

# MASTER THESIS



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## **Renormalization Group Studies of relativistic Luttinger Fermions**

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# 1 Introduction

In recent years studies of electronic band touching have received increasingly much attention. Due to special behavior arising from higher symmetry states occurring on these band junctions, investigations of materials exhibiting these structures intensified as they promise interesting and complex optical and electronic properties. Models with linear band touching or crossings, such as mono-layer graphene or certain d-wave superconductors, have since been investigated using relativistic Gross-Neveu type models and evince a quantum phase transition at finite interactions [1, 2, 3]. In those cases spontaneous symmetry breaking occurs at a critical coupling leading to the bands separating at finite interaction strengths. For quadratic band touchings (QBT) however, a different behavior emerges. Due to their higher momentum dependence, spontaneous symmetry breaking can already occur for infinitesimal couplings. This phenomenon leads to varying properties depending on the dimensionality of the system. For example in 2D bi-layer graphene, non vanishing densities remain in these quadratic band touchings causing the non-interacting ground state to be unstable towards repulsive interactions [4, 5, 6, 7]. On the other hand, those characteristics differ when examining 3D QBTs which have been observed in, for example, grey tin or mercury telluride [8, 9, 10]. State densities now vanish at the Fermi-point resulting in long range electron-electron interactions attaining a dominant role. In either case, investigations on QBTs raise more and more interest in fields like material science, optics and electronics. Therefore, a good description of fermions with a quadratic dispersion relation is of increasing concern. Many works have been published which present a Euclidean quadratic Hamiltonian in order to model these phenomena. Rooted in the works of J.M. Luttinger, the corresponding particles are referred to as *Luttinger fermions* [11].

Another topic that has gained increasing attention in recent years is the study of ultraviolet complete quantum field theories [12]. The standard model exhibits problematic behavior in that it is not necessarily well defined on all scales. When approaching a finite UV - limit couplings diverge, for example in the hypercharge  $U(1)$  sector, due to phenomena like Landau poles [13, 14, 15]. However, in order to avoid these complications we can only set the coupling to zero yielding a non-interacting theory. This is often referred to as the *triviality problem*. Studying whether such poles arise and are actually relevant or lie outside the physical parameter range, as it is the case in QED, has become a major focus. As couplings diverge, perturbative methods fail to describe the theory accurately, demanding non-perturbative approaches to be used. Functional renormalization group (FRG) methods have been at the forefront

in these investigations [16, 14, 17, 18]. Especially the study of fixed points in the theory space of couplings transpires to show promising results as these fixed points may attract the renormalization group flow in the UV-regime towards finite values, rendering theories well defined at all scales. Theories exhibiting these fixed points are called *asymptotically safe* whereas Gaussian fixed points mark a special case in which the theory is not only UV-finite but *asymptotically free*. Studying these phenomena provides insights into the applicability of quantum field theories on different scales. In light of both of these intriguing topics we want to investigate a theory which bridges the gap between those two areas by establishing a relativistic quantum theory of Luttinger fermions and exploring its dynamics using FRG methods which will lead us to an asymptotically free theory. In order to do this, this thesis is organized into three parts. We start by considering the necessary algebraic structure underlying a quadratic fermionic theory and construct a relativistic Luttinger action. We continue by giving an introduction to functional renormalization group techniques such as the Wetterich equation [18]. In chapter 4.1 we apply those methods to the derived action and interpret the resulting flow equations in special cases. Lastly, we examine the dynamics for a partially bosonized action and discuss spontaneous symmetry breaking as well as potential mass generation in chapter 4.2. For this we use standard quantum field theory conventions which are specified in Appendix D.

## 2 Relativistic Luttinger Fermions

### 2.1 Spin Coupling Matrix $G_{\mu\nu}$

To start off, we aim to find a non-trivial way of constructing a second order fermionic theory. In a fashion similar to the Dirac equation, we search for a kinetic operator the square of which is equivalent to the square of the d'Alembert operator:

$$\hat{K}^2 = \square^2, \quad (2.1)$$

where  $\hat{K}$  is the operator in question. To achieve this, we first construct the most general form of an operator quadratic in the derivative by introducing  $G_{\mu\nu}$ :

$$\hat{K} = -G_{\mu\nu} \partial^\mu \partial^\nu. \quad (2.2)$$

To derive some properties of  $G_{\mu\nu}$ , we follow the steps of [10] where this has been done for the Euclidean case. First, let us consider the operator  $\hat{K}$  in a momentum representation:

$$\hat{K} = G_{\mu\nu} p^\mu p^\nu. \quad (2.3)$$

Now we demand  $\hat{K}^2 = p^4 \mathbb{1}$ , so that only even powers of the components of  $p$  remain in  $\hat{K}^2$ . In order to fulfill this condition, the matrices  $G_{\mu\nu}$  have to satisfy the following anticommutation relations which, can be obtained directly (see Appendix A) by decomposing the Einstein summation of the indices in (2.3):

$$\{G_{\mu\mu}, G_{\kappa\lambda}\} = \{G_{\mu\nu}, G_{\kappa\lambda}\} = 0, \quad (2.4)$$

with  $\mu \neq \nu, \kappa \neq \lambda$  and  $(\mu, \nu) \neq (\kappa, \lambda)$  so that any off-diagonal element anticommutes with all others. Using these two identities, the squared kinetic operator reads<sup>1</sup>:

$$\hat{K}^2 = \sum_{\mu=1}^D G_{\mu\mu}^2 (p^\mu)^4 + \sum_{\mu < \nu} (4G_{\mu\nu}^2 + \{G_{\mu\mu}, G_{\nu\nu}\}) (p^\mu)^2 (p^\nu)^2. \quad (2.5)$$

By normalizing all  $G_{\mu\mu}$  to 1 our imposed condition of  $H^2 = p^4 \mathbb{1}$  is satisfied provided

$$4G_{\mu\nu}^2 + \{G_{\mu\mu}, G_{\nu\nu}\} = 2g_{\mu\mu} g_{\nu\nu} \quad (2.6)$$

for  $\mu \neq \nu$ . We note that we can decompose  $G_{\mu\nu}$  with respect to its Lorentz indices into

<sup>1</sup>We want to emphasize that in equation (2.5) Einstein summation convention is not used.

$$G_{\mu\nu} = G_{\mu\nu}^{\text{TL}} + \frac{g^{\mu\nu}}{D} G, \quad (2.7)$$

where  $G = g^{\mu\nu} G_{\mu\nu}$  and  $G_{\mu\nu}^{\text{TL}}$  is the traceless part. Since  $\hat{K}$  is supposed to be a kinetic operator for a field theory, the trace part will only contribute a Klein-Gordon type term in  $K$  proportional to  $G$  which can be added later on if desired. For now, we only want to focus on the irreducible derivative tensor structure  $\partial^\mu \partial^\nu - \frac{g^{\mu\nu}}{D} \partial^2$  by setting the trace of  $G$  to zero. From now on,  $G_{\mu\nu}$  will therefore correspond only to the trace-less part i.e.  $G_{\mu\nu} \equiv G_{\mu\nu}^{\text{TL}}$  in this work. Without a trace, another restriction on  $G_{\mu\nu}$  has been imposed, leading to yet another anti-commutation relation:

$$\begin{aligned} 0 &= \{G_{\mu\mu}, \sum_{\nu=1}^D g_{\nu\nu} G_{\nu\nu}\} = 2 + \{G_{\mu\mu}, \sum_{\mu \neq \nu} g_{\nu\nu} G_{\nu\nu}\} \\ \{G_{\mu\mu}, G_{\nu\nu}\} &= -\frac{2}{D-1} g_{\mu\mu} g_{\nu\nu} \quad \text{for } \mu \neq \nu. \end{aligned} \quad (2.8)$$

Together with (2.4) and a normalization, we have four anti-commutation relations which can be used to construct a general Clifford algebra. Because the tensor structure must respect Lorentz invariance and has to be symmetric, the algebra can only be spanned by the metric. Using products of  $g_{\mu\nu}$  for an ansatz, the general form reads:

$$\{G_{\mu\nu}, G_{\kappa\lambda}\} = -\frac{2}{D-1} g_{\mu\nu} g_{\kappa\lambda} + \frac{D}{D-1} (g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa}). \quad (2.9)$$

This algebra is fundamental for further investigations in this quadratic theory. Therefore, some more basic properties should be discussed before moving on to the construction of a quadratic fermionic action. Since  $G_{\mu\nu}$  is a  $D$ -dimensional, trace-less, symmetric matrix in terms of its Lorentz indices, it has  $\frac{1}{2}D(D-1)$  off-diagonal and  $(D-1)$  diagonal independent elements. Thus, in order to span the space of all  $G_{\mu\nu}$

$$d_e = \frac{1}{2}D^2 + \frac{1}{2}D - 1 \quad (2.10)$$

anticommuting elements are needed. These are available in a  $d_\gamma$ -dimensional Dirac-Clifford algebra. For convenience, we choose it to be Euclidean:

$$\{\gamma_m, \gamma_n\} = 2\delta_{mn}. \quad (2.11)$$

It is spanned by  $d_e$  different matrices  $\gamma_m$ . In order to keep the full structure of (2.9), the  $G_{\mu\nu}$  can be spanned by:



$$G_{\mu\nu} = a_{\mu\nu}^m \gamma_m, \quad (2.12)$$

where  $a_{\mu\nu}^m$  are symmetric second rank Lorentz tensors for each value of  $m \in (1, \dots, d_e)$ . Since  $G_{\mu\nu}$  is traceless with respect to the Lorentz indices, the  $a_{\mu\nu}^m$  must fulfill the same restriction. Inserting (2.12) into (2.9) and contracting with  $g^{\mu\kappa}$  and  $g^{\nu\lambda}$  yields:

$$a_m^{\mu\nu} a_{\mu\nu}^m = \frac{D(D+2)}{2}. \quad (2.13)$$

Furthermore, other restrictions on the  $a_{\mu\nu}^m$  can be taken from (2.9). Especially the normalization conditions of different  $G_{\mu\nu}$  lead to:

$$\begin{aligned} \sum_m (a_{\mu\mu}^m)^2 &= 1 \\ \sum_m (a_{ij}^m)^2 &= \frac{1}{2} \frac{D}{D-1} \\ \sum_m (a_{0i}^m)^2 &= -\frac{1}{2} \frac{D}{D-1}. \end{aligned} \quad (2.14)$$

The construction of an explicit representation of the  $G_{\mu\nu}$  in 3+1 dimensions is straight forward: Due to the large freedom in the parameters of all  $a_{\mu\nu}^m$ , a simple ansatz can be chosen. All the off-diagonal elements can be represented by  $(G_{01}, G_{02}, G_{03}) = \frac{\sqrt{2}}{3} i(\gamma_1, \gamma_2, \gamma_3)$  and  $(G_{12}, G_{13}, G_{23}) = \frac{\sqrt{2}}{3} (\gamma_4, \gamma_5, \gamma_6)$ . Since we need  $d_e = 9$  different anti-commuting matrices to span all  $G_{\mu\nu}$  in 4 dimensions, we have 3 elements left to construct the diagonal parts:

$$\begin{aligned} G_{00} &= \gamma_7 & G_{11} &= \frac{1}{3} \gamma_7 + \frac{\sqrt{8}}{3} \gamma_8 \\ G_{22} &= \frac{1}{3} \gamma_7 - \frac{\sqrt{2}}{3} \gamma_8 + \sqrt{\frac{2}{3}} \gamma_9 & G_{33} &= \frac{1}{3} \gamma_7 - \frac{\sqrt{2}}{3} \gamma_8 - \sqrt{\frac{2}{3}} \gamma_9. \end{aligned} \quad (2.15)$$

From this representation the  $a_{\mu\nu}^m$  are fixed and of the form:

$$\begin{aligned} a^1 &= \frac{\sqrt{2}}{3} i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & a^2 &= \frac{\sqrt{2}}{3} i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & a^3 &= \frac{\sqrt{2}}{3} i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ a^4 &= \frac{\sqrt{2}}{3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & a^5 &= \frac{\sqrt{2}}{3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} & a^6 &= \frac{\sqrt{2}}{3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ a^7 &= \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & a^8 &= \frac{\sqrt{2}}{3} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & a^9 &= \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned} \quad (2.16)$$

This is only one out of many possible choices for the Lorentz-factors. However, the  $\gamma$  matrices themselves can have different forms. In accordance with standard Dirac theory's spin base invariance, a similar symmetry is found in the Clifford algebra (2.11). Thus, even more different full representations of  $G_{\mu\nu}$  exist apart from those that emerge through the ambiguities of  $a_{\mu\nu}^m$ . More precisely, the Luttinger algebra (2.9) is invariant under  $\text{GL}(d_\gamma, \mathbb{C})$  with the subgroup  $\text{SL}(d_\gamma, \mathbb{C})$  acting non-trivially on the  $G_{\mu\nu}$ . This spin base symmetry will be used later on in chapter 2.3 to define a conjugate spinor hence it is important to point it out. Having constructed a representation for all  $G_{\mu\nu}$  in 4 dimensions, it is time to try and incorporate it into an action that can be used to study the behavior of quadratic fermions in more detail. But before constructing a relativistic Luttinger fermion theory with the derived  $G_{\mu\nu}$ , another matter has to be addressed carefully.

## 2.2 Dimensionality of $G_{\mu\nu}$

Due to the fact that we only consider the trace-less part of  $G_{\mu\nu}$ ,  $d_e$  is odd for  $D = 4$ . This leads to  $\gamma_m$  being at least 16x16 matrices. In general, this dimensionality can be computed straightforwardly. Depending on whether  $d_e$  is even or odd,  $d_\gamma$  is obtained directly for the irreducible representation[19]:

$$\text{even: } d_\gamma = 2^{\frac{d_e}{2}} \quad \text{odd: } d_\gamma = 2^{\frac{d_e-1}{2}}. \quad (2.17)$$

For even dimensions  $d_e$ , this can be compared to a Dirac algebra. Whereas in an odd number of dimensions  $d'_e$  the space of all  $G_{\mu\nu}$  can still be spanned by  $d_e = d'_e - 1$  independent matrices  $\gamma_m$  if we include  $\gamma_* \equiv \gamma_{d'_e} = \alpha \prod_{m=1}^{d_e} \gamma_m$  as a final element. The matrix  $\gamma_*$  is often called chiral element of the algebra and anticommutes with all its factors  $\gamma_m$ :

$$\begin{aligned}
\gamma_* \cdot \gamma_m &= \alpha \prod_{n=1}^{d_e} \gamma_n \cdot \gamma_m \\
&= \alpha \prod_{n=1}^{m-1} \gamma_n \cdot \gamma_m \cdot \prod_{n=m+1}^{d_e} \gamma_n \cdot \gamma_m \\
&\stackrel{(2.11)}{=} \alpha \prod_{n=1}^{m-1} \gamma_n \cdot \gamma_m \cdot \gamma_m \cdot \prod_{n=m+1}^{d_e} \gamma_n \cdot (-1)^{d_e-(m+1)} \\
&= \alpha (-1)^{m-1-1} \gamma_m \cdot \prod_{n=1}^{m-1} \gamma_n \cdot \gamma_m \cdot \prod_{n=m+1}^{d_e} \gamma_n \cdot (-1)^{d_e-(m+1)} \\
&= \alpha (-1)^{d_e-3} \gamma_m \cdot \gamma_* \\
&\stackrel{d_e \text{ even}}{\implies} \{\gamma_*, \gamma_m\} = 0.
\end{aligned} \tag{2.18}$$

To determine the prefactor  $\alpha$ , it suffices to demand  $\gamma_*^2 = \mathbb{1}$  so that it obeys equation (2.11). It turns out that  $\alpha = i$  is needed since the number of  $\gamma$ -matrices produces a negative sign in the squaring process. Moreover, it causes  $\gamma_*$  to be hermitian. Altogether,  $\gamma_*$  then reads:

$$\begin{aligned}
\gamma_* &= i\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8\gamma_9\gamma_{10} \\
\gamma_*^2 &= \mathbb{1} \quad \text{and} \quad \gamma_*^\dagger = \gamma_*
\end{aligned} \tag{2.19}$$

meaning that for even dimensions  $d_e$  an element in the Clifford algebra exists which anticommutes with the whole set of generators apart from the identity. In the context of Dirac theory, this leads to the emergence of chirality. In contrast, for odd dimensions  $d_e'$  the additional matrix  $\gamma_*$  does not exist since it is used up as yet another generator, and the product of all  $\gamma_m$  in that case just ends up to be the identity (up to a factor). Hence, in odd dimensions anticommutativity is completely exhausted within the generators. However, to construct a chiral theory we need an extra anticommuting element or else it cannot be defined [20]. This is often expressed in terms of the statement that chirality does not exist in odd dimensional spacetime. In spite of that, generalized definitions of chirality become possible using reducible representations. Closely connected to this problem are the existence of a parity operator as well as Kramer's theorem [21] which establishes a connection between spin and the squared time reversal operator. In particular, as shown in [10] for the Euclidean case, in four space dimensions the  $G_{\mu\nu}$  give rise to a spinless theory. Similar problems occur for a generalization to the Minkowski metric. All of these complications arise due to

the underlying Lorentz-symmetry and its spin representation which depends on the Clifford algebra structure in (2.9). To circumvent this and to construct a chiral, quadratic theory which describes fermions with half integer spin, an adjustment needs to be made. In particular, the root of the problem is the absence of a chiral element in odd dimensions  $d_e$ . Thus, we can just expand the dimensionality of  $d_e$  to the next even number to prevent the problem. This, however, comes at the price of doubling the number of spinor components for odd  $d_e$  but allows us to define a relativistic theory as we will see later on. Explicitly, this means that for  $D = 4$  we have  $d_e = 9$ , which we expand to  $d_e = 10$ . By doing so,  $d_\gamma$  is raised to 32 and the Clifford algebra (2.11) has 10 anticommuting generators. But since  $G_{\mu\nu}$  still only needs 9 independent parts to be spanned in four dimensions, nothing changes from chapter 2.1. The only difference is that  $\gamma_{10}$  and also  $\prod_{m=1}^{10} \gamma_m = \gamma_*$  exist and both of them anticommute with  $\gamma_{1\dots 9}$ . As a result, we are not only able to define a chiral theory but we even have two different options at our disposal.

## 2.3 Construction of a quadratic action

### 2.3.1 Spin metric $h$ and conjugate spinor

After establishing the basic aspects of a kinetic operator for relativistic Luttinger fermions, it is now time to incorporate said operator into an action. For the theory to be consistent, we want it to be Lorentz invariant, spin base invariant and of course unitary. Within the spin base invariance, there exists the subset of transformations that will correspond to chiral symmetry. We start by studying the kinetic part:

$$S_{\text{kin}} = \int d^D x \bar{\Psi} G_{\mu\nu} (-i\partial^\mu)(-i\partial^\nu) \Psi. \quad (2.20)$$

Firstly, we have to clarify how  $\bar{\Psi}$  is defined. Just like with regular Dirac fermions, we can use  $\bar{\Psi}$  as an abbreviation for  $\Psi^\dagger h$ , where  $h$  is often referred to as the spin metric [22, 23, 24, 25]. This spin metric has to satisfy certain conditions. For example,  $h$  should be hermitian because when a mass term is introduced it should be real:

$$\begin{aligned} \left(\frac{m}{2} \bar{\Psi} \Psi\right)^* &= \left(\frac{m}{2} \Psi^\dagger h \Psi\right)^* \\ &= \frac{m}{2} \Psi^\dagger h^\dagger \Psi \stackrel{!}{=} \frac{m}{2} \Psi^\dagger h \Psi = \frac{m}{2} \bar{\Psi} \Psi \\ &\rightarrow h = h^\dagger. \end{aligned} \quad (2.21)$$

Secondly, for the action to be real we can insert  $\bar{\Psi} = \Psi^\dagger h$  into (2.20) to see that  $h$  must satisfy:

$$\begin{aligned}
S_{\text{kin}} &\stackrel{!}{=} S_{\text{kin}}^* \\
&= \int d^D x (i\partial^\mu)(i\partial^\nu) \Psi^\dagger G_{\mu\nu}^\dagger \bar{\Psi}^\dagger \\
&= \int d^D x \Psi^\dagger h h^{-1} G_{\mu\nu}^\dagger (i\partial^\mu)(i\partial^\nu) h^\dagger \Psi \\
&= \int d^D x \bar{\Psi} h^{-1} G_{\mu\nu}^\dagger h^\dagger (-i\partial^\mu)(-i\partial^\nu) \Psi \\
&\rightarrow G_{\mu\nu} \stackrel{!}{=} h^{-1} G_{\mu\nu}^\dagger h^\dagger,
\end{aligned} \tag{2.22}$$

where in the third line we used integration by parts and inserted  $\mathbb{1} = h h^{-1}$ . With these constraints and the representation derived in chapter 2.1, the following relations must hold:

$$\begin{aligned}
G_{0i} &= h^{-1} G_{0i}^\dagger h^\dagger = -h^{-1} G_{0i} h^\dagger \\
\rightarrow h G_{0i} &= -G_{0i} h^\dagger = -G_{0i} h \\
&\rightarrow \{h, G_{0i}\} = 0,
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
G_{ij} &= h^{-1} G_{ij}^\dagger h^\dagger = h^{-1} G_{ij} h^\dagger \\
\rightarrow h G_{ij} &= G_{ij} h^\dagger = G_{ij} h \\
\rightarrow [h, G_{ij}] &= 0 \quad , \quad [h, G_{\mu\mu}] = 0.
\end{aligned} \tag{2.24}$$

Due to  $G_{\mu\nu}$  being traceless, we find that  $h$  has to obey 9 equations. In order to find a solution, let us examine the algebra in a more granular matter. Since we have 10 basis elements  $\gamma_i$  along with the identity matrix, we can create a set of generators  $\Xi$  for the algebra (2.11). It consists of the identity, the basis elements and all possible products of them, including  $\gamma_*$ . Any matrix  $S \in \text{GL}(32, \mathbb{C})$  can then be represented as a linear combination of certain elements in  $\Xi$ :

$$\begin{aligned}
\Xi &= \{\mathbb{1}, \gamma_1, \dots, \gamma_{10}, \gamma_1 \gamma_2, \dots, \gamma_9 \gamma_{10}, \gamma_1 \gamma_2 \gamma_3, \dots, \gamma_*\}. \\
S &= e^{\omega_A \Xi^A} \quad \text{with } \omega_A \in \mathbb{C} \text{ and } \Xi^A \in \Xi.
\end{aligned} \tag{2.25}$$

It may be of interest to examine the size of  $\Xi$ , in order to derive which integer values  $A$  can take. To construct  $\Xi$ , not all possible products of  $\gamma_i$  need to be considered. Since (2.9) holds, the order of the products does not matter up to possibly prefactors of  $(-1)$ . Moreover, products with repeated  $\gamma_i$  are also unnecessary, as they can be broken down to some other element  $\Xi^A$  by switching the order according to (2.9) and contracting said index  $i$ . Hence, it suffices to consider only products where every index  $\alpha_i$  is unique and no order is relevant. Due to these simplifications, calculating the size of  $\Xi$  boils down to a combinatorics problem. When looking at the number of possible products of length  $k$ , the answer is just given by  $\binom{n}{k}$ . Thus, the total number of products in  $\Xi$  equals:

$$\#\Xi = \sum_{k=0}^{10} \binom{10}{k} = \sum_{k=0}^{10} \binom{10}{k} 1^{10-k} 1^k = (1+1)^{10} = 2^{10} = 1024. \quad (2.26)$$

The case  $k = 0$  denotes a product of zero  $\gamma$ -matrices, i.e., the identity. This result goes to show that we need 1024 generators to span  $\text{GL}(32, \mathbb{C})$ , which is in one to one correspondence to the number of complex elements within a  $32 \times 32$  matrix. But with the set of generators in mind, the space of solutions to (2.23) and (2.24) must be a subspace of  $\Xi$ . In order to determine this subspace, we can start by assuming that any  $h$  fulfilling (2.23) and (2.24) can be represented by a basis element of  $\Xi$ , meaning it is of the form:

$$h = \prod_{j=1}^n \gamma_{\alpha_j}, \quad (2.27)$$

where  $\alpha_i \in (1, \dots, 10)$  are  $n$  different indices, so that  $h$  is an element of  $\Xi$  up to possibly a sign. We proceed by considering the equations one by one and ask whether  $h$  contains the corresponding basis element. Beginning by (2.23), we get two cases: Either  $h$  contains  $\gamma_i$  where  $i \in (1, 2, 3)$  or it does not. The first case leads to:

$$\begin{aligned}
\left\{ \prod_{j=1}^n \gamma_{\alpha_j}, G_{0i} \right\} &= \frac{\sqrt{2}}{3} i \left\{ \prod_{j=1}^n \gamma_{\alpha_j}, \gamma_i \right\} \\
&\sim \prod_{j=1}^n \gamma_{\alpha_j} \cdot \gamma_i + \gamma_i \prod_{j=1}^n \gamma_{\alpha_j} \\
&= \prod_{j=1}^{i-1} \gamma_{\alpha_j} \cdot \gamma_{\alpha_i} \cdot \prod_{j=i+1}^n \gamma_{\alpha_j} \cdot \gamma_i + \gamma_i \prod_{j=1}^n \gamma_{\alpha_j} \\
\gamma_{\alpha_i} = \gamma_i &= (-1)^{n-1} \gamma_i \prod_{j=1}^{i-1} \gamma_{\alpha_j} \cdot \gamma_{\alpha_i} \cdot \prod_{j=i+1}^n \gamma_{\alpha_j} + \gamma_i \prod_{j=1}^n \gamma_{\alpha_j} \\
&= (-1)^{n-1} \gamma_i \prod_{j=1}^n \gamma_{\alpha_j} + \gamma_i \prod_{j=1}^n \gamma_{\alpha_j} \stackrel{!}{=} 0 \\
&\rightarrow n \text{ is even.}
\end{aligned} \tag{2.28}$$

