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– Masterarbeit –

Wilsonian Perspective on Generic Self-Interacting U(1) Gauge Theories

Global Existence of Fixed Point Actions

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Zusammenfassung

Ziel. In dieser Arbeit nehmen wir eine dem Titel nach generische selbst-wechselwirkende, lokal U(1)-invariante und materiefreie Theorie im Rahmen der durch die funktionale Renormierungsgruppe bereitgestellten Methoden in den Blick. Für dieses Modell werden wir unser Interesse anschließend auf die Suche nach global existierenden Fixpunktwirkungen ausrichten.

Methoden. Die zugrundeliegende effektive Lagrangedichte $\mathscr{L}$ wird aus der Quantenelektrodynamik heraus motiviert und zunächst im $d = 4 - \epsilon$ dimensionalen euklidischen Raum als beliebige Funktion in den manifest lokal U(1)-invarianten Kontraktionen des (dualen) Feldstärketensors, $\mathscr{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ und $\mathscr{G} = \frac{1}{4}F_{\mu\nu}\left(\star F\right)^{\mu\nu}$, für einen betragsmäßig kleinen Parameter ϵ formuliert. Mit Hilfe nicht-perturbativer Techniken der exakten Renormierungsgruppe werden homogene Feldkonfigurationen mit als konstant angenommenen Invarianten $\mathscr{F}$ und $\mathscr{G}$ betrachtet und zunächst die, auf das skalenabhängige effektive Potential $\mathscr{W}_k$ projizierte, Flussgleichung detailliert abgeleitet. Für die daraus resultierende Fixpunktgleichung erfolgt unter Ausschluss der Invariante $\mathscr{G}$ und Beschränkung auf den Fall der Selbstdualität, $\mathbf{F} = \star\mathbf{F}$, eine gemäß den Methoden der Klein- und Großfeldentwicklung geleitete analytische Sichtung nach global existierenden Fixpunktpotentialen in $d = 4$ raumzeitlichen Dimensionen.

Resultate. Die gefundenen Ergebnisse offenbaren unter den hier getroffenen Kernannahmen, im Einzelnen; vernachlässigter Materiesektor, trunkierter Feldraum und selbstduale Feldkonfigurationen, sich stabilisierende Lösungen für Kleinfeldapproximationen der Fixpunktgleichung mit Konvergenzradien der Größenordnung $r \sim 10^{-3}$. Demgegenüber versagen Großfeldentwicklungen infolge divergierender Koeffizienten und ein Anschluss an den Bereich kleiner Feldamplituden erweist sich als ausgeschlossen. Wir schließen mit der Erkenntnis, dass global existierende Fixpunktwirkungen für das in dieser Arbeit demonstrierte, eingeschränkte U(1)-Eichmodell nicht konstruierbar sind.

Abstract

Aim. In this thesis, we consider a generic self-interacting, locally U(1)-invariant and matter-free theory within the methodological framework provided by the functional renormalisation group. For this model, we will subsequently focus our interest on the search for globally-existing fixed point actions.

Methods. The underlying effective Lagrangian $\mathscr{L}$ is motivated from quantum electrodynamics and is formulated in $d = 4 - \epsilon$ dimensional Euclidean space as an arbitrary function in the manifest locally U(1)-invariant contractions of the (dual) field strength tensor, $\mathscr{F} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $\mathscr{G} = \frac{1}{4}F_{\mu\nu}\left(\star F\right)^{\mu\nu}$, for a small parameter ϵ. Using non-perturbative techniques of the exact renormalisation group, homogeneous field configurations with constant invariants $\mathscr{F}$ and $\mathscr{G}$ are considered and the projected flow equation with respect to the scale-dependent effective potential $\mathscr{W}_k$ is derived in detail. For the resulting fixed point equation, an analytical investigation, guided by the methods of small- and large-field expansion, is carried out under exclusion of the invariant $\mathscr{G}$ and restriction to the case of self-duality, $\mathbf{F} = \star\mathbf{F}$, in $d = 4$ spacetime dimensions

Results. Under the influence of our core assumptions, including a neglected matter sector, a truncated field space and self-dual field configurations, the results of this work reveal stabilising solutions for small-field approximations of the fixed point equation with corresponding radii of convergence of order $r \sim 10^{-3}$. In contrast, large-field evaluations fail due to diverging expansion coefficients and thus contact to the region of small-field amplitudes proves to be precluded. We will conclude that globally-existing fixed point actions are not constructible, at least for the restricted U(1) gauge model demonstrated in this thesis.

Contents

List of Abbreviations

QFT	Quantum Field Theory
QED	Quantum Electrodynamics
QCD	Quantum Chromodynamics
YM	Yang-Mills
RG	Renormalisation Group
FRG	Functional Renormalisation Group
EAA	Effective Average Action
ERGE	Exact Renormalisation Group Equation
FPE	Fixed Point Equation
GFP	Gaussian Fixed Point
NGFP	Non-Gaussian Fixed Point
IR	Infrared
UV	Ultraviolet
SFE	Small-Field Expansion
LFE	Large-Field Expansion
LHS	Left-Hand Side
RHS	Right-Hand Side
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
1PI	One-Particle Irreducible

Notation & Conventions

Units

nu — We agree on a *natural unit system* in which both the reduced Planck constant and the speed of light are set to value 1, i.e. $\hbar = 1 \wedge c = 1$.

Symbols

$\triangleq$ — For two expressions E_1 and E_2 which are *equal by definition* we write $E_1 \triangleq E_2$.

$\mathrel{\hat{=}}$ — An object O furnished with a collection of structure elements may be represented by another object $\tilde{O} \equiv F(O)$ that exhibits the same properties as the original object. Here, F describes a specific *representation* and in this situation we write $O \mathrel{\hat{=}} F(O)$.

