \frac{m}{2}\omega^2 q^2 - jq, \tag{3.27}$$

where the time-dependent source $j(s)$ is proportional to the external force. The classical action entering the continuum path integral (2.32) reads

$$S_j[w] = S[w] + (j,w), \quad \text{where} \quad (j,w) = \int_{t'}^{t} ds\, j(s)w(s) \tag{3.28}$$

and $S$ denotes the action (3.1) of the oscillator without external force. By considering the forced oscillator we shall encounter several problems which one comes across in various approximations to more realistic and complicated systems. In addition, the resulting path integral yields the generating functional for the Greenfunctions and thus will be of use when we derive the perturbation expansion for interacting quantum system.

Classical solutions are extremal points of the action and fulfill the equation of motion

$$-\frac{\delta S[w]}{\delta w(s)}\Big|_{w_{\rm cl}} = m\ddot{w}_{\rm cl}(s) + m\omega^2(s)w_{\rm cl}(s) = j(s). \tag{3.29}$$

Similarly as for the discrete path integral considered in the previous section we expand an arbitrary path about the classical trajectory,

$$w(s) \longrightarrow w_{\rm cl}(s) + \xi(s), \quad \text{where} \quad w_{\rm cl}(t') = q' \quad \text{and} \quad w_{\rm cl}(t) = q. \tag{3.30}$$

The classical path $w_{\rm cl}$ obeys the boundary conditions such that the fluctuations $\xi$ vanishes at the endpoints, $\xi(t')=\xi(t)=0$. With

$$S_j[w_{\rm cl}+\xi]=S_j[w_{\rm cl}]+S[\xi], \tag{3.31}$$

the path integral for the propagator reads

$$K_\omega(t,q,t',q';j)=\int\mathcal{D}w\;e^{iS_j[w]/\hbar}=e^{iS_j[w_{\rm cl}]/\hbar}\int\limits_{\xi(t')=0}^{\xi(t)=0}\mathcal{D}\xi\;e^{iS[\xi]/\hbar}. \tag{3.32}$$

The path integral factorizes into a classical part depending on the source and the endpoints and a path integral over the fluctuations. The latter is just the propagator $K_\omega$ of the force-free oscillator (3.21) for the propagation from $q'=0$ to $q=0$. For vanishing endpoints the action $S[w_{\rm cl}]$ entering $K_\omega$ in (3.21) is zero and we obtain the simple formula

$$K_\omega(t,q,t',q';j)=\sqrt{\frac{m}{2\pi\hbar i}}\frac{1}{\sqrt{D(t,t')}}\;e^{iS_j[w_{\rm cl}]/\hbar}, \tag{3.33}$$

where the $D$-function solves the initial value problem (3.18).

Let us finally isolate the part of the classical action depending on the source $j$. To that aim we decompose the classical path $w_{\rm cl}$ into the classical path $w^0_{\rm cl}$ starting and ending at the origin and the solution $w_h$ of the homogeneous equation of motion (without source) starting at $q'$ and ending at $q$,

$$\begin{aligned} w_{\rm cl}(s)=w^0_{\rm cl}(s)+w_h(s),\qquad &\frac{\delta S}{\delta w}\Big|_{w^0_{\rm cl}}=-j,\quad w^0_{\rm cl}(t')=0,\quad w^0_{\rm cl}(t)=0\\ &\frac{\delta S}{\delta w}\Big|_{w_h}=0,\qquad w_h(t')=q',\quad w_h(t)=q. \end{aligned} \tag{3.34}$$

Without external source an oscillator at the origin stays at the origin such that $w^0_{\rm cl}(s)=0$ for a vanishing source. On the other hand , for $q'=q=0$ the homogeneous solution $w_h(s)$ vanishes. The action of $w_{\rm cl}$ decomposes as

$$S_j[w_{\rm cl}]=S_j[w^0_{\rm cl}]+S_j[w_h]+m\int\dot w^0_{\rm cl}\dot w_h-m\int\omega^2w^0_{\rm cl}w_h.$$

