Bachelor Thesis

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Spin-Base Symmetry and Noether current of the Dirac Equation

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1 Introduction

The Dirac equation is one of the most well-known equations in physics. It was derived by Paul Dirac and is still one of the most credited achievements in physics as it was the first successful attempt at describing a theory for a massive spin-$\frac{1}{2}$ particle which is consistent with Einsteins special relativity. The discovery of this equation brought heaps of new insight into the field of atomic and particle physics and is an important ingredient in the construction of the standard model - the most accurate description of modern physics to date. Extensive study of this equation is therefore essential for the understanding of modern physics. Over the course of history theoretical physics used different approaches to unveil the mysteries of the universe. The Noether Theorem is considered as one of the best approaches for analyzing a theory by considering its symmetries and was discovered and proved by Emmy Noether [1]. These symmetries, if continuous, give rise to conservation laws. Establishing this relation was one of Noether's greatest achievements and layed the foundation for many following works in modern physics. Her theorem is imperative to this thesis and leads to the results concluded at the end of this paper.

The symmetries of the Dirac equation are diverse and wide ranged but also well known, which is why this thesis considers a less well studied symmetry. The Dirac equation has a rather large GL(4,$\mathbb{C}$) symmetry which can be achieved by transforming not only the spinor field but also the Dirac matrices. This spin-base invariance and the corresponding formalism has been known since 1932 when it was discovered by Schrödinger and Bargmann [2],[3]. The invariance of the Clifford algebra under similarity transformations (also SL(4,$\mathbb{C}$)) and the resulting equivalence of the Dirac matrices, independent of their representation, was firstly done by Pauli [4]. It is the goal of this thesis to establish all the necessary mathematical tools to handle Noether's Theorem in conjunction with fermions in flat space-time and consequently to derive the corresponding conservations that arise from the spin-base symmetry by calculating the Noether current.

A few conventions which are used should be clarified first as it might lead to confusion otherwise:

- Greek indices refer to space-time indices and run from 0 to 3 with 0 representing the time coordinate and 1, 2 and 3 representing the space coordinates.
- Latin indices refer to other indices (for example a spinor index) and will be specified, if used.
- The spacetime metric will be represented as $g^{\mu\nu}$ and reads: $\mathrm{diag}(1,-1,-1,-1)$.
- Equations in which there is any importance of units are formulated in natural units meaning $\hbar = c = 1$.
- The Einstein summation convention is used meaning that identical upper and lower indices are summed over

$$p_\mu x^\mu = g_{\mu\nu} p^\nu x^\mu = p^0 x^0 - p^1 x^1 - p^2 x^2 - p^3 x^3. \tag{1.1}$$

2 Classic fermionic field theory

2.1 Dirac Lagrangian and spinor fields

In order to analyze the behavior of fermions and describe them accurately, one finds that they cannot be characterized by a standard vector or tensor field. The need for spinor fields arises, as they obey the Pauli principle and meet all the mathematical needs emerging from the observations of fermions like electrons. With the help of these spinor fields which are represented as 4-component complex spinors and the 4×4 complex Dirac matrices satisfying their central and defining property, the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}, \tag{2.1}$$

electrons can be described in a rather simple way using the action principle and a corresponding Lagrangian density which reads

$$\mathscr{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi. \tag{2.2}$$

Where $\overline{\psi} = \psi^\dagger\gamma^0$ is the Dirac conjugate of ψ. It follows the equation of motion for a fermion by using the Euler-Lagrange equation

$$0 = \frac{\partial\mathscr{L}}{\partial\overline{\psi}} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\overline{\psi})}. \tag{2.3}$$

This in turn results in the famous Dirac equation [5] which describes the relativistic motion of spin-$\frac{1}{2}$ particles with mass m.

$$0 = (i\gamma^\mu\partial_\mu - m)\psi. \tag{2.4}$$

As fermions make up an important part of standard model particles, in particular the electron, it is almost obligatory to study this equation in a detailed manner and try to understand its underlying physics. This has been done many times, deepening our insight on these kinds of particles. Because of that this thesis is not focused on the Dirac equation itself and its consequences but instead it is centered on its symmetries. Still one should get a glimpse of classic fermion fields to better understand the analysis of its symmetries.

Taking another look at the Lagrangian density and trying to obtain the equation of motion for the Dirac-conjugated spinor $\overline{\psi}$ it becomes clear that one can easily achieve this by using the chain rule

$$i\overline{\psi}\gamma^\mu\partial_\mu\psi = i\partial_\mu(\overline{\psi}\gamma^\mu\psi) - i\partial_\mu\overline{\psi}\gamma^\mu\psi. \tag{2.5}$$

Inserting this into the Lagrangian density and using the fact that the total derivative term modifies the action only by an irrelevant surface term one obtains the equivalent Lagrangian

$$\mathscr{L} = -i\partial_\mu\overline{\psi}\gamma^\mu\psi - m\overline{\psi}\psi. \tag{2.6}$$

Considering the action of the field

$$S = \int_{-\infty}^{\infty} \mathrm{d}^4 x \mathscr{L}, \tag{2.7}$$

the total derivative term in the action becomes a surface term which vanishes for physical fields. With this considering only the other part of the chain rule is sufficient. It then follows the equation of motion for the Dirac-conjugated spinor using the Euler-Lagrange-equation on (2.6)

$$0 = \frac{\partial \mathscr{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi)}, \tag{2.8}$$

which directly leads to

$$0 = \overline{\psi}(i\gamma^\mu \partial_\mu + m). \tag{2.9}$$

With equation (2.4) and (2.9) the nature of fermionic behavior becomes a little clearer since the combination of these equations leads to a rather well known result.

$$0 = \overline{\psi}(i\gamma^\mu \partial_\mu + m)(i\gamma^\mu \partial_\mu - m)\psi, \tag{2.10}$$

which by means of the Clifford algebra simplifies to

$$0 = \overline{\psi}(\partial_\mu \partial^\mu - m^2)\psi. \tag{2.11}$$

This means that fermionic motion must also satisfy the famous Klein-Gordon equation componentwise. Solutions to the Klein-Gordon equation are well known and can be described by plane waves and interpreted as wavelike massive particles [6],[7]. Thus the solutions to the Dirac equation should also have a plane wave part [8]. The full extent of solving the Dirac equation is not necessary to understand the following parts which is why only the final solutions will be presented. The solutions for ψ and $\overline{\psi}$ in the Dirac equation then yield

$$\begin{array}{ll} \psi = u(E,\vec{p}) \cdot e^{-ip_\mu x^\mu} & \psi = v(E,\vec{p}) \cdot e^{ip_\mu x^\mu} \\ \overline{\psi} = \overline{u}(E,\vec{p}) \cdot e^{ip_\mu x^\mu} & \overline{\psi} = \overline{v}(E,\vec{p}) \cdot e^{-ip_\mu x^\mu} \end{array}, \tag{2.12}$$

where $\vec{p}$ describes the standard 3-dimensional momentum and E represents the Energy of field excitations. The $u(E,\vec{p})$ and $v(E,\vec{p})$ are arbitrary constant spinors that satisfy

$$(i\gamma^\mu \partial_\mu - m)u(E,\vec{p}) = 0 \qquad (i\gamma^\mu \partial_\mu + m)v(E,\vec{p}) = 0. \tag{2.13}$$

Concerning the nature of fermions it should also be mentioned that the scalar product of two spinors is antisymmetric, meaning the spinors anticommute. This must be the case as it is the direct consequence of the Pauli principle which fermions do obey.