$\mathbb{N}$ — The set of *natural numbers without zero*, i.e. $\{1, 2, 3, \ldots\}$. Additionally we have $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

$\mathbf{1}_A$ — The *indicator function* with regard to a set A. If A is a subset of a set X, then by definition:

$$\mathbf{1}_A : X \to \{0,1\};\, x \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} .$$

$\Re[\,\cdot\,], \Im[\,\cdot\,]$ — *Real and imaginary part* of a complex number. Consider $z \in \mathbb{C}$, then in canonical formulation we write $z \mathrel{\hat{=}} \Re[z] + \imath\Im[z]$, where $\Re[z], \Im[z] \in \mathbb{R}$ respectively designate the real and imaginary part of z, and $\imath$ denotes the *imaginary unit* which satisfies $\imath^2 = -1$.

$\imath\mathbb{R}$ — The set of *imaginary numbers*, that is the vertical axis of the complex plane, i.e. $\imath\mathbb{R} \equiv \{z \in \mathbb{C} \,|\, \Re[z] = 0\}$.

$\mathbb{Z}_2$ — The *cyclic group of order 2* which is isomorphic to the group $\big(\{1, -1\}, \cdot\big)$, that is the two-element set $\{1, -1\} \subset \mathbb{Z}$ furnished with the ordinary multiplication $\cdot$ as the group operation.

$\star$ Let $\bigwedge(V)$ be the exterior algebra of a d-dimensional vector space V. The *Hodge star operator*, $\star$, transforms a p-form, $\boldsymbol{\alpha} \in \bigwedge^p(V)$, into a $(d-p)$-form $\star\boldsymbol{\alpha} \in \bigwedge^{d-p}(V)$ which is called the *Hodge dual* to $\boldsymbol{\alpha}$.

$\otimes$ Given two elements $u, w \in V$ of a d-dimensional vector space V, the *dyadic product* of u and w, denoted as $u \otimes w$, forms a $d \times d$-matrix with components $(u \otimes w)_{ij} = u_i w_j$ for all $(i,j) \in \{1,\dots,d\}^2$.

Kronecker and Dirac delta

δ The Greek letter δ is used for both the discrete *Kronecker delta* and its continuum continuation; the *Dirac delta distribution*. We distinguish them by their index structure: the Kronecker delta always appears with two indices, $\delta_{ab} = 1$ for $a = b$ and $\delta_{ab} = 0$ otherwise, whereas the Dirac delta distribution in d dimensions is indicated as $\delta^{(d)}(x,y)$ for a pair of points (x,y) that belong to some (pseudo-)Riemannian manifold $\mathcal{M}$. The defining property of $\delta^{(d)}(x,y)$ is for any suitable test function f on $\mathcal{M}$ given by:

$$\int_{\mathcal{M}} f(x)\delta^{(d)}(x,x_0)\mathrm{d}\mathcal{V}(x) = f(x_0),$$

where $\mathrm{d}\mathcal{V}$ is the natural volume form on $\mathcal{M}$.

Transformations

Legendre Let f be a convex function, then its *Legendre transform* is denoted by f^*.

Fourier The *Fourier transform* $\mathcal{F}f$ of a function f, which is initially defined on some coordinate space taking values $f(x)$, is a function in momentum space with values $f(p) \equiv (\mathcal{F}f)(p)$. The distinction between the coordinate and momentum space representation of f will mostly be made by its argument. Factors of 2π appear exclusively in the momentum integral. Transitions between d-dimensional coordinate and momentum space are carried out due to the following relations:

$$f(x) = \int f(p)\mathrm{e}^{\imath p\cdot x}\frac{\mathrm{d}^d p}{(2\pi)^d} \qquad \Big| \qquad f(p) = \int f(x)\mathrm{e}^{-\imath p\cdot x}\mathrm{d}^d x.$$

Einstein notation — *Summation over pairs of identical raised and lowered indices* that appear within a single term is automatically understood.

Conjugation — Let $z \in \mathbb{C}$ be a complex number, then we identify $\bar{z}$ as the *complex conjugate* of z. That is, if $z \mathrel{\hat{=}} \Re[z] + \imath\Im[z]$, then $\bar{z} \mathrel{\hat{=}} \Re[z] - \imath\Im[z]$. For any complex valued object $\mathbf{z}$, the *Hermitian conjugate* of $\mathbf{z}$ is denoted by $\mathbf{z}^\dagger \equiv \bar{\mathbf{z}}^T$, i.e. a complex conjugation in combination with transposition.

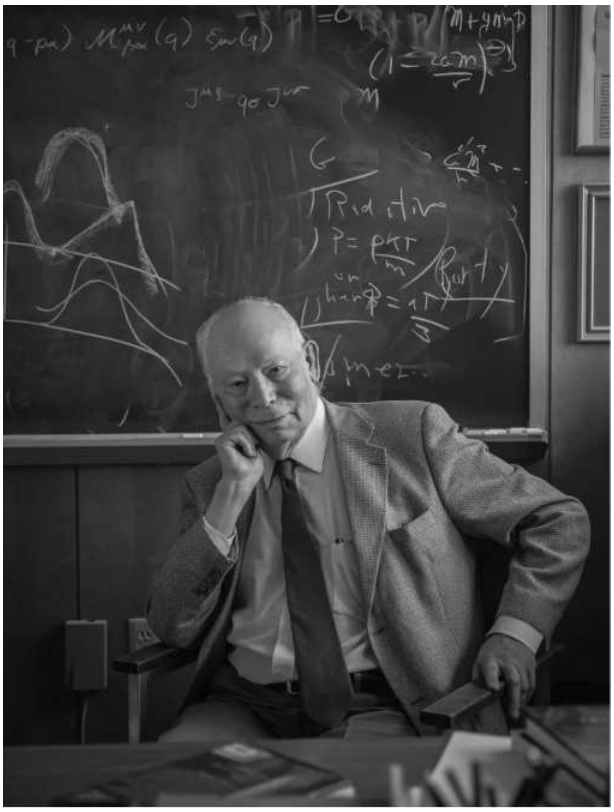
Derivatives — Let O be an object, for instance a function, which depends on a single argument $a \in \mathcal{D}$, i.e. $O \mathrel{\hat{=}} \{O(a) \,|\, a \in \mathcal{D}\}$, where $\mathcal{D}$ denotes the domain of O. The *derivative* of any object with respect to its argument is *marked by a prime*: $O' \equiv \frac{\mathrm{d}O}{\mathrm{d}a}$. The number of primes indicates the number of derivatives which are taken of O. Alternatively, the number of derivatives can also appear as a parenthesised superscript, so that the n-th derivative of O is denoted as $O^{(n)}$.