After a partial integration in the integral of $\dot w_h\dot w^0_{\rm cl}$ the last two term can be written as

$$m\int_{t'}^t\frac{d}{ds}(w^0_{\rm cl}\dot w_h)-m\int_{t'}^t w^0_{\rm cl}(\ddot w_h+\omega^2w_h)=0.$$

The first term is zero because $w^0_{\rm cl}$ vanishes at the endpoints and the second term is zero because $w_h$ obeys the homogeneous equation of motion. Thus we obtain

$$S_j[w_{\rm cl}]=S_j[w^0_{\rm cl}]+S[w_h]+\int ds\,j(s)w_h(s). \tag{3.35}$$

When the source is switches off then the action reduces to the source-independent term $S[w_h]$ and the propagator reduces to the kernel $K_\omega$ in (3.21), such that

$$K_\omega(t,q,t',q';j) = K_\omega(t,q,t',q')\, e^{iW_\omega[j]/\hbar}, \tag{3.36}$$

where we introduced the *Schwinger functional* for the harmonic oscillator

$$\begin{aligned} W_\omega[j] &= \int ds\, j(s) w_h(s) + S_j[w^0_{\rm cl}] \\ &= \int dsj(s)w_h(s) + \frac{1}{2}\int dsj(s)w^0_{\rm cl}(s). \end{aligned} \tag{3.37}$$

To prove the last identity one uses the equation of motion for the classical path $w^0_{\rm cl}$. In order to find the explicit source dependence of the Schwinger functional we introduce the Greensfunction $G_D$ with respect to *Dirichlet boundary conditions*,

$$m\left(\frac{d}{ds^2} + \omega^2(s)\right) G_D(s,s') = \delta(s,s'). \tag{3.38}$$

As Greenfunction of a selfadjoint and real operator $G_D$ is symmetric in its arguments and vanishes at the endpoints,

$$G_D(s,s') = G_D(s',s) \quad \text{and} \quad G_D(t,s) = G_D(s,t') = 0. \tag{3.39}$$

Now we can construct the solution $w^0_{\rm cl}$ with the help of this Greensfunction as follows,

$$w^0_{\rm cl}(s) = \int_{t'}^{t} G_D(s,s')j(s')ds'. \tag{3.40}$$

Inserting this result into (3.37) yields the following expression for the Schwinger functional,

$$W_\omega[j] = \int ds\, j(s)w_h(s) + \frac{1}{2}\int dsds'\, j(s)G_D(s,s')j(s'). \tag{3.41}$$

The first term is linear and the second is quadratic in the source. Note that according to (2.55) and (2.52) the kernel in (3.36) generates all Greenfunctions of time-ordered products of the position operators at different times. For example, the correlator of two positions for the oscillator without source is

$$\begin{aligned} \langle q,t|\,\mathbf{T}\,\hat q(t_1)\hat q(t_2)|q'\rangle &= \left(\frac{\delta W_\omega}{\delta j(t_1)}\frac{\delta W_\omega}{\delta j(t_2)} + \frac{h}{i}\frac{\delta^2 W_\omega}{\delta j(t_1)\delta j(t_2)}\right)\Big|_{j=0} K_\omega(t,q,t',q') \\ &= \left(w_h(t_1)w_h(t_2) + \frac{\hbar}{i}G_D(t_1,t_2)\right) K_\omega(t,q,t',q'). \end{aligned} \tag{3.42}$$

Next we calculate the kernel and in particular the Schwinger functional for the free particle and for the oscillator with constant frequency.