2.2 Dirac matrices

To construct a mathematically consistent theory not only the spinors should obey an anti-commutation relation as this is fundamental to the behavior of fermions. That is why a closer look at them is essential. The γ or Dirac matrices are a specific set of complex 4×4 matrices that satisfy the already presented anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}. \tag{2.14}$$

This relation is the defining property of them as it demands an underlying structure: the clifford algebra. It dictates how spin properties connect to special relativity. To better see their connection to spin it is important to realise that these Dirac matrices can be expressed by using the well-known Pauli matrices which are used in nonrelativistic descriptions of spin. One representation of the γ^μ is called the Dirac basis and reads

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{2.15}$$

where i runs from 1 to 3 and the σ refer to the Pauli matrices. There are other known representations that satisfy the given anticommutation relation, for example the Weyl basis. But the actual matrix representation is not needed to handle these objects as their structural characteristics resulting from the Clifford algebra are enough. Especially important is the following property which directly leads to the hermicity of the zeroth Dirac matrix and the antihermicity of the others [9].

$$\left(\gamma^\mu\right)^\dagger = \gamma^0\gamma^\mu\gamma^0. \tag{2.16}$$

Another well defined term connected to the Dirac matrices is the so called γ_5 matrix

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.17}$$

which simplifies many expressions and has some interesting properties as it anticommutes with all other Dirac matrices, is hermitian and its square is the identity as proven in appendix A (7.1)

$$\{\gamma_5, \gamma^\mu\} = 0 \qquad \left(\gamma_5\right)^\dagger = \gamma_5 \qquad \left(\gamma_5\right)^2 = \mathbb{1}. \tag{2.18}$$

The γ_5 matrix is often used to discuss chirality phenomena and it can be utilized to define left and right handed projection operators but it also makes its appearance elsewhere as will be seen later on. Another interesting term arises when trying to commute two Dirac matrices. The result is a totally antisymmetric tensor which forms a linearly independent set of matrices together with γ_5, γ^μ and $\gamma^\mu\gamma_5$

$$\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]. \tag{2.19}$$

A special relation between the Dirac matrices proven in appendix A (7.1) should be established as well since it is rather useful in the calculations of the Noether current

$$\gamma^\mu\gamma^\nu\gamma^\sigma = g^{\mu\nu}\gamma^\sigma + g^{\nu\sigma}\gamma^\mu - g^{\mu\sigma}\gamma^\nu + i\epsilon^{\mu\nu\sigma\kappa}\gamma_\kappa\gamma_5, \tag{2.20}$$

where ϵ is the antisymmetric Levi-Civita tensor.

3 Noether's theorem

As the basics of fermionic field theory in flat space-time have been presented the next part focuses on Nother's Theorem [1]. Since the following section is solely centered around establishing the mathematical methodology it is not necessary to consider spinor fields in particular. Hence an arbitrary field ϕ with an arbitrary Lagrangian is used. Noethers Theorem is usually derived from the classical action principle without special relativity.

$$S = \int \mathrm{d}t L. \tag{3.1}$$

This easily carries over to a relativistic format where space and time are treated equally by looking not at the standard Lagrangian but at the Lagrangian density

$$L = \int \mathrm{d}^3x \mathscr{L}(\phi, \partial_\mu\phi). \tag{3.2}$$

With this the action becomes a Lorentz-invariant scalar if the Lagrangian density is Lorentz-invariant as the measure d^4x already satisfies this symmetry

$$S = \int \mathrm{d}^4x \mathscr{L}(\phi, \partial_\mu\phi). \tag{3.3}$$

Now one can consider symmetries by looking at infinitesimal transformations of the field.

$$\phi \longrightarrow \phi' = \phi + \delta\phi, \tag{3.4}$$

where $\delta\phi$ is an infinitesimal continuous deformation. As we are looking at symmetry transformations the field equations must remain invariant. Hence the Lagrangian density can only change by a total derivative since these only contribute a surface term to the action as explained in chapter 2.1

$$\mathscr{L} \longrightarrow \mathscr{L}' = \mathscr{L} + \delta\mathscr{L} \qquad \delta\mathscr{L} = \partial_\mu K^\mu. \tag{3.5}$$

The invariance of the action and the equations of motion can be related to a conserved quantity which is precisely what Noether's Theorem does. It states:

Let $\phi \longrightarrow \phi + \delta\phi$ and $\mathscr{L} \longrightarrow \mathscr{L} + \delta\mathscr{L} = \mathscr{L} + \partial_\mu K^\mu$ be a symmetry transformation. Then there is a conserved current

$$J^{\mu} = \frac{\partial \mathscr{L}}{\partial(\partial_{\mu}\phi)}\delta\phi + \partial_{\mu}K^{\mu}$$

In order to prove this one can take a look at the change of the Lagrangian density

$$\delta\mathscr{L} = \partial_{\mu}K^{\mu} = \frac{\partial \mathscr{L}}{\partial \phi}\delta\phi + \frac{\partial \mathscr{L}}{\partial(\partial_{\mu}\phi)}\delta\partial_{\mu}\phi. \tag{3.6}$$

Because functional derivatives and the space-time derivatives interchange and the chain rule can be applied the following can be obtained

$$\partial_{\mu}K^{\mu} = \left(\frac{\partial \mathscr{L}}{\partial \phi} - \partial_{\mu}\frac{\partial \mathscr{L}}{\partial(\partial_{\mu}\phi)}\right)\delta\phi + \partial_{\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_{\mu}\phi)}\delta\phi\right), \tag{3.7}$$

where it immediately becomes clear that a current is conserved since the first expression in parentheses vanishes as it is the Euler-Lagrange equation. The rest then simplifies to

$$0 = \partial_{\mu}\left(\frac{\partial \mathscr{L}}{\partial(\partial_{\mu}\phi)}\delta\phi - K^{\mu}\right) = \partial_{\mu}J^{\mu}, \tag{3.8}$$

which leaves J^{μ} as a conserved 4-current. Not only is this a simple proof but it also provides a straightforward way to calculate the Noether current. Moreover, there is another quantity of interest. One can obtain the so-called Noether charge by integrating the zeroth component of the Noether current over space

$$Q = \int \mathrm{d}^3 x J^0. \tag{3.9}$$

It directly follows that Q is conserved with respect to time meaning $\dot{Q} = 0$ since time and space derivatives interchange. Using these mathematical tools will prove indispensable in chapter 5.