Tensors & Forms — *Bold letters* are used for all objects that exhibit a non-trivial index structure. This includes in particular *tensors of all kinds and differential forms*. However, the *components* of these objects are denoted by *light italic letters* equipped with the appropriate number of indices. Exceptions to this agreement are the Kronecker delta, and - for reasons of convenience - all sorts of one-index objects. For instance, a vector, as an object with one index, could appear as u with components u_i. Likewise, a tensor of degree 2 could be designated by $\mathbf{T}$ with components T_{ij}.

$\boldsymbol{\varepsilon}$ — The usual rules for the totally antisymmetric *Levi-Civita tensor* $\boldsymbol{\varepsilon}$ apply. In d dimensions we agree on the following "initial condition": $\varepsilon^{12\cdots d} = 1$.

Signature — For the *Minkowski metric tensor* $\boldsymbol{\eta}$, acting on d-dimensional Minkowski space, we assume a signature $\sigma(\boldsymbol{\eta}) \equiv (r_+(\boldsymbol{\eta}), r_-(\boldsymbol{\eta})) = (d-1, 1)$. The *Euclidean metric tensor* is denoted by $\boldsymbol{\delta}$ and has signature $\sigma(\boldsymbol{\delta}) = (d, 0)$.

Fixed Points — If the renormalisation flow of a given theory contains a *fixed point*, then flow quantities c_k evaluated at the fixed point are marked by an *asterisk subscript*, i.e. $c_k \to c_*$ as the scale parameter k approaches its fixed point value.



"The effort to understand the universe is one of the very few things which lifts human life a little above the level of farce and gives it some of the grace of tragedy."

$\sim$ Steven Weinberg

[image source]
Loeb McClain, D. (July 25, 2021).
"Steven Weinberg, Groundbreaking Nobelist in Physics, Dies at 88",
The New York Times.

[quote source]
Weinberg, S. (1976).
"The First Three Minutes",
(1st edition), p. 149. Fontana Paperbacks.

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Preface

From a historical perspective on physics, the past century was predominantly characterised through the paradigmatic shift that was induced by the formation of two novel scientific disciplines in the field of physics whose horizons of application are designed rather contrary to each other, but yet enjoy pronounced acknowledgement and major acceptance within the modern scientific physics community.

In the years from 1905 to 1915, the essential foundations of the *theory of relativity* were decisively developed and published by Albert Einstein, which until today represent the most resistant attempt for a classical description of mechanical phenomena and the gravitational interaction [1]-[3]. On the other side, the substantial principles of *quantum mechanics* were worked out just a few years later in the period from 1925 to 1932 by historical personalities such as M. Born, P. Dirac, W. Heisenberg, W. Pauli and E. Schrödinger among others [4]-[6]. Since then, both branches have developed rather independently from each other, not least because of their significant conceptual differences and quite separated scopes.

To be more specific, general relativity, whose core is supported by the Einstein field equations, provides a deterministic description of the mutual interplay between the way matter behaves under curvature variations of spacetime, and conversely, how curvature responds to matter dynamics. In contrast, the physical reality of the microcosm experiences a quantum theoretical portrayal, which is driven by indeterminacy of physical observables and granularity at fundamental scales.

In the mid 20th century, a prominent cornerstone of quantum physics beyond ordinary quantum mechanics has been established with the successful construction of a quantum theory of the electromagnetic field, which is nowadays known as *quantum electrodynamics* and is still one of the most efficacious theories to date [7]-[9]. In the meantime, further of so-called *quantum field theories* have been shaped with the esperance to expand the triumph of quantum electrodynamics to the other known fundamental forces of nature, followed by *quantum chromodynamics* that covers the strong interaction [10]-[12], as well as the unification of the electromagnetic and weak force in the Glashow-Salam-Weinberg model of the *electroweak interaction* [13] & [14]. Eventually, the ambitious efforts have culminated in one of the most accurate theoretical framework known today as the *standard model of particle physics*, which essentially summarises the evolution of quantum theory in the realm of particle physics in one key word.

However, despite the striking achievement of the standard model, the strenuous times in modern theoretical physics are not over yet, so that much of current endeavour happens in rather recent fields of research which are often phrased as "*physics beyond the standard model*". Among them are the attempts to add also the strong interaction to the amalgamation of the electromagnetic and weak interaction in order to arrive at so-called *grand unified theories*, which are represented by combined symmetry groups of their constituents, such as SU(5) or SO(10) [15].

Aside from the well-known interactions exposed above, of whom we already have successful quantum descriptions at hand, the gravitational interaction still awaits its quantum depiction. The reason for this circumstance is that, conceptually, gravity does not match with standard quantum field theoretical methods, such as perturbative renormalisation [16]. In fact, a multitude of research groups worldwide are still facing the question of how to reconcile the probabilistic nature of quantum theory with the deterministic classical reality encoded in the principles of general relativity.