Analogously, this can be done for the second case where it then leads to  $n$  being odd. Continuing with the first case, now knowing that an even number of  $\gamma$  matrices is needed, we shall have a look at the other 6 equations from (2.24). Since they all share the same structure, it suffices to consider  $[h, G_{ij}] = 0$ :

$$\begin{aligned}
\left[ \prod_{j=1}^n \gamma_{\alpha_j}, G_{ij} \right] &= \frac{\sqrt{2}}{3} \left[ \prod_{j=1}^n \gamma_{\alpha_j}, \gamma_k \right] \\
&\sim \prod_{j=1}^n \gamma_{\alpha_j} \cdot \gamma_k - \gamma_k \prod_{j=1}^n \gamma_{\alpha_j} \\
\text{Appendix A} &= \left( 1 - 2 \sum_{i=1}^n \delta_{\alpha_i, k} \right) (-1)^n \gamma_k \prod_{j=1}^n \gamma_{\alpha_j} - \gamma_k \prod_{j=1}^n \gamma_{\alpha_j} \\
n \text{ is even} &= 2 \gamma_k \prod_{j=1}^n \gamma_{\alpha_j} \sum_{i=1}^n \delta_{\alpha_i, k} \stackrel{!}{=} 0 \\
&\rightarrow \delta_{\alpha_i, k} = 0 \quad \forall i \in (1, \dots, n).
\end{aligned} \tag{2.29}$$

Therefore, none of the matrices that comprise  $h$  are found in  $G_{ij}$  or, due to a similar argument, in  $G_{\mu\mu}$ . With representation (2.15),  $h$  is then restricted to  $\gamma_i$ , where  $i \in (1, 2, 3)$  and  $\gamma_{10}$ . For  $h$  to fulfill all 3 equations from (2.23), it must contain  $\gamma_1, \gamma_2$  and  $\gamma_3$ . But this product does not consist of an even number of  $\gamma$  matrices. Thus,  $\gamma_{10}$  has to be included leading to a first solution of  $h$ :

$$h = \gamma_1 \gamma_2 \gamma_3 \gamma_{10}. \tag{2.30}$$

But similarly, the second case, where one starts with no  $\gamma_i$  for  $i \in (1,2,3)$  in  $h$ , leads to a solution that is, in a sense, "conjugate" to the first one. Following the same calculations, we arrive at:

$$\tilde{h} = i\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8\gamma_9\gamma_{10}. \quad (2.31)$$

The prefactor of  $i$  is needed to ensure that  $h$  is hermitian. Now we are left with two independent options which span a space of solutions. Due to the linearity of anticommutators and commutators, equations (2.23) and (2.24) allow any linear combination of  $h$  and  $\tilde{h}$ . Not only this, but we ignored the fact that  $\gamma_*$  could also be used as a forth part in  $h$  or a seventh part in  $\tilde{h}$ :

$$h' = \gamma_1\gamma_2\gamma_3\gamma_* \quad \text{and} \quad \tilde{h}' = i\gamma_4\gamma_5\gamma_6\gamma_7\gamma_8\gamma_9\gamma_*. \quad (2.32)$$

But if  $\gamma_*$  is expanded, both alternative solutions relate to the former ones like:

$$h' = -\tilde{h} \quad \text{and} \quad \tilde{h}' = h, \quad (2.33)$$

meaning the space of solutions is spanned only by  $h$  and  $\tilde{h}$ . Although technically any pre-factors are allowed, only real ones keep a composite solution  $H$  hermitian. Furthermore, demanding  $H^2 = 1$  restricts the solutions even more:

$$\begin{aligned} H &= \alpha h + \beta \tilde{h} \quad \alpha, \beta \in \mathbb{R} \\ H^2 &= \alpha^2 \underbrace{h^2}_{=1} + \beta^2 \underbrace{\tilde{h}^2}_{=1} + \alpha\beta \underbrace{\{h, \tilde{h}\}}_{=0} \\ &= \alpha^2 + \beta^2 \stackrel{!}{=} 1. \end{aligned} \quad (2.34)$$

Still, there are infinitely many  $H$  in the span of  $h$  and  $\tilde{h}$  fulfilling all 9 conditions (2.23), (2.24). This ambiguity is introduced by the necessary increase of  $d_e$ . It hints at the theory not being irreducible, but diving deeper into this topic goes beyond the scope of the thesis. Whereas we do have infinite options for the spin metric, from now on we will only be using arguably the simplest representation, namely  $h$ , for convenience. To finish the studies on the spin metric, we want to address one more interesting feature. Let us start by attempting to find a parity operator. As usual, we can begin by using the definition of the parity operator and try to find a matrix representation in spinor space:



$$\begin{aligned}
\mathcal{P} &= \text{diag}(1, -1, -1, -1) = \Lambda^\mu_\nu \\
G_{\kappa\lambda} \Lambda^\kappa_\mu \Lambda^\lambda_\nu &= S \cdot G_{\mu\nu} \cdot S^{-1} \\
a^m_{\kappa\lambda} \Lambda^\kappa_\mu \Lambda^\lambda_\nu \gamma_m &= a^n_{\mu\nu} S \cdot \gamma_n \cdot S^{-1} \\
\text{with } a^m \text{ from (2.16): } \Lambda^T a^m \Lambda &= -a^m \quad m \in (1, 2, 3) \\
\Lambda^T a^m \Lambda &= a^m \quad m \in (4, \dots, 10) \\
\rightarrow \{S, G_{0i}\} &= 0 \tag{2.35} \\
\rightarrow [S, G_{ij}] = 0 \text{ and } [S, G_{\mu\mu}] &= 0. \tag{2.36}
\end{aligned}$$

It becomes apparent that the spinor representation  $S$  of the parity operator has to fulfill the exact requirements that apply to  $h$  ((2.23), (2.24)) meaning  $\mathcal{P}$  is also part of the span of  $h$  and  $\tilde{h}$ . Therefore, also  $\mathcal{P}$  technically has an infinite number of representations as long as its usual properties (hermitian, unitary, involutory) are kept [26]. Thus, the parity operator is of the form presented in (2.34).

$$\mathcal{P} = \alpha h + \beta \tilde{h} \quad \text{with } (\alpha, \beta) \in \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}. \tag{2.37}$$

To keep the formalism simple, we propose to just use  $\alpha = 1$  and  $\beta = 0$  i.e. setting  $\mathcal{P} = h$ . Although, we should not forget that any other representation is a parity operator as well. Since  $h$  and  $\mathcal{P}$  are part of the same subspace of the algebra, we can conclude that  $h$  has a nice physical interpretation. It changes the parity of the spinor it acts on.

### 2.3.2 Luttinger action, symmetries and interaction terms

After establishing the conjugate spinor by introducing the spin metric  $h$  which coincidentally also is a parity operator, we are now ready to introduce the final kinetic action:

$$S_{\text{kin}} = - \int d^D x \, \bar{\Psi} G_{\mu\nu} \partial^\mu \partial^\nu \Psi. \tag{2.38}$$

By construction this action is real. But with the new interpretation of  $h$  as a parity operator, it can be shown quite easily as well by considering:

$$\bar{\Psi}(t, \vec{x}) = \Psi^\dagger(t, \vec{x}) \cdot h = \mathcal{P}(\Psi^\dagger(t, \vec{x})) = \Psi^\dagger(t, -\vec{x}). \tag{2.39}$$

Inserting this identity into (2.38) will help simplify the calculation of  $S_{\text{kin}}^*$  to show that

$S_{\text{kin}}$  is in fact a real quantity:

$$\begin{aligned}
S_{\text{kin}}^* &= - \int d^D x \partial^\mu \partial^\nu \Psi^\dagger(t, \vec{x}) G_{\mu\nu}^\dagger \Psi(t, -\vec{x}) \\
\text{i.b.p.} &= - \int d^D x \Psi^\dagger(t, \vec{x}) G_{\mu\nu}^\dagger \partial^\mu \partial^\nu \Psi(t, -\vec{x}) \\
&= - \int d^D x \Psi^\dagger(t, \vec{x}) \left[ G_{00}^\dagger \partial^0 \partial^0 + 2G_{0i}^\dagger \partial^0 \partial^i + G_{ij}^\dagger \partial^i \partial^j \right] \Psi(t, -\vec{x}) \\
(2.15), (2.16) &= - \int d^D x \Psi^\dagger(t, \vec{x}) \left[ G_{00} \partial^0 \partial^0 - 2G_{0i} \partial^0 \partial^i + G_{ij} \partial^i \partial^j \right] \Psi(t, -\vec{x}) \quad (2.40) \\
\text{now let } \vec{x} \mapsto -\vec{x} \text{ i.e. } \int d^D x &\mapsto \int d^D x, \partial^0 \partial^i \mapsto -\partial^0 \partial^i \\
&= - \int d^D x \Psi^\dagger(t, -\vec{x}) \left[ G_{00} \partial^0 \partial^0 + 2G_{0i} \partial^0 \partial^i + G_{ij} \partial^i \partial^j \right] \Psi(t, \vec{x}) \\
(2.39) &= - \int d^D x \bar{\Psi} G_{\mu\nu} \partial^\mu \partial^\nu \Psi(t, \vec{x}) \\
&= S_{\text{kin}}.
\end{aligned}$$

This action is not only real and obviously Lorentz invariant but also shares a very large symmetry with the standard Dirac action, namely a spin base symmetry, which historically dates back to Schrödinger and Bargmann [27, 22, 23]. In the year 2001, it was picked up by Weldon to describe fermions in curved spacetime without the use of vierbeins [24]. In our case, it is part of a larger symmetry of type  $\text{GL}(32, \mathbb{C})$ . Meaning any complex  $32 \times 32$  matrix is valid for the following transformation:

$$\Psi \rightarrow \Psi' = S\Psi \quad S \in \text{GL}(32, \mathbb{C}). \quad (2.41)$$

Additionally, the adjoint spinor and  $G_{\mu\nu}$  transform analogously to the Dirac case in in [24, 25] since  $\bar{\Psi}$  is also defined by a spin metric  $h$  and equation (2.12) illustrates the similarity to the Dirac matrices. The only difference is that the whole space of solutions for  $h$  transforms as well according to  $h \rightarrow (S^\dagger)^{-1} h S^{-1}$ . Nonetheless, the mapping reads:

$$\bar{\Psi} \rightarrow \bar{\Psi}' = \bar{\Psi} S^{-1} \quad \text{and} \quad G_{\mu\nu} \rightarrow G'_{\mu\nu} = S G_{\mu\nu} S^{-1}. \quad (2.42)$$

Now it can be shown quite easily that under this transformation the algebra (2.9) and the action remain invariant:

$$\begin{aligned}
\{G'_{\mu\nu}, G'_{\kappa\lambda}\} &= SG_{\mu\nu}S^{-1}SG_{\kappa\lambda}S^{-1} + SG_{\kappa\lambda}S^{-1}SG_{\mu\nu}S^{-1} \\
&= SG_{\mu\nu}G_{\kappa\lambda}S^{-1} + SG_{\kappa\lambda}G_{\mu\nu}S^{-1} \\
&= S\{G_{\mu\nu}, G_{\kappa\lambda}\}S^{-1} \\
&= S\left(-\frac{2}{D-1}g_{\mu\nu}g_{\kappa\lambda} + \frac{D}{D-1}(g_{\mu\kappa}g_{\nu\lambda} + g_{\mu\lambda}g_{\nu\kappa})\right)\mathbb{1}S^{-1} \\
&= \left(-\frac{2}{D-1}g_{\mu\nu}g_{\kappa\lambda} + \frac{D}{D-1}(g_{\mu\kappa}g_{\nu\lambda} + g_{\mu\lambda}g_{\nu\kappa})\right)SS^{-1} \\
&= \{G_{\mu\nu}, G_{\kappa\lambda}\}
\end{aligned} \tag{2.43}$$

and

$$\begin{aligned}
S' &= -\int d^Dx \bar{\Psi}S^{-1}SG_{\mu\nu}S^{-1}\partial^\mu\partial^\nu S\Psi \\
&= -\int d^Dx \bar{\Psi}G_{\mu\nu}\partial^\mu\partial^\nu\Psi \\
&= S.
\end{aligned} \tag{2.44}$$

With this spin base symmetry in place, we start to examine it in order to recognize certain parts, in particular a chiral transformation. Since  $GL(32, \mathbb{C})$  is isomorphic to  $SL(32, \mathbb{C}) \times L^2$ , it decomposes into the spin base transformations  $SL(32, \mathbb{C})$ , a  $U(1)$  symmetry and a dilatation symmetry represented by multiplication with a real factor  $\lambda \in \mathbb{R}^+$ . It is important to note that  $SL(32, \mathbb{C})$  covers the similarity transformations of the Clifford Algebra (2.9) completely [25]. Hence, the representation of a chiral transformation must also be a part of this group. To show this more explicitly, it is helpful to look at the generation of any element in  $GL(32, \mathbb{C})$  as presented in (2.25):

$$S = e^{\omega_A \Xi^A}. \tag{2.45}$$

By taking the determinant on both sides, it becomes apparent that  $\mathbb{1}$  generates the residual symmetry  $L$  while all other  $\Xi^A$  are responsible for the similarity transformations from  $SL(32, \mathbb{C})$ :

$$\begin{aligned}
\det(S) &= \det\left(e^{\omega_A \Xi^A}\right) = e^{\omega_A \text{Tr}(\Xi^A)} \\
&= e^{\omega_1 \text{Tr}(\mathbb{1}) + \omega_{A'} \text{Tr}(\Xi^{A'})} = e^{32(a+ib)} e^{\omega_{A'} \text{Tr}(\Xi^{A'})}
\end{aligned} \tag{2.46}$$

$$\text{Appendix (A)} = \lambda e^{i\varphi},$$

---

<sup>2</sup> $L$  is a residual symmetry composed of arbitrary complex numbers.

where  $A'$  denotes the residual summation from 2 to 1024 and  $\lambda = e^{32a}$  as well as  $32b = \varphi$ . We have used that  $\text{Tr}(\Xi^{A'}) = 0$  which is proven in Appendix A. Therefore,  $\mathbb{1}$  is the generator responsible for changing the determinant of  $S$  from 1 to be any complex number. Hence, removing  $\mathbb{1}$  from the generators leaves us with only the symmetry group of  $\text{SL}(32, \mathbb{C})$ . In analogy to the Dirac case, chiral symmetry can be defined using an element of the  $\text{SL}(32, \mathbb{C})$  algebra which anticommutes with all basis elements spanning  $G_{\mu\nu}$ . But due to the reducibility of our theory, we have a choice to make: either we use  $\gamma_{10}$  or  $\gamma_*$ . Both matrices should work because  $G_{\mu\nu}$  does not include either one. But the choice for the spin metric  $h$  induces a choice for chirality. We want the chiral transformation to read:

$$\Psi \rightarrow \Psi' = S\Psi = e^{i\alpha\gamma_{chiral}}, \quad \bar{\Psi} \rightarrow \bar{\Psi}' = \bar{\Psi}e^{i\alpha\gamma_{chiral}}, \quad (2.47)$$

where  $\gamma_{chiral}$  refers to either  $\gamma_{10}$  or  $\gamma_*$  and  $\alpha$  is a real number. It becomes apparent that the chiral symmetry is not a part of the spin base symmetry  $\text{SL}(32, \mathbb{C})$  as (2.47) does not correspond to (2.42). In order to work, chiral symmetry acts only on the fields  $\bar{\Psi}$  and  $\Psi$  and is generated by an element of the  $\text{SL}(32, \mathbb{C})$  algebra. For (2.47) to hold,  $h$  must anticommute with  $\gamma_{chiral}$ :

$$\{\gamma_{chiral}, h\} = 0. \quad (2.48)$$

Due to our choice of  $h = \gamma_1\gamma_2\gamma_3\gamma_{10}$ , only  $\gamma_{10}$  fulfills (2.48). Thus, defining our chiral transformation as:

$$\Psi \rightarrow \Psi' = S\Psi = e^{i\alpha\gamma_{10}}, \quad \bar{\Psi} \rightarrow \bar{\Psi}' = \bar{\Psi}e^{i\alpha\gamma_{10}}. \quad (2.49)$$

Had we chosen  $\tilde{h}$ , chirality would have been established using  $\gamma_*$ . Similarly, for any composite  $H$  as the spin metric the chiral element would have been a linear combination of  $\gamma_{10}$  and  $\gamma_*$ . Either way, chirality is defined and the kinetic action remains invariant under chiral transformations:

$$\begin{aligned} S' &= - \int d^Dx \bar{\Psi}' G_{\mu\nu} \partial^\mu \partial^\nu \Psi' \\ &= - \int d^Dx \bar{\Psi} e^{i\alpha\gamma_{10}} G_{\mu\nu} \partial^\mu \partial^\nu e^{i\alpha\gamma_{10}} \Psi \\ &= - \int d^Dx \bar{\Psi} G_{\mu\nu} e^{-i\alpha\gamma_{10}} e^{i\alpha\gamma_{10}} \partial^\mu \partial^\nu \Psi \\ &= S. \end{aligned} \quad (2.50)$$

By contrast, a mass term of the form  $\frac{m^2}{2}\bar{\Psi}\Psi$  is not chirally symmetric. Studying the behavior of this symmetry breaking in more detail reveals that the discrete chiral transformation  $\Psi \rightarrow \Psi\gamma_{10}$  leaves the mass term almost symmetric up to a factor of  $-1$ . But this motivates the introduction of an interaction term of a form  $\sim (\bar{\Psi}\Psi)^2$  since it remains invariant under the said transformation. This symmetry is analogous to the ones found in Gross-Neveu models in which spontaneous chiral symmetry breaking occurs due to fermionic self interactions which induce a finite vacuum expectation value  $\langle\bar{\Psi}\Psi\rangle$  leading to the generation of a fermion mass [28]. Closely related to this are studies concerning the generation of massless bosonic modes which align with field excitations along the remaining unbroken symmetries, so called Goldstone bosons [29, 30]. But our studies will mainly be focused on the mass generation. This mechanism is of interest to us since it gives further insight into the behavior of relativistic Luttinger fermions. Hence, our focus from now on shall be on an action of the form:

$$S = \int d^Dx \left[ -Z_\Psi \bar{\Psi} G_{\mu\nu} \partial^\mu \partial^\nu \Psi + \frac{\lambda_0}{2} (\bar{\Psi}\Psi)^2 \right], \quad (2.51)$$

where we introduced  $Z_\Psi$  as the fermionic wave function renormalization also called field strength renormalization ([31]) and  $\lambda_0$  as a coupling constant. Further terms not violating the invariance under  $\Psi \rightarrow \Psi\gamma_{10}$  can also be added. Especially, another term, quartic in fermions, is of interest:  $(\bar{\Psi} G_{\mu\nu} \Psi)^2$ . This interaction will become very important later on as it is dynamically generated by the  $\lambda_0$  channel which we will see in chapter 4. But for now it suffices to simply add it to (2.51):

$$S = \int d^Dx \left[ -Z_\Psi \bar{\Psi}_a G_{\mu\nu} \partial^\mu \partial^\nu \Psi^a + \frac{\lambda_0}{2} (\bar{\Psi}_a \Psi^a)^2 + \frac{\lambda_t}{2} (\bar{\Psi}_a G_{\mu\nu} \Psi^a)^2 \right]. \quad (2.52)$$

Moreover, in (2.52) another generalization is established. The index  $a$  represents different flavours of the Luttinger fermions and runs from one to  $N_f$ , the total number of flavours. This expands the model to multiple flavours and will allow us to introduce a mean field approximation in chapter 4. On the other hand  $\lambda_t$  is introduced as a coupling for the tensor channel. It turns out that this channel is actually symmetric under the total chiral transformation in (2.49). In view of a generalized Gross Neveu model [28, 32], there might also be other interaction terms, for example  $(\bar{\Psi}\gamma_{10}G_{\mu\nu}\Psi)^2$ , that could be used to dynamically generate our two channels, or  $(\bar{\Psi}\gamma_{10}\Psi)^2$ , which together with  $(\bar{\Psi}\Psi)^2$  would result in a theory fully symmetric under (2.49). But this goes beyond the scope of this thesis and shall be of interest to further studies on relativistic Luttinger fermions.

## 3 Functional Renormalization Group - Concept

### 3.1 Introduction to functional RG

Let us now introduce the concept of the functional renormalization group. This method aims to combine functional approaches to quantum theories with techniques of the renormalization group (RG).

In the former, the physical information of the system can be extracted from a generating functional. The approach has similarities to statistical physics and has since been widely used in modern quantum theories as it provides a way to extract any  $n$ -point function by differentiation. Hence, once the generating functional is obtained, all relevant physical quantities can be derived arguably easier compared to other computational methods. The crux lies in calculating the generating functional in the first place. But still, it remains a useful method for analytic computations. For a more detailed introduction to this area, it suffices to look into any standard literature in quantum field theory like [31] or others.

The RG aims to bridge the gap between microscopic quantum theories and their behavior at large scales. Renormalization is often viewed as a standard tool to deal with arising infinities in all kinds of quantum theories. Often debated to be just a mathematical means in order to remove diverging behavior of perturbative calculations, it has since become a much more physically relevant topic over the years. Especially, the meaning of a cutoff  $\Lambda$  has evolved from a mathematical necessity to a self-evident idea. As physics changes across different scales, we do not know the behavior of a theory beyond certain energies. Emergent properties like temperature arise only at large scales as a macroscopic phenomenon whereas quantum effects such as tunneling occur (in a significant manner) exclusively in the smallest of regimes. Hence, establishing a cutoff  $\Lambda$  for quantum theories seems only natural as we cannot just assume the quantum interactions to play the same role on every scale. Furthermore, even below  $\Lambda$  a change of dynamics is to be expected. Firstly proposed by Kadanoff to describe the Ising model near a critical temperature [33], the idea to group nearby variables to "cells" which are microscopically large in order to just use the collective variable of each cell already paints a picture of changing scales. Further developed by Wilson [16, 14], Wegner and Houghton [34] and later on Polchinsky [17] as well as Wetterich [18], the notion of scale-dependent Lagrangians consolidates its physical pertinence. This scale-dependence is realized through parameters in the theory, i.e. coupling "constants". By renormalizing these couplings, they can be adjusted to represent the physics of a model correctly on a given scale. We are then able to go

one step further and imagine the set of all renormalizations which change the theory to fit a certain scale - the renormalization group. This set of transformations can then be used to extrapolate the theory from a given scale to any other one. Moreover, studies on phase transitions have become a central aspect of the renormalization group as it can be used to find fix points in the flow of couplings which can relate to characteristic phase transitions of a system.

Amalgamating both approaches yields a method for obtaining an understanding of a theory's macroscopic dynamics purely in terms of the description of its microscopic interactions or vice versa. More specifically, we can take the average action as a generating functional describing the macroscopic expectation values of the fields and examine it in the presence of microscopic sources. We can then relate this action to a "bare action" corresponding to the classical theory at a large scale cutoff  $\Lambda$  and the full quantum action involving the sources at a limit where the scale approaches zero,  $k \rightarrow 0$ , by introducing a regulator function that allows us to integrate out the fluctuations momentum shell by momentum shell. Then the RG flow equations can be determined to derive a flow function  $\beta$  for the couplings which governs their evolution through scales. At last, we can use these flow equations to examine the behavior of the theory for  $k \rightarrow 0$  and study the full quantum action. This approach has been used widely in the literature to analyze different quantum theories and even theories of quantum gravity [14, 10, 35, 36, 37, 38, 39]. The great advantage of this method compared to perturbative calculations lies in its applicability to the case of strong interactions. We will therefore use it to analyze the relativistic Luttinger action and its self interaction in a more detailed manner. For this, we closely follow the construction of the Wetterich equation which in turn can be used to compute  $\beta$  functions [18, 40, 35]. As a starting point, let us consider a UV-regularized generating functional  $Z[J]$  where  $\Lambda$  denotes that we only integrate over fields for which  $p < \Lambda$ :

$$Z[J] = \int_{\Lambda} \mathcal{D}\Psi(p) e^{iS[\Psi] + iJ^T \cdot \Psi}. \quad (3.1)$$

Here,  $p$  denotes a dependency on all  $p_i$  where  $i \in (1, \dots, D)$ .  $\Psi$  and  $J$  should be understood as vectors in field space just like it is done in [35]:

$$\Psi(p) = \begin{pmatrix} \Psi(p) \\ \bar{\Psi}^T(-p) \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} \bar{J}^T \\ J \end{pmatrix}. \quad (3.2)$$

Furthermore, the scalar product in (3.1) is used as a short hand notation to write  $\int d^Dx \bar{J}\Psi + J^T\bar{\Psi}^T$ . At this point, we want to address a common problem: The complex-valuedness of the generating functional. As the imaginary phase leaves  $Z[J]$  not well

defined, let us instead have a look at a Euclidean version. In order to do this, we need to employ a Wick rotation  $t \rightarrow i\tau$  rendering the exponential real. A careful approach has to be taken since the algebra also changes under this transformation, which is calculated in Appendix B. The Euclidean generating functional then reads:

$$Z[J] = \int_{\Lambda} \mathcal{D}\Psi(p) e^{-S[\Psi] + J^T \cdot \Psi}. \quad (3.3)$$

### 3.2 IR regularization

From here onward, we want to introduce a mechanism which ensures that our theory is also infrared finite. This can be achieved by inserting an additional term  $\Delta S_k$ :

$$\Delta S_k[\Psi] = \frac{1}{2} \int \frac{d^D p}{(2\pi)^2} \Psi_i^T(-p) R_k^{ij}(p) \Psi_j(p) = \frac{1}{2} \Psi^T \cdot R_k \cdot \Psi, \quad (3.4)$$

where we sum over  $i, j$  which both run from 1 to 2. In the last step we used our previously defined scalar product together with  $\Psi^T \equiv \Psi^T(-p)$  to shorten the notation. The term  $R_k$  can be viewed as an IR regulator function suppressing modes with momenta  $|p| < k$ , hence it should be of a similar form as the kinetic term in our action. Furthermore,  $R_k$  is matrix valued in field space which permits the following representation:

$$R_k = \begin{pmatrix} 0 & -G_{\mu\nu}^T p^\mu p^\nu \\ G_{\mu\nu} p^\mu p^\nu & 0 \end{pmatrix} Z_\Psi r_\Psi, \quad (3.5)$$

where  $Z_\Psi$  is again the wave function renormalization and  $r_\Psi$  represents a dimensionless regulator shape function [35]. We can now insert (3.3) into (3.1) in order to arrive at a generating functional  $Z_k[J]$  that is UV and IR regularized:

$$Z_k[J] = \int_{\Lambda} \mathcal{D}\Psi(p) e^{-S[\Psi] - \Delta S_k[\Psi] + J^T \cdot \Psi}. \quad (3.6)$$

Now we want to have a closer look at  $R_k$  and see if we can find some sensible constraints. Firstly, it should obey:

$$\lim_{\substack{p^2 \rightarrow 0 \\ k^2 \rightarrow 0}} R_k > 0. \quad (3.7)$$

This implements the IR regularization. Additionally, demanding that we can retrieve  $Z[J]$  in the IR-limit  $Z_{k \rightarrow 0}[J] = Z[J]$  as well as sending  $R_k$  to infinity for  $k \rightarrow \Lambda$  will provide a way to interpolate between the bare action without any sources and the full quantum action as we will see later. Therefore, two more conditions can be posed:



$$\lim_{\frac{k^2}{p^2} \rightarrow 0} R_k = 0 \quad \text{and} \quad \lim_{\substack{k \rightarrow \Lambda \\ \Lambda \rightarrow \infty}} R_k \rightarrow \infty. \quad (3.8)$$

Since  $R_k$  can be represented like (3.5), these three conditions translate directly to  $r_\Psi$ . For simplicity we will use a regulator function of Litim form [41]:

$$r_\Psi = \left( \sqrt{\frac{k^2}{p^2}} - 1 \right) \theta(k^2 - p^2). \quad (3.9)$$

With this  $R_k$  in place, we can now immediately see that  $Z_k[J]$  approaches the full quantum action  $Z[J]$  for  $k \rightarrow 0$ . Moreover  $Z_{k \rightarrow \Lambda}[J]$  leads to the exponential being heavily suppressed. This justifies a saddle point approximation which results in leading order to just the bare action being quantized:  $Z_{k \rightarrow \Lambda}[J] \simeq e^{-S[\Psi_{\min}]}$ . For now, this regulator shape function will suffice as it satisfies all necessary conditions. In future studies potentially better choices may be found such that the theory converges more rapidly towards the full quantum action. We are now one step closer to obtaining the coarse grained effective action  $\Gamma_k$  which interpolates our theory from a bare action to the full quantum action. But beforehand, we need a few more mathematical ingredients.