4 GL(4,ℂ) symmetry representation

4.1 Dirac-Algebra and Action Invariance

The symmetry of interest is called spin-base symmetry and dates back to Schrödinger and Bargmann [2],[3]. It remained largely unnoticed until Weldon picked it up again and used it for his description of fermions in curved spacetime to show that there is no necessity for vierbeins involved [9]. The spin-base formalism refers to the following transformations

$$\psi \longrightarrow \psi' = S\psi \quad \text{and} \quad \overline{\psi} \longrightarrow \overline{\psi}' = \overline{\psi}S^{-1} \quad \text{as well as} \quad \gamma^{\mu} \longrightarrow \gamma^{\mu\prime} = S\gamma^{\mu}S^{-1}. \tag{4.1}$$

As not only the spinors transform but also the Dirac matrices themselves the transforming matrices can be elements of the GL(4,ℂ) group. In order to see that this is actually a symmetry transformation one should not only check the Lagrangian but also the defining property of the Dirac matrices as they also transform. The invariance of both of them can easily be shown. For the anticommutation relation this proof reads

$$\begin{aligned}
\{\gamma^{\mu},\gamma^{\nu}\} \longrightarrow \{\gamma^{\mu\prime},\gamma^{\nu\prime}\} &= \{S\gamma^{\mu}S^{-1}, S\gamma^{\nu}S^{\cdot -1}\} \\
&= S\gamma^{\mu}S^{-1}S\gamma^{\nu}S^{-1} + S\gamma^{\nu}S^{-1}S\gamma^{\mu}S^{-1} \\
&= S\gamma^{\mu}\gamma^{\nu}S^{-1} + S\gamma^{\nu}\gamma^{\mu}S^{-1} \\
&= S\{\gamma^{\mu},\gamma^{\nu}\}S^{-1} = S2g^{\mu\nu}\mathbb{1}S^{-1} = 2g^{\mu\nu}\mathbb{1} \\
&= \{\gamma^{\mu},\gamma^{\nu}\}.
\end{aligned} \tag{4.2}$$

Similarly the proof for the Lagrangian density reads

$$\begin{aligned}
\mathscr{L} \longrightarrow \mathscr{L}' &= \overline{\psi}'\gamma^{\mu\prime}\partial_{\mu}\psi' - m\overline{\psi}'\psi' \\
&= \overline{\psi}S^{-1}S\gamma^{\mu}S^{-1}\partial_{\mu}(S\psi) - m\overline{\psi}S^{-1}S\psi
\end{aligned} \tag{4.3}$$

as S is not spacetime dependent all S and S^{-1} cancel leading to the result

$$\mathscr{L}' = \mathscr{L}, \tag{4.4}$$

which shows that this not only is a symmetry transformation. But since it only rotates the field in spinor space there is no $\delta\mathscr{L}$ thus simplifying the Noether current to

$$J^{\mu} = \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\psi)}\delta\psi + \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\overline{\psi})}\delta\overline{\psi}. \tag{4.5}$$

The second part comes from the fact that ψ and $\overline{\psi}$ can be viewed as independent fields, both of them contributing to the Noether current. Since this expression makes no sense dimension-wise as the first term is a scalar in spinor space while the second one is a matrix, the $\delta\overline{\psi}$ term is supposed to be prior to the canonical momentum term creating a scalar as well. Because of the fermionic nature of ψ and $\overline{\psi}$, the rearrangement causes a sign change resulting in

$$J^\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi)}\delta\psi - \delta\overline{\psi}\frac{\partial \mathscr{L}}{\partial(\partial_\mu \overline{\psi})}. \tag{4.6}$$

4.2 Decomposition of GL(4,ℂ)

In order to determine the terms used to calculate the Noether current it is necessary to find a way to express the infinitesimal change that arises in ψ and $\overline{\psi}$. Finite changes, as long as continuous, can always be reduced to consecutive infinitesimal changes. For a group the size of GL(4,ℂ) it seems reasonable to decompose it into smaller and better known groups. As matrices with a determinant of one are much easier to handle it should be natural to try and factorize GL(4,ℂ) locally into SL(4,ℂ) and a residual symmetry group L. This process can be done using an isomorphism

$$\mathrm{GL}(4,\mathbb{C}) \simeq \mathrm{SL}(4,\mathbb{C}) \times \mathrm{L}. \tag{4.7}$$

Since the only thing distinguishing GL(4,ℂ) from SL(4,ℂ) is having an arbitrary complex-valued determinant, the leftover group can only be composed of complex-valued numbers. These can be written in the form of an exponential spanning the whole complex plane

$$c = e^{\lambda + i\varphi} \in \mathrm{L}. \tag{4.8}$$

For simplicity's sake and easier computation of the Noether current later on this residual group L can be factorized even further by treating the phase and amplitude of the complex numbers separately, leading to two smaller leftover symmetries: an $\mathbb{R}^+$ and a U(1) symmetry

$$\mathrm{GL}(4,\mathbb{C}) \simeq \mathrm{SL}(4,\mathbb{C}) \times \mathrm{U}(1) \times \mathbb{R}^+. \tag{4.9}$$

In fact, SL(4,ℂ) is the group of all spin-base transformations that transform the Dirac matrices nontrivially and cover the similarity transformations of the Clifford algebra completely [10].