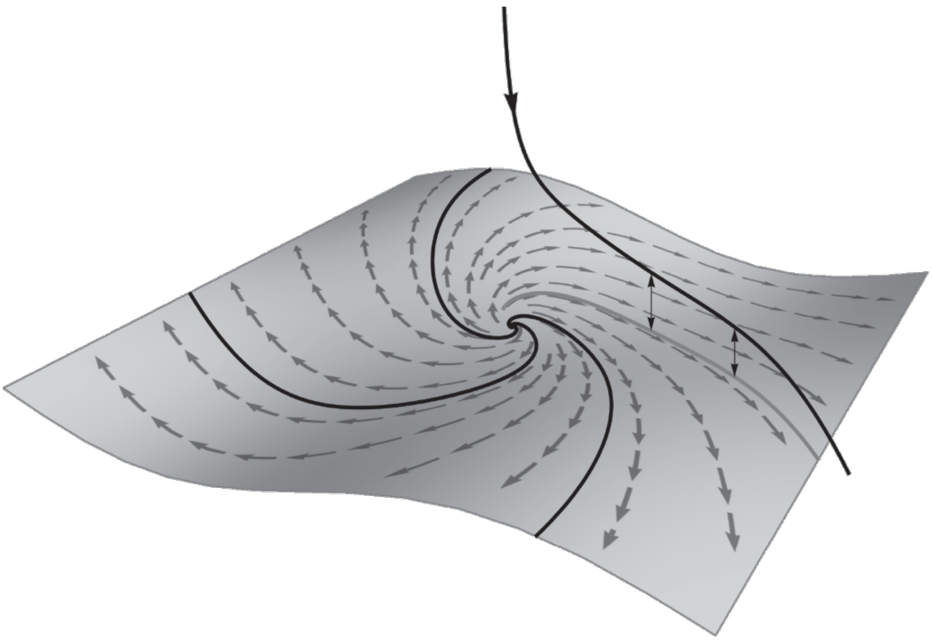
Ideas for theories of *quantum gravity* are numerous and diverse, some of them even leave the approved tracks of quantum field theory, the probably most popular of them being string theory [17] and loop quantum gravity [18]. As one of the concepts closest to quantum field theory stands the non-perturbative treatment of gravity using a scenario known as *asymptotic safety*, which originated from a seminal 1979 paper published by Steven Weinberg [19]. He conjectured the existence of a *non-trivial fixed point* in the exact renormalisation flow of gravity. This statement is presently known as the "*asymptotic safety hypothesis*", which is still considered as being unproven. However, a non-negligible collection of indications provide evidence for its validity [20]-[26].

In fact, it has already been shown that a non-trivial fixed point of the gravitational renormalisation flow exists within the naturally given *Einstein-Hilbert truncation* in $d = 3+1$ dimensions [24]. Furthermore, promising analyses which assume truncations of increased complexity, involving additional powers of the Ricci scalar R within so-called $f(R)$ gravity have already taken place and contributed further instances in favor of the asymptotic safety hypothesis [26].

As it was already mentioned, a general proof for the asymptotic safety hypothesis remains elusive. Progress in this direction is primarily dampened by hard technical demands and peculiarities inherent to gravity. The latter involves a careful implementation of *diffeomorphism invariance* - at least in the framework of quantum Einstein gravity - and moreover, a precise integration of *background independence*, which is one of the key properties every theory of quantum gravity is expected to go along with. For this purpose, the background field method provides a common tool in which the metric $\mathbf{g}$, that initially serves as the dynamical variable, is decomposed into a fixed background component, $\bar{\mathbf{g}}$, and a dynamical fluctuating part, $\mathbf{h}$, leading to $\mathbf{g} = \bar{\mathbf{g}} + \mathbf{h}$. Here, $\mathbf{h}$ now acts as the actual dynamical variable and the background $\bar{\mathbf{g}}$ enables the applicability of quantum field theoretical methods. Following this route, background independence is restored by keeping $\bar{\mathbf{g}}$ arbitrary to some sufficient extent [28].

With the aspiration of approximating the difficile complex regarding asymptotic safety in quantum gravity, we follow a basically untained path and substitute the common pattern by a reduced model which exhibits structures technically similar to gravity, but is simple enough to make a thorough investigation of its fixed point sector accessible. These structures include a selected type of gauge invariance as well as the inclusion of tensorially associated degrees of freedom, propagating on the background of a flat space. Specifically, we consider a generic theory of a *massless vector boson with local* U(1) *symmetry*. The degrees of freedom are captured by the most general set of independent U(1) invariants which are related to the boson's field strength tensor and are consistent with the canonical requirement of Lorentz invariance.

We will proceed as follows: In the subsequent chapter, ch. 2, the basic technical understanding of the utilised methodology - the *functional renormalisation group* - will be mediated among other relevant concepts, including aspects of asymptotic safety and global existence of fixed point actions. All relevant terms are introduced and explained there. Ch. 3 then begins with the core analysis that consists of two principal parts: the derivation of the central equations and identities will be carried out in the first part, sec. 3.1, whereas the presentation and corresponding discussion of their consequences concerning the existence of non-trivial and globally defined fixed point actions is devoted to the second part, sec. 3.2. All considerations receive their final comments in ch. 4 where we additionally give an outlook that refers in particular to possible extensions of the work presented here and is intended to make possible points of contact for further investigation transparent.



"Indeed, everything in nature is changing,
but behind the changing rests an eternal."
∼ Johann Wolfgang von Goethe

[image source]
Adapted from p. 79 of ref. [28].