### 3.3 The effective action $\Gamma$

Now let us introduce the generating functional of connected correlators [31, 42]:

$$W[J] = \ln(Z[J]). \quad (3.10)$$

In simple terms,  $W$  represents a more efficient way of storing the physical information in our system. With this at hand we can define a new action  $\Gamma$ :

$$\Gamma[\Phi] = \sup_J (J^T \cdot \Phi - W[J]). \quad (3.11)$$

Here,  $\Phi$  is a vector in field space and is related to the expectation value analogue of (3.2) as we will see now. If we apply a functional derivative of  $J$  at  $J = J_{\text{sup}}$  to equation (3.11) it yields:

$$\begin{aligned} \frac{\overrightarrow{\delta}}{\delta J_i^T(y)} \Gamma[\Phi] &= \frac{\overrightarrow{\delta}}{\delta J_i^T(y)} \left( \int d^D x J_a^T(x) \Phi^a(x) - W[J] \right) \\ 0 &= \Phi_i(y) - \frac{\overrightarrow{\delta} W[J]}{\delta J_i^T(y)} \\ \rightarrow \Phi_i(y) &= \frac{\overrightarrow{\delta} W[J]}{\delta J_i^T(y)} = \frac{1}{Z[J]} \frac{\overrightarrow{\delta} Z[J]}{\delta J_i^T(y)} = \langle \Psi_i(y) \rangle_{J_i^T}. \end{aligned} \quad (3.12)$$

$\langle \Psi \rangle_J$  can be understood as the expectation value of each component of  $\Psi$  that arises in presence of each component of  $J$  respectively. We want to remark that this functional derivative acts toward the right. The direction needs to be specified when working with Grassmann valued fields as sign changes may arise otherwise. With this in mind, we can construct a quantum analogue of equations of motion for the field expectation values  $\Phi_i$  at  $J = J_{\text{sup}}$ :

$$\begin{aligned} \Gamma[\Phi] \frac{\overleftarrow{\delta}}{\delta \Phi_i(x)} &= \int d^D y J_a^T(y) \Phi^a(y) \frac{\overleftarrow{\delta}}{\delta \Phi_i(x)} - W[J] \frac{\overleftarrow{\delta}}{\delta \Phi_i(x)} \\ &= J_i^T(x) + \int d^D y \frac{\delta J_a^T(y)}{\delta \Phi_i(x)} \Phi^a(y) - \int d^D y \frac{\delta J_a^T(y)}{\delta \Phi_i(x)} \frac{\delta W[J]}{\delta J_a^T(y)} \\ \xrightarrow{(3.12)} \frac{\delta \Gamma[\Phi]}{\delta \Phi_i(x)} &= J_i^T(x), \end{aligned} \quad (3.13)$$

here we inserted a functional derivative analogue of the chain rule in the second line

and simplified the notation of the functional derivatives<sup>3</sup>. From these equations of motion (3.13) we can interpret  $\Gamma[\Phi]$  as an *effective action* governing the dynamics of the field expectation values including all quantum fluctuations.

### 3.4 Derivation of the Wetterich equation

With this formalism in place we can now go back to (3.3) and combine the IR regularization from (3.6) with our effective action model. For this, let us perform a shift of the integration variable:  $\Psi \rightarrow \Psi + \Phi$ . This leads to:

$$\begin{aligned} Z[J] &= e^{W[J]} = \int_{\Lambda} \mathcal{D}\Psi e^{-S[\Psi+\Phi] + J^T \cdot \Psi + J^T \cdot \Phi} \\ &\rightarrow e^{W[J] - J^T \cdot \Phi} = \int_{\Lambda} \mathcal{D}\Psi e^{-S[\Psi+\Phi] + J^T \cdot \Psi} \\ &\xrightarrow{J=J_{\text{sup}}} e^{-\Gamma[\Phi]} = \int_{\Lambda} \mathcal{D}\Psi e^{-S[\Psi+\Phi] + \frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} \cdot \Psi} \end{aligned} \quad (3.14)$$

At this point, a way of calculating  $\Gamma$  does not present itself as an easy matter as equation (3.14) shows the effective action to be determined by a non linear functional differential equation nested into a functional integral. Thus, solving it exactly for generic cases is out of the question. Nevertheless, there are functional methods that feature approximation schemes. It turns out that using a vertex expansion of  $\Gamma[\Phi]$  and inserting it into (3.14) allows a comparison of coefficients leading to an infinite tower of coupled differential equations, namely the Dyson-Schwinger equations [43]. Solving these equations corresponds to integrating out all fluctuations at once yielding a solution to the full quantum action. But for our purposes, we want to use another approach which relies on RG techniques that we have already set up in equations (3.4) to (3.9). For this to work, we have to include the IR regulator term again by modifying our calculation in (3.14). We will start again with  $Z_k[J]$  and derive a similar form for  $\Gamma_k[\Phi]$ , the coarse grained average effective action:

$$Z_k[J] = e^{W_k[J]} = \int_{\Lambda} \mathcal{D}\Psi e^{-S[\Psi+\Phi] - \Delta S_k[\Psi+\Phi] + J^T \cdot (\Psi+\Phi)}. \quad (3.15)$$

Since we performed the shift again we have to deal with  $\Delta S_k[\Psi + \Phi]$ . We can do this by using its definition:  $\Delta S_k[\Psi] = \frac{1}{2} \Psi^T R_k \Psi$ . Inserting it into (3.15) allows us to split the terms and rearrange those that are not dependent on  $\Psi$ :

<sup>3</sup>We want to remark that for the rest of this thesis functional derivatives with respect to a transposed term always act from the left whereas functional derivatives with no transposition act from the right

$$\begin{aligned}
e^{W_k[J] - J^T \cdot \Phi + \Delta S_k[\Phi]} &= \int_{\Lambda} \mathcal{D}\Psi \, e^{-S[\Psi + \Phi] - \Delta S_k[\Psi] - \frac{1}{2} \Psi^T \cdot R_k \cdot \Phi + J^T \cdot \Psi - \frac{1}{2} \Phi^T \cdot R_k \cdot \Psi} \\
&= \int_{\Lambda} \mathcal{D}\Psi \, e^{-S[\Psi + \Phi] - \Delta S_k[\Psi] - \frac{1}{2} \Psi^T \cdot R_k \cdot \Phi + (J^T - \frac{1}{2} \Phi^T \cdot R_k + \frac{1}{2} \Phi^T \cdot R_k^T) \cdot \Psi} \quad (3.16) \\
&\xrightarrow{J \rightarrow J_{\text{sup}}} e^{-\Gamma_k[\Phi]} = \int_{\Lambda} \mathcal{D}\Psi \, e^{-S[\Psi + \Phi] - \Delta S_k[\Psi] + \frac{\delta \Gamma_k}{\delta \Phi} \Psi}.
\end{aligned}$$

In the last line we have used the fact that the coarse grained action  $\Gamma_k$  now obeys the equations:

$$\begin{aligned}
\Gamma_k[J] &= \sup_J (J^T \cdot \Phi - W_k[J]) - \Delta S_k[\Phi] \\
\text{and } \frac{\delta \Gamma_k[\Phi]}{\delta \Phi(x)} &= J^T(x) - \frac{1}{2} \Phi^T \cdot R_k + \frac{1}{2} \Phi^T \cdot R_k^T. \quad (3.17)
\end{aligned}$$

Since we are working with fermionic fields, we want to address a special property of scalar terms in spinor space like  $\Psi^T \cdot R_k \cdot \Phi$ . As they are Grassmann-valued they generate a minus sign under transposition:

$$\Psi^T \cdot R_k \cdot \Phi = (\Psi^T \cdot R_k \cdot \Phi)^T = -\Phi^T \cdot R_k^T \cdot \Psi. \quad (3.18)$$

Nonetheless, this formalism can be extended to also contain various other fields within  $\Psi$  and  $\Phi$  which are not Grassmann-valued. But for our studies it suffices to only consider fermionic components. We can see with (3.5) that  $R_k$  transposes to itself up to a minus sign. Inserting this fact into (3.17) simplifies our modified quantum equations of motion to:

$$\Gamma_k[\Phi] \frac{\overleftarrow{\delta}}{\delta \Phi(x)} = \frac{\delta \Gamma_k[\Phi]}{\delta \Phi(x)} = J^T(x) - \Phi^T \cdot R_k. \quad (3.19)$$

Concerning (3.19), we see  $\Gamma_k$  still governs the dynamics of field expectation values and converges towards (3.13) for  $k \rightarrow 0$ . We can also deduce another relationship from (3.19) by applying another functional derivative, this time acting to the right:

$$\frac{\overrightarrow{\delta} J^T(x)}{\delta \Phi^T(y)} = \frac{\overrightarrow{\delta} \Gamma_k[\Phi] \overleftarrow{\delta}}{\delta \Phi^T(y) \delta \Phi(x)} + R_k(x, y). \quad (3.20)$$

This term relates the functional derivative of  $J$  with respect to the field expectation values to the second functional derivative of  $\Gamma_k$  with the added regulator term. The right hand side has an important physical meaning which we will establish now. As

we saw in (3.12) field expectation values can be calculated by a functional derivative acting on  $W$ . This obviously still holds for  $W_k$ . By applying another functional derivative of  $J$  to the left we obtain:

$$\frac{\Phi(x) \overleftarrow{\delta}}{\delta J(y)} = \frac{\overrightarrow{\delta} W_k[J] \overleftarrow{\delta}}{\delta J^T(x) \delta J(y)} = K(x, y). \quad (3.21)$$

Here we have defined  $K$  which can be understood as the propagator for connected correlators (see Appendix B). Now we can combine (3.20) and (3.21) to achieve an important identity (in operator notation) which gives meaning to the right hand side of (3.20):

$$\mathbb{1} = \frac{\delta \Phi(x)}{\delta \Phi(x')} = \frac{\delta \Phi(x)}{\delta J(y)} \cdot \frac{\delta J(y)}{\delta \Phi(x')} = K(x', y) \cdot \left( \frac{\overrightarrow{\delta} \Gamma_k[\Phi] \overleftarrow{\delta}}{\delta \Phi^T(x') \delta \Phi(y)} + R_k(y, x') \right) = K \cdot \left( \Gamma_k^{(2)} + R_k \right). \quad (3.22)$$

Here we introduced a short hand notation for the second functional derivative of  $\Gamma_k$ , where both ones indicate that a derivative is applied from left and right once. With this equation in place we can see that the right hand side of equation (3.20) can be interpreted to be the inverse of the connected propagator:

$$K = \left( \Gamma_k^{(2)} + R_k \right)^{-1}. \quad (3.23)$$

Now we have arrived at a point where all ingredients are in place to derive the flow equation of  $\Gamma_k$  at  $J = J_{\text{sup}}$  with  $t = \ln\left(\frac{k}{\Lambda}\right)$  as the "RG" time [35]:

$$\begin{aligned} \partial_t \Gamma_k &= -\partial_t W_k[J] - \partial_t \Delta S_k[\Phi] \\ &\xrightarrow{\text{Appendix B}} = \frac{1}{2} \text{STr}[\partial_t R_k \cdot K] \\ &\xrightarrow{(3.23)} = \frac{1}{2} \text{STr} \left[ \partial_t R_k \cdot \left( \Gamma_k^{(2)} + R_k \right)^{-1} \right], \end{aligned} \quad (3.24)$$

where  $\text{STr}$  stands for the supertrace. Since we used generalized field vectors, we need to distinguish between bosonic and fermionic contributions as the latter are Grassmann-valued. The supertrace enforces this distinction by assigning a negative sign to fermionic parts while bosonic terms remain positive [44, 45]. The derived equation in (3.24) is called the Wetterich equation [18]. It is of major importance to modern FRG studies and will be the main tool to analyze relativistic Luttinger

fermions in chapter 4. For this reason, let us discuss some of its intricacies now before we move on to manipulate it further.

Although (3.24) has a one-loop structure similar to those in standard QFT calculations, it is an exact equation and not an approximation which is indicated by the appearance of  $K$ , the exact propagator, in the denominator. This structure is a consequence of  $\Delta S_k$  being quadratic in  $\Phi$ . Moreover,  $\Delta S_k$  implements IR regularization as  $R_k$  screens any modes for which the momentum is much smaller than  $k$ . Due to this and  $R_k$  vanishing for  $p \rightarrow \infty$ , the regulator has to drop off around  $p^2 \approx k^2$ . This drop off leads to the derivative of  $R_k$  with respect to  $k$  (or  $t$ ) peaking around  $p^2 \approx k^2$  and being small anywhere else. The exact form depends, of course, on the choice of  $R_k$  but the general behavior stays the same. This ensures that we can integrate out field fluctuations at  $p^2 \approx k^2$  while those of different  $p$  are suppressed. Hence, we do not have to include all fluctuations at once but are able to deal with them momentum shell by momentum shell. For each of these shells we have to consider a different theory as the dynamics changes with the scale. The set of all these theories spans a space of action functionals. Within this space there is not only the bare action  $S$ , which we know, but also the full quantum action  $\Gamma$ . By starting at the bare action  $S = \Gamma_\Lambda$  and integrating out momentum shell by momentum shell, we can end up with the full quantum action  $\Gamma$ . The set of all points in between are characterized by  $\Gamma_k$  which provide a trajectory from the bare action to the full theory. Solving the flow equation then corresponds to finding this trajectory by starting at  $k = \Lambda$  and traveling along until we hit the endpoint at which we arrive at  $\Gamma$ . The exact shape of this path depends on the form of  $R_k$  but the endpoints do not as long as the Wetterich equation is solved exactly. This is to be expected since non-universal quantities do in fact depend on the renormalization scheme but the final result, i.e. the full quantum action is invariant under change of  $R_k$  which we ensured with (3.7) and (3.8). Therefore, equation (3.24) yields a method of extracting all relevant physical information of a theory. It produces results that respect all quantum fluctuations arising in presence of a source.

### 3.5 Approximative solutions

However, we first need to solve (3.24) in order to obtain this information. To accomplish this, let us manipulate the equation so that it is more reminiscent to a one-loop perturbative correction term to the effective action by introducing a formal partial derivative  $\tilde{\partial}_t$  that acts only on the scale dependence in  $R_k$  :

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \text{STr} \left( \tilde{\partial}_t \ln \left( \Gamma_k^{(2)}[\Phi] + R_k \right) \right). \quad (3.25)$$

It turns out that with this form the necessary calculations become more transparent. By separating the propagator into a field-dependent part  $\mathcal{I}_k$  and a field-independent part  $\mathcal{D}_k$  it is possible to construct a power series expansion in the fields:

$$\begin{aligned} \partial_t \Gamma_k[\Phi] &= \frac{1}{2} \text{STr} \left( \tilde{\partial}_t \ln \left( \Gamma_k^{(2)}[\Phi] + R_k \right) \right) \\ &= \frac{1}{2} \text{STr} \left( \tilde{\partial}_t \ln (\mathcal{I}_k + \mathcal{D}_k[\Phi]) \right) \end{aligned} \quad (3.26)$$

$$= \frac{1}{2} \text{STr} \left( \tilde{\partial}_t \left( \frac{1}{\mathcal{I}_k} \mathcal{D}_k[\Phi] \right) \right) - \frac{1}{4} \text{STr} \left( \tilde{\partial}_t \left( \frac{1}{\mathcal{I}_k} \mathcal{D}_k[\Phi] \right)^2 \right) + \frac{1}{6} \text{STr} \left( \tilde{\partial}_t \left( \frac{1}{\mathcal{I}_k} \mathcal{D}_k[\Phi] \right)^3 \right) - \dots \quad (3.27)$$

These definitions might at first seem redundant as  $R_k$  is already a field-independent part, but since there are parts of  $\Gamma_k^{(2)}$  that do not depend on  $\Phi$  as well, we can group them together to form  $\mathcal{I}_k$ . If we now represent  $\Gamma_k$  in terms of field operators with respective couplings that encode the scale dependence we are able to extract flow equations for those couplings by simply comparing coefficients of the operators on the left hand side and right hand side of equation (3.27). In order to better understand this, let us have a look at a simplified example where  $\Gamma_k$  is of the form:

$$\Gamma_k[\phi] \sim \partial_\mu \phi \partial^\mu \phi + g_k \phi^4, \quad (3.28)$$

where  $\phi$  is an arbitrary scalar field. The left hand side of (3.27) would now contain a term  $\sim \partial_t g_k \phi^4$  whereas a similar potential appears on the right hand side as well, however without any derivative involved. Instead, a function of  $g_k$  will appear as a pre-factor which occurs due to the interaction of  $\mathcal{I}_k$  with  $\mathcal{D}_k$  leading to a term  $\sim f(g_k) \phi^4$ . A comparison of coefficients of  $\phi^4$  terms leads to a flow equation:

$$\partial_t g_k = f(g_k). \quad (3.29)$$

This method can easily be extended to multiple couplings and has a major advantage: If we know the form of  $\Gamma_k$  in terms of field operators we can already see which parts of the logarithmic power series expansion will produce the desired terms. Due to  $\Gamma_k^{(2)}$  being a second derivative of  $\Gamma_k$  it is comprised of operators containing exactly two fields less. This means we can predict which powers of fields occur on the right hand side of (3.27) allowing us to neglect any terms that will not produce the same operators as found in the original  $\Gamma_k$ . However, for all of this to work we need an appropriate ansatz for  $\Gamma_k$  as discussed after (3.27). Such an ansatz can be achieved in terms of an operator expansion. In particular, an expansion of derivative operators is of interest since we can use them as a systematic expansion scheme starting with the Luttinger action in (2.52) as lowest order. The average action can then be expressed as a sum of an effective potential term  $V_k^{\text{eff}}$  followed by kinetic terms of increasing mass dimension:

$$\Gamma_k[\Psi] = \int d^Dx \left[ V_k^{\text{eff}}(\Psi) - Z_\Psi \Psi^T G_{\mu\nu} \partial^\mu \partial^\nu \Psi + \mathcal{O}(\partial^4) \right]. \quad (3.30)$$

In this case, the scale dependence and therefore the quantum fluctuations are encoded in the coupling "constants" within  $V_k^{\text{eff}}$  as well as the wave function renormalization  $Z_\Psi$  and its higher order analogues in  $\mathcal{O}(\partial^4)$ . Using this approach of course demands a truncation to be implemented since we cannot hope to solve an infinite number of coupled flow equations arising from (3.30) inserted into (3.27). By ignoring higher order derivative terms as well as terms in  $V_k^{\text{eff}}$  exceeding a certain power in  $\Psi$  we truncate the ansatz which necessarily introduces a source of errors. These errors may in fact be grievous depending on the theory since we neglect the effect of higher order  $n$ -point functions on the ones we chose to keep, possibly leading to instabilities in the IR regime [35]. In order to make sure that results do not deviate too much from the true behavior, we have to check that the expansion scheme converges towards a stable solution. This can be done in a variety of ways. One of which is to rely on the fact that physical observables are independent of our regularization scheme, i.e. independent of  $R_k$ . Hence, we can vary  $R_k$  and check whether results of the truncated theory stay the same or vary only weakly. If that is the case, this truncation might be a useful candidate for further investigations. A physical way of deciding whether a certain ansatz is at least potentially a good candidate to approximate the flow equation is to make sure that all relevant degrees of freedom (in terms of field operators) in  $S$  are represented in the ansatz for  $\Gamma_k$ . But still, there will be a discrepancy between a full representation of  $\Gamma_k$  and a truncated one. However, discussing this mismatch in detail goes beyond the scope of this thesis. Assuming



that the truncation is able to represent aspects of the full quantum action, we will therefore proceed without any higher derivative orders beyond the quadratic one. Furthermore, the potential part  $V_k^{\text{eff}}$  is not considered to contain any terms that are of a higher order in  $\Psi$  as the bare action  $S$ .

After this discussion, we can now apply the derivative operator expansion to the Luttinger action (2.52). Due to our definition of  $\Psi$  in (3.2), we seemingly obtain two kinetic terms after truncating any parts containing four or more derivatives. But since we can rewrite terms of the form  $\partial^\mu \Psi^T G_{\mu\nu}^T \partial^\nu \bar{\Psi}^T$  into  $\bar{\Psi} G_{\mu\nu} \partial^\mu \partial^\nu \Psi$  by transposing and integrating by parts, the kinetic term is unique and it suffices to consider an effective average action of the form:

$$\Gamma_k[\Psi] = \int d^D x \left[ -Z_\Psi \bar{\Psi}_a G_{\mu\nu} \partial^\mu \partial^\nu \Psi^a + \frac{\bar{\lambda}_0}{2} (\bar{\Psi}_a \Psi^a)^2 + \frac{\bar{\lambda}_t}{2} (\bar{\Psi}_a G_{\mu\nu} \Psi^a)^2 \right]. \quad (3.31)$$

Introduced in (3.31),  $\bar{\lambda}_0$  and  $\bar{\lambda}_t$  are the scale-dependent couplings for which flow equations need to be determined. Together with the  $k$  dependence of  $Z_\Psi$  they encode all quantum fluctuations induced by the Luttinger action (2.52). Of course a flow equation for  $Z_\Psi$  can also be determined but we will see later that it vanishes within our ansatz (3.31). Additionally, the index  $a$  refers to a flavour index and is being summed over as established in (2.52). With this truncation at our disposal, we are ready to study the behavior of relativistic Luttinger fermions by computing said flow equations.