4.3 Generators of GL(4,ℂ)

Because GL(4,ℂ) decomposes into SL(4,ℂ), U(1) and $\mathbb{R}^+$ symmetries their currents in turn can be calculated separately as well using their respective generators. The U(1) symmetry is simply generated by an arbitrary angle φ. This manifests itself in a phase shift of ψ and $\overline{\psi}$ which immediately satisfies the symmetry condition $\mathscr{L}' = \mathscr{L}$ thus leaving the equations of motion invariant

$$\psi \longrightarrow \psi' = e^{i\varphi}\psi \quad \text{and} \quad \overline{\psi} \longrightarrow \overline{\psi}' = \overline{\psi}e^{-i\varphi}. \tag{4.10}$$

Analogously the $\mathbb{R}^+$ symmetry can be represented by their generators λ which correspond to a dilation of ψ and $\overline{\psi}$. The only difference being the exponent not featuring an i since it does not have a rotational nature and only represents the real part of the residual symmetry L

$$\psi \longrightarrow \psi' = e^{\lambda}\psi \quad \text{and} \quad \overline{\psi} \longrightarrow \overline{\psi}' = \overline{\psi} e^{-\lambda}. \tag{4.11}$$

More interesting becomes the task of finding the generators of SL(4,ℂ) since it is a much larger symmetry group. One method of spanning this group is to use a generelization of the Gellmann matrices [11]. These would come with an internal structure corresponding to a Lie algebra making it easy to handle them. But an arguably better group of generators can be obtained by using the Dirac matrices as they span an algebra with a basis of 16 independent elements as well

$$\tilde{\Gamma}^A = \{\mathbb{1}, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \sigma^{\mu\nu}\}, \tag{4.12}$$

which together with complex prefactors can also span SL(4,ℂ). But it has the advantage of coming with the already used Clifford algebra structure. To avoid unnecessary factoring and provide an easily usable algebra one can normalize the elements of $\tilde{\Gamma}^A$ to Γ^A by setting a normalization condition

$$\mathrm{tr}\left(\Gamma^A\Gamma^B\right) = 4\delta^{AB}. \tag{4.13}$$

In appendix B (7.2) it is verified that

$$\Gamma^A = \left\{\mathbb{1}, \gamma_5, \gamma^0, i\gamma^i, i\gamma^0\gamma_5, \gamma^i\gamma_5, i\sigma^{0i}, \sigma^{ij}\right\}, \tag{4.14}$$

satisfies this condition and thus can be used as a normalized basis. Here i and j denote latin indices running from 1 to 3. Since $\sigma^{\mu\nu}$ is antisymmetric, it only has 6 independent entries. In order to not count twice only $i > j$ should be regarded. This basis together with 16 independent complex factors spans SL(4,ℂ) therefore also providing a set of generators for the group meaning every element S of SL(4,ℂ) can be expressed as an exponential

$$S = e^{\omega_A\Gamma^A}, \tag{4.15}$$

where ω_A describes the complex factors and can in turn be represented like this:

$$\omega_A = \left\{s, p, v_0, -i\,v_i, a_0, i\,a_i, t_{0i}, i\,t_{ij}\right\}. \tag{4.16}$$

The sum in equation (4.15) then reads

$$\omega_A\Gamma^A = s\mathbb{1} + p\gamma_5 + v_\mu\gamma^\mu + a_\mu\gamma^\mu\gamma_5 + \tfrac{i}{2}t_{\mu\nu}\sigma^{\mu\nu}. \tag{4.17}$$

Since the ω_A have extra i terms introduced to their spacial components, the sum can be expressed kovariantly. The factor of $\frac{1}{2}$ in the tensor part is used to not count the terms twice as $\sigma^{\mu\nu}$ is antisymmetric and thus only 6 components contribute to the basis Γ^A. The choice of variables here is not arbitrary as it has been used in literature before [9]. The factors are named after the kind of object they result in later on. A scalar, pseudoscalar, vector, axial vector and tensor part arise naming the variables. Due to the i in the normalization of Γ^A some of the ω_A have a factor i as well to preserve the nature of the corresponding type of current. It should be noted that the span of $e^{\omega_A \Gamma^A}$ is actually much larger than SL(4,$\mathbb{C}$) due to ω_A being arbitrary and Γ^A containing $\mathbb{1}$. This leads to

$$\det S = e^{\mathrm{tr}\left(\omega_A \Gamma^A\right)} = e^{\mathrm{tr}\left(s\mathbb{1}\right)+\mathrm{tr}\left(R\right)}, \tag{4.18}$$

where R represents $\omega_A \Gamma^A$ without $\mathbb{1}$. Using the identities proven in appendix B (7.2) the trace of R vanishes leaving only

$$\det S = e^{4s}. \tag{4.19}$$

As s can be any complex number the determinant of S can be as well. Since in this case the identity is the generator of the residual symmetry L leaving it out of Γ^A to not cover L twice ensures that all Noether currents can be calculated independently.

5 Noether current for the Spin-Base-Symmetry

5.1 Electric current

The U(1) symmetry transformation can be used to determine a corresponding deformation $\delta\psi$ as explained in chapter 3. As it is an infinitesimal change in ψ a first-order approximation of the transformation is sufficient

$$\begin{aligned} \psi \longrightarrow \psi' &= e^{i\varphi}\psi \\ &= (1+i\varphi)\psi \\ &= \psi + \delta\psi. \end{aligned} \tag{5.1}$$

Analogously $\overline{\psi}$ is transformed thus providing $\delta\psi$ and $\delta\overline{\psi}$. With these terms, the Noether current for the U(1) symmetry can be calculated using the equations (2.2), (2.6) and (4.6)

$$\begin{aligned} J^\mu &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \psi)}\delta\psi - \delta\overline{\psi}\frac{\partial \mathscr{L}}{\partial(\partial_\mu \overline{\psi})} \\ &= \left(-i\overline{\psi}\gamma^\mu\right)\left(i\varphi\psi\right) - \left(\overline{\psi}(-i\varphi)\right)\left(-i\gamma^\mu\psi\right) \\ &= 2\varphi\overline{\psi}\gamma^\mu\psi. \end{aligned} \tag{5.2}$$

The anticommutativity of ψ is giving rise to a sign when letting $\frac{\partial}{\partial(\partial_\mu \psi)}$ act on the kinetic term of $\mathscr{L}$ as $\overline{\psi}$ comes before $\partial_\mu\psi$. To prove that this current is actually conserved, making use of the equations of motion turns out to be of help

$$\begin{aligned} \partial_\mu J^\mu &= 2\varphi\left(\partial_\mu\overline{\psi}\gamma^\mu\psi + \overline{\psi}\gamma^\mu\partial_\mu\psi\right) \\ \overset{\text{using (2.4) and (2.9)}}{\Longrightarrow} \quad &= 2\varphi\left(im\overline{\psi}\psi + \overline{\psi}(-im\psi)\right) = 0. \end{aligned} \tag{5.3}$$

It is also interesting to take a look at the Noether charge

$$Q = \int \mathrm{d}^3x\, J^0 = 2\varphi \int \mathrm{d}^3x\, \overline{\psi}\gamma^0\psi = 2\varphi \int \mathrm{d}^3x\, \psi^\dagger\psi. \tag{5.4}$$