[quote source]
"Goethe: Lektüre für Augenblicke",
(1st edition from 1982),
p. 46, Insel Verlag.
>translated from German by the author<

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Conceptual and Technical Foundations

In the course of this chapter, we describe the elementary preconditions for the upcoming analyses, starting in ch. 3. Beginning with sec. 2.1, we develop a basic understanding about the central method of this work - the *functional renormalisation group* (FRG) - which combines features of Wilson's renormalisation group as well as of functional aspects originating from the path integral formalism of quantum field theory (QFT). The FRG technique enables us to probe quantum fluctuations over all energy scales and constitute an opportunity for general considerations of QFTs at a non-perturbative level. Specifically, fixed point structures can be explored accurately and rather straightforwardly using this method.

All the relevant information about *fixed points* and their status within the generic asymptotic safety scenario will be outlined in sec. 2.2, including classification schemes and general computation patterns provided by the FRG.

Conclusively, in sec. 2.3, we discuss an essential property of fixed points that goes under the name "*global existence*" and give an illustration of this abstract notion by means of a worked example afterwards. Simultaneously, we will portray two crucial procedures which help to systematically construct globally existing fixed point solutions. They are known as small- and large-field expansion (SFE & LFE).

2.1

The Functional Renormalisation Group

Remark: the content of this section decisively follows refs. [27]-[29] and partly [31]. Moreover, another common FRG reference is [30].

A. From the Partition Functional to the Effective Average Action

Apart from more specific field quantisation procedures such as algebraic or loop quantisation, the common way to construct QFTs from classical templates is commonly either *canonical* or *path integral quantisation*. They are connected through the *n-point functions*, G_n, which carry the correlation data of a system containing n quantum fields, thus representing the key objects of any QFT in which all the relevant physical information is encoded. To avoid unnecessary technical and structural complications, let us stipulate on a scalar field theory with a corresponding action S, that depends on a real scalar field, φ, which propagates on a d-dimensional flat Euclidean background manifold, $\left(\mathbb{R}^d, \boldsymbol{\delta}\right)$. Here, $\boldsymbol{\delta}$ denotes the Euclidean metric tensor whose associated matrix representation in Cartesian coordinates is given by the d-dimensional unit matrix; $\boldsymbol{\delta} \mathrel{\hat{=}} \mathbb{1}_{d\times d}$.

Within canonical quantisation, the classical fields are promoted to field operators, $\varphi \to \hat{\varphi}$, which act upon an appropriate Fock space.

In contrast to this pattern, we could also keep the classical fields as functions, but instead introduce a functional integral formalism to obtain a quantum description of physical quantities by integrating over all possible field configurations, which ultimately leads to the path integral algorithm.

The link between both perspectives becomes visible at the level of the n-point functions, where we have the following relation which connects field operators and path integrals [31]:

$$\left\langle \mathrm{T}\left[\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\right]\right\rangle_{\mathrm{vac}} \equiv G_n\left(\{x_i\}_{i=1}^n\right) \equiv \frac{1}{\mathcal{N}}\int \varphi(x_1)\cdots\varphi(x_n)\mathrm{e}^{-S[\varphi]}\,[\mathcal{D}\varphi]\,. \tag{2.1}$$

On the left, T is the time-ordering operator and $\langle\cdot\rangle_{\mathrm{vac}}$ gives the vacuum expectation value of its argument. The right-hand side represents a functional integral expression that contains a normalisation factor $\mathcal{N}^{-1}$ - which is determined by the condition $G_n(\emptyset) = 1$, i.e. where no fields are considered at all - as well as a weight factor, e^{-S}, such that the total expression behaves similar to an expectation value for continuous functions. The integral measure, $[\mathcal{D}\varphi]$, is generally not well-defined unless a scale regulator is implemented, e.g. a momentum cutoff.

Now, let us stick to the path integral formalism for what is to come. Beside eq. (2.1), the n-point functions can also be obtained from a generating functional, $\mathcal{Z}$, which is known as the *partition functional*, or simply *partition function*, of QFT. It is defined as:

$$\mathcal{Z}\,[J] \equiv \mathrm{e}^{\mathcal{W}[J]} := \int \exp\left(-S\,[\varphi] + \int \varphi(x)J(x)\mathrm{d}^d x\right)[\mathcal{D}\varphi]\,. \tag{2.2}$$

The argument of $\mathcal{Z}$ describes an arbitrary smooth source function that manifests itself in the *source action* "$\int \varphi J$". With this construction, G_n is regained by taking n functional derivatives with respect to the source J:

$$G_n\left(\{x_i\}_{i=1}^n\right) = \frac{1}{\mathcal{N}}\left(\left.\frac{\delta^n \mathcal{Z}\,[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0}\right). \tag{2.3}$$

In the middle of eq. (2.2) we have incidentally introduced the generating functional for the *connected n-point functions*, $\mathcal{W}$, which is sometimes also referred to as the *Schwinger functional*. Speaking in terms of perturbative methods, $\mathcal{W}$ includes only connected Feynman diagrams when calculating the n-point functions explicitly in terms of a perturbation series. These reduced n-point functions are generated in accordance to eq. (2.3) after $\mathcal{Z}$ was replaced by $\mathcal{W}$.