## 4 Dynamics of Luttinger fermions

### 4.1 RG-flow equations

With the methods established in the previous chapter, we now want to calculate flow equations for  $\bar{\lambda}_0$  and  $\bar{\lambda}_t$  as well as  $Z_\Psi$ . For this, we evaluate the effective average action for constant homogeneous background fields  $\psi$ :

$$\begin{aligned} \Psi &= \begin{pmatrix} \Psi_n(p) \\ \bar{\Psi}_n^\top(-p) \end{pmatrix} & \text{with} & \quad \bar{\Psi}(p) = \bar{\psi}(2\pi)^D \delta^{(D)}(p) \\ \Psi^\top &= (\Psi_m^\top(-p), \bar{\Psi}_m(p)) & & \quad \Psi(p) = \psi(2\pi)^D \delta^{(D)}(p). \end{aligned} \quad (4.1)$$

$\Gamma_k^{(2)}$  will also be evaluated for those fields. But first, we need compute it. We proceed by taking a look at its definition once again:

$$\Gamma_k^{(2)} = \frac{\overrightarrow{\delta}}{\delta \Psi_m^\top(-p)} \Gamma_k \frac{\overleftarrow{\delta}}{\delta \Psi_n(p')}, \quad (4.2)$$

where the arrows denote the direction in which the derivative acts as established in chapter 3. In order to compute this directly using the operator expansion in (3.31), we need to perform a Fourier transformation on the kinetic and potential parts. Doing so results in:

$$\Gamma_{\text{kin}} = \int \frac{d^D q}{(2\pi)^D} Z_\Psi \bar{\Psi}_a(q) G_{\mu\nu} q^\mu q^\nu \Psi^a(q) \quad (4.3)$$

$$\Gamma_0 = \frac{\bar{\lambda}_0}{2} \prod_{i=1}^3 \int \frac{d^D q_i}{(2\pi)^D} \bar{\Psi}_a(q_1) \Psi^a(q_2) \bar{\Psi}_b(q_3) \Psi^b(q_1 - q_2 + q_3) \quad (4.4)$$

$$\Gamma_t = \frac{\bar{\lambda}_t}{2} \prod_{i=1}^3 \int \frac{d^D q_i}{(2\pi)^D} \bar{\Psi}_a(q_1) G_{\mu\nu} \Psi^a(q_2) \bar{\Psi}_b(q_3) G^{\mu\nu} \Psi^b(q_1 - q_2 + q_3). \quad (4.5)$$

Here we decomposed  $\Gamma_k = \Gamma_{\text{kin}} + \Gamma_0 + \Gamma_t$  into three separate parts: The kinetic term, a scalar channel term and a tensor channel term. This will become useful later on for our calculations. Concerning the derivative of  $\Gamma_k$  with respect to the transposed fields with negative momentum, we can simply transpose  $\Gamma_k$  as it is a scalar in spinor space. Thus, using properties of Grassmann-valued fields together with a substitution  $q \rightarrow -q$  for the kinetic term, equations (4.3) - (4.5) read:

$$\Gamma_{\text{kin}} = - \int \frac{d^D q}{(2\pi)^D} Z_\Psi \Psi^{aT}(-q) G_{\mu\nu}^T q^\mu q^\nu \bar{\Psi}_a^T(-q) \quad (4.6)$$

$$\Gamma_0 = \frac{\bar{\lambda}_0}{2} \prod_{i=1}^3 \int \frac{d^D q_i}{(2\pi)^D} \Psi^{aT}(q_2) \bar{\Psi}_a^T(q_1) \Psi^{bT}(q_1 - q_2 + q_3) \bar{\Psi}_b^T(q_3) \quad (4.7)$$

$$\Gamma_t = \frac{\bar{\lambda}_t}{2} \prod_{i=1}^3 \int \frac{d^D q_i}{(2\pi)^D} \Psi^{aT}(q_2) G_{\mu\nu}^T \bar{\Psi}_a^T(q_1) \Psi^{bT}(q_1 - q_2 + q_3) G^{\mu\nu T} \bar{\Psi}_b^T(q_3). \quad (4.8)$$

By splitting  $\Gamma_k$  into these three parts we can immediately see that  $\Gamma_{\text{kin}}$  will contribute a field-independent part in  $\Gamma_k^{(2)}$  whereas  $\Gamma_0$  and  $\Gamma_t$  remain field-dependent after their respective functional differentiation. Therefore, we split the computation of  $\Gamma_k^{(2)}$  into three different parts as well. Furthermore, we can construct the corresponding regulator function in accordance with chapter 3 but modified to include the flavour indices. Computing the functional derivatives for  $\Gamma_k$  then yields (Appendix C):

$$\Gamma_{\text{kin}}^{(2)} = Z_\Psi \begin{pmatrix} 0 & -G_{\mu\nu}^T p^\mu p^\nu \\ G_{\mu\nu} p^\mu p^\nu & 0 \end{pmatrix} (2\pi)^D \delta_{mn} \delta^{(D)}(p - p') \quad (4.9)$$

$$R_k = Z_\Psi \begin{pmatrix} 0 & -G_{\mu\nu}^T p^\mu p^\nu \\ G_{\mu\nu} p^\mu p^\nu & 0 \end{pmatrix} r_\Psi (2\pi)^D \delta_{mn} \delta^{(D)}(p - p') \quad (4.10)$$

$$\Gamma_0^{(2)} = -\bar{\lambda}_0 \begin{pmatrix} \mathcal{D}_0^{11} & \mathcal{D}_0^{12} \\ \mathcal{D}_0^{21} & \mathcal{D}_0^{22} \end{pmatrix} (2\pi)^D \delta^{(D)}(p - p') \quad (4.11)$$

$$\Gamma_t^{(2)} = -\bar{\lambda}_t \begin{pmatrix} \mathcal{D}_t^{11} & \mathcal{D}_t^{12} \\ \mathcal{D}_t^{21} & \mathcal{D}_t^{22} \end{pmatrix} (2\pi)^D \delta^{(D)}(p - p'), \quad (4.12)$$

where

$$\mathcal{D}_0 = \begin{pmatrix} \mathcal{D}_0^{11} & \mathcal{D}_0^{12} \\ \mathcal{D}_0^{21} & \mathcal{D}_0^{22} \end{pmatrix} = \begin{pmatrix} \bar{\psi}_m^T \bar{\psi}_n & (\bar{\psi}\psi)\delta_{mn} - \bar{\psi}_m^T \psi_n^T \\ -(\bar{\psi}\psi)\delta_{mn} - \psi_m \bar{\psi}_n & \psi_m \psi_n^T \end{pmatrix},$$

and

$$\begin{aligned} \mathcal{D}_t &= \begin{pmatrix} \mathcal{D}_t^{11} & \mathcal{D}_t^{12} \\ \mathcal{D}_t^{21} & \mathcal{D}_t^{22} \end{pmatrix} \\ &= \begin{pmatrix} G_{\mu\nu}^T \bar{\psi}_m^T \bar{\psi}_n G^{\mu\nu} & (\bar{\psi} G_{\mu\nu} \psi) G^{\mu\nu T} \delta_{mn} - G_{\mu\nu}^T \bar{\psi}_m^T \psi_n^T G^{\mu\nu T} \\ -(\bar{\psi} G_{\mu\nu} \psi) G^{\mu\nu} \delta_{mn} - G_{\mu\nu} \psi_m \bar{\psi}_n G^{\mu\nu} & G_{\mu\nu} \psi_m \psi_n^T G^{\mu\nu T} \end{pmatrix}. \end{aligned}$$

Here we have introduced  $(\bar{\psi}\psi) = \bar{\psi}_a \psi^a$  in order to simplify expressions and emphasize that terms in brackets are scalars in field space. Moreover,  $\mathcal{D}_{0/t}$  is used to summarize field-dependent parts of  $\Gamma_k^{(2)}$ . In fact, the separation of the propagator as proposed in (3.27) can now be implemented immediately:

$$\mathcal{I}_k = \Gamma_{\text{kin}}^{(2)} + R_k \quad \mathcal{D}_k = \mathcal{D}_0 + \mathcal{D}_t. \quad (4.13)$$

We also want to note that due to the definition of  $\Psi$  and its transpose  $\mathcal{I}_k$  and  $\mathcal{D}_k$  both have a special property: Their off diagonal elements are negative transposes of each other. In order to continue the calculations, we need to find the inverse of  $\mathcal{I}_k$ . It can easily be checked that

$$\mathcal{I}_k^{-1} = \begin{pmatrix} 0 & G_{\mu\nu} p^\mu p^\nu \\ -G_{\mu\nu}^T p^\mu p^\nu & 0 \end{pmatrix} \frac{1}{Z_\Psi(1+r_\Psi)p^4} (2\pi)^D \delta_{mn} \delta(p-p'), \quad (4.14)$$

does in fact act as an inverse of  $\mathcal{I}_k$  (Appendix C). Furthermore, we notice that potential terms consisting of four fermion fields as in the scalar and tensor channels  $\bar{\lambda}_0, \bar{\lambda}_t$  in  $\Gamma_k$  can only be generated by  $(\mathcal{I}_k^{-1} \mathcal{D}_k)^2$  on the right hand side of (3.27). Hence, it suffices to calculate this term of the expansion and neglect all others. Moreover, as we apply a trace operator we do not need any off diagonal elements. By squaring

$$(\mathcal{I}_k^{-1} \mathcal{D}_k) = \begin{pmatrix} G_{\mu\nu} \mathcal{D}_{21} & G_{\mu\nu} \mathcal{D}_{22} \\ -G_{\mu\nu}^T \mathcal{D}_{11} & -G_{\mu\nu}^T \mathcal{D}_{12} \end{pmatrix} p^\mu p^\nu \frac{(2\pi)^D \delta^{(D)}(p-p')}{Z_\Psi(1+r_\Psi)p^4}, \quad (4.15)$$

we arrive at:

$$(\mathcal{I}_k^{-1} \mathcal{D}_k)^2 = \mathcal{M} p^\mu p^\nu p^\kappa p^\lambda \frac{(2\pi)^D \delta^{(D)}(p-p')}{Z_\Psi^2(1+r_\Psi)^2 p^8}$$

with  $\mathcal{M} = \begin{pmatrix} G_{\mu\nu} \mathcal{D}_{21} G_{\kappa\lambda} \mathcal{D}_{21} - G_{\mu\nu} \mathcal{D}_{22} G_{\kappa\lambda}^T \mathcal{D}_{11} & \dots \\ \dots & G_{\mu\nu}^T \mathcal{D}_{12} G_{\kappa\lambda}^T \mathcal{D}_{12} - G_{\mu\nu}^T \mathcal{D}_{11} G_{\kappa\lambda} \mathcal{D}_{22} \end{pmatrix},$

(4.16)

where the dots indicate the remaining matrix entries which are not of interest to us as they will not contribute to the trace. Now we can proceed by differentiating those parts with respect to  $t$  which come from the regulator, i.e.  $Z_\Psi r_\Psi$ . Since the matrix does not contain any of these terms we just focus on the scalar factor

$$\tilde{\partial}_t \frac{1}{Z_\Psi^2(1+r_\Psi)^2} = -2 \frac{\partial_t r_\Psi - \eta_\Psi r}{Z_\Psi^2(1+r_\Psi)^3}, \quad (4.17)$$

where we introduced the so called anomalous dimension  $\eta_\Psi = -\partial_t \ln Z_\Psi$ . This term is the associated "small" parameter of the operator expansion of  $\Gamma_k$  [35]. As long as  $\eta_\Psi$  stays reasonably small our expansion is justified. As we will see later, we can in fact show that  $\eta_\Psi$  vanishes in the present approximation. Moving on, we need to apply the trace operator. Some caution is needed as this supertrace not only acts on the generalized matrix but also on the flavour indices and momenta:

$$\begin{aligned}\partial_t \Gamma_k &= -\frac{1}{4} \text{Tr} \left( \tilde{\partial}_t (\mathcal{I}_k^{-1} \mathcal{D}_k)^2 \right) \\ &= \frac{1}{2} \sum_{i=1}^{N_f} \int d^D p \frac{\delta^{(D)}(0)}{p^8} \frac{\partial_t r_\Psi - \eta_\Psi r}{Z_\Psi^2 (1 + r_\Psi)^3} p^\mu p^\nu p^\kappa p^\lambda (-\text{Tr} \mathcal{M}_{11} - \text{Tr} \mathcal{M}_{22}).\end{aligned}\quad (4.18)$$

Here,  $\mathcal{M}$  is again the matrix from equation (4.16). The delta function might seem problematic at first as it diverges due to the trace acting on the momenta. But if we did this calculation more rigorously and in position space, it would turn out that this factor corresponds to the space time volume  $\Omega$ . When inserting the background fields into the left hand side of (3.27) such a factor arises as well. Of course, one has to be careful when working with diverging quantities but in principle the whole calculation could be done for a finite spacetime volume such that this factor cancels on both sides of the equation leading to a converging calculation. Then the limit for  $\Omega \rightarrow \infty$  can be implemented. But for our purposes it suffices to just compare coefficients and cancel both of the arising divergences with one another. Therefore, the only thing left to compute is the trace of  $\mathcal{M}$ . A nice simplification can be made by exploiting properties of the trace as well as the fact that the off-diagonal elements of  $\mathcal{D}_k$  are negative transposes of each other:

$$\begin{aligned}\text{Tr} \mathcal{M}_{22} &= \text{Tr} \left( G_{\mu\nu}^T \mathcal{D}_{12} G_{\kappa\lambda}^T \mathcal{D}_{12} - G_{\mu\nu}^T \mathcal{D}_{11} G_{\kappa\lambda} \mathcal{D}_{22} \right) \\ &= \text{Tr} \left( G_{\mu\nu}^T \mathcal{D}_{21}^T G_{\kappa\lambda}^T \mathcal{D}_{21}^T \right) - \text{Tr} \left( G_{\mu\nu}^T \mathcal{D}_{11} G_{\kappa\lambda} \mathcal{D}_{22} \right) \\ &= \text{Tr} \left( \mathcal{D}_{21} G_{\mu\nu} \mathcal{D}_{21} G_{\kappa\lambda} \right) - \text{Tr} \left( G_{\mu\nu} \mathcal{D}_{22} G_{\kappa\lambda}^T \mathcal{D}_{11} \right) \\ &= \text{Tr} \left( G_{\mu\nu} \mathcal{D}_{21} G_{\kappa\lambda} \mathcal{D}_{21} - G_{\mu\nu} \mathcal{D}_{22} G_{\kappa\lambda}^T \mathcal{D}_{11} \right) \\ &= \text{Tr} \mathcal{M}_{11}.\end{aligned}\quad (4.19)$$

Therefore, the calculation of the flow equations boils down to just one of those traces:

$$\partial_t \Gamma_k = -\Omega \sum_{i=1}^{N_f} \int d^D p \frac{\partial_t r_\Psi - \eta_\Psi r}{Z_\Psi^2 (1 + r_\Psi)^3 p^8} p^\mu p^\nu p^\kappa p^\lambda \text{Tr} (\mathcal{M}_{11}). \quad (4.20)$$

We calculate the trace in Appendix C and also use the fact that we can replace the momenta  $p^\mu p^\nu p^\kappa p^\lambda$  in the integral by  $\frac{2p^4}{D(D+2)}$  together with a totally symmetric Lorentz index contraction of corresponding  $G_{\mu\nu}$  in the trace of  $\mathcal{M}$  (see also Appendix C). Furthermore, we include the factor  $\frac{2}{D(D+2)}$  in the trace which then, together with the sum over flavour indices, yields:

$$\begin{aligned} \text{Tr}(\mathcal{M}_{11}) &= (\bar{\psi}\psi)^2 \left[ \bar{\lambda}_0^2 (N_f d_\gamma - 2) - \bar{\lambda}_0 \bar{\lambda}_t \frac{D}{D-1} (2d_e + 4) \right] \\ &\quad + (\bar{\psi} G_{\mu\nu} \psi)^2 \left[ -\bar{\lambda}_0^2 \frac{4}{D(D+2)} + \bar{\lambda}_0 \bar{\lambda}_t \left( 2 - \frac{4}{d_e} \right) \right. \\ &\quad \left. - \bar{\lambda}_t^2 \frac{D}{d_e(D-1)} (N_f d_\gamma (d_e - 2) + 2(d_e^2 - d_e + 2)) \right] \\ &= (\bar{\psi}\psi)^2 \bar{f}(N_f, D, \bar{\lambda}_0, \bar{\lambda}_t) + (\bar{\psi} G_{\mu\nu} \psi)^2 \bar{g}(N_f, D, \bar{\lambda}_0, \bar{\lambda}_t), \end{aligned} \quad (4.21)$$

where we have introduced  $d_e$  and  $d_\gamma$  again in correspondence to (2.10) and (2.17). Moreover, the functions  $\bar{f}$  and  $\bar{g}$  are shorthand notations where the bar indicates their dependence on  $\bar{\lambda}_0$  and  $\bar{\lambda}_t$ . Let us now introduce renormalized, dimensionless couplings:

$$\lambda_0 = \bar{\lambda}_0 Z_\Psi^{-2} k^{D-4} \quad \text{and} \quad \lambda_t = \bar{\lambda}_t Z_\Psi^{-2} k^{D-4}. \quad (4.22)$$

Using these new parameters and their corresponding functions  $f(N_f, D, \lambda_0, \lambda_t)$  and  $g(N_f, D, \lambda_0, \lambda_t)$  as well as the calculated trace, (4.20) simplifies a lot once we cancel some prefactors depending on  $k$  and  $Z_\Psi$  which emerge from our substitution (4.22) on the left hand side of (3.27). This leads to a flow equation of the form:

$$\partial_t \Gamma_k = -\Omega \left[ (\bar{\psi}\psi)^2 f + (\bar{\psi} G_{\mu\nu} \psi)^2 g \right] \frac{\partial_D}{2} \int_0^\infty dx x^{\frac{D-6}{2}} \frac{\partial_t r_\Psi - \eta_\Psi r}{(1 + r_\Psi)^3}, \quad (4.23)$$

where  $\partial_D$  denotes the  $D$ -dimensional solid angle and  $x = \frac{p^2}{k^2}$ . The integral over  $x$  is often called a threshold function and encodes the entirety of our regularization and therefore corresponds to a scheme-dependent factor. Depending on the choice of  $r_\Psi$  this factor changes. For our purposes, we evaluate the integral for a regulator of Litim form (3.9) in Appendix C. Meanwhile, the left hand side of (3.27) can also be calculated straightforwardly. Since the derivative acts only on the renormalized couplings and not on any fields, we can immediately perform the integration as the delta functions in our background fields produce another divergent term corresponding to the spacetime volume. We can also ignore the kinetic term as none such operator

is on the right hand side. Canceling remaining factors of  $k$  and  $Z_\Psi$  with the ones on the right, the left side reads:

$$\partial_t \Gamma_k = \frac{(2\pi)^D}{2} \Omega \left[ (\bar{\psi}\psi)^2 (\partial_t \lambda_0 - (D-4+2\eta_\Psi)\lambda_0) + (\bar{\psi} G_{\mu\nu} \psi)^2 (\partial_t \lambda_t - (D-4+2\eta_\Psi)\lambda_t) \right]. \quad (4.24)$$

Now comparing coefficients of the field operator terms leads directly to flow equations for  $\lambda_0$  and  $\lambda_t$ . Inserting the result of the threshold function from Appendix C yields:

$$\begin{aligned} \partial_t \lambda_0 &= (D-4+2\eta_\Psi)\lambda_0 - \frac{\partial_D}{(2\pi)^D} \left( \frac{2}{D-2} - \frac{2\eta_\Psi}{(D-2)(D-1)} \right) f(N_f, D, \lambda_0, \lambda_t) \\ \partial_t \lambda_t &= (D-4+2\eta_\Psi)\lambda_t - \frac{\partial_D}{(2\pi)^D} \left( \frac{2}{D-2} - \frac{2\eta_\Psi}{(D-2)(D-1)} \right) g(N_f, D, \lambda_0, \lambda_t). \end{aligned} \quad (4.25)$$

The result is a system of two coupled differential equations. We can group them into a total flow function  $\beta$  by treating them as part of a vector

$$\beta = \partial_t \begin{pmatrix} \lambda_0 \\ \lambda_t \end{pmatrix}, \quad (4.26)$$

which results in a vector field that we can analyze. But before that, let us discuss the flow of the wave function renormalization  $Z_\Psi$  in a little more detail. Its equation can be derived by considering a background field that corresponds to incoming and outgoing particles in the form of plain waves. In momentum space this corresponds to a shift in the delta functions of (4.1) by an external momentum  $U$ :

$$\bar{\Psi}(p) = \bar{\psi}(2\pi)^D \delta^{(D)}(p-U) \quad \text{and} \quad \Psi(p) = \psi(2\pi)^D \delta^{(D)}(p+U). \quad (4.27)$$

However, this leads to the left hand side of (3.27) being dependent on  $U$  whereas the right hand side will not produce any  $U$  dependence in the bi-linear field terms. Therefore, comparing coefficients leads to a vanishing flow of  $Z_\Psi$ :

$$\partial_t Z_\Psi = 0. \quad (4.28)$$

Hence, we can assume the wave function renormalization to be constant. This has the additional benefit of a vanishing anomalous dimension justifying our expansion of  $\Gamma_k$ . Since the flow vanishes we can simply set  $Z_\Psi = 1$  for the rest of our analysis.

Next, we want to focus on the  $\beta$  function (4.26). With the definition of the solid angle  $\vartheta_D = 2\pi^{D/2}\Gamma(D/2)^{-1}$  and the aforementioned fact that  $\eta_\Psi = 0$  we can evaluate the flow for the special case where we set  $D = 4$  and  $N_f = 1$ . This then yields:

$$\beta = \partial_t \begin{pmatrix} \lambda_0 \\ \lambda_t \end{pmatrix} = -\frac{1}{(12\pi)^2} \begin{pmatrix} 540\lambda_0^2 - 528\lambda_0\lambda_t \\ -3\lambda_0^2 + 28\lambda_0\lambda_t - 992\lambda_t^2 \end{pmatrix}. \quad (4.29)$$

This equation represents the flow of both quartic couplings. It encodes the dynamics of relativistic Luttinger fermions on different scales. For this reason, let us study the  $\beta$  function in a little more detail and discuss its properties.

The  $\beta$  function consists of two coupled first order differential equations. While similar flow functions have been derived for other theories, for example the Gross Neveu model [28], they normally contain a linear term in their respective channels. In  $D = 4$  this is not the case for Luttinger fermions which is due to the quadratic derivative term in the kinetic operator. By construction, this leads to the flow function only containing purely quadratic terms in the couplings rendering them perturbatively renormalizable. Moreover, there is no contribution to the flow of  $\lambda_0$  which does not depend on  $\lambda_0$  itself. Therefore, the scalar channel cannot be generated dynamically by the tensor channel, but has to be excited in the first place in order to contribute to the total quantum fluctuations. In contrast to this, the flow of the tensor channel  $\lambda_t$  does contain a term quadratic in  $\lambda_0$ . This means even if the system starts with only the scalar channel, i.e.  $\lambda_t = 0$ , quantum fluctuations in form of the tensor channel will arise and contribute to the total dynamics since  $\partial_t \lambda_t$  does not vanish:

$$\partial_t \begin{pmatrix} \lambda_0 \\ \lambda_t \end{pmatrix} = -\frac{1}{(12\pi)^2} \begin{pmatrix} 540\lambda_0^2 \\ -3\lambda_0^2 \end{pmatrix}. \quad (4.30)$$

Here, the contribution to the tensor channel is comparatively small in relation to the scalar one. But as we will see later on this does not necessarily mean that the  $\lambda_0$  channel stays dominant. On the other hand, when we start with no scalar channel the system remains in a state without  $\lambda_0$  terms. Thus, the differential equations decouple into two analytically solvable equations:

$$\partial_t \begin{pmatrix} \lambda_0 \\ \lambda_t \end{pmatrix} = -\frac{1}{(12\pi)^2} \begin{pmatrix} 0 \\ -992\lambda_t^2 \end{pmatrix} \quad (4.31)$$

$$\rightarrow \partial_t \lambda_0 = 0 \quad \text{and} \quad \partial_t \lambda_t = \frac{62}{9\pi^2} \lambda_t^2. \quad (4.32)$$



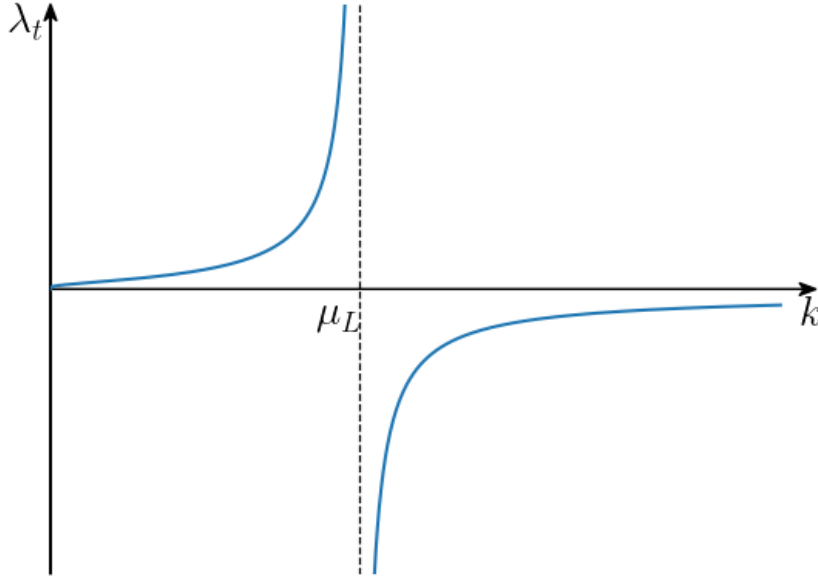
In this case  $\lambda_0$  is constant with respect to  $t$  or  $k$  and remains zero if we start with it being zero.  $\lambda_t$  is a little more interesting as its corresponding axis forms an RG invariant subspace within our truncation. Its differential equation can now be solved analytically. Fixing the coupling at some scale  $\mu_0$ ,  $\lambda_{t,\mu_0} = \lambda_t(\mu_0)$ , yields

$$\begin{aligned} \int_{\lambda_{t,\mu_0}}^{\lambda_t} d\lambda'_t \frac{1}{\lambda'^2_t} &= \frac{62}{9\pi^2} \int_{\mu_0}^k dk' \frac{1}{k'} \\ \rightarrow \lambda_t(k) &= \frac{\lambda_{t,\mu_0}}{1 - \frac{62}{9\pi^2} \lambda_{t,\mu_0} \ln\left(\frac{k}{\mu_0}\right)}. \end{aligned} \quad (4.33)$$

This evolution equation has some interesting properties and is illustrated in Figure 1. We observe two distinct cases which we discuss separately.

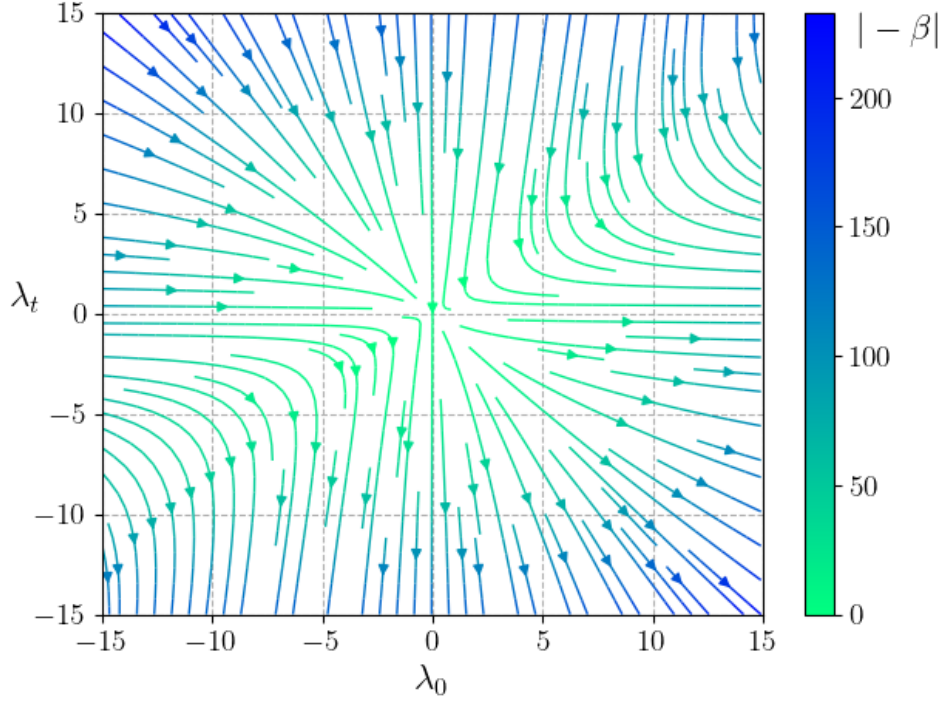
In the case of  $\lambda_{t,\mu_0} > 0$  the theory becomes non-interacting in the IR-limit as the coupling tends to zero. For increasing  $k$  on the other hand the coupling approaches a Landau pole. This pole signals a break down of the theory since we cannot safely let the UV- cutoff approach infinity as the corresponding coupling diverges as well. Therefore we are unable to obtain a UV-complete theory. This behavior is similar to QED where it was established that such a pole could spell trouble for the theory [46]. In order to fix this problem multiple studies have been focused on detailed investigations around non perturbative methods to mend QED. Nowadays it is clear that such a pole, at least in QED, does not impede on the validity of said theory as it was shown that non perturbative dynamics such as spontaneous symmetry breaking or non Gaussian RG fixed points cause the Landau pole to remain outside the reach of physical parameters [47, 48]. Similar analytic methods may be applied here but for our purposes it suffices to attain only a general insight into the behavior of the system without digging too deep into its details.