This term, apart from 2φ, can, similarly to nonrelativistic quantum mechanics, be interpreted as a probability since it is always positive, real and nonzero for nontrivial ψ. On the other hand, once a coupling with a photon field A^μ is introduced to the Lagrangian density of the fermion field, it is of interest to interpret the current in a different way

$$\mathscr{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi. \tag{5.5}$$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the kinetic term of the photon field and $D_\mu = \partial_\mu - ieA_\mu$ summarizes the kinetic term of the spinor field and the coupling term. The theory has a problem though as it is obviously not invariant under local gauge transformations $A_\mu \longrightarrow A_\mu + \partial_\mu\lambda(x)$ because it has a single A_μ term found in the coupling term. But this can be fixed by transforming the spinor field as well

$$\begin{aligned} A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu\lambda(x) \quad &\text{and} \quad \psi \longrightarrow \psi' = e^{ie\lambda(x)}\psi \\ \Longrightarrow \mathscr{L} \longrightarrow \mathscr{L}' = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} &+ i\overline{\psi}e^{-ie\lambda}\gamma^\mu D_\mu\big(e^{ie\lambda}\psi\big) - m\overline{\psi}e^{-ie\lambda}e^{ie\lambda}\psi. \end{aligned} \tag{5.6}$$

The mass term for the spinor field obviously stays the same and the kinetic photon term is invariant under gauge transformations anyway. But by choosing the D_μ as stated, the middle part of the Lagrangian stays invariant as well since the first and last term cancel and the exponentials nullify

$$i\overline{\psi}e^{-ie\lambda}\gamma^\mu\big(ie\partial_\mu\lambda e^{ie\lambda}\psi + e^{ie\lambda}\partial_\mu\psi - ieA_\mu e^{ie\lambda}\psi - ie\partial_\mu\lambda e^{ie\lambda}\psi\big) = i\overline{\psi}\gamma^\mu D_\mu\psi. \tag{5.7}$$

Hence e can be seen as a coupling term connecting the spinor and photon field. Now calculating the Noether current for this symmetry yields an almost identical term as before

$$J^\mu = 2e\lambda\overline{\psi}\gamma^\mu\psi, \tag{5.8}$$

the only difference being the coupling constant e. Checking the dimension of e and acknowledging the connection of both fields in previous theories leads to the interpretation of this Noether current being the electric current with

$$Q = e\int \mathrm{d}^3x\psi^\dagger\psi, \tag{5.9}$$

representing the electric charge. Furthermore, the Lagrangian density in equation (5.5) can then be viewed as a theory of quantum electrodynamics.

5.2 Dilatation current

Analogously to the electric current, the $\mathbb{R}^+$ symmetry yields a corresponding current of high resemblance. It is identified as a dilatation current, since its transformation is dilatating the spinor. The transformation is very similar to the one for the electric current

$$\psi \longrightarrow \psi' = e^{\lambda}\psi \Longrightarrow \delta\psi = \lambda\psi. \tag{5.10}$$

Again $\delta\overline{\psi}$ is calculated in the same way. Once more the Noether current follows from equation (4.6) resulting in the dilatation current

$$J^{\mu} = -2i\lambda\overline{\psi}\gamma^{\mu}\psi, \tag{5.11}$$

which is almost identical to the electric current. The verification of its conservation is identical to (5.3). The Noether charge reads

$$Q = -2i\lambda\int \mathrm{d}^3x\,\psi^{\dagger}\psi. \tag{5.12}$$

This is an imaginary equivalent to the electric charge without the coupling term. As it is not real and cannot become real by any means since the integrant is always real and λ is an element of $\mathbb{R}$, it is not observable, thus being more of a mathematical result than a physical one. It is a structural consequence of GL(4,$\mathbb{C}$) allowing arbitrary absolute values for the matrix determinants. Furthermore it is, apart from the i term identical to the electric current thus not providing any independent information.

5.3 Current for SL(4,$\mathbb{C}$)-symmetry

In contrast to the dilatation and electric current the SL(4,$\mathbb{C}$) current is a little more complex. The computation of $\delta\psi$ follows the same process as in the sections above using the generators found in section 4.3

$$\begin{aligned} \psi \longrightarrow S\psi &= e^{\omega_A\Gamma^A}\psi & \overline{\psi} \longrightarrow \overline{\psi}S^{-1} &= \overline{\psi}e^{-\omega_A\Gamma^A} \\ &= \left(1+\omega_A\Gamma^A\right)\psi & &= \overline{\psi}\left(1-\omega_A\Gamma^A\right) \\ &= \psi + \delta\psi & &= \overline{\psi} + \delta\overline{\psi}. \end{aligned} \tag{5.13}$$

Having calculated the deformations the Noether current is again computed using (4.6) resulting in

$$\begin{aligned} J^{\mu} &= -i\overline{\psi}\gamma^{\mu}\omega_A\Gamma^A\psi - i\overline{\psi}\omega_A\Gamma^A\gamma^{\mu}\psi \\ &= -i\omega_A\overline{\psi}\left\{\gamma^{\mu},\Gamma^A\right\}\psi. \end{aligned} \tag{5.14}$$

Inserting the generators from (4.14) excluding $\mathbb{1}$ as it would cover the residual symmetry L again, the current reads

$$J^\mu = -ip\overline{\psi}\{\gamma^\mu,\gamma_5\}\psi - iv_0\overline{\psi}\{\gamma^\mu,\gamma^0\}\psi - i(-iv_i)\overline{\psi}\{\gamma^\mu,i\gamma^i\}\psi \\ - ia_0\overline{\psi}\{\gamma^\mu,i\gamma^0\gamma_5\}\psi - i(ia_i)\overline{\psi}\{\gamma^\mu,\gamma^i\gamma_5\}\psi - t_{0i}\overline{\psi}\{\gamma^\mu,i\sigma^{0i}\}\psi - \tfrac{i}{2}(it_{ij})\overline{\psi}\{\gamma^\mu,\sigma^{ij}\}\psi. \tag{5.15}$$

Using the calculations in appendix C (7.3) the anticommutators simplify and the equation reads

$$J^\mu = -2iv_\nu g^{\mu\nu}\overline{\psi}\psi - 2a_\nu\overline{\psi}\sigma^{\mu\nu}\gamma_5\psi - t_{\nu\rho}\epsilon^{\mu\nu\rho\sigma}\overline{\psi}\gamma_\sigma\gamma_5\psi, \tag{5.16}$$

where $v_\nu = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $a_\nu = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $t_{\nu\rho} = \begin{pmatrix} 0 & t_{01} & t_{02} & t_{03} \\ t_{10} & 0 & t_{12} & t_{13} \\ t_{20} & t_{21} & 0 & t_{23} \\ t_{30} & t_{31} & t_{32} & 0 \end{pmatrix}$.