**Free particle**

For simplicity we take $t' = 0$ as initial propagation time of the free particle. The Greenfunction and homogeneous solution are

$$G_D(s > s') = \frac{1}{mt}(s-t)s' \quad \text{and} \quad w_h(s) = \frac{1}{t}[sq + (t-s)q']. \tag{3.43}$$

The quadratic Schwinger functional (3.41) for the free particle has the explicit form

$$W_0[j] = \frac{1}{t}\int_0^t ds\,(sq + (t-s)q')j(s) + \frac{1}{mt}\int_0^t ds \int_0^s ds'\,(s-t)s' j(s)j(s') \tag{3.44}$$

and it enters the propagator in the presence of an external source

$$K_0(t,q,q';j) = K_0(t,q,q')\, e^{iW_0[j]/\hbar}. \tag{3.45}$$

Note that for vanishing endpoints we arrive at the simpler formula

$$K_0(t,0,0;j) = \left(\frac{m}{2\pi i\hbar t}\right)^{1/2} \exp\left\{\frac{i}{\hbar}\int_0^t ds \int_0^s ds'\, \frac{(s-t)s'}{mt} j(s)j(s')\right\}. \tag{3.46}$$

**Harmonic oscillator with constant frequency**

Again we take as initial time $t' = 0$. For a constant frequency $\omega$ the Greenfunction and solution of the source-free oscillator read

$$\begin{aligned} G_D(s > s') &= \frac{1}{m\omega\sin\omega t}\sin\omega(s-t)\sin\omega s' \\ w_h(s) &= \frac{1}{\sin\omega t}\{q\sin\omega s + q'\sin\omega(t-s)\}. \end{aligned} \tag{3.47}$$

Hence the Schwinger function of the oscillator has the explicit form

$$\begin{aligned} W_\omega[j] &= \frac{1}{\omega\sin\omega t}\int_0^t ds\,(q\sin\omega s + q'\sin\omega(t-s)q)j(s) \\ &+ \frac{1}{m\omega\sin\omega t}\int_0^t ds \int_0^s ds'\, \sin\omega(s-t)\sin\omega s'\, j(s)j(s'), \end{aligned} \tag{3.48}$$

and for a vanishing frequency is converges to the Schwinger functional of the free particle. The functional $W_\omega$ enters the formula for the propagator of the oscillator with constant frequency

$$K_\omega(t,q,q';j) = K_\omega(t,q,q')\, e^{iW_\omega[j]/\hbar}. \tag{3.49}$$

For vanishing endpoints the evolution kernel for $j = 0$ on the right hand side simplifies further and we obtain the simple formula

$$K_\omega(t,0,0;j) = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}}\, \exp\left\{\frac{i}{\hbar}\int_0^t ds \int_0^s ds'\, \frac{\sin\omega(s-t)\sin\omega s'}{m\omega\sin\omega t}\, j(s)j(s')\right\}. \tag{3.50}$$

It generates all correlations of time-ordered products of oscillator positions at different times.

## 3.3 Mode expansion

The path integral (3.32) factorizes into a factor containing the action of the classical trajectory $w_{\rm cl}$ with prescribed initial and final positions and a factor containing the path integral over the fluctuations $\xi$. The latter is independent of the endpoints since the fluctuations vanish for $t'$ and $t$ and for its computation we need the explicit form of the action

$$S[\xi]=\frac{1}{2}(\xi,S''\xi)\quad\text{with}\quad S''=-m\left(\frac{d^2}{ds^2}+\omega^2(s)\right). \tag{3.51}$$

The operator $S''$ is called *fluctuation operator* since it acts on the fluctuations about $w_{\rm cl}$. It is a self-adjoint operator on functions vanishing at times $t'$ and $t$. Hence we can diagonalize it

$$S''\xi_n=\lambda_n\xi_n,\quad\text{where}\quad \xi_n(t')=\xi_n(t)=0. \tag{3.52}$$