In the second case, where  $\lambda_{t,\mu_0} < 0$ , the dynamics differ. For increasing  $k$  the coupling approaches the UV-limit not only safely but even vanishes rendering the theory non-interacting. But now the Landau pole is situated in the IR-regime. This behavior can be compared to QCD where quarks are asymptotically free in the UV-regime but only occur in confined states for smaller scales. The IR-pole does not indicate a breakdown of the theory but can be removed using higher twist corrections and instead only hints at critical behavior in the form a phase transition toward confinement [49]. In order to perform a similar analysis in our case, we would need to employ higher order truncations. Therefore, we want to finish the discussion on the lone tensor channel for now and move on toward the total flow (4.29).



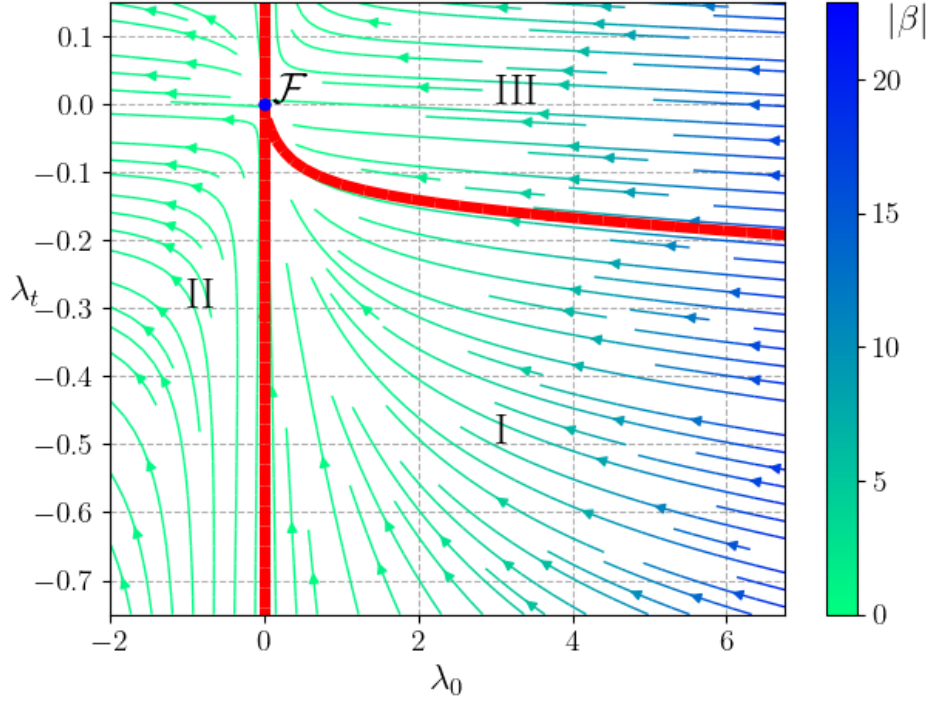
**Figure 1:** Evolution of  $\lambda_t(k)$  for vanishing scalar channel. Here  $\mu_L = \Lambda \exp\left(\frac{9\pi^2}{62\lambda_{t,\mu_0}}\right)$  marks a Landau pole which is a UV pole for  $\lambda_{t,\mu_0} > 0$  marking the UV incompleteness of the theory, and an IR-pole for  $\lambda_{t,\mu_0} < 0$  indicating potentially the onset of a phase transition.

When not setting the scalar channel to 0, we have no method of solving the  $\beta$ -function analytically. But we can numerically approximate its behavior and therefore still gain a lot of insight into the dynamics. As discussed above, the scalar channel provides a negative sign in its quadratic term as well as many mixed terms and contributions to the tensor channel thereby inducing a complex interplay between the couplings. In order to analyze these dynamics, we can simulate  $\beta$  numerically by plotting a vector field (Figure 2). It is useful to plot  $-\beta$  in order to visualize the flow towards the IR regime.



**Figure 2:**  $-\beta$ -function (4.29) plotted as a vector field. The vertical axis represents the tensor channel while the horizontal one illustrates the scalar channel coupling. A colour gradient shows values of  $|\beta| = \sqrt{(\partial_t \lambda_0)^2 + (\partial_t \lambda_t)^2}$  emphasizing the existence of a Gaussian fixed point at  $(0,0)$ .

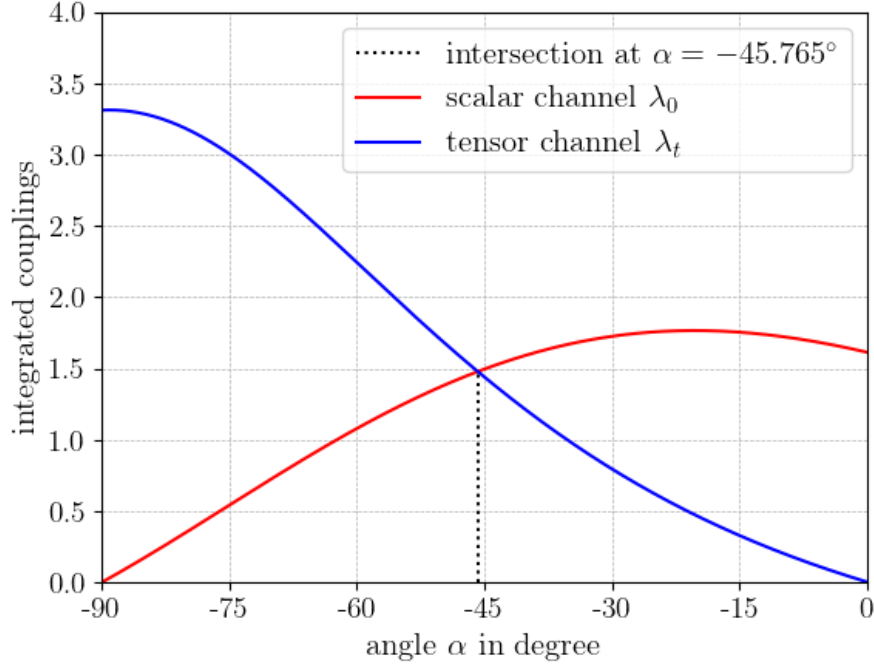
Due to this, we can clearly see that  $(0,0)$  is an asymptotically free Gaussian fixed point in the UV regime for starting values  $\lambda_0$  and  $\lambda_t$  in the forth quadrant. Since the flow diverges in all other quadrants towards the UV we can disregard these areas as we want to focus on a theory that provides asymptotic safety. Therefore, we will only consider positive values for  $\lambda_0$  and negative ones for  $\lambda_t$ . However, this does not entirely ensure asymptotic freedom since there remains a zone just below the  $\lambda_t = 0$  axis where the tensor channel is able to induce enough fluctuations to force  $\beta$  across the axis resulting in  $\lambda_t$  becoming positive and for large  $k$  divergent. In order to avoid UV divergences which render our theory unphysical, we need to find separatrices that divide phase space into distinct domains in which different asymptotic behaviors emerge. By fine-tuning initial conditions, the boundaries of these domains can be approximated and are shown in Figure 3.



**Figure 3:**  $\beta$  function plotted with approximated separatrices (red lines). The Gaussian fixed point is indicated by a blue dot at (0,0) and areas I, II and III mark domains of different behavior for  $k \rightarrow \infty$

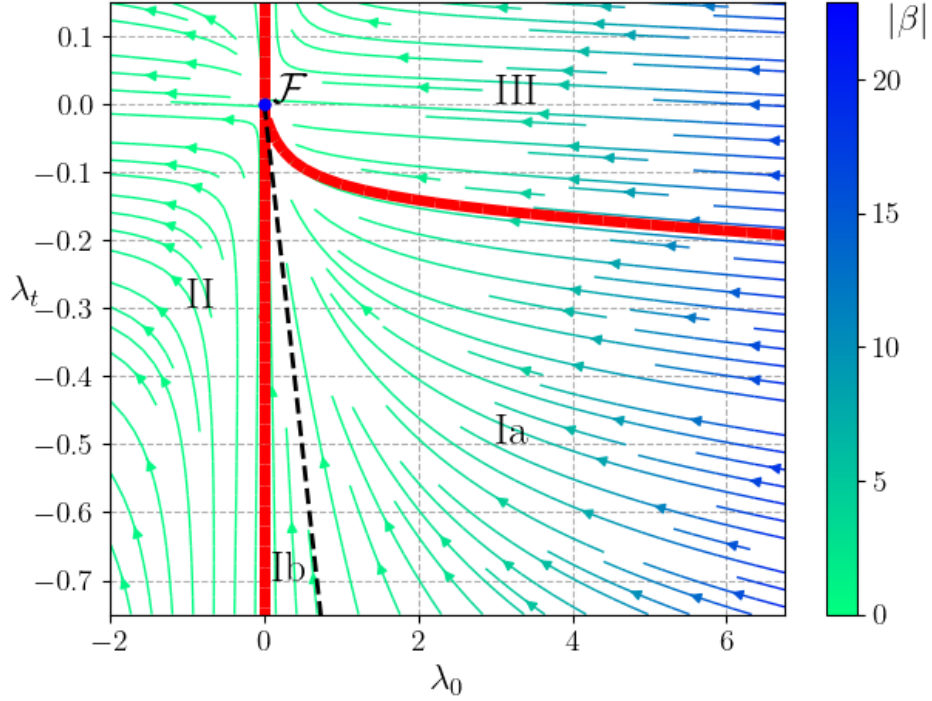
One such separatrix is given by the invariant subspace  $\lambda_0 = 0$ . It divides phase space into 2 parts which never interfere with one another. For initial conditions left of this boundary, i.e.  $\lambda_0 < 0$  (II) in Figure 3), the UV behavior always shows the scalar channel approaching negative infinity whereas the tensor channel flows towards positive infinity. Even for negative initial  $\lambda_t$  this holds, since for any  $\lambda_0 < 0$  the flow of  $\lambda_t$  will always remain positive which follows from  $3\lambda_0^2 - 28\lambda_0\lambda_t + 992\lambda_t^2 < 0$  having no solutions for  $\lambda_0 < 0$ . The phase space in which  $\lambda_0 > 0$  is split into two other domains by the second separatrix. In domain III both channels diverge towards positive infinity. Moreover,  $\lambda_t$  always ends up dominating the scalar channel. However, this is not the case in domain I. For any initial pair of couplings starting in this area the theory is UV-finite. Not only this but the UV flow always converges towards a Gaussian fixed point  $\mathcal{F}$  guaranteeing not only asymptotic safety but actually asymptotic freedom thereby ensuring a consistent model at any scale. For this reason we want to dedicate the rest of this thesis on domain I and restrict the Luttinger theory to suitable initial conditions.

With our model now being a free theory in the UV limit, we still have strong interactions occurring in the IR regime. In general, such a difference of dynamics on different scales often leads to critical behavior or phase transitions. For example, in QCD quarks are free particles in a high energy limit whereas their behavior drastically changes towards lower energy scales as they form condensates and bound states such as mesons or even protons and neutrons. This, of course, occurs due to them being coupled with gluons [50]. It induces a strong fermionic self interaction which may cause spontaneous symmetry breaking leading to critical changes in the dynamics [29]. In order to study these phenomena, we can try to express the action (2.52) in a partially bosonized form. This method allows us to disregard four fermion interactions at the cost of introducing an auxiliary field  $\phi$  which is in turn of bi-fermionic nature  $\bar{\Psi}\Psi$ . Such a pairing often occurs when dealing with strongly interacting fermions like in QCD, or in condensed matter physics where it leads to superconductivity and superfluidity [51, 52]. The new field  $\phi$  then carries information about the field expectation value of a bound state  $\bar{\Psi}\Psi$ . Since the Luttinger model does in fact feature strongly interacting fermions in the IR regime, it may be of interest to apply a partial bosonization in order to gain more insight into the macroscopic dynamics of the theory. However, to make use of this method we cannot keep both interaction channels. If we did introduce, for example,  $\phi \sim \bar{\Psi}\Psi$  to get rid of the four fermion scalar channel but kept the tensor channel it could simply induce new fluctuations in the scalar channel as they are coupled to each other. This renders the bosonization only approximate since as soon as we would change  $k$  the flow equation could generate a scalar channel term through changes in  $\lambda_t$  and vice versa. In principle, this problem can be solved by the technique of dynamical bosonization [53]. For simplicity, we will confine our investigations in the following on the scalar channel and ignore the tensor channel. To achieve good approximate results nonetheless, we need the impact of the tensor channel to be comparatively small meaning  $\lambda_0$  should be the dominant part. By integrating out both couplings numerically, we can approximate for which initial conditions in phase space  $\lambda_0$  stays dominant when approaching the IR regime. We choose initial conditions lying on the unit circle in phase space and integrate out all of them for  $k \in [0,1]$  as the couplings rapidly tend to infinity. Using the last values of the integrated couplings we can compare both channels with respect to the angle  $\alpha = \arctan\left(\frac{\lambda_t}{\lambda_0}\right)$  at which their initial conditions were chosen. By doing so, we can define a susceptibility of  $\beta$  in domain I as done in Figure 4.



**Figure 4:** Susceptibility of  $\beta$  displaying the coupling strengths depending on the angle  $\alpha = \arctan\left(\frac{\lambda_t}{\lambda_0}\right)$  of initial conditions in the fourth quadrant of figure 2.

From this graph we can infer that for any initial conditions in the forth quadrant of Figure 2 for which  $\alpha < -45.765^\circ$  the tensor channel coupling stays dominant when being integrated out towards the IR regime. On the other hand, for angles greater than  $-45.765^\circ$  the scalar channel takes over. Furthermore, we observe an interesting property of  $\lambda_0$  at an angle around  $20^\circ$  where its graph hits a maximum. This phenomenon occurs due to  $\beta$  always having a non-zero second component  $\partial_t \lambda_t$ . The flow of  $\lambda_t$  starts to pull  $\lambda_0$  towards zero for smaller values of  $\lambda_t$ . However, this should not be confused with the separatrix which separates domains I and II. Its appearance occurs much later at an angle of approximately  $-0.72^\circ$ . But the influence of the tensor channel on the scalar one kicks in much earlier. Nonetheless, the scalar channel stays dominant for  $-45.765 < \alpha < 0$  meaning we can neglect the impact of the tensor channel and still approximate the theory reliably. Therefore, we can combine the results of Figure 3 with this angle dependent behavior in order to find a domain of initial conditions which not only exerts asymptotic freedom but can also be approximated using partial bosonization to determine macroscopic dynamics.



**Figure 5:**  $\beta$  function with its domains of different asymptotic behavior I, II and III. Red solid lines represent separatrices as boundaries of those domains. The black dashed line indicates at which angle ( $-45.765^\circ$ ) of initial values  $\lambda_0$  (Ia) or  $\lambda_t$  (Ib) dominates the IR-regime. Due to the scaling of the axes the line appears to be at a wider angle.  $\mathcal{F}$  is the Gaussian fixed point which attracts the flow in domain I.

In Figure 5 this domain is precisely represented by area Ia. This concludes our discussion on the  $\beta$  function of relativistic Luttinger fermions. In the next section, we will focus on applying a partial bosonization which may provide insights on mass generation, spontaneous symmetry breaking, and possible stability issues.

## 4.2 Partial bosonization

In this section, we introduce the well-established method of partial bosonization by applying a Hubbard-Stratonovich transformation [54, 55]. This method provides a transition from four-fermionic interactions to a scalar field, to which our fermions are coupled. In principle, this can be done analogously for a theory that is completely chirally symmetric by adding a potential term of the form  $-(\bar{\Psi}\gamma_{10}\Psi)^2$  [28]. In doing so, we would need to establish the modified flow equation of  $\lambda'_0$  as well which would likely lead to a dynamical generation of an axial tensor channel  $(\bar{\Psi}\gamma_{10}G_{\mu\nu}\Psi)^2$ . Therefore, we would have to start with such a term in our ansatz and compute the total flow. After that we would again need to consider in which domain the tensor channels could be neglected. However, this goes beyond the scope of this thesis. Thus, a minimal approach suffices for now, where we only work with a partial chiral symmetry as in action (2.51):

$$S_0[\bar{\Psi}, \Psi] = \int d^Dx \left[ -Z_\Psi \bar{\Psi} G_{\mu\nu} \partial^\mu \partial^\nu \Psi + \frac{\lambda_0}{2} (\bar{\Psi}\Psi)^2 \right]. \quad (4.34)$$

This Luttinger action is symmetric under  $\Psi \rightarrow \gamma_{10}\Psi$ . Such a symmetry may be broken spontaneously according to the Goldstone theorem leading to the generation of massless bosons [29]. This happens only if the vacuum expectation value  $\langle \bar{\Psi}\Psi \rangle$  becomes finite. But as we do not have a continuous symmetry we do not expect any massless bosons to be generated as there will not be any residual symmetry left after the spontaneous breaking. Hence, we only focus on the potential mass generation. The corresponding Euclidean generating functional for (4.34) reads<sup>4</sup>:

$$Z = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi e^{-S_0[\bar{\Psi}, \Psi]}. \quad (4.35)$$

We can again set  $Z_\Psi = 1$ . Now we introduce an auxiliary field  $\phi$  to bosonize the scalar four fermion interaction by multiplying our generating functional with a constant factor:

$$1 = \frac{1}{\mathcal{N}} \int \mathcal{D}\phi e^{-\int d^Dx \frac{m^2}{2} \phi^2}, \quad (4.36)$$

where  $\mathcal{N}$  is a normalization factor. We can now introduce a coupling term that induces interactions between  $\phi$  and  $\Psi$  to arrive at a theory of the form:

$$Z = \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \mathcal{D}\phi e^{-S_{\text{FB}}[\bar{\Psi}, \Psi, \phi]} \quad (4.37)$$

<sup>4</sup>We omit the source terms here for convenience.



with

$$S_{\text{FB}}[\bar{\Psi}, \Psi, \phi] = \int d^D x \left[ -\bar{\Psi} G_{\mu\nu} \partial^\mu \partial^\nu \Psi + h_0 \bar{\Psi} \phi \Psi + \frac{m^2}{2} \phi^2 \right], \quad (4.38)$$

where  $h_0$  is the corresponding coupling constant and the subscript FB is chosen to illustrate that this action contains fermionic and bosonic fields. This new theory can now be related to (4.35) if we consider the equations of motions of  $\phi$ . Since we do not have any kinetic term for this scalar field, the e.o.m. are trivial and read:

$$\begin{aligned} m^2 \phi + h_0 \bar{\Psi} \Psi &= 0 \\ \phi &= -\frac{h_0}{m^2} \bar{\Psi} \Psi. \end{aligned} \quad (4.39)$$

Inserting this result back into (4.38) yields

$$S_{\text{FB}} = \int d^D x \left[ -\bar{\Psi} G_{\mu\nu} \partial^\mu \partial^\nu \Psi - \frac{h_0^2}{2m^2} (\bar{\Psi} \Psi)^2 \right]. \quad (4.40)$$

Therefore,  $S_{\text{FB}}$  is equivalent to (4.34) if we demand:

$$\lambda_0 = -\frac{h_0^2}{m^2}. \quad (4.41)$$

This partially bosonized action is now symmetric under the combined transformations  $\Psi \rightarrow \gamma_{10} \Psi$  and  $\phi \rightarrow -\phi$ . If this was done with a continuous chiral symmetry, meaning an additional term  $-(\bar{\Psi} \gamma_{10} \Psi)^2$  was in the action to begin with, we would have introduced two real or one complex scalar field which would result in a continuous symmetry transformation. Nonetheless, the generating functional of this partially bosonized action now has one great advantage over the one in (4.35). It is only quadratic in fermionic fields and therefore we can straightforwardly perform the integration over them. This assumes that we do not have any bosonic fluctuations and that the bosonic ground state is identical to the one of the classical bosonic action. These assumptions correspond to a *mean field approximation* where  $\phi = \langle \phi \rangle$ . Furthermore, we shift our view from the generating functional towards an effective action of  $\phi$ . We can approximate this effective action to one loop order as:

$$\Gamma[\phi] = S[\phi] + \Gamma^{1\text{-loop}}[\phi], \quad (4.42)$$

where  $\Gamma^{1\text{-loop}}[\phi]$  is the fermionic one-loop correction which we will derive from (4.37). Assuming  $\phi$  to be homogeneous, we take  $\phi = \text{const.}$  which reduces  $\Gamma[\phi]$  to an effective potential, where  $S[\phi]$  is only of the form:

$$S[\phi] = \int d^4x \frac{m^2}{2} \phi^2. \quad (4.43)$$

Now we are left with the task to calculate  $\Gamma^{1\text{-loop}}$ . Starting out with (4.37) and using (3.11), while omitting source terms, we arrive at:

$$\Gamma^{1\text{-loop}}[\phi] = -\ln \left[ \mathcal{N} \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \exp \left( - \int \frac{d^D p}{(2\pi)^D} \bar{\Psi} (G_{\mu\nu} p^\mu p^\nu + h_0 \phi) \Psi \right) \right], \quad (4.44)$$

where we performed a Fourier transform in the exponent. But this path integral is analytically solvable with standard QFT procedure since the exponent is quadratic in  $\Psi$ . Therefore, we only need to evaluate a Gaussian integral resulting in:

$$\Gamma^{1\text{-loop}}[\phi] = -\ln [\mathcal{N} \det (G_{\mu\nu} p^\mu p^\nu + h_0 \phi)]. \quad (4.45)$$

This expression can still be highly non-local due to the complex structure involving the natural logarithm of a determinant. However, since we chose  $\phi$  to be homogeneous it does not contain any spacetime dependencies leaving us with an expression that we are able to evaluate analytically. Inserting  $\gamma_{10}^2$  into the determinant simplifies the expression as calculated in Appendix C:

$$\Gamma^{1\text{-loop}}[\phi] = -\frac{1}{2} \text{Tr} [\ln (\mathcal{N} (-p^4 + h_0^2 \phi^2))]. \quad (4.46)$$

Moreover, since  $\Gamma^{1\text{-loop}}[\phi]$  depends only on  $\phi$  once we have performed the trace operation, we can fix the normalization condition such that for a field expectation value of  $\phi = 0$  the generating functional  $Z$  vanishes as well. Applying this normalization yields:

$$\Gamma^{1\text{-loop}}[\phi] = -\frac{1}{2} \text{Tr} \left[ \ln \left( \frac{p^4 - h_0^2 \phi^2}{p^4} \right) \right]. \quad (4.47)$$

Now we would like to use Frullani's formula [56] in order to rewrite the logarithm and calculate  $\Gamma^{1\text{-loop}}[\phi]$  analytically as it is done in standard QFT. But we cannot apply this method since the numerator in the logarithm may very well be negative ruling out Frullani's formula. We address this problem by examining equation (4.41). Since  $\lambda_0$  has already been restricted to positive values in the previous section and  $m^2$ , representing a mass term, needs to be positive as well,  $h_0^2$  is forced to be negative. Hence, we can replace  $h_0$  with  $i|h_0|$ . The absolute value ensures that we cover all cases since  $(\pm i h_0)^2$  both yield  $-h_0^2$ . Inserting this substitution leads to a slightly

different form of (4.47). The numerator has an overall negative sign which together with the normalization condition yields:

$$\Gamma^{1\text{-loop}}[\phi] = -\frac{1}{2} \text{Tr} \left[ \ln \left( \frac{p^4 + |h_0|^2 \phi^2}{p^4} \right) \right]. \quad (4.48)$$

We can now apply Frullani's formula as the numerator and denominator are real and strictly positive. This allows us to calculate the one-loop correction explicitly (Appendix C):

$$\Gamma^{1\text{-loop}}[\phi] = \frac{(VT)}{2\pi^2} \left( |h_0|^2 \phi^2 (\gamma - 1) + |h_0|^2 \phi^2 \ln \left( \frac{|h_0|^2 \phi^2}{\Lambda^4} \right) \right), \quad (4.49)$$

where  $\gamma$  is the Euler-Mascheroni-constant,  $(VT)$  is a factor representing the space-time volume and  $\Lambda \sim p$  represents a UV cutoff. Combining this correction with  $S[\phi]$  and performing the spacetime integral in  $S[\phi]$  leads to

$$\begin{aligned} \Gamma[\phi] &= (VT) \left( \frac{m^2}{2} + \frac{|h_0|^2}{2\pi^2} \left( (\gamma - 1) + \ln \left( \frac{|h_0|^2 \phi^2}{\Lambda^4} \right) \right) \right) \phi^2, \\ &= (VT) V_{\text{eff}}[\phi], \end{aligned} \quad (4.50)$$

where we defined  $V_{\text{eff}}[\phi]$  as an effective potential [31]. Let us now discuss some properties of  $V_{\text{eff}}$ . It has the form of a Coleman-Weinberg potential [57] which has been studied in multiple contexts as it arises, for example, in  $\phi^4$  theories. It exhibits two non-trivial minima which determine the field expectation value. By inserting  $\frac{|h_0|^2}{|h_0|^2}$  in the first term we can identify  $\frac{m^2}{|h_0|^2}$  with equation (4.41) and the substitution we made  $h_0 \rightarrow i|h_0|$  to find a dependence on  $\lambda_0$ :

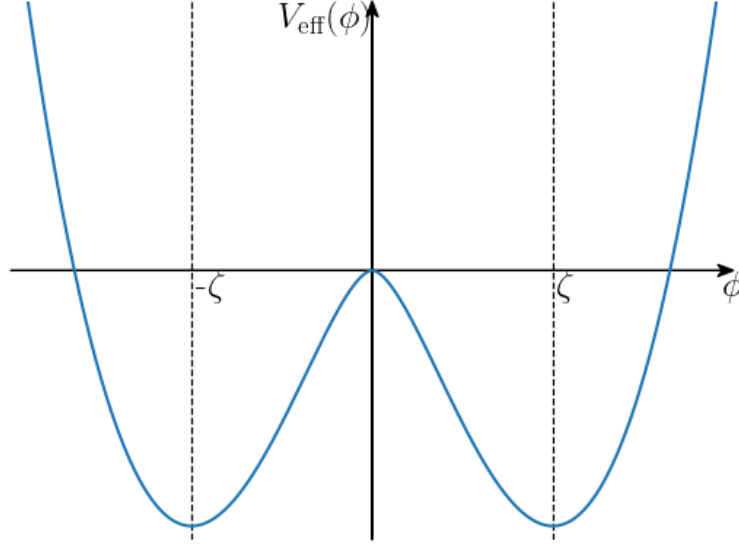
$$V_{\text{eff}}[\phi] = \frac{|h_0|^2 \phi^2}{2\lambda_0} + \frac{|h_0|^2 \phi^2}{2\pi^2} \left( \gamma - 1 + \ln \left( \frac{|h_0|^2 \phi^2}{\Lambda^4} \right) \right). \quad (4.51)$$

Now we can determine both minima as they follow from

$$\left. \frac{dV_{\text{eff}}}{d\phi} \right|_{\langle \phi \rangle} = 0. \quad (4.52)$$

Apart from a trivial solution  $\langle \phi \rangle = 0$ , the two non-trivial solutions read:

$$|h_0| \langle \phi \rangle = \pm \Lambda^2 \exp \left( -\frac{\pi^2}{2\lambda_0} - \frac{\gamma}{2} \right). \quad (4.53)$$



**Figure 6:** Effective potential  $V_{\text{eff}}(\phi)$  over  $\phi$ . Characteristic minima appear at  $\pm\zeta = \pm \frac{\Lambda^2}{|h_0|} \exp\left(-\frac{\pi^2}{2\lambda_0} - \frac{\gamma}{2}\right)$ .