It should be noted that $t_{\nu\rho}$ is antisymmetric. The result is the Noether current for the SL(4,ℂ) symmetry containing a vector, an axial vector and a tensor part, all of them being conserved separately because the ω_A are completely independent

$$\begin{aligned} \text{vector current:} \quad & J_V^{\mu\nu} = ig^{\mu\nu}\overline{\psi}\psi \\ \text{axial vector current:} \quad & J_A^{\mu\nu} = \overline{\psi}\sigma^{\mu\nu}\gamma_5\psi \\ \text{tensor current:} \quad & J_T^{\mu\nu\rho} = \epsilon^{\mu\nu\rho\sigma}\overline{\psi}\gamma_\sigma\gamma_5\psi. \end{aligned} \tag{5.17}$$

Showing that J^μ is conserved can be done by using the equations of motion

$$i\gamma^\mu\partial_\mu\psi = m\psi. \tag{5.18}$$

But as γ^μ was transformed as well its equations of motion have to be fulfilled as well

$$\frac{\partial\mathscr{L}}{\partial\gamma^\mu} = 0. \tag{5.19}$$

Since the left hand side has 2 spinor indices from the Dirac matrix the equation of motion should also have 2 spinor indices. It is also of use to contract the derivative with $\left(\gamma^\mu\right)^N{}_L$ resulting in

$$\left(\gamma^\mu\right)^N{}_L \frac{\partial\mathscr{L}}{\partial\left(\gamma^\mu\right)^M{}_L} = i\overline{\psi}_M\left(\gamma^\mu\right)^N{}_L\partial_\mu\psi^L. \tag{5.20}$$

This can be written as a tensor product producing the equations of motion for the Dirac matrices

$$i\overline{\psi}\otimes\gamma^\mu\partial_\mu\psi = 0. \tag{5.21}$$

Combining both equation (5.18) and (5.21) leads to

$$m\overline{\psi} \otimes \psi = 0. \tag{5.22}$$

This result sets all possible combinations of products of the spinor components equal to zero. Due to that all terms of the form $\overline{\psi}A\psi$ must equal zero as well since they break down to sums of products of the spinor components multiplied with the matrix entries. This leads to the whole Noether current vanishing as it only consists of those terms. The surprisingly simple result originates from the equations of motion for the Dirac matrices. As they are not spacetime dependent their behavior is not dynamic which leads to very restrictive equations of motion. These restrictions manifest themselves in constraints for the spinor field that are too severe to yield non trivial results as seen in equation (5.22). But since the condition in (5.22) can be satisfied as well by setting the mass of the theory equal to zero it immediatly raises the question what happens for a massless spin-$\frac{1}{2}$ particle. In this case the Noether current itself remains the same. Only the equations of motion change. They read

$$i\gamma^\mu \partial_\mu \psi = 0 \quad \text{and} \quad \overline{\psi} \otimes \partial_\mu \psi = 0. \tag{5.23}$$

In this case the Noether current does not vanish directly. Showing that it remains conserved now demands to look at the divergence

$$\begin{aligned} \partial_\mu J^\mu = &-2i v_\nu g^{\mu\nu}\left(\partial_\mu \overline{\psi}\psi + \overline{\psi}\partial_\mu \psi\right) - 2a_\nu\left(\partial_\mu \overline{\psi}\sigma^{\mu\nu}\gamma_5\psi + \overline{\psi}\sigma^{\mu\nu}\gamma_5\partial_\mu\psi\right) \\ &- 2t_{\nu\rho}\epsilon^{\mu\nu\rho\sigma}\left(\partial_\mu\overline{\psi}\gamma_\sigma\gamma_5\psi + \overline{\psi}\gamma_\sigma\gamma_5\partial_\mu\psi\right). \end{aligned} \tag{5.24}$$

But all of these terms vanish now because of the second part in 5.23. The equations of motion of the Dirac matrices again prove to be very restrictive resulting in the divergence of J^μ to vanish using the same logic as before. Still, the massless theory at least has a conserved current consisting of a vector, an axial vector and a tensor part. This leads one to believe that the mass term in the Dirac equation is somehow responsible for the vanishing. It can indeed be interpreted in this way as especially the tensor part has a structure that closely resembles a chiral current that is usually derived by looking at a chiral transformation $\psi \longrightarrow e^{i\alpha\gamma_5}\psi$. This chiral symmetry is directly broken by a mass term since γ_5 anticommutes with γ^0 which appears in the Dirac conjugation of ψ leading to

$$\overline{\psi}\psi \longrightarrow m\psi^\dagger\gamma^0\gamma^0 e^{-i\alpha\gamma_5}\gamma^0 e^{i\alpha\gamma_5}\psi \overset{(2.18)}{=} m\overline{\psi}e^{2i\alpha\gamma_5}\psi \neq m\overline{\psi}\psi. \tag{5.25}$$

On the other hand, the spin-base symmetry does not exert such a behavior directly since it is not broken by a mass term. But as this term seems to be responsible for the vanishing Noether current taking a look at other field theories without masses peaks the interest. The simplest non trivial model that can be investigated is the Thirring model which introduces a self interaction in the Lagrangian [12]. The Lagrangian density of the theory then reads

$$\mathscr{L} = i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - \left(\overline{\psi}\gamma_{\mu}\psi\right)\left(\overline{\psi}\gamma^{\mu}\psi\right). \tag{5.26}$$

Since the symmetry remains the same and the kinetic term for ψ and $\overline{\psi}$ does not change the Noether current stays invariant. But the equations of motion are vastly different. For ψ they yield

$$\frac{\partial\mathscr{L}}{\partial\overline{\psi}} = i\gamma^{\mu}\partial_{\mu}\psi - \gamma_{\mu}\psi\left(\overline{\psi}\gamma^{\mu}\psi\right) - \left(\overline{\psi}\gamma_{\mu}\psi\right)\gamma^{\mu}\psi = 0. \tag{5.27}$$

Using the anticommutation of the spinor fields and adjusting the indices this simplifies to

$$i\gamma^{\mu}\partial_{\mu}\psi = 2\left(\overline{\psi}\gamma_{\mu}\psi\right)\gamma^{\mu}\psi. \tag{5.28}$$

The equations of motion for the Dirac matrices can also be calculated resulting in

$$\frac{\partial\mathscr{L}}{\partial\gamma^{\mu}} = i\overline{\psi}\otimes\partial_{\mu}\psi - 2\left(\overline{\psi}\gamma_{\mu}\psi\right)\overline{\psi}\otimes\psi = 0. \tag{5.29}$$

Inserting (5.28) in (5.29) yields 0 meaning both equations contain the same information. But the equations of motion of $\overline{\psi}$ can be used as well

$$i\partial_{\mu}\overline{\psi}\gamma^{\mu} = -2\left(\overline{\psi}\gamma_{\mu}\psi\right)\overline{\psi}\gamma^{\mu}. \tag{5.30}$$