It is a common fact that Feynman diagrams can be dismantled down to one-particle irreducible (1PI) diagrams where the core physical data is stored, and in fact, there is also a generating functional only for this type of diagrams which is known as the *effective action* Γ. The construction procedure of Γ follows a Legendre transformation of $\mathcal{W}$, or equivalently of $\ln(\mathcal{Z})$:

$$\Gamma\,[\phi] := \ln(\mathcal{Z})^*\,[\phi] \triangleq \sup_J\left(\int \phi(x)J(x)\mathrm{d}^d x - \ln\left(\mathcal{Z}\,[J]\right)\right). \tag{2.4}$$

Here, the new variable ϕ - that is induced from the Legendre transformation - can be interpreted as the vacuum expectation value of the field operator $\hat{\varphi}$ in the presence of the "supremum source", $\tilde{J}$, which is singled out by the supremum operation in eq. (2.4), that is:

$$\phi(x) = \langle\hat{\varphi}_{\tilde{J}}(x)\rangle_{\mathrm{vac}} \equiv \left.\left(\frac{1}{\mathcal{Z}\,[J]}\frac{\delta\mathcal{Z}\,[J]}{\delta J(x)}\right)\right|_{J=\tilde{J}}. \tag{2.5}$$

Applying the principle of least action to Γ yields the *quantum equations of motion*, that is, the equations of motion for the vacuum expectation value of the field operators that accounts for all quantum fluctuations. For a proof of this statement and eq. (2.5) see either Lemma D.1 & Theorem D.3 of app. D, or ref. [27].

In order to review the above ideas, the generating functionals presented so far are pictorially compared in fig. 2.1.

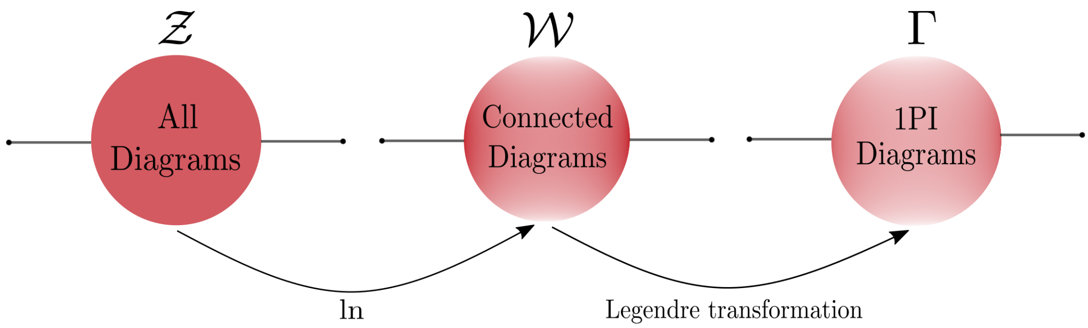

Figure 2.1: *Generating functionals in terms of Feynman diagrams. The partition functional $\mathcal{Z}$ includes the whole spectrum of possible diagrams. A piecewise exclusion of specific diagram classes, namely disconnected and then also one-particle reducible diagrams, leads respectively to the Schwinger functional $\mathcal{W}$ and the effective action Γ.*

There are various options to calculate the effective action, however, a rather prolific approach uses elements of Wilson's renormalisation group (RG) in a fairly modern kind. For this, we introduce a smooth interpolating action, Γ_k, that mediates between the microscopic bare action, S - which appears for instance in eq. (2.2) - and the effective action Γ. The continuous parameter $k \in \mathbb{R}_0^+$ describes a momentum scale, such that Γ_k can be construed as an average action that accounts for all quantum fluctuations in the ultraviolet (UV) region beyond k; that includes momenta $p \in [k, \infty)$, whereas fluctuations of the corresponding infrared (IR) sector are entirely suppressed. This concept, which can be interpreted as a combination of the RG together with functional aspects, is labelled as the **functional renormalisation group** (FRG). Although this description suggests a sharp momentum cutoff, the formulas provided by the FRG basically allow for a wide range of more general cutoff profiles, see eq. (2.10) below. The interpolating action Γ_k is specified such that we obtain the following limits [27]:

$$\lim_{k\to\infty} \Gamma_k \equiv \Gamma_\infty \simeq S \qquad \& \qquad \lim_{k\to 0} \Gamma_k \equiv \Gamma_0 = \Gamma. \tag{2.6}$$

Because of its proximity to the effective action as well as its averaging character, Γ_k is often called **effective average action** (EAA).

An explicit expression for Γ_k can be found by tracing back eq. (2.4) and consequently define a modified partition functional, $\mathcal{Z}_k$, such that:

$$\Gamma_k\,[\phi] = \ln\left(\mathcal{Z}_k\right)^*[\phi] + \mathsf{F}_k\,[\phi]\,, \tag{2.7}$$

where we have used the freedom to add any functional, F_k, to the Legendre transformation of $\ln(\mathcal{Z}_k)$, as long as it vanishes for $k \to 0$ to recover the IR limit, that is the effective action, given in (2.4). We will see that there is actually a preferred choice for F_k, once we have introduced the regulator concept below. First of all, we obtain the path integral representation of $\mathcal{Z}_k$ essentially from eq. (2.2), in which the bare action is extended by an IR regulator term, such that $\mathcal{Z}_k$ gets equipped with a built-in IR suppression [27]:

$$\mathcal{Z}_k\,[J] = \int \exp\left(-S\,[\varphi] - \Delta S_k\,[\varphi] + \int \varphi(x) J(x) \mathrm{d}^d x\right) [\mathcal{D}\varphi]\,. \tag{2.8}$$

For any given $k \in \mathbb{R}_0^+$, the extra weight factor, $\mathrm{e}^{-\Delta S_k}$, where ΔS_k is called the *cutoff action*, precisely realises the inhibition of low momentum modes, i.e. of those momenta p with a smaller squared norm than k; $p^2 < k^2$. This is done through a mass-like damping behaviour of ΔS_k as p^2 drops below the scale k^2. In contrast, the high momentum modes, i.e. modes in momentum space with $p^2 > k^2$, are straightforwardly integrated out, unaffected of any additional suppression.