Them being non-trivial corresponds to spontaneous symmetry breaking of the partially bosonized action. The transformation  $\Psi \rightarrow \gamma_{10}\Psi$  together with  $\phi \rightarrow -\phi$  does leave the action invariant but the field expectation value of  $\phi$  attaining a non-zero value breaks this invariance. Usually, this would induce a fermionic mass as a result of strong coupling quantum interactions. However, we have to be careful with such interpretations as we modified  $h_0 \rightarrow i|h_0|$  to be imaginary. In order to better understand what kind of physical consequences the spontaneous symmetry breaking may have for the theory, let us consider the partially bosonized propagator since it indicates whether masses are generated by its pole structure. Returning to the partially bosonized action with the substituted  $i|h_0|$ , we switch to a momentum representation which allows us to find the propagator directly:

$$\begin{aligned}
 S_{\text{FB}}[\bar{\Psi}, \Psi, \phi] &= \int d^D x \left[ -\bar{\Psi} G_{\mu\nu} \partial^\mu \partial^\nu \Psi + i|h_0| \bar{\Psi} \phi \Psi + \frac{m^2}{2} \phi^2 \right] \\
 &= \int \frac{d^D p}{(2\pi)^D} \left[ \bar{\Psi} (G_{\mu\nu} p^\mu p^\nu + i|h_0| \phi) \Psi + \frac{m^2}{2} \phi^2 \right] \\
 \rightarrow K^{-1} &= G_{\mu\nu} p^\mu p^\nu + i|h_0| \phi.
 \end{aligned} \tag{4.54}$$

Inverting the expression in (4.54) yields:

$$\begin{aligned}
 K &= \frac{G_{\mu\nu} p^\mu p^\nu - i|h_0|\phi}{p^4 + |h_0|^2 \phi^2} \\
 K &= \frac{G_{\mu\nu} p^\mu p^\nu - i|h_0|\phi}{(p^2 + i|h_0|\phi)(p^2 - i|h_0|\phi)},
 \end{aligned} \tag{4.55}$$

which results in the propagator having not a real, masslike pole but two conjugate complex poles instead. Therefore, the substitution from  $h_0$  to  $i|h_0|$  to ensure a real scalar field mass  $m$  and positive  $\lambda_0$  in (4.41) leads to the poles in the propagator being shifted off the real axis and into the complex plane. To be precise, we can identify these poles as two conjugate imaginary mass poles  $im_\Psi^2$  which is obvious when deriving the equations of motion for  $\Psi$  from the partially bosonized action  $S_{\text{FB}}$  (4.38):

$$\begin{aligned}
 \delta S_{\text{FB}} &= 0 \\
 0 &= (G_{\mu\nu} p^\mu p^\nu + h_0 \phi) \Psi \\
 \xrightarrow[h_0 = i|h_0|]{\phi = \langle \phi \rangle} 0 &= (G_{\mu\nu} p^\mu p^\nu \pm im_\Psi^2) \Psi,
 \end{aligned} \tag{4.56}$$

where in the last step we assigned the imaginary fermion mass term  $im_\Psi^2 = i|h_0|\langle \phi \rangle$ . Hence, we could also write  $m_\Psi^2$  on the left hand side of equation (4.53). The two propagator poles at  $im_\Psi^2$  and  $-im_\Psi^2$  exhibit an asymptotically free behavior meaning for  $\lambda_0 \rightarrow 0$  they both collapse and vanish leaving us with a non-interacting theory. This corresponds to the Gaussian fixed point we found in the previous section and confirms our results, as we concluded to obtain asymptotic freedom there as well, which of course implies that there are no interactions left which could cause spontaneous symmetry breaking in the first place. Therefore, the  $\gamma_{10}$  symmetry is restored in the UV limit and at any finite scale the fermions stay in a broken phase. However, as we discussed above, the fermions do not acquire a mass in said phase. What we discovered instead is a structure reminiscent of a Gribov-Stingl type propagator [58]. These objects also contain imaginary poles and have been studied in a QCD context when working, for example, with gluon self interactions. In this scenario it has been shown that the Lehmann-Källén spectral function of the propagator does not exhibit positive definiteness, thereby not describing any asymptotically propagating, physical degrees of freedom [59, 60]. The same could be done in our case rendering

the Luttinger fermions not being part of the asymptotic spectrum. In view of QCD, an interesting interpretation emerges. Similar to gluons, Luttinger fermions might also be confined such that they could only appear as part of composite particles (like glueballs [61]) or else decay rapidly as seen in, for example, gluon jets [58]. However, to confirm that Luttinger fermions may in fact show such confinement properties, much more work is needed. Especially, a more in depth analysis of the propagator is required as well as possible couplings to other quantum fields in order to describe potential decaying processes. However, this exceeds the scope of this thesis but it may be of interest to future studies. Nonetheless, we can still continue with some examinations of the effective potential. By identifying  $\phi$  as a constant bi-spinor condensate again, we can resubstitute terms in the effective potential naively using (4.39), (4.41) and again  $h_0 \rightarrow i|h_0|$ , to arrive at a potential for the bi-spinor  $(\bar{\Psi}\Psi)$ . Some caution is needed because after the resubstitution the coupling term  $i|h_0|\bar{\Psi}\phi\Psi$  becomes  $\lambda_0(\bar{\Psi}\Psi)^2$ , thereby contributing to the total effective potential:

$$V_{\text{eff}}[\bar{\Psi}\Psi] = \frac{\lambda_0}{2}(\bar{\Psi}\Psi)^2 - \frac{\lambda_0^2}{2\pi^2}(\bar{\Psi}\Psi)^2 \left( (\gamma - 1) + \ln \left( \frac{-\lambda_0^2(\bar{\Psi}\Psi)^2}{\Lambda^4} \right) \right). \quad (4.57)$$

A negative sign arises when replacing the fields and coupling terms which, at first glance, causes instabilities in the log term by introducing complex parts. However, this arises only due to the imaginary coupling  $i|h_0|\bar{\Psi}\phi\Psi$  which forces a complex contribution to appear. In spite of this, the potential does provide interesting insights in the scaling dependence. That is why we neglect any concerns for these complex parts for now, as this remains only a naive investigation. By artificially inserting a scale dependence  $k$  and regrouping terms with similar field dependence, we can define a renormalized coupling  $\lambda_R$ :

$$\begin{aligned} V_{\text{eff}}[\bar{\Psi}\Psi] &= \left( \frac{\lambda_0}{2} - \frac{\lambda_0^2}{2\pi^2} \left( \gamma - 1 + \ln \left( \frac{k^4}{\Lambda^4} \right) + \ln(\lambda_0^2) \right) \right) (\bar{\Psi}\Psi)^2 - \frac{\lambda_0^2}{2\pi^2} (\bar{\Psi}\Psi)^2 \ln \left( \frac{-(\bar{\Psi}\Psi)^2}{k^4} \right) \\ &= \frac{\lambda_R}{2} (\bar{\Psi}\Psi)^2 - \frac{\lambda_0^2}{2\pi^2} (\bar{\Psi}\Psi)^2 \ln \left( \frac{-(\bar{\Psi}\Psi)^2}{k^4} \right). \end{aligned} \quad (4.58)$$

This renormalized coupling now depends on the arbitrary scale term  $k$ . We derive the flow equation for  $\lambda_R$  by simple differentiation with respect to the RG-time  $t = \ln \left( \frac{k}{\Lambda} \right)$ :

$$\partial_t \lambda_R = -\frac{4}{\pi^2} \lambda_0^2. \quad (4.59)$$

This result is equivalent to the one from section 4.1 when we consider a large flavour limit. It can be seen immediately when we introduce multiple flavours in our mean field approach as this would simply lead to a total prefactor  $N_f$  in equation (4.44) since the different flavours are not interacting with each other in the partially bosonized action  $S_{\text{FB}}$ . Hence, all the different path integrals  $\mathcal{D}\bar{\Psi}_i \mathcal{D}\Psi_i$  can be reduced to  $(\mathcal{D}\bar{\Psi} \mathcal{D}\Psi)^{N_f}$ . Using logarithm rules, this leads to the aforementioned prefactor  $N_f$ . Thus, we modify equation (4.59) to contain this factor:

$$\partial_t \lambda_R = -\frac{4N_f}{\pi^2} \lambda_0^2. \quad (4.60)$$

On the other hand, the flow equation in (4.25) for  $D = 4, \eta_\Psi = 0$  and vanishing tensor channel reads:

$$\partial_t \lambda_0(k) = -\frac{4(N_f - \frac{2}{d_f})}{\pi^2} \lambda_0^2, \quad (4.61)$$

which converges towards (4.60) for large  $N_f$ . Therefore, the renormalized versions of  $\lambda_0$  do in fact match up as expected, again confirming the results of the previous section. Since we did not involve the tensor channel flow or an axial scalar channel, this analysis is incomplete. Especially with respect to even higher order channels there may be induced quantum fluctuations that change the dynamics of the system compared to these results. But as a first minimal approach this should suffice to show that relativistic Luttinger fermions may spontaneously break chiral symmetry with some analogies to a Gross-Neveu model [28], but also major differences, as there is no non-trivial fixed point in the coupling dynamics leading to the system residing in a broken phase for all scales. Additionally, imaginary mass poles, similar to a Gribov-Stingl-type propagator, arise leading to behavior vastly different from a standard Gross-Neveu model as no mass is generated but instead other phenomena like confinement might occur. Yet, in order to describe those dynamics and interpret the physics correctly, more detailed investigations are needed. These results shall conclude our first studies on relativistic Luttinger fermions for now.

## 5 Conclusion

The goal of this thesis was to establish a relativistic theory on Luttinger fermions and elaborate the first studies on their dynamics. We started by constructing a kinetic operator  $\hat{K}$ :

$$\hat{K} = -G_{\mu\nu}\partial^\mu\partial^\nu, \quad (5.1)$$

which is quadratic in derivatives. For the theory to be Lorentz-invariant, we introduced a spin coupling matrix  $G_{\mu\nu}$  in analogy to Dirac matrices. Moreover, we determined the algebra structure of  $G_{\mu\nu}$  and decomposed  $G_{\mu\nu}$  into Lorentz tensors  $a_{\mu\nu}^m$  and  $\gamma_m$  matrices:  $G_{\mu\nu} = a_{\mu\nu}^m\gamma_m$ . These  $\gamma$  matrices satisfy a Euclidean Clifford algebra. By means of dimensional analysis, we concluded  $m \in (1, \dots, 9)$  in four space-time dimensions. However, we also realised that an odd dimensional Clifford algebra is not suitable for the description of fermions with half integer spin. Therefore, we implemented a reducible representation where the algebra is spanned by 10 independent  $\gamma$  matrices instead. Due to this, we were able to derive a spin metric  $h = \gamma_1\gamma_2\gamma_3\gamma_{10}$  which provided a way to define a conjugate spinor  $\bar{\Psi} = \Psi^\dagger h$  that allowed us to devise a real action for relativistic Luttinger fermions. Since the theory is reducible, we actually found a whole subspace of the algebra to act as such a spin metric. Furthermore, we discovered that the same subspace also spans the set of all parity operators meaning the spin metrics are acting as parity transformations. By fixing the spin metric  $h$ , a unique chiral element was established. We continued by constructing four fermion interaction terms. In particular, we found  $(\bar{\Psi}\Psi)^2$  and  $(\bar{\Psi}G_{\mu\nu}\Psi)^2$  to be of our interest as both of these interaction channels interact with each other through quantum fluctuations. We forwent the inclusion of axial scalar and axial tensor channels, which would have left the theory totally chirally symmetric, for simplicity. After an introduction to functional renormalization group techniques we used a minimal ansatz for an action of the form:

$$S = \int d^Dx \left[ -Z_\Psi \bar{\Psi}_a G_{\mu\nu} \partial^\mu \partial^\nu \Psi^a + \frac{\lambda_0}{2} (\bar{\Psi}_a \Psi^a)^2 + \frac{\lambda_t}{2} (\bar{\Psi}_a G_{\mu\nu} \Psi^a)^2 \right], \quad (5.2)$$

and analyzed it with the aforementioned methods. This lead us to the flow equations  $\beta$ . Evaluated at  $D = 4$  and  $N_f = 1$  we found

$$\beta = \partial_t \begin{pmatrix} \lambda_0 \\ \lambda_t \end{pmatrix} = -\frac{1}{(12\pi)^2} \begin{pmatrix} 540\lambda_0^2 - 528\lambda_0\lambda_t \\ -3\lambda_0^2 + 28\lambda_0\lambda_t - 992\lambda_t^2 \end{pmatrix}. \quad (5.3)$$

We established a special case in which the scalar channel coupling vanishes and is



not dynamically generated, which decoupled the two equations. Analyzing only the tensor channel lead to an evolution of  $\lambda_t$  reminiscent of QED and QCD, featuring the appearance of a Landau pole in the IR or UV regime depending on whether  $\lambda_t < 0$  or  $\lambda_t > 0$ . Since the tensor channel flow is always dynamically generated there is no such uncoupled case for  $\lambda_0$ . However, we found the total flow to be asymptotically free for a certain domain of initial conditions. This is due the appearance of a Gaussian fixed point which attracts the flow in the UV-regime. Depending on the UV-behavior, we divided phase space into three distinct domains which are partitioned by two separatrices. Moreover, we computed a susceptibility of  $\beta$  in the UV-finite region to determine for which initial conditions each coupling dominates. We found that for initial conditions at an angle  $\alpha = \arctan\left(\frac{\lambda_t}{\lambda_0}\right)$  the integrated scalar channel dominates in the interval  $-45.765^\circ < \alpha < -0.72^\circ$  where the upper bound is given by the numerically approximated separatrix. Hence, we were able to neglect the contributions from the tensor channel in this region, such that a partial bosonization is justified. We proceeded to analyze the dynamics of the scalar channel by implementing a mean field approximation for a partially bosonized action. We discovered that the theory undergoes spontaneous symmetry breaking of its  $\gamma_{10}$  symmetry for any finite coupling. However, no fermion mass was generated as the poles that arose in the propagator were of a conjugate, imaginary structure reminiscent of Gribov-Stingl propagators which have been discussed in the QCD context. Those poles might lead to the propagator not having a positive definite spectral function in the Lehmann-Källen representation. Therefore, Luttinger fermions might not be part of the asymptotically propagating physical spectrum. In view of the similarity to the Gribov-Stingl propagators we suggested that Luttinger fermions may thus be confined in the broken symmetry phase. But this interpretation needs further investigations to be validated. Furthermore, in our minimal approach we only considered the discrete chiral symmetry as the introduction of an axial scalar channel might dynamically generate an axial tensor channel as well rendering the flow analysis incomplete. Continuous symmetries may be of interest to future studies. Moreover, we derived a renormalized scalar channel coupling  $\lambda_R$  from the effective potential and were able to show that this coupling matches with the renormalized coupling from the flow equation studies in a large  $N_f$  limit thereby confirming the previous results.

This concluded our studies on relativistic Luttinger fermions. Future investigations may develop more intricate models by including axial channels or even other tensor terms. Higher order truncations or different methods, like rebosonization, will

provide further insights into the dynamics of relativistic Luttinger fermions. The asymptotic freedom, which is part of this model, might be an exciting property when introducing gravity into the picture. On the other hand, comprehending the dynamics of the Luttinger fermions comprehensively may provide additional insights into quadratic band touchings in general which are currently of major interest in fields like material science and electronics. Besides, understanding the physical mechanisms that occur when coupling the theory to other quantum fields may be of interest. Those are only a few examples for a multitude of conceivable research areas surrounding Luttinger fermions. We can, without any doubt, be excited for what is to come.

## Appendices

### A Luttinger algebra calculations

**On the summation of  $(G_{\mu\nu}p^\mu p^\nu)^2$**

writing the index summation explicitly, the square of the kinetic operator reads:

$$\begin{aligned}
 \hat{K}^2 = p^4 &= \left( \sum_{\mu} p^{\mu} \left( \sum_{\nu} G_{\mu\nu} p^{\nu} \right) \right)^2 \\
 &= \sum_{\mu} (p^{\mu})^2 \left( \sum_{\nu} G_{\mu\nu} p^{\nu} \right)^2 + \sum_{\substack{\mu, \kappa \\ \mu \neq \kappa}} p^{\mu} p^{\kappa} \left( \sum_{\nu} G_{\mu\nu} p^{\nu} \right) \left( \sum_{\lambda} G_{\kappa\lambda} p^{\lambda} \right) \\
 &= \sum_{\mu} (p^{\mu})^2 \left( \sum_{\nu} G_{\mu\nu}^2 (p^{\nu})^2 + \sum_{\substack{\nu, \sigma \\ \nu \neq \sigma}} G_{\mu\nu} G_{\mu\sigma} p^{\nu} p^{\sigma} \right) + \dots \\
 &= \sum_{\mu, \nu} (G_{\mu\nu} p^{\mu} p^{\nu})^2 + \sum_{\substack{\mu, \nu, \sigma \\ \nu \neq \sigma}} (p^{\mu})^2 p^{\nu} p^{\sigma} G_{\mu\nu} G_{\mu\sigma} + \sum_{\substack{\mu, \nu, \kappa, \lambda \\ \mu \neq \kappa}} p^{\mu} p^{\kappa} p^{\nu} p^{\lambda} G_{\mu\nu} G_{\kappa\lambda}.
 \end{aligned} \tag{4}$$

Now we realize that in order for  $\hat{K}^2$  to contain only even powers of  $p$  the second sum must vanish completely. For this to be the case, we have to demand  $\{G_{\mu\nu}, G_{\mu\sigma}\} = 0$  for  $\nu \neq \sigma$  which is part of (2.4). The remainder comes from the third sum in (4). There we see that only even powers of  $p$  remain when either  $\mu = \nu \wedge \kappa = \lambda$  or  $\mu = \lambda \wedge \nu = \kappa$  holds. Thus, the rest of the sum must vanish as well which is the case when  $\{G_{\mu\mu}, G_{\kappa\lambda}\} = 0$  with  $\kappa \neq \lambda$  and when  $\{G_{\mu\nu}, G_{\kappa\lambda}\} = 0$  with  $\mu \neq \nu \wedge \kappa \neq \lambda \wedge (\mu, \nu) \neq (\kappa, \lambda)$ . Altogether, this leads to the statement in (2.4). We can then proceed by simplifying (4) with these anticommutators which leads to (2.5).

**product relations of  $\gamma$  matrices**

In order to simplify the second line in equation (2.29), we start in the first term by bringing  $\gamma_k$  to the left. Swapping positions of  $\gamma$  matrices by using (2.11) yields the formula in equation (2.29):

$$\begin{aligned}
\prod_{j=1}^n \gamma_{\alpha_j} \cdot \gamma_k &= \prod_{j=1}^{n-1} \gamma_{\alpha_j} (2\delta_{\alpha_n, k} - \gamma_k \gamma_{\alpha_n}) \\
&= 2\delta_{\alpha_n, k} \prod_{j=1}^{n-1} \gamma_{\alpha_j} - \prod_{j=1}^{n-2} \gamma_{\alpha_j} (2\delta_{\alpha_{n-1}, k} - \gamma_k \gamma_{\alpha_{n-1}}) \gamma_{\alpha_n} \\
&= 2\delta_{\alpha_n, k} \prod_{\substack{j=1 \\ j \neq n}}^n \gamma_{\alpha_j} - 2\delta_{\alpha_{n-1}, k} \prod_{\substack{j=1 \\ j \neq n-1}}^n \gamma_{\alpha_j} + \prod_{j=1}^{n-2} \gamma_{\alpha_j} \cdot \gamma_k \cdot \prod_{j=n-1}^n \gamma_{\alpha_j} \\
\text{repeat } n \text{ times} &= 2 \sum_{i=0}^{n-1} \left( (-1)^i \delta_{\alpha_{n-i}, k} \prod_{\substack{j=1 \\ j \neq n-i}}^n \gamma_{\alpha_j} \right) + (-1)^n \gamma_k \prod_{j=1}^n \gamma_{\alpha_j} \\
&= 2 \sum_{i=0}^{n-1} \left( (-1)^i \delta_{\alpha_{n-i}, k} \underbrace{\gamma_k \gamma_{\alpha_{n-i}}}_{=\mathbb{1}} \prod_{\substack{j=1 \\ j \neq n-i}}^n \gamma_{\alpha_j} \right) + (-1)^n \gamma_k \prod_{j=1}^n \gamma_{\alpha_j} \\
\text{all } \alpha_j \text{ are different} &= 2 \sum_{i=0}^{n-1} \left( (-1)^i \delta_{\alpha_{n-i}, k} \gamma_k (-1)^{n-i-1} \prod_{j=1}^n \gamma_{\alpha_j} \right) + (-1)^n \gamma_k \prod_{j=1}^n \gamma_{\alpha_j} \\
&= 2(-1)^{n-1} \gamma_k \prod_{j=1}^n \gamma_{\alpha_j} \sum_{i=0}^{n-1} \left( \delta_{\alpha_{n-i}, k} \right) + (-1)^n \gamma_k \prod_{j=1}^n \gamma_{\alpha_j} \\
&= (1 - 2 \sum_{i=1}^n \delta_{\alpha_i, k}) (-1)^n \gamma_k \prod_{j=1}^n \gamma_{\alpha_j}.
\end{aligned} \tag{5}$$

This identity holds for any set of  $\gamma$ -matrices  $\alpha_i$  as long as  $\alpha_i \neq \alpha_j$  for all  $i \neq j$ .

Using the results of (5) we can show that the trace of any generator  $\Xi^A$  vanishes except for that of  $\mathbb{1}$ :

$$\begin{aligned}
\text{Tr}(\Xi^{A'}) &= \text{Tr}\left(\prod_{i=1}^n \gamma_{\alpha_i}\right) \\
&= \text{Tr}\left(\left[\prod_{i=1}^{n-1} \gamma_{\alpha_i}\right] \cdot \gamma_{\alpha_n}\right) \\
(5), \text{ all } \alpha_i \text{ different} &= \text{Tr}\left((-1)^{n-1} \gamma_{\alpha_n} \prod_{i=1}^{n-1} \gamma_{\alpha_i}\right) \\
\text{cyclicity of Tr} &= (-1)^{n-1} \text{Tr}(\Xi^{A'}) \\
\rightarrow (1 - (-1)^{n-1}) \text{Tr}(\Xi^{A'}) &= 0 \\
\stackrel{n \text{ is even}}{\longrightarrow} \text{Tr}(\Xi^{A'}) &= 0.
\end{aligned} \tag{6}$$

This covers the case where we have an even number of  $\gamma$  matrices. The case of an odd numbered product is way simpler since we can just insert a  $\mathbb{1}$  to show that the trace must vanish:

$$\begin{aligned}
\text{Tr}(\Xi^{A'}) &= \text{Tr}\left(\prod_{i=1}^m \gamma_{\alpha_i}\right) \\
&= \text{Tr}\left(\prod_{i=1}^m \gamma_{\alpha_i} \mathbb{1}\right) \\
&= \text{Tr}\left(\prod_{i=1}^m \gamma_{\alpha_i} \gamma_* \gamma_*\right) \\
\text{using (3.18)} &= \text{Tr}\left((-1)^m \gamma_* \prod_{i=1}^m \gamma_{\alpha_i} \gamma_*\right) \\
\text{cyclicity of Tr} &= (-1)^m \text{Tr}(\Xi^{A'}) \\
\stackrel{m \text{ is odd}}{\longrightarrow} \text{Tr}(\Xi^{A'}) &= 0.
\end{aligned} \tag{7}$$

Together, (6) and (7) show the trace of any generator of  $\text{GL}(32, \mathbb{C})$  vanishes except for the first one, namely the trace of  $\mathbb{1}$ .