Together with equation (5.28) and (5.30) the conservation of the Noether current is easily proven by showing that the vector, axial vector and tensor current vanish in equation (5.24)

$$\begin{aligned}
-2iv_{\nu}g^{\mu\nu}\left(\partial_{\mu}\overline{\psi}\psi + \overline{\psi}\partial_{\mu}\psi\right) &= -2v_{\nu}g^{\mu\nu}\left(-2\left(\overline{\psi}\gamma_{\mu}\psi\right)\overline{\psi}\psi + \overline{\psi}2\left(\overline{\psi}\gamma_{\mu}\psi\right)\psi\right)\\
-2a_{\nu}\left(\partial_{\mu}\overline{\psi}\sigma^{\mu\nu}\gamma_5\psi + \overline{\psi}\sigma^{\mu\nu}\gamma_5\partial_{\mu}\psi\right) &= -2a_{\nu}\left(2i\left(\overline{\psi}\gamma_{\mu}\psi\right)\overline{\psi}\sigma^{\mu\nu}\gamma_5\psi + \overline{\psi}\sigma^{\mu\nu}\gamma_5\left(-2i\left(\overline{\psi}\gamma_{\mu}\psi\right)\psi\right)\\
-2t_{\nu\rho}\epsilon^{\mu\nu\rho\sigma}\left(\partial_{\mu}\overline{\psi}\gamma_{\sigma}\gamma_5\psi + \overline{\psi}\gamma_{\sigma}\gamma_5\partial_{\mu}\psi\right) &= -2t_{\nu\rho}\epsilon^{\mu\nu\rho\sigma}\left(2i\left(\overline{\psi}\gamma_{\mu}\psi\right)\overline{\psi}\gamma_{\sigma}\gamma_5\psi + \overline{\psi}\gamma_{\sigma}\gamma_5\left(-2i\left(\overline{\psi}\gamma_{\mu}\psi\right)\psi\right).
\end{aligned} \tag{5.31}$$

Since $\left(\overline{\psi}\gamma_{\mu}\psi\right)$ is just a scalar in spinor space all of the terms in parentheses on the right hand side cancel each other, thus showing that $\partial_{\mu}J^{\mu}$ equals 0 resulting in the Noether current being conserved.

6 Conclusion

After establishing the mathematical foundation of fermionic field theory and introducing Noether's theorem in a methodic manner, both were used to derive a conservation law for the spin-base symmetry of the Dirac equation. The result consists of three different currents: An electric current, a dilatation current and an SL(4,$\mathbb{C}$) current. The electric current comes with charge conservation and its divergence can be interpreted as a continuity equation. The electric nature arises when coupled to a photon field which results in a theory of quantum electrodynamics. The second conserved current is identical to the electric current apart from a factor i. Therefore it does not contain any independent information. Lastly the SL(4,$\mathbb{C}$) current was calculated. It was confirmed that the current not only is conserved but completely vanishes, leaving no conserved quantity at all. J^{μ} equaling zero can be traced back to the equations of motion of the Dirac matrices. Due to their non-dynamic nature in flat spacetime their equations yield severely restrictive conditions for the spinor field which in fact, leads to the current completely vanishing if the mass term is non-zero (5.22). This peaked the interest in other theories. Checking not only the massless Dirac equation but also the Thirring model resulted in a conserved current of the same form as before, this time non-vanishing. Other theories could be considered in the future. Especially those that have a spacetime dependent Dirac matrix $\gamma^{\mu}(x)$ since this would most likely result in less trivial behavior of the Noether current. For example, generalizing the theory to curved space time using the spin-base formulism again could yield interesting results because in that case a spin connection term couples fermions to gravity. This spin connection can be expressed in terms of derivatives of γ^{μ} [9], [10], leading to non-trivial equations of motion of γ^{μ} .

7 Appendix

7.1 A

With the definition of γ_5 follow the relations:

$$\begin{aligned}\{\gamma_5, \gamma^\mu\} &= \gamma_5\gamma^\mu + \gamma^\mu\gamma_5 \\ &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3.\end{aligned} \tag{7.1}$$

As γ^μ must be exactly one of the other $\gamma^{0...3}$ it can be anticommuted with the other matrices until it is next to an identical Dirac matrix. They contract to $\mathbb{1}$ for $\mu = 0$ or $-\mathbb{1}$ for the other values. If γ^μ has to anticommute with the other matrices n times to reach its identical counterpart, the second addend of $\{\gamma_5, \gamma^\mu\}$ has to anticommute the γ^μ exactly $3-n$ times since there are always five Dirac matrices in the products of the addends of $\{\gamma_5, \gamma^\mu\}$. This causes a difference in sign for the first and second term for any value of μ meaning the anticommutator always equals 0 proving the first equation in (2.18).

In order to show that γ_5 is hermitian making use of equation (2.16) proves useful

$$\begin{aligned}(\gamma_5)^\dagger &= -i(\gamma^0\gamma^1\gamma^2\gamma^3)^\dagger \\ &= -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger \\ &= -i\gamma^0\gamma^3\gamma^0\gamma^0\gamma^2\gamma^0\gamma^0\gamma^1\gamma^0\gamma^0\gamma^0\gamma^0 \\ \overset{\text{using (2.14)}}{\Longrightarrow} &= -i\gamma^0\gamma^3\gamma^2\gamma^1 \\ &= i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5\end{aligned} \tag{7.2}$$

Proving the third part of the equations in (2.18) is easily done by just using equation (2.14)

$$\begin{aligned}(\gamma_5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= -\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^2\gamma^3\gamma^3 \\ &= -\mathbb{1}(-\mathbb{1})(-\mathbb{1})(-\mathbb{1}) = \mathbb{1}\end{aligned} \tag{7.3}$$

Calculating the Noether current for the Sl(4,$\mathbb{C}$) involves a special relation which is stated in equation (2.20). To prove it one can simply check the different cases that arise. If all indices are equal the ϵ term vanishes thus leaving

$$\gamma^\mu\gamma^\mu\gamma^\mu = g^{\mu\mu}\gamma^\mu, \tag{7.4}$$

which is obiviously correct since $(\gamma^\mu)^2$ is $\mathbb{1}$ for $\mu = 0$ and $-\mathbb{1}$ for the other cases matching the sign of $g^{\mu\mu}$ in either case. If on the other hand all the indices are different then the first three terms vanish and only the Levi-Civita tensor part remains. Both sides are completely antisymmetric. Therefore it suffices to prove the equation for the cases $\gamma^0\gamma^1\gamma^2$, $\gamma^0\gamma^1\gamma^3$,

$\gamma^0\gamma^2\gamma^3$ and $\gamma^1\gamma^2\gamma^3$. As these proofs are completely analogous it is reasonable to just show it once

$$i\epsilon^{012\sigma}\gamma_\sigma i\gamma^0\gamma^1\gamma^2\gamma^3 = -\epsilon^{0123}\gamma_3\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^1\gamma^2(-\mathbb{1}) = \gamma^0\gamma^1\gamma^2. \tag{7.5}$$