The typical shape of a cutoff action is best presented for fields φ defined in momentum space by means of the Fourier transformation. In this situation we have [27]:

$$\Delta S_k[\varphi] = \frac{1}{2}\int \varphi(-p)\mathcal{R}_k(p)\varphi(p)\frac{\mathrm{d}^d p}{(2\pi)^d}, \tag{2.9}$$

and even for more complicated field space structures, such as vector-like degrees of freedom, the cutoff action appears often in the form (2.9), containing a characteristic quadratic field dependence and its essential component, the **regulator function** $\mathcal{R}_k$. For completeness, the coordinate space representation of the regulator $\mathcal{R}_k$ is derived in Lemma D.2 of app. D, however, eq. (2.9) is sufficient for what follows.

The concrete appearance of the regulator function $\mathcal{R}_k$ is almost unrestricted up to some necessary but mild properties [27]:

$$\lim_{\frac{p^2}{k^2}\to 0} \mathcal{R}_k(p) > 0 \qquad \& \qquad \lim_{\frac{p^2}{k^2}\to\infty} \mathcal{R}_k(p) = 0. \tag{2.10}$$

In the simplest but common situation, we have $\mathcal{R}_k(p) \sim k^2$ for $p^2 \ll k^2$, i.e. in the IR regime. To get a better grasp on the regulator function, an archetypal example is visualised in fig. 2.2.

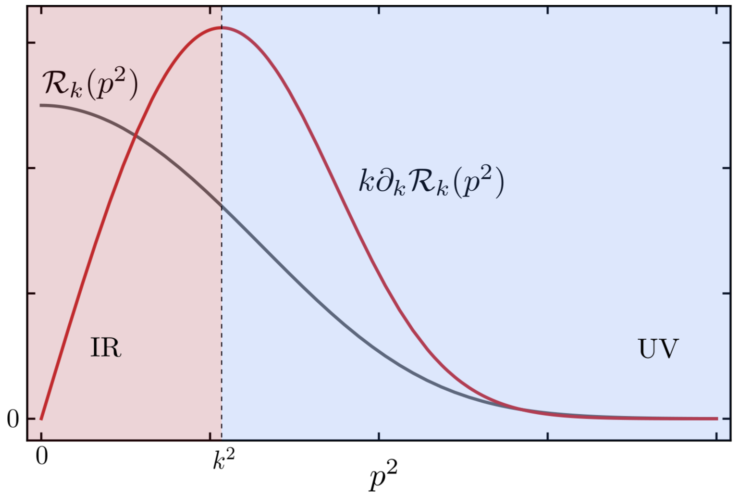

Figure 2.2: *A sketch of a generic regulator function $\mathcal{R}_k$ (black curve) and its "RG time" derivative, $\partial_t\mathcal{R}_k \triangleq k\partial_k\mathcal{R}_k$ (red curve), see eq. (2.12) below. The significance of the latter will become clear in paragraph* ***B*** *of this section. As it is obvious, the profile of $\mathcal{R}_k$ satisfies the conditions given in (2.10) and as a consequence, the term $k\partial_k\mathcal{R}_k$ develops a pronounced peak around the boundary between the UV and IR regions. It is this property that implements Wilson's concept of a momentum shell with respect to a fixed momentum scale k.*

Let us finally turn back to eq. (2.7). With ΔS_k from eq. (2.9), together with the properties (2.10), we just found an excellent candidate for F_k and can conclude with the full expression for the EAA [27]:

$$\boxed{\Gamma_k[\phi] = \sup_J \left(\int \phi(x)J(x)\mathrm{d}^d x - \ln\big(\mathcal{Z}_k[J]\big)\right) - \Delta S_k[\phi]\,.} \tag{2.11}$$

B. Renormalisation Flow and Theory Space

There exists a "dynamical" equation that offers the effective average action as its solution once a set of initial data has been fixed. A detailed derivation of this so-called *flow equation* can be found in various references, e.g. [27]-[29], but also in Theorem D.5 of app. D. However, here we are instead satisfied with a brief sketch, which is sufficient for our purposes.

One starts by introducing a UV cutoff scale, $\Lambda < \infty$, to render all functional integrals well-defined, and additionally establishes an alternative parameter that replaces the scale k:

$$t := \ln\left(\frac{k}{k_0}\right) \qquad \Rightarrow \qquad \frac{\partial}{\partial t} = k\frac{\partial}{\partial k}, \tag{2.12}$$

where $k_0 \in \mathbb{R}_0^+$ is an arbitrary *reference scale* setting the transition between negative and positive values for t. Since the new parameter acquires its values from the whole real axis $\mathbb{R}$, it behaves similar to a time coordinate and is therefore called *renormalisation group time*. Next, one would take the time derivative of Γ_k using eq. (2.11), for the right-hand side being evaluated at the now scale dependent "supremum source" $J = \tilde{J}_k$. The result is essentially of the form: $\partial_t \Gamma_k[\phi] = -\partial_t \ln\left(\mathcal{Z}_k[\tilde{J}_k(\phi)]\right) - \partial_t \Delta S_k[\phi]$. The t-derivative of the first term can be obtained from eq. (2.8), where one parallely introduces a symbol for the *scale and source dependent connected propagator*, $G_k(x,y) := \frac{\delta^2 \ln(\mathcal{Z}_k[J])}{\delta J(x)\delta J(y)}$. This quantity satisfies the rather important operator relation:

$$\left(\Gamma_k^{(2)} + \mathcal{R}_k\right) G_k = \mathbb{1}, \tag{2.13}$$

where we have introduced a common shorthand:

$$\Gamma_k^{(n)}[\phi] \equiv \frac{\delta^n \Gamma_k[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)}. \tag{2.14}$$

From the results above, one can conclusively deduce the flow equation. Here, we will directly present its general form which is valid even beyond the scalar field approach. The flow equation reads [28]:

$$\boxed{\partial_t \Gamma_k[\Phi] = \frac{1}{2}\mathbf{STr}\left[\left(\Gamma_k^{(2)}[\Phi] + \mathcal{R}_k\right)^{-1} \partial_t \mathcal{R}_k\right].} \tag{2.15}$$

In this expression, Φ represents a collection of fields, possibly also of different kinds. The *supertrace operation*, **STr**, performs summations over all incorporated index structures, e.g. including an integration over coordinate or momentum space and summation over Lorentz indices, spinor indices and so forth, considering a minus sign for fermionic degrees of freedom.