## B Functional calculations

### Wick rotation

Employing a Wick rotation  $t \rightarrow \pm i\tau$  in order to go from a term  $iS$  in the exponent to  $-S$  demands that we change  $G_{\mu\nu}$  as well:

$$\begin{aligned}
iS &= i \int d^D x - \bar{\Psi} G_{\mu\nu} \partial^\mu \partial^\nu \Psi \\
&= i \int d^D x - \bar{\Psi} \left[ g^{00} g^{00} G_{00} \partial_0 \partial_0 + 2g^{00} g^{i'i} G_{0i'} \partial_0 \partial_i + g^{i'i} g^{j'j} G_{i'j'} \partial_i \partial_j \right] \Psi \\
\frac{t \rightarrow \pm i\tau}{g_{\mu\nu} \rightarrow \delta_{\mu\nu}} &\rightarrow i \int d(\pm i\tau) d^{D-1} x - \bar{\Psi} G'_{\mu'\nu'} \partial^{\mu'} \partial^{\nu'} \Psi \\
&= \mp \int d\tau d^{D-1} x - \bar{\Psi} \left[ \delta^{00} \delta^{00} G'_{00} (\mp i \partial_0) (\mp i \partial_0) \right. \\
&\quad \left. + 2\delta^{00} \delta^{i'i} G'_{0i'} (\mp i \partial_0) \partial_i + \delta^{i'i} \delta^{j'j} G'_{i'j'} \partial_i \partial_j \right] \Psi \stackrel{!}{=} -S \\
\frac{g^{ii} = -1}{g^{00} = \delta^{\mu\mu} = 1} &\rightarrow \mp G'_{00} = G_{00} \quad \mp i G'_{0i} = G_{0i} \quad \pm G'_{ij} = G_{ij}.
\end{aligned} \tag{8}$$

The upper sign always stands for the case where we employ the positive Wick rotation  $t \rightarrow i\tau$ . These new  $G'_{\mu\nu} = G_{\mu\nu}^E$  can also be understood as the euclidean versions of  $G_{\mu\nu}$ . It can now easily be shown that these  $G_{\mu\nu}^E$  do in fact satisfy a euclidean analogue to (2.9) where we had the Minkowski versions  $G_{\mu\nu}^M$ :

$$\begin{aligned}
\{G_{\mu\nu}^E, G_{\kappa\lambda}^E\} &= -\frac{2}{D-1} \delta_{\mu\nu} \delta_{\kappa\lambda} + \frac{D}{D-1} (\delta_{\mu\kappa} \delta_{\nu\lambda} + \delta_{\mu\lambda} \delta_{\nu\kappa}) \\
\left. \begin{aligned} \{G_{00}^M, G_{00}^M\} &= 2 \\ \{G_{00}^E, G_{00}^E\} &= 2 \end{aligned} \right\} &\rightarrow G_{00}^M = \mp G_{00}^E \\
\left. \begin{aligned} \{G_{00}^M, G_{ii}^M\} &= -\frac{2}{D-1} g_{00} g_{ii} = \frac{2}{D-1} \\ \{G_{00}^E, G_{ii}^E\} &= -\frac{2}{D-1} \delta_{00} \delta_{ii} = -\frac{2}{D-1} \end{aligned} \right\} &\rightarrow G_{ii}^M = \pm G_{ii}^E \\
\left. \begin{aligned} \{G_{0i}^M, G_{0i}^M\} &= \frac{D}{D-1} g_{00} g_{ii} = -\frac{D}{D-1} \\ \{G_{0i}^E, G_{0i}^E\} &= \frac{D}{D-1} \delta_{00} \delta_{ii} = \frac{D}{D-1} \end{aligned} \right\} &\rightarrow G_{0i}^M = \mp i G_{0i}^E \\
\left. \begin{aligned} \{G_{ij}^M, G_{ij}^M\} &= \frac{D}{D-1} g_{ii} g_{jj} = \frac{D}{D-1} \\ \{G_{00}^E, G_{ii}^E\} &= \frac{D}{D-1} \delta_{ii} \delta_{jj} = \frac{D}{D-1} \end{aligned} \right\} &\rightarrow G_{ij}^M = \pm G_{ij}^E.
\end{aligned} \tag{9}$$

This goes to show that we are free to choose which Wick rotation to employ as long as we only consider the kinetic term. This differs from Dirac theory where we need to use the negative rotation to obtain a negative sign in front of the action. This results from the spin metric changing as well under the Wick rotation which is not the case

for Luttinger fermions. There is no need for  $h$  to change as it is just comprised of  $\gamma$  matrices and not of any prefactors  $a_{\mu\nu}^m$ . But these are the only objects which have to transform under Wick rotations as they alone encode the spacetime metric whereas in Dirac theory the  $\gamma$  matrices themselves contain this information forcing them to transform under Wick rotation. However, if we introduce interaction terms we need to be careful. For a scalar channel of the form  $(\bar{\Psi}\Psi)^2$  the action transforms as follows:

$$\begin{aligned} iS &= i \int d^D x (\bar{\Psi}\Psi)^2 \\ \xrightarrow{t \rightarrow \pm i\tau} &= \mp \int d\tau d^{D-1}x (\bar{\Psi}\Psi)^2 = \mp S. \end{aligned} \quad (10)$$

Therefore, we have to use the positive Wick rotation. We also want to remark that an interaction channel of the form  $(\bar{\Psi}G_{\mu\nu}\Psi)^2$  does in fact not change under both Wick rotations and thus also needs a positive rotation to achieve the negative sign in front of  $S$ .

#### derivative of $W_k$

In (3.24) some simplifications were made that will be detailed in the following calculations. In particular, the derivative of  $W_k$  with respect to  $t$  has to be computed. In order to do this, let us establish another relation concerning (3.21):

$$\begin{aligned} \frac{\delta^2 W[J]}{\delta J^T(x) \delta J(y)} &= \frac{\delta}{J^T(x)} \left( \frac{1}{Z_k} \frac{\delta Z_k}{\delta J(y)} \right) = \frac{1}{Z_k} \frac{\delta^2 Z_k}{\delta J^T(x) \delta J(y)} - \frac{1}{Z_k^2} \frac{\delta Z_k}{\delta J^T(x)} \frac{\delta Z_k}{\delta J(y)} \\ &= \langle \Psi \cdot \Psi^T \rangle - \langle \Psi \rangle \cdot \langle \Psi^T \rangle = \langle \Psi \cdot \Psi^T \rangle - \Phi \cdot \Phi^T = K(x, y). \end{aligned} \quad (11)$$

Here the  $\cdot$  stands for a generalized outer product in contrast to the scalar product and  $K$  is again the propagator of connected correlators. Additionally, we need to consider how to take the derivative of  $W_k$  with respect to  $k$  since the scale dependence lies within the matrix valued regulator function. In general, for a scalar function  $g(\mathbf{U})$  depending on the matrix  $\mathbf{U}(t)$  the derivative with respect to  $t$  reads [62]:

$$\frac{\partial g(\mathbf{U})}{\partial t} = \text{Tr} \left( \frac{\partial g(\mathbf{U})}{\partial \mathbf{U}} \cdot \frac{\partial \mathbf{U}}{\partial t} \right). \quad (12)$$

Furthermore, introducing the scalar product  $A^T \cdot \mathbf{U} \cdot B$  where  $A$  and  $B$  are vectors we can take the derivative with respect to  $U$  [62]:

$$\frac{\partial(A^T \cdot \mathbf{U} \cdot B)}{\partial \mathbf{U}} = A \cdot B^T \quad (13)$$

Therefore, we can differentiate  $W_k$  with respect to  $t$  as follows:

$$\begin{aligned} \partial_t W_k &= \text{Tr} \left( \frac{\partial W_k}{\partial R_k} \cdot \frac{\partial R_k}{\partial t} \right) \\ &= \text{Tr} \left( -\frac{1}{2} \frac{1}{Z_k} \int \mathcal{D}\Psi e^{-S[\Psi] - \Delta S_k[\Psi] + \mathbf{J}^T \cdot \Psi} \Psi \cdot \Psi^T \cdot \partial_t R_k \right) \\ &= \text{Tr} \left( \left( \left( -\frac{1}{2} \frac{1}{Z_k} \int \mathcal{D}\Psi e^{-S[\Psi] - \Delta S_k[\Psi] + \mathbf{J}^T \cdot \Psi} \Psi \cdot \Psi^T \right) + \frac{1}{2} \Phi \cdot \Phi^T - \frac{1}{2} \Phi \cdot \Phi^T \right) \cdot \partial_t R_k \right) \\ &= \text{Tr} \left( -\frac{1}{2} K \cdot \partial_t R_k - \frac{1}{2} \Phi \cdot \Phi^T \cdot \partial_t R_k \right) \\ &= -\frac{1}{2} \text{Tr} (K \cdot \partial_t R_k) - \frac{1}{2} \text{Tr} (\Phi \cdot \Phi^T \cdot \partial_t R_k) \\ &= -\frac{1}{2} \text{Tr} (K \cdot \partial_t R_k) - \partial_t \Delta S_k[\Phi]. \end{aligned} \quad (14)$$

Inserting this into the first equation of (3.24) leads to  $\Delta S_k$  being canceled resulting in the Wetterich equation [18].



## C Flow equation and mean field calculations

### Functional derivative of $\Gamma_k$

To evaluate the second derivatives of (4.6) for constant background fields (4.1), we start by considering:

$$\frac{\delta}{\delta \Psi_n(p)} \Psi_a(q) = (2\pi)^D \delta^{(D)}(p - q) \delta_{na}. \quad (15)$$

Furthermore, noticing that we can always transpose  $\Gamma_k$  as it is a scalar we arrive at expressions for  $\Gamma_{\text{kin}}$ ,  $\Gamma_0$  and  $\Gamma_t$  which contain the transposed spinors. Additionally, we can substitute  $q \rightarrow -q$  in the case of  $\Gamma_{\text{kin}}$ :

$$\Gamma_{\text{kin}} = \int \frac{d^D q}{(2\pi)^D} -Z_\Psi \Psi^{aT}(-q) G_{\mu\nu}^T q^\mu q^\nu \bar{\Psi}_a^T(-q) \quad (16)$$

$$\Gamma_0 = \frac{\bar{\lambda}_0}{2} \prod_{i=1}^3 \int \frac{d^D q_i}{(2\pi)^D} \Psi^{bT}(q_1 - q_2 + q_3) \bar{\Psi}_b^T(q_3) \Psi^{aT}(q_2) \bar{\Psi}_a^T(q_1) \quad (17)$$

$$\Gamma_t = \frac{\bar{\lambda}_t}{2} \prod_{i=1}^3 \int \frac{d^D q_i}{(2\pi)^D} \Psi^{bT}(q_1 - q_2 + q_3) G^{\mu\nu T} \bar{\Psi}_b^T(q_3) \Psi^{aT}(q_2) G_{\mu\nu}^T \bar{\Psi}_a^T(q_1). \quad (18)$$

For  $\Gamma_0$  and  $\Gamma_t$  we could also just transpose one of the scalar products  $\bar{\Psi}_i \Psi^i$  and receive an overall minus sign. With these transposed versions in place we can easily apply the functional derivatives. Starting with  $\Gamma_{\text{kin}}$  we get:

$$\begin{aligned} \frac{\overrightarrow{\delta} \Gamma_{\text{kin}} \overleftarrow{\delta}}{\delta \Psi_m^T(-p) \delta \Psi_n(p')} &= 0 \\ \frac{\overrightarrow{\delta} \Gamma_{\text{kin}} \overleftarrow{\delta}}{\delta \Psi_m^T(-p) \delta \bar{\Psi}_n^T(-p')} &= -G_{\mu\nu}^T p^\mu p^\nu Z_\Psi (2\pi)^D \delta^{(D)}(p - p') \delta_{mn} \\ \frac{\overrightarrow{\delta} \Gamma_{\text{kin}} \overleftarrow{\delta}}{\delta \bar{\Psi}_m(p) \delta \bar{\Psi}_n^T(-p')} &= 0 \\ \frac{\overrightarrow{\delta} \Gamma_{\text{kin}} \overleftarrow{\delta}}{\delta \bar{\Psi}_m(p) \delta \Psi_n(p')} &= G_{\mu\nu} p^\mu p^\nu Z_\Psi (2\pi)^D \delta^{(D)}(p - p') \delta_{mn}, \end{aligned} \quad (19)$$

which we can write in matrix representation to arrive exactly at (4.9). Similarly, we can apply the derivatives to the scalar and tensor channels. There we have to be a bit more careful as the product rule has to be taken care of. We now perform this calculation for  $\Gamma_t$ :

$$\begin{aligned}
 \frac{\vec{\delta} \Gamma_t \overleftarrow{\delta}}{\delta \Psi_m^T(-p) \delta \Psi_n(p')} &= \frac{\bar{\lambda}_t}{2} \prod_{i=1}^3 \int \frac{d^D q_i}{(2\pi)^D} \\
 &\cdot \left[ -(2\pi)^{2D} \delta^{(D)}(q_1 - q_2 + q_3 + p) \delta_{bm} G_{\mu\nu}^T \bar{\Psi}_b^T(q_3) \bar{\Psi}_a(q_1) G^{\mu\nu} \delta^{(D)}(q_2 - p') \delta_{an} \right. \\
 &\quad \left. - (2\pi)^{2D} \delta^{(D)}(q_2 - p) \delta_{am} G_{\mu\nu}^T \bar{\Psi}_a^T(q_1) \bar{\Psi}_b(q_3) G^{\mu\nu} \delta^{(D)}(q_1 - q_2 + q_3 + p') \delta_{bn} \right]. \quad (20)
 \end{aligned}$$

Here we can now insert the background field definition and resolve the delta functions with the integrations over  $q_i$ . Furthermore, we can simplify the Kronecker deltas from the flavours with the remaining fields leading to:

$$\frac{\vec{\delta} \Gamma_t \overleftarrow{\delta}}{\delta \Psi_m^T(-p) \delta \Psi_n(p')} = -\bar{\lambda}_0 G_{\mu\nu}^T \bar{\Psi}_m^T \bar{\Psi}_n G^{\mu\nu} (2\pi)^D \delta^{(D)}(p - p'). \quad (21)$$

Moreover, we can compute the next part:

$$\begin{aligned}
 \frac{\vec{\delta} \Gamma_t \overleftarrow{\delta}}{\delta \bar{\Psi}_m(p) \delta \Psi_n(p')} &= \frac{\bar{\lambda}_t}{2} \prod_{i=1}^3 \int \frac{d^D q_i}{(2\pi)^D} \\
 &\cdot \left[ (2\pi)^{2D} \delta^{(D)}(q_1 - p) \delta_{am} G_{\mu\nu} \Psi_a(q_2) \bar{\Psi}_b(q_3) G^{\mu\nu} \delta^{(D)}(q_1 - q_2 + q_3 - p') \delta_{bn} \right. \\
 &\quad + (2\pi)^{2D} \delta^{(D)}(q_1 - p) \delta^{(D)}(q_2 - p') \delta_{mn} G_{\mu\nu} \bar{\Psi}_b(q_3) G^{\mu\nu} \Psi^b(q_1 - q_2 + q_3) \\
 &\quad + (2\pi)^{2D} \bar{\Psi}_a(q_1) G_{\mu\nu} \Psi^a(q_2) G^{\mu\nu} \delta^{(D)}(q_3 - p) \delta^{(D)}(q_1 - q_2 + q_3 - p') \delta_{mn} \\
 &\quad \left. + (2\pi)^{2D} \delta^{(D)}(q_3 - p) \delta_{bm} G_{\mu\nu} \Psi_b(q_1 - q_2 + q_3) \bar{\Psi}_a(q_1) G^{\mu\nu} \delta^{(D)}(q_2 - p') \delta_{an} \right]. \quad (22)
 \end{aligned}$$

Again, we can insert the background fields and contract the flavour indices as well as perform the integrations leading to:

$$\frac{\vec{\delta} \Gamma_t \overleftarrow{\delta}}{\delta \bar{\Psi}_m(p) \delta \Psi_n(p')} = -\bar{\lambda}_t \left( -\delta_{mn} (\bar{\Psi} G_{\mu\nu} \Psi) G^{\mu\nu} - G_{\mu\nu} \Psi_m \bar{\Psi}_n G^{\mu\nu} \right) (2\pi)^D \delta^{(D)}(p - p'), \quad (23)$$

where the brackets in  $(\bar{\Psi} G_{\mu\nu} \Psi)$  indicate that we sum over the flavour indices leading to a scalar in spinor space.

Analogously, we can compute the rest of the the tensor channel derivatives:

$$\frac{\overrightarrow{\delta} \Gamma_t \overleftarrow{\delta}}{\delta \bar{\Psi}_m(p) \delta \bar{\Psi}_n^T(-p')} = -\bar{\lambda}_0 G_{\mu\nu} \psi_m \psi_n^T G^{\mu\nu T} (2\pi)^D \delta^{(D)}(p - p') \quad (24)$$

$$\frac{\overrightarrow{\delta} \Gamma_t \overleftarrow{\delta}}{\delta \Psi_m^T(-p) \delta \bar{\Psi}_n^T(-p')} = -\bar{\lambda}_t \left( \delta_{mn} (\bar{\psi} G_{\mu\nu} \psi) G^{\mu\nu} - G_{\mu\nu}^T \bar{\psi}_m^T \psi_n^T G^{\mu\nu T} \right) (2\pi)^D \delta^{(D)}(p - p'). \quad (25)$$

Altogether, equations (21), (23), (24) and (25) can be arranged in the form of (4.12). Moreover,  $\Gamma_0^{(2)}$  yields the same results just without any of the  $G_{\mu\nu}$  and  $G^{\mu\nu}$  terms and, of course, with the scalar channel coupling instead.

### Inverse of $\mathcal{I}_k$

Checking that  $\mathcal{I}_k^{-1}$  from (4.14) is the inverse to  $\mathcal{I}_k$  can be done by matrix multiplication:

$$\begin{aligned} \mathcal{I}_k^{-1} \cdot \mathcal{I}_k &= \sum_i^{N_f} \int \frac{d^D \tilde{p}}{(2\pi)^D} (2\pi)^{2D} \delta_{mi} \delta_{in} \delta^{(D)}(\tilde{p} - p) \delta^{(D)}(\tilde{p} - p') \frac{1}{p^4} \\ &\quad \cdot \begin{pmatrix} 0 & G_{\mu\nu} \tilde{p}^\mu \tilde{p}^\nu \\ -G_{\mu\nu}^T \tilde{p}^\mu \tilde{p}^\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & -G_{\mu\nu}^T \tilde{p}^\mu \tilde{p}^\nu \\ G_{\mu\nu} \tilde{p}^\mu \tilde{p}^\nu & 0 \end{pmatrix} \\ &= (2\pi)^D \delta_{mn} \delta^{(D)}(p - p') \mathbb{1}, \end{aligned} \quad (26)$$

which shows that  $\mathcal{I}_k^{-1}$  is the sought inverse.

### Trace computation

A more challenging task is the calculation of the trace of  $\mathcal{M}_{11}$ . In order to compute it, let us split this matrix into three components using the definition of  $\mathcal{D}_0$  and  $\mathcal{D}_t$  in equations (4.11) and (4.12):

$$\begin{aligned} \mathcal{M}_{11} &= G_{\mu\nu} (\mathcal{D}_{21}^0 + \mathcal{D}_{21}^t) G_{\kappa\lambda} (\mathcal{D}_{21}^0 + \mathcal{D}_{21}^t) - G_{\mu\nu} (\mathcal{D}_{22}^0 + \mathcal{D}_{22}^t) G_{\kappa\lambda}^T (\mathcal{D}_{11}^0 + \mathcal{D}_{11}^t) \\ &= \underbrace{G_{\mu\nu} \mathcal{D}_{21}^0 G_{\kappa\lambda} \mathcal{D}_{21}^0 - G_{\mu\nu} \mathcal{D}_{22}^0 G_{\kappa\lambda}^T \mathcal{D}_{11}^0}_{:= \mathcal{M}_{11}^{00}} + \underbrace{G_{\mu\nu} \mathcal{D}_{21}^t G_{\kappa\lambda} \mathcal{D}_{21}^t - G_{\mu\nu} \mathcal{D}_{22}^t G_{\kappa\lambda}^T \mathcal{D}_{11}^t}_{:= \mathcal{M}_{11}^{tt}} \\ &\quad + \underbrace{G_{\mu\nu} \mathcal{D}_{21}^0 G_{\kappa\lambda} \mathcal{D}_{21}^t + G_{\mu\nu} \mathcal{D}_{21}^t G_{\kappa\lambda} \mathcal{D}_{21}^0 - G_{\mu\nu} \mathcal{D}_{22}^0 G_{\kappa\lambda}^T \mathcal{D}_{11}^t - G_{\mu\nu} \mathcal{D}_{22}^t G_{\kappa\lambda}^T \mathcal{D}_{11}^0}_{:= \mathcal{M}_{11}^{0t}}. \end{aligned} \quad (27)$$

The superscripts of  $\mathcal{M}_{11}^{00}$ ,  $\mathcal{M}_{11}^{tt}$  and  $\mathcal{M}_{11}^{0t}$  are used to indicate their dependence on the couplings meaning all parts containing a  $\lambda_0^2$  are found within  $\mathcal{M}_{11}^{00}$ , mixed terms  $\lambda_0\lambda_t$  are in  $\mathcal{M}_{11}^{0t}$  and  $\lambda_t^2$  parts are in  $\mathcal{M}_{11}^{tt}$ . We can now calculate each part separately. But before that, let us observe one more thing. In equation (4.20) we need to perform a momentum integral over  $\sim \frac{p^\mu p^\nu p^\kappa p^\lambda}{p^8}$  which is an integral of the form  $\int d^D p f(p^2) p^\mu p^\nu p^\kappa p^\lambda$ . We can rewrite this integral in terms of the metric tensors instead:

$$\int d^D p p^\mu p^\nu p^\kappa p^\lambda f(p^2) \stackrel{!}{=} c \int d^D p p^4 f(p^2) (g^{\mu\nu} g^{\kappa\lambda} + g^{\mu\kappa} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\kappa}), \quad (28)$$

where  $c$  is a constant that we need to determine. In order to calculate  $c$ , we can contract the left side of (28) with  $g_{\mu\nu} g_{\kappa\lambda}$  to arrive at:

$$g_{\mu\nu} g_{\kappa\lambda} \int d^D p p^\mu p^\nu p^\kappa p^\lambda f(p^2) = \int d^D p p^4 f(p^2), \quad (29)$$

whereas contracting the right hand side of equation (28) yields:

$$g_{\mu\nu} g_{\kappa\lambda} c \int d^D p p^4 f(p^2) (g^{\mu\nu} g^{\kappa\lambda} + g^{\mu\kappa} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\kappa}) = c(D^2 + 2D) \int d^D p p^4 f(p^2). \quad (30)$$

Therefore, we can conclude that

$$c = \frac{1}{D(D+2)}. \quad (31)$$

With this information we can simplify some terms in the integral (4.20). We can contract  $(g^{\mu\nu} g^{\kappa\lambda} + g^{\mu\kappa} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\kappa})$  with the matrices  $G_{\mu\nu}$  and  $G_{\kappa\lambda}$  in  $\mathcal{M}_{11}$  which yields a factor of two since the first addend vanishes due to  $G_{\mu\nu}$  being trace-less whereas the other two contractions have the same result:  $G_{\mu\nu}$  and  $G_{\kappa\lambda}$  terms in  $\mathcal{M}_{11}$  are being contracted. Therefore, we can overall replace  $p^\mu p^\nu p^\kappa p^\lambda$  in (4.20) with  $\frac{2p^4}{D(D+2)}$  and contract the matrices  $G_{\mu\nu}$ ,  $G_{\kappa\lambda}$  or  $G_{\kappa\lambda}^T$  in  $\mathcal{M}_{11}$  with one another. Now we can go back to computing the trace and we start with  $\mathcal{M}_{11}^{00}$ :

$$\begin{aligned} & \text{Tr}(\mathcal{M}_{11}^{00}) \\ &= \text{Tr} \left( G_{\mu\nu} \mathcal{D}_{21}^0 G^{\mu\nu} \mathcal{D}_{21}^0 - G_{\mu\nu} \mathcal{D}_{22}^0 G^{\mu\nu T} \mathcal{D}_{11}^0 \right) \\ &= \bar{\lambda}_0^2 \text{Tr} \left[ G_{\mu\nu} (-\bar{\psi}\psi) \delta_{mi} - \psi_m \bar{\psi}_i \right] G^{\mu\nu} (-\bar{\psi}\psi) \delta_{in} - \psi_i \bar{\psi}_n - G_{\mu\nu} \psi_m \psi_i^T G^{\mu\nu T} \bar{\psi}_i^T \bar{\psi}_n \Big], \end{aligned} \quad (32)$$

where the brackets around bi-spinors again indicate that their flavor indices are being summed over. Since there is a sum over flavour indices in (4.20) which arose from the total trace, we can perform it here now in order to simplify. Furthermore, using properties of the trace and transposing some scalar terms we end up with:

$$\begin{aligned}
 & \text{Tr}(\mathcal{M}_{11}^{00}) \\
 &= \bar{\lambda}_0^2 \text{Tr} \left[ (\bar{\psi}\psi)^2 G_{\mu\nu} G^{\mu\nu} \delta_{mn} \mathbb{1} + 2(\bar{\psi}\psi) G_{\mu\nu} G^{\mu\nu} \mathbb{1} \psi_m \bar{\psi}_n + 2G_{\mu\nu} \psi_m (\bar{\psi} G^{\mu\nu} \psi) \bar{\psi}_n \right] \\
 &= \bar{\lambda}_0^2 \left[ N_f d_\gamma \frac{D(D+2)}{2} (\bar{\psi}\psi)^2 - D(D+2) (\bar{\psi}\psi)^2 - 2(\bar{\psi} G_{\mu\nu} \psi)^2 \right] \\
 &= \bar{\lambda}_0^2 \frac{D(D+2)}{2} \left[ (N_f d_\gamma - 2) (\bar{\psi}\psi)^2 - \frac{4}{D(D+2)} (\bar{\psi} G_{\mu\nu} \psi)^2 \right],
 \end{aligned} \tag{33}$$

where we used

$$\text{Tr}(\delta_{mn} \mathbb{1}) = N_f d_\gamma \tag{34}$$

$$\text{Tr}(\psi_m \bar{\psi}_n) = -(\bar{\psi}\psi) \tag{35}$$

$$\text{Tr}(\psi_m G_{\mu\nu} \bar{\psi}_n) = -(\bar{\psi} G_{\mu\nu} \psi) \tag{36}$$

$$G_{\mu\nu} G^{\mu\nu} = \frac{D(D+2)}{2}, \tag{37}$$

to simplify the terms. Now we can include the factor  $\frac{2}{D(D+2)}$  in order to simplify even further:

$$\bar{\lambda}_0^2 \left[ (N_f d_\gamma - 2) (\bar{\psi}\psi)^2 - \frac{4}{D(D+2)} (\bar{\psi} G_{\mu\nu} \psi)^2 \right]. \tag{38}$$

With this first expression calculated, we can move forward to  $\text{Tr}(\mathcal{M}_{11}^{tt})$ . For this, let us first establish some relations between different  $G_{\mu\nu}$  which we need to compute some expressions. They both result from (2.9):

$$G_{\mu\nu} G_{\kappa\lambda} = \left( \frac{-2}{D-1} g_{\mu\nu} g_{\kappa\lambda} + \frac{D}{D-1} (g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa}) \right) \mathbb{1} - G_{\kappa\lambda} G_{\mu\nu} \tag{39}$$

$$\begin{aligned}
 \rightarrow \text{Tr}[G_{\mu\nu} G_{\kappa\lambda}] &= \frac{1}{2} \text{Tr}[\{G_{\mu\nu}, G_{\kappa\lambda}\}] = \frac{d_\gamma}{2} \left( \frac{-2}{D-1} g_{\mu\nu} g_{\kappa\lambda} + \frac{D}{D-1} (g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa}) \right).
 \end{aligned} \tag{40}$$