7.2 B

To satisfy the normalization condition the values of $\tilde{\Gamma}^A$ are simply inserted. For $\mathbb{1}$ this is obviously true. The other results read

$$\begin{aligned}\mathrm{tr}(\gamma^\mu\gamma^\mu) &= \mathrm{tr}\left(\tfrac{1}{2}\{\gamma^\mu,\gamma^\mu\}\right)\\ &= \mathrm{tr}\left(\tfrac{1}{2}g_{\mu\mu}\mathbb{1}\right)\\ &= 4g^{\mu\mu} \Longrightarrow \gamma^0 \longrightarrow \gamma^0 \text{ and } \gamma^i \longrightarrow i\gamma^i\end{aligned} \tag{7.6}$$

For the γ_5 matrix follows

$$\begin{aligned}\mathrm{tr}(\gamma_5\gamma_5) &= \mathrm{tr}(-\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3)\\ &= \mathrm{tr}(-\gamma^0\gamma^0\gamma^1\gamma^1\gamma^2\gamma^2\gamma^3\gamma^3)\\ &= \mathrm{tr}(\mathbb{1})\\ &= 4 \Longrightarrow \gamma_5 \longrightarrow \gamma_5.\end{aligned} \tag{7.7}$$

In the same way the $\gamma^\mu\gamma_5$ term can be normalized leading to

$$\begin{aligned}\mathrm{tr}(\gamma^\mu\gamma_5\gamma^\mu\gamma_5) &= \mathrm{tr}(-\gamma_5\gamma^\mu\gamma^\mu\gamma_5)\\ &= -g^{\mu\mu}\mathrm{tr}(\gamma_5\gamma_5)\\ &= -4g^{\mu\mu} \Longrightarrow \gamma^0\gamma_5 \longrightarrow i\gamma^0\gamma_5 \text{ and } \gamma^i\gamma_5 \longrightarrow \gamma^i\gamma_5.\end{aligned} \tag{7.8}$$

Similarly calculating the normalizing factor for $\sigma^{\mu\nu}$ can be done. Due to the antisymmetry of this term considering only cases $\mu > \nu$ ensures to not count them twice. Because of this the cases $\mu = \nu$ can also be disregarded as they are zero.

$$\begin{aligned}\mathrm{tr}(\sigma^{\mu\nu}\sigma^{\mu\nu}) &= \mathrm{tr}\left(\tfrac{-1}{4}[\gamma^\mu,\gamma^\nu][\gamma^\mu,\gamma^\nu]\right)\\ &= \tfrac{-1}{4}\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\nu\gamma^\mu - \gamma^\nu\gamma^\mu\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\gamma^\nu\gamma^\mu\right)\\ \xrightarrow[\text{cyclicity}]{\mu\neq\nu} &= \tfrac{-1}{4}\mathrm{tr}(4\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu)\\ &= \mathrm{tr}(\gamma^\mu\gamma^\mu\gamma^\nu\gamma^\nu)\\ &= \mathrm{tr}(g^{\mu\mu}g^{\nu\nu}\mathbb{1})\\ &= 4g^{\mu\mu}g^{\nu\nu} \Longrightarrow \sigma^{0i} \longrightarrow i\sigma^{0i} \text{ and } \sigma^{ij} \longrightarrow \sigma^{ij}.\end{aligned} \tag{7.9}$$

Now it just remains to be shown that this basis is orthogonal

$$\begin{aligned} & \mathrm{tr}\big((\mathrm{odd}\#)\gamma^\mu\big) = \mathrm{tr}\big((\mathrm{odd}\#)\gamma^\mu\gamma_5\gamma_5\big) \\ \xrightarrow[\text{using (7.1)}]{\text{cyclicity}} \quad & = \mathrm{tr}\big(\gamma_5(\mathrm{odd}\#)\gamma^\mu\gamma_5\big) = -\mathrm{tr}\big(\gamma_5(\mathrm{odd}\#)\gamma^\mu\gamma_5\big) \\ \Longrightarrow \quad & = 0. \end{aligned} \tag{7.10}$$

This also holds for any additional number of γ_5 in the trace . Additionally we have

$$\begin{aligned} & \mathrm{tr}\big(\gamma_5\big) = \mathrm{tr}\big(\gamma_5\gamma^0\gamma^0\big) \\ \xrightarrow[\text{using (7.1)}]{\text{cyclicity}} \quad & = \mathrm{tr}\big(\gamma^0\gamma_5\gamma^0\big) = -\mathrm{tr}\big(\gamma^0\gamma_5\gamma^0\big) \\ \Longrightarrow \quad & = 0. \end{aligned} \tag{7.11}$$

This works for $\gamma^\mu\gamma^\nu\gamma_5$ as well which immediatly covers $\sigma^{\mu\nu}\gamma_5$. Now considering

$$\begin{aligned} \mathrm{tr}\big(\gamma^\mu\gamma^\nu\big) & \overset{\text{cyclicity}}{=} \frac{1}{2}\mathrm{tr}\big(\{\gamma^\mu,\gamma^\nu\}\big) \\ & \overset{2.14}{=} 4g^{\mu\nu}, \end{aligned} \tag{7.12}$$

and all the cases for $\mathrm{tr}\big(\Gamma^A\Gamma^B\big)$ are covered. This results in (4.14) satisfying the normaliziation condition.

7.3 C

In order to simplify the expressions in equation (5.15) calculating the anticommutators proves to be beneficial

$$\begin{aligned} \{\gamma^\mu,\gamma_5\} & \overset{7.1}{=} 0 \\ \{\gamma^\mu,\gamma^\nu\} & \overset{2.14}{=} 2g^{\mu\nu}\mathbb{1} \\ \{\gamma^\mu,\gamma^\nu\gamma_5\} & = \gamma^\mu\gamma^\nu\gamma_5 + \gamma^\nu\gamma_5\gamma^\mu = \gamma^\mu\gamma^\nu\gamma_5 - \gamma^\nu\gamma^\mu\gamma_5 = [\gamma^\mu,\gamma^\nu]\gamma_5 \\ & = \frac{2}{i}\sigma^{\mu\nu}\gamma_5 \\ \{\gamma^\mu,\sigma^{\nu\rho}\} & = \frac{i}{2}\{\gamma^\mu,[\gamma^\nu,\gamma^\rho]\} = \frac{i}{2}\big(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\mu\gamma^\rho\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\rho\gamma^\nu\gamma^\mu\big) \\ & \overset{2.20}{=} -2\epsilon^{\sigma\mu\rho\nu}\gamma_\sigma\gamma_5 = -2\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma_5 \end{aligned} \tag{7.13}$$

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