The eigenmodes may be chosen to be orthonormal

$$(\xi_n,\xi_m)\equiv\int_{t'}^{t}ds\,\xi_n(s)\xi_m(s)=\delta_{n,m}, \tag{3.53}$$

and an arbitrary fluctuation $\xi(s)$ can be expanded in terms of these modes,

$$\xi(s)=\sum_n a_n\xi_n(s). \tag{3.54}$$

Since the map $\xi(s)\longrightarrow\{a_n\}$ is a *unitary map* form $L_2$ to $\ell_2$ the 'measure' in $\mathcal{D}\xi$ is equal to the 'measure' $\prod da_n$. Inserting the expansion into the exponent in (3.32) we obtain

$$\int\limits_{\xi(t')=0}^{\xi(t)=0}\mathcal{D}\xi\;e^{i(\xi,S''\xi)/2\hbar}=\int\prod da_n\;e^{i\lambda_n a_n^2/2\hbar}=\prod\left(\frac{2\pi i\hbar}{\lambda_n}\right)^{1/2}. \tag{3.55}$$

The product of the eigenvalues $\lambda_n$ is the determinant of the fluctuation operator $S''$ and thus the path integral leads to an inverse square root of the determinant of $S''$,

$$K_\omega(t,q,t',q')=\frac{\mathcal{N}}{\sqrt{\det(\partial^2+\omega^2)}}\;e^{iS[w_{\rm cl}]/\hbar}. \tag{3.56}$$

For simplicity we assumed that the external source has been switched off. The divergent normalization factor $\mathcal{N}$ can be fixed a posteriori by considering the ratio of two path integrals. This is sufficient in quantum mechanics where the ratio of two fluctuation determinants is finite. It is not sufficient in field theory where an additional regularization may be necessary. Before considering the ratio of determinants we quote a classical result of WEYL [23], according to which the eigenvalues in (3.52) grow asymptotically as

$$|\lambda_n|\sim\text{const}\cdot\left(\frac{n}{t-t'}\right)^2, \tag{3.57}$$

implying that the determinant does not exist. This is not surprising since already in the regularized path integral on the time lattice (3.17) $\det C \sim 1/\epsilon$ also tends to infinity in the continuum limit. The problem with this harmless divergence is resolved as follows: imagine that we repeat the same steps leading to (3.56) for the free particle instead of the oscillator. We obtain

$$K_0(t,0,t',0) = \frac{\mathcal{N}}{\sqrt{\det(\partial^2)}}, \tag{3.58}$$

since the classical trajectory starting and ending at the origin is just $w_{\rm cl}(s) = 0$ and hence the action $S[w_{\rm cl}]$ in (3.56) vanishes in this case. On the other hand we know from (2.21) that

$$K_0(t,0,t',0) = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}. \tag{3.59}$$

Now we divide the evolution kernel in (3.56) by $K_0$ as in (3.58) and multiply again by $K_0$ as in (3.59). The unknown constant $\mathcal{N}$ chancels in the quotient and we obtain

$$K_\omega(t,q,t',q') = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}\left(\det\frac{\partial^2+\omega^2(.)}{\partial^2}\right)^{-1/2} e^{iS[w_{\rm cl}]/\hbar}. \tag{3.60}$$

According to (3.17) the ratios of the determinants are given by the ratios of the $D$-functions of the corresponding fluctuation operators. The $D$-function of $\partial^2$ is $D(s,t') = s - t'$, such that

$$K_\omega(t,q,t',q') = \sqrt{\frac{m}{2\pi i\hbar}}\frac{1}{\sqrt{D(t,t')}}\, e^{iS[w_{\rm cl}]/\hbar}. \tag{3.61}$$

Alternatively we could divide and multiply (3.56) with the evolution kernel $K_\omega$ of the oscillator with constant $\omega$, as given in (3.25). One finds

$$K_\omega(t,q,t',q') = \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega(t-t')}}\left(\det\frac{\partial^2+\omega^2(.)}{\partial^2+\omega^2}\right)^{-1/2} e^{iS[w_{\rm cl}]/\hbar}, \tag{3.62}$$

where $\omega$ and $\omega(.)$ are the constant and time-dependent frequencies. Inserting the $D$-function $1/\omega\cdot\sin\omega(t-t')$ of the oscillator with constant frequency again leads to the result (3.61).