Using these expressions, we can now compute  $\text{Tr}(\mathcal{M}_{11}^{tt})$ :

$$\begin{aligned}
 & \text{Tr}(\mathcal{M}_{11}^{tt}) \\
 &= \bar{\lambda}_t^2 \text{Tr}[G_{\mu\nu} \mathcal{D}_{21}^t G_{\kappa\lambda} \mathcal{D}_{21}^t - G_{\mu\nu} \mathcal{D}_{22}^t G_{\kappa\lambda}^T \mathcal{D}_{11}^t] \\
 &= \text{Tr} \left[ G_{\kappa\lambda} ((\bar{\psi} G_{\mu\nu} \psi) G^{\mu\nu} \delta_{mi} + G_{\mu\nu} \psi_m \bar{\psi}_i G^{\mu\nu}) G^{\kappa\lambda} ((\bar{\psi} G_{\mu\nu} \psi) G^{\mu\nu} \delta_{in} + G_{\mu\nu} \psi_i \bar{\psi}_n G^{\mu\nu}) \right. \\
 &\quad \left. - G_{\kappa\lambda} G_{\mu\nu} \psi_m \psi_i^T G^{\mu\nu T} G^{\kappa\lambda T} G_{\rho\sigma}^T \bar{\psi}_i^T \bar{\psi}_n G^{\rho\sigma} \right] \bar{\lambda}_t^2 \\
 &= \text{Tr} \left[ \delta_{mn} G_{\mu\nu} G^{\kappa\lambda} G^{\mu\nu} G^{\rho\sigma} (\bar{\psi} G_{\kappa\lambda} \psi) (\bar{\psi} G_{\rho\sigma} \psi) + (\bar{\psi} G_{\kappa\lambda} \psi) G_{\mu\nu} G^{\kappa\lambda} G^{\mu\nu} G_{\rho\sigma} \psi_m \bar{\psi}_n G^{\rho\sigma} \right. \\
 &\quad + G_{\mu\nu} G_{\kappa\lambda} \psi_m \bar{\psi}_n G^{\kappa\lambda} G^{\mu\nu} G^{\rho\sigma} (\bar{\psi} G_{\rho\sigma} \psi) + G_{\mu\nu} G_{\kappa\lambda} \psi_m (\bar{\psi} G^{\kappa\lambda} G^{\mu\nu} G_{\rho\sigma} \psi) \bar{\psi}_n G^{\rho\sigma} \\
 &\quad \left. + G_{\mu\nu} G_{\kappa\lambda} \psi_m (\bar{\psi} G_{\rho\sigma} G^{\mu\nu} G^{\kappa\lambda} \psi)^T \bar{\psi}_n G^{\rho\sigma} \right] \bar{\lambda}_t^2 \\
 &= \bar{\lambda}_t^2 \left[ N_f d_\gamma \alpha \frac{D}{D-1} (\bar{\psi} G_{\mu\nu} \psi)^2 - 2\alpha^2 (\bar{\psi} G_{\mu\nu} \psi)^2 - (\bar{\psi} G^{\kappa\lambda} G^{\mu\nu} G_{\rho\sigma} \psi) (\bar{\psi} G^{\rho\sigma} G_{\mu\nu} G_{\kappa\lambda} \psi) \right. \\
 &\quad \left. - (\bar{\psi} G_{\rho\sigma} G^{\mu\nu} G^{\kappa\lambda} \psi) (\bar{\psi} G^{\rho\sigma} G_{\mu\nu} G_{\kappa\lambda} \psi) \right], \tag{41}
 \end{aligned}$$

where we introduced  $\alpha = \left( \frac{2D}{D-1} - \frac{D(D+2)}{2} \right)$  to shorten the expressions. We see that tensor terms of higher order arise. These are problematic since they are dynamically induced fluctuations that we did not cover in our action. However, we can manipulate them algebraically and see that they actually cancel out, leaving only contributions toward the tensor channel  $(\bar{\psi} G_{\mu\nu} \psi)$ . To perform the cancellation let us only focus on the last term in (41) and try to change the order of matrices to match up with the other term. For that, let us simply study how to change the order of the 3 matrices. After that, we can insert the result in the last term in (41):

$$\begin{aligned}
 G_{\rho\sigma} G^{\mu\nu} G^{\kappa\lambda} &= \left( \frac{-2}{D-1} g_{\rho\sigma} g^{\mu\nu} + \frac{D}{D-1} (g_\rho^\mu g_\sigma^\nu + g_\rho^\nu g_\sigma^\mu) \right) G^{\kappa\lambda} - G^{\mu\nu} G_{\rho\sigma} G^{\kappa\lambda} \\
 &= \left( \frac{-2}{D-1} g_{\rho\sigma} g^{\mu\nu} + \frac{D}{D-1} (g_\rho^\mu g_\sigma^\nu + g_\rho^\nu g_\sigma^\mu) \right) G^{\kappa\lambda} \\
 &\quad - \left( \frac{-2}{D-1} g_{\rho\sigma} g^{\kappa\lambda} + \frac{D}{D-1} (g_\rho^\kappa g_\sigma^\lambda + g_\rho^\lambda g_\sigma^\kappa) \right) G^{\mu\nu} + G^{\mu\nu} G^{\kappa\lambda} G_{\rho\sigma} \\
 &= \left( \frac{-2}{D-1} g_{\rho\sigma} g^{\mu\nu} + \frac{D}{D-1} (g_\rho^\mu g_\sigma^\nu + g_\rho^\nu g_\sigma^\mu) \right) G^{\kappa\lambda} \\
 &\quad - \left( \frac{-2}{D-1} g_{\rho\sigma} g^{\kappa\lambda} + \frac{D}{D-1} (g_\rho^\kappa g_\sigma^\lambda + g_\rho^\lambda g_\sigma^\kappa) \right) G^{\mu\nu} \\
 &\quad + \left( \frac{-2}{D-1} g^{\mu\nu} g^{\kappa\lambda} + \frac{D}{D-1} (g^{\mu\kappa} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\kappa}) \right) G_{\rho\sigma} - G^{\kappa\lambda} G^{\mu\nu} G_{\rho\sigma}. \tag{42}
 \end{aligned}$$

We can now insert this result into the last term in (41). Contracting all the indices from the metrics yields:

$$\begin{aligned}
 & (\bar{\psi} G_{\rho\sigma} G^{\mu\nu} G^{\kappa\lambda} \psi) (\bar{\psi} G^{\rho\sigma} G_{\mu\nu} G_{\kappa\lambda} \psi) \\
 &= \left( \bar{\psi} \left( \frac{-2}{D-1} g_{\rho\sigma} g^{\mu\nu} + \frac{D}{D-1} (g_\rho^\mu g_\sigma^\nu + g_\rho^\nu g_\sigma^\mu) \right) G^{\kappa\lambda} \psi \right) (\bar{\psi} G^{\rho\sigma} G_{\mu\nu} G_{\kappa\lambda} \psi) \\
 &\quad - \left( \bar{\psi} \left( \frac{-2}{D-1} g_{\rho\sigma} g^{\kappa\lambda} + \frac{D}{D-1} (g_\rho^\kappa g_\sigma^\lambda + g_\rho^\lambda g_\sigma^\kappa) \right) G^{\mu\nu} \psi \right) (\bar{\psi} G^{\rho\sigma} G_{\mu\nu} G_{\kappa\lambda} \psi) \\
 &\quad + \left( \bar{\psi} \left( \frac{-2}{D-1} g^{\mu\nu} g^{\kappa\lambda} + \frac{D}{D-1} (g^{\mu\kappa} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\kappa}) \right) G_{\rho\sigma} \psi \right) (\bar{\psi} G^{\rho\sigma} G_{\mu\nu} G_{\kappa\lambda} \psi) \\
 &\quad - (\bar{\psi} G^{\kappa\lambda} G^{\mu\nu} G_{\rho\sigma} \psi) (\bar{\psi} G^{\rho\sigma} G_{\mu\nu} G_{\kappa\lambda} \psi) \\
 &= \frac{2D^2(D+2)}{D-1} (\bar{\psi} G_{\mu\nu} \psi)^2 - \frac{2D}{D-1} (\bar{\psi} G_{\mu\nu} \psi) (\bar{\psi} G^{\kappa\lambda} G_{\mu\nu} G_{\kappa\lambda} \psi) \\
 &\quad - (\bar{\psi} G^{\kappa\lambda} G^{\mu\nu} G_{\rho\sigma} \psi) (\bar{\psi} G^{\rho\sigma} G_{\mu\nu} G_{\kappa\lambda} \psi) \\
 &= \frac{2D}{D-1} [D(D+2) - \alpha] (\bar{\psi} G_{\mu\nu} \psi)^2 - (\bar{\psi} G^{\kappa\lambda} G^{\mu\nu} G_{\rho\sigma} \psi) (\bar{\psi} G^{\rho\sigma} G_{\mu\nu} G_{\kappa\lambda} \psi).
 \end{aligned} \tag{43}$$

Therefore, the tensor terms of higher order cancel and the total trace of  $\mathcal{M}_{11}^{tt}$  reads:

$$\text{Tr}(\mathcal{M}_{11}^{tt}) = \bar{\lambda}_t^2 \left[ N_f d_\gamma \alpha \frac{D}{D-1} - 2\alpha^2 - \frac{2D}{D-1} (D(D+2) - \alpha) \right] (\bar{\psi} G_{\mu\nu} \psi)^2. \tag{44}$$

Again, we can include the factor  $\frac{2}{D(D+2)}$  from the integration. By using  $d_e = \frac{1}{2}D^2 + \frac{1}{2}D - 1$  from chapter 2.1 (equation (2.10)) we can simplify by noticing two relations:

$$\frac{2}{D(D+2)} \alpha = \frac{2}{d_e} - 1 \tag{45}$$

$$\frac{D-1}{D} \alpha = \frac{1}{d_e} \left( \frac{2}{d_e} - 1 \right). \tag{46}$$

Now we simplify (44) by multiplication with  $\frac{2}{D(D+2)}$  and putting  $\frac{D}{d_e(D-1)}$  outside the brackets:

$$\text{Tr}(\mathcal{M}_{11}^{tt}) = \bar{\lambda}_t^2 \frac{D}{d_e(D-1)} \left[ N_f d_\gamma (2 - d_e) - 2(d_e^2 - d_e + 2) \right]. \tag{47}$$

Taking out another minus sign leads to the desired expression in (4.21). Lastly, we have to compute  $\text{Tr}(\mathcal{M}_{11}^{0t})$ . For that we need to know that the trace of a single  $G_{\mu\nu}$  vanishes. This can easily be checked by expanding  $G_{\mu\nu}$  as  $a_{\mu\nu}^m \gamma_m$  and using the trace relations from Appendix A (equation (7)). Now we are ready to take on  $\text{Tr}(\mathcal{M}_{11}^{0t})$ :

$$\begin{aligned}
 & \text{Tr}(\mathcal{M}_{11}^{0t}) \\
 &= \bar{\lambda}_0 \bar{\lambda}_t \text{Tr} \left[ 2G_{\mu\nu} \mathcal{D}_{21}^0 G^{\mu\nu} \mathcal{D}_{21}^t - G_{\mu\nu} \mathcal{D}_{22}^0 G^{\mu\nu T} \mathcal{D}_{11}^t - G_{\mu\nu} \mathcal{D}_{22}^t G^{\mu\nu T} \mathcal{D}_{11}^0 \right] \\
 &= \text{Tr} \left[ 2G_{\mu\nu} ((\bar{\psi}\psi)\delta_{mi} + \psi_m \bar{\psi}_i) G^{\mu\nu} \left( (\bar{\psi} G_{\kappa\lambda} \psi) G^{\kappa\lambda} \delta_{in} + G_{\kappa\lambda} \psi_i \bar{\psi}_n G^{\kappa\lambda} \right) \right. \\
 &\quad \left. - G_{\mu\nu} \psi_m \bar{\psi}_i^T G^{\mu\nu T} G_{\kappa\lambda}^T \bar{\psi}_i^T \bar{\psi}_n G^{\kappa\lambda} - G_{\mu\nu} G_{\kappa\lambda} \psi_m \psi_i^T G^{\kappa\lambda T} G^{\mu\nu T} \bar{\psi}_i^T \bar{\psi}_n \right] \bar{\lambda}_0 \bar{\lambda}_t \\
 &= \text{Tr} \left[ 2G_{\mu\nu} G^{\mu\nu} (\bar{\psi}\psi) (\bar{\psi} G_{\kappa\lambda} \psi) G^{\kappa\lambda} \delta_{mn} + 2(\bar{\psi}\psi) G_{\mu\nu} G^{\mu\nu} G_{\kappa\lambda} \psi_m \bar{\psi}_n G^{\kappa\lambda} \right. \\
 &\quad + 2G_{\mu\nu} \psi_m \bar{\psi}_n G^{\mu\nu} (\bar{\psi} G_{\kappa\lambda} \psi) G^{\kappa\lambda} + 2G_{\mu\nu} \psi_m (\bar{\psi} G^{\mu\nu} G_{\kappa\lambda} \psi) \bar{\psi}_n G^{\kappa\lambda} \\
 &\quad \left. + G_{\mu\nu} \psi_m (\bar{\psi} G_{\kappa\lambda} G^{\mu\nu} \psi)^T \bar{\psi}_n G^{\kappa\lambda} + G_{\mu\nu} G_{\kappa\lambda} \psi_m (\bar{\psi} G^{\mu\nu} G^{\kappa\lambda} \psi)^T \bar{\psi}_n \right] \bar{\lambda}_0 \bar{\lambda}_t \\
 &= - \left[ \frac{1}{2} D^2 (D+2)^2 (\bar{\psi}\psi)^2 + 2\alpha (\bar{\psi} G_{\mu\nu} \psi)^2 + 2(\bar{\psi} G_{\kappa\lambda} G_{\mu\nu} \psi) (\bar{\psi} G^{\kappa\lambda} G^{\mu\nu} \psi) \right. \\
 &\quad \left. + 2(\bar{\psi} G_{\kappa\lambda} G_{\mu\nu} \psi) (\bar{\psi} G^{\mu\nu} G^{\kappa\lambda} \psi) \right] \bar{\lambda}_0 \bar{\lambda}_t.
 \end{aligned} \tag{48}$$

We see that again higher order tensor channels are generated. Let us again manipulate the last term such that the order of  $G_{\mu\nu}$  matches the second to last term:

$$\begin{aligned}
 & (\bar{\psi} G_{\kappa\lambda} G_{\mu\nu} \psi) (\bar{\psi} G^{\mu\nu} G^{\kappa\lambda} \psi) \\
 &= (\bar{\psi} G_{\kappa\lambda} G_{\mu\nu} \psi) (\bar{\psi} \left( \frac{-2}{D-1} g^{\mu\nu} g^{\kappa\lambda} + \frac{D}{D-1} (g^{\mu\kappa} g^{\nu\lambda} + g^{\mu\lambda} g^{\nu\kappa}) - G^{\kappa\lambda} G^{\mu\nu} \right) \psi) \\
 &= \frac{D^2 (D+2)}{D-1} (\bar{\psi}\psi)^2 - (\bar{\psi} G_{\kappa\lambda} G_{\mu\nu} \psi) (\bar{\psi} G^{\kappa\lambda} G^{\mu\nu} \psi).
 \end{aligned} \tag{49}$$

This time the higher order tensor channels also cancel out. The rest then reads:

$$\text{Tr}(\mathcal{M}_{11}^{0t}) = -\bar{\lambda}_0 \bar{\lambda}_t \left[ \left( \frac{1}{2} D^2 (D+2)^2 + 2 \frac{D^2 (D+2)}{D-1} \right) (\bar{\psi}\psi)^2 + 2\alpha (\bar{\psi} G_{\mu\nu} \psi)^2 \right], \tag{50}$$

which turns into

$$\text{Tr}(\mathcal{M}_{11}^{0t}) = -\bar{\lambda}_0 \bar{\lambda}_t \left[ \frac{D}{D-1} (2d_e + 4) (\bar{\psi}\psi)^2 - \left( 2 - \frac{4}{d_e} \right) (\bar{\psi} G_{\mu\nu} \psi)^2 \right], \tag{51}$$

when we include the factor  $\frac{2}{D(D+2)}$  again and write some parts in terms of  $d_e$ . Having computed all of the trace we can now rearrange the terms according to their channel. After doing so, we end up with the total result of (4.21).



### Threshold function

We can manipulate the momentum integral after the introduction of  $\lambda_0 = \bar{\lambda}_0 Z_\Psi^{-2} k^{D-4}$  and  $\lambda_t = \bar{\lambda}_t Z_\Psi^{-2} k^{D-4}$ . Due to the couplings being quadratic on the right hand side of (4.20) and only linear on the left hand side, we are left with a factor  $k^{4-D}$  as well as  $Z_\Psi^2$  in the momentum integral.  $Z_\Psi$  cancels nicely. Additionally, we can manipulate the variables such that a substitution of  $x = \frac{p^2}{k^2}$  simplifies the rest:

$$\begin{aligned} \int d^D p \frac{k^{4-D}}{p^4} \frac{\partial_t r_\Psi - \eta_\Psi r_\Psi}{(1 + r_\Psi)^3} &= \nu_D \int_0^\infty dp \frac{p}{k^2} \frac{p^{D-6}}{k^{D-6}} \frac{\partial_t r_\Psi - \eta_\Psi r_\Psi}{(1 + r_\Psi)^3} \\ &\xrightarrow{\frac{p^2}{k^2} = x} = \frac{\nu_D}{2} \int_0^\infty dx x^{\frac{D-6}{2}} \frac{\partial_t r_\Psi - \eta_\Psi r_\Psi}{(1 + r_\Psi)^3}. \end{aligned} \quad (52)$$

The regulator from (3.9) can easily be expressed in terms of  $x$ :

$$r_\Psi = \left( x^{-\frac{1}{2}} - 1 \right) \theta(1 - x). \quad (53)$$

Furthermore, we can express the derivative with respect to  $t$  in terms of  $x$ :

$$\partial_t = k \frac{d}{dk} = k \frac{dx}{dk} \frac{d}{dx} = -2x \frac{d}{dx}. \quad (54)$$

Inserting these relations into (52) yields:

$$\begin{aligned} &\int_0^\infty dx x^{\frac{D-6}{2}} \frac{\partial_t r_\Psi - \eta_\Psi r_\Psi}{(1 + r_\Psi)^3} \\ &= \int_0^\infty dx x^{\frac{D-6}{2}} \frac{2x\delta(1-x) \left( x^{-\frac{1}{2}} - 1 \right) + x^{-\frac{1}{2}}\theta(1-x) - \eta_\Psi \left( x^{-\frac{1}{2}} - 1 \right) \theta(1-x)}{\left( 1 + \left( x^{-\frac{1}{2}} - 1 \right) \theta(1-x) \right)^3} \\ &= \int_0^1 dx x^{\frac{D-4}{2}} - \eta_\Psi \int_0^1 dx \left( x^{\frac{D-4}{2}} - x^{\frac{D-3}{2}} \right) \\ &= \frac{2}{D-2} - \frac{2\eta_\Psi}{(D-2)(D-1)}. \end{aligned} \quad (55)$$

### Simplifications on the one-loop effective action correction

We want to simplify the expression in equation (4.47). As mentioned we achieve this by inserting  $\gamma_{10}^2 = \mathbb{1}$ :

$$\begin{aligned}
 & -\ln[\mathcal{N} \det(G_{\mu\nu} p^\mu p^\nu + h_0 \phi)] \\
 &= -\frac{1}{2} \ln[\det(G_{\mu\nu} p^\mu p^\nu + h_0 \phi)^2] - \ln[\mathcal{N}] \\
 &= -\frac{1}{2} \ln[\det((G_{\mu\nu} p^\mu p^\nu + h_0 \phi)(G_{\mu\nu} p^\mu p^\nu + h_0 \phi) \gamma_{10}^2)] - \ln[\mathcal{N}] \\
 &= -\frac{1}{2} \ln[\det((G_{\mu\nu} p^\mu p^\nu + h_0 \phi) \gamma_{10} (-G_{\mu\nu} p^\mu p^\nu + h_0 \phi) \gamma_{10})] - \ln[\mathcal{N}] \\
 &= -\frac{1}{2} \ln[\det((G_{\mu\nu} p^\mu p^\nu + h_0 \phi) (-G_{\mu\nu} p^\mu p^\nu + h_0 \phi))] - \ln[\mathcal{N}] \\
 &= -\frac{1}{2} \text{Tr}[\ln(\mathcal{N}(-p^4 + h_0^2 \phi^2))],
 \end{aligned} \tag{56}$$

where we made use of some determinant properties as well as  $\ln[\det(\dots)] = \text{Tr}[\ln(\dots)]$ .

### Calculation of the one-loop correction

Using Frullanis formula [56], we can express the logarithm as:

$$-\frac{1}{2} \text{Tr} \left[ \ln \left( \frac{p^4 + h_0^2 \phi^2}{p^4} \right) \right] = -\frac{1}{2} \text{Tr} \left[ \int_0^\infty \frac{dS}{S} \left( e^{-Sp^4} - e^{-S(p^4 + |h_0|^2 \phi^2)} \right) \right]. \tag{57}$$

Writing the trace as the spacetime volume and a momentum integral leads to:

$$\begin{aligned}
 -\frac{1}{2} \text{Tr} \left[ \ln \left( \frac{p^4 + h_0^2 \phi^2}{p^4} \right) \right] &= -\frac{(VT)d_\gamma}{2} \int_0^\infty \frac{dS}{S} \int \frac{d^D p}{(2\pi)^D} e^{-Sp^4} \left( 1 - e^{-S|h_0|^2 \phi^2} \right) \\
 &= -\frac{(VT)d_\gamma \vartheta_D}{2(2\pi)^D} \int_0^\infty \frac{dS}{S} \left( 1 - e^{-S|h_0|^2 \phi^2} \right) \int_0^\infty dp p^{D-1} e^{-Sp^4},
 \end{aligned} \tag{58}$$

where  $\vartheta_D$  again denotes the solid angle in  $D$  dimensions. We can evaluate the momentum integral by employing the substitution  $t = Sp^4$ :

$$\begin{aligned}
 \int_0^\infty dp p^{D-1} e^{-Sp^4} &= \frac{1}{4} \frac{1}{S^{\frac{D}{4}}} \int_0^\infty dt t^{\frac{D}{4}-1} e^{-t} \\
 &= \frac{\Gamma(D/4)}{4S^{D/4}}.
 \end{aligned} \tag{59}$$

Thus, the trace simplifies to

$$-\frac{1}{2} \text{Tr} \left[ \ln \left( \frac{p^4 + h_0^2 \phi^2}{p^4} \right) \right] = -\frac{(VT) d_\gamma \vartheta_D \Gamma(D/4)}{8(2\pi)^D} \int_0^\infty \frac{dS}{S^2} \left( 1 - e^{-S|h_0|^2 \phi^2} \right). \quad (60)$$

We perform this integral by using integration by parts twice yielding:

$$\begin{aligned} \int_0^\infty \frac{dS}{S^2} \left( 1 - e^{-S|h_0|^2 \phi^2} \right) &= \frac{e^{-S|h_0|^2 \phi^2} - 1}{S} \Big|_0^\infty + |h_0|^2 \phi^2 \int_0^\infty \frac{dS}{S} e^{-S|h_0|^2 \phi^2} \\ &= \frac{e^{-S|h_0|^2 \phi^2} - 1}{S} \Big|_0^\infty + |h_0|^2 \phi^2 \ln(S) e^{-S|h_0|^2 \phi^2} \Big|_{1/\Lambda^4}^\infty \\ &\quad + |h_0|^4 \phi^4 \int_0^\infty dS \ln(S) e^{-S|h_0|^2 \phi^2}, \end{aligned} \quad (61)$$

where we introduced a cutoff  $\frac{1}{\Lambda^4}$  in order to regularize the integral. Again, we employ a substitution,  $|h_0|^2 \phi^2 S = x$  in order to evaluate the last part:

$$\begin{aligned} |h_0|^4 \phi^4 \int_0^\infty dS \ln(S) e^{-S|h_0|^2 \phi^2} &= |h_0|^2 \phi^2 \left[ \int_0^\infty dx \ln(x) e^{-x} - \ln(|h_0|^2 \phi^2) \int_0^\infty dx e^{-x} \right] \\ &= -|h_0|^2 \phi^2 [\gamma + \ln(|h_0|^2 \phi^2)]. \end{aligned} \quad (62)$$

Only the first two addends from (61) remain. In both cases the upper bound is exponentially suppressed. In the first term the lower bound can be evaluated using a Taylor expansion

$$\begin{aligned} \frac{e^{-S|h_0|^2 \phi^2} - 1}{S} \Big|_0^\infty &= -\lim_{S \rightarrow 0} \frac{1 - S|h_0|^2 \phi^2 + \mathcal{O}(S^2) - 1}{S} \\ &= |h_0|^2 \phi^2. \end{aligned} \quad (63)$$

The lower bound of the second addend can also be simplified yielding

$$\begin{aligned} |h_0|^2 \phi^2 \ln(S) e^{-S|h_0|^2 \phi^2} \Big|_{1/\Lambda^4}^\infty &= -|h_0|^2 \phi^2 \ln \left( \frac{1}{\Lambda^4} \right) e^{-\frac{|h_0|^2 \phi^2}{\Lambda^4}} \\ &= |h_0|^2 \phi^2 \ln(\Lambda^4) + \mathcal{O}(\Lambda^{-4}). \end{aligned} \quad (64)$$

Collecting the results from (62), (63) and (64) and omitting  $\mathcal{O}(\Lambda^{-4})$ , we can arrange them to the form:

$$-\frac{1}{2} \text{Tr} \left[ \ln \left( \frac{p^4 + h_0^2 \phi^2}{p^4} \right) \right] = -\frac{(VT) d_\gamma \vartheta_D \Gamma(D/4)}{8(2\pi)^D} |h_0|^2 \phi^2 \left[ 1 - \gamma - \ln \left( \frac{|h_0|^2 \phi^2}{\Lambda^4} \right) \right], \quad (65)$$

which results in equation (4.49) for  $D = 4$ .

## D Conventions

### Units

In this thesis we study a the relativistic quantum field theory for Luttinger fermions. Since it is common practice to employ natural units in works on quantum fields, we abide by these conventions and set

$$\hbar = c = k_B = 1, \quad (66)$$

thereby relating SI units to an energy scale which in turn corresponds to a mass scale as well.

### Metric

We employ the "-2" convention where the Minkowski metric reads:

$$g_{\mu\nu} = \text{diag}(+, -, -, -). \quad (67)$$

### Fourier transformation

We use a negative sign for the primary transformation from position to momentum space and employ the normalization factor as follows:

$$\int d^D x e^{-i p_\mu x^\mu} = (2\pi)^D \delta^{(D)}(p) \quad (68)$$

$$\Psi(x) = \int \frac{d^D p}{(2\pi)^D} \Psi(p) e^{i p_\mu x^\mu} \quad (69)$$

$$\bar{\Psi}(x) = \int \frac{d^D p}{(2\pi)^D} \bar{\Psi}(p) e^{-i p_\mu x^\mu}. \quad (70)$$

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