Eq. (2.15) is alternatively well known as the **Wetterich equation**, named after Christof Wetterich who derived it in 1993 [32]. Seminal publications in this regard are in addition [33] & [34]. Moreover, it is an **exact renormalisation group equation** (ERGE), i.e. that the implicit dependence on the UV cutoff Λ can be removed by taking the limit $\Lambda \to \infty$. Furthermore, the formal limit of the solution Γ_k as $k \to 0$ yields the exact effective action Γ. This is properly justified from the Wilsonian structure invoked through the term $\partial_t \mathcal{R}_k$ (cf. fig. 2.2).

In preparation for ch. 3, we will now work out a step by step recipe of how to apply the FRG method in order to inspect natural phenomena and their corresponding field theoretical quantum description using the Wetterich equation.

(1). As a first step, we need to decide on a set of concrete *physical degrees of freedom* and *symmetry conditions*. In simple cases, these decisions are actually anticipated by already existing classical field theories.

If we turn for a moment back to the scalar field theory, one could for instance ask for a relativistic $\mathbb{Z}_2$ symmetric model. Then we would address this requirement with an action functional that includes all independent Lorentz covariant products of the field and its derivatives which are invariant under parity transformations in field space. Then, an appropriate set of "basis elements" could contain: $\{\mathbb{1}, \phi^{2n}, (\partial_\mu \phi)^{2m}, \phi^{2n}(\partial_\mu \phi)^{2m}, \dots\} \equiv \mathcal{B}_\phi$, with $n, m \in \mathbb{Z}$. Consequently, the desired action can be constructed by means of the linear span of $\mathcal{B}_\phi$ with in general yet unknown coefficients.

Let us now universalise this perspective and consider a collection $\mathcal{B}$, which is assumed to be a maximally extended set of "basis functionals", i.e. actions containing operator products that are consistent with a given choice of symmetry constraints just as desribed in the scalar field example above[1]. A typical element $b \in \mathcal{B}$ could look like: $b[\Phi] = \int (\partial_\mu \Phi^a(x))^2 \, \mathrm{d}^d x$. This suggests to define the space of all actions which are elements of the span of $\mathcal{B}$ and refer to it as **theory space**, $\mathcal{T} := \mathrm{span}(\mathcal{B})$. Indeed, also the searched for EAA, Γ_k, counts as an element of $\mathcal{T}$ for all $k \in \mathbb{R}_0^+$, hence it can be expanded using the basis $\mathcal{B}$:

$$\Gamma_k[\Phi] = \sum_{\alpha \in I} u_\alpha(k) b^\alpha[\Phi], \tag{2.16}$$

for $b^\alpha \in \mathcal{B}$ and I being an appropriate index set. For the components u_α we motivate the term *generalised couplings*.

In order to concisely summarise the above discussion, step **(1)** basically consists of the task to specify a theory space that respects our structural boundary conditions.

(2). We need one more ingredient for the Wetterich equation (2.15), that is the regulator function $\mathcal{R}_k$. In principle, all momentum and scale dependent functions which fulfil the demands (2.10) do the job. Since part of the ERGE is based on $\mathcal{R}_k$, different choices accordingly lead to different solutions. In fact, one can show that under a slight variation $\mathcal{R}_k \to \mathcal{R}_k + \epsilon \tilde{\mathcal{R}}_k$, where ϵ is viewed as a small parameter and $\tilde{\mathcal{R}}_k$ is any suitable regulator, the EAA is correspondingly deformed; $\Gamma_k \to \Gamma_k + \epsilon \tilde{\Gamma}_k$ with [28]:

$$\tilde{\Gamma}_k[\Phi] = \frac{1}{2} \mathbf{STr}\left[\left(\Gamma_k^{(2)}[\Phi] + \mathcal{R}_k\right)^{-1} \tilde{\mathcal{R}}_k\right]. \tag{2.17}$$

However, for $k \to 0$, the deformation eventually vanishes and reproduces the effective action without any modification. Therefore, the technical details of mode suppression are rather insubstantial in a full RG flow analysis as long as no truncations of theory space were made beforehand [28]. However, fixed points, that are yet to be discussed in the next section, do depend on the choice of $\mathcal{R}_k$. In this sense, explorations of fixed point sectors go along with an optimisation process due to different options regarding the particularities of regularisation.

In ch. 3, we will adhere to the conventional choice for $\mathcal{R}_k$, that is the *linear cutoff*, or sometimes also called the *optimised cutoff* [35]:

$$\mathcal{R}_k^{\mathrm{op}}(p) = \left(k^2 - p^2\right) \mathbf{1}_{[0,k^2)}(p^2) \equiv p^2 R^{\mathrm{op}}\left(\frac{p^2}{k^2}\right). \tag{2.18}$$

In the last step we have introduced the *regulator shape function*, R^{op}, by explicitly factorising the p-dependence such that the argument of R^{op} is dimensionless per construction[2].

(3). Once Γ_k and $\mathcal{R}_k$ are fixed, the RHS of eq. (2.15) is now computable. We need the second functional derivative of the EAA, $\Gamma_k^{(2)}$, the time derivative of the regulator function, $\partial_t \mathcal{R}_k$, and the regularised full propagator, $\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} \equiv G_k$. The latter usually marks the most intricated stage.

[1]In typical situations it is $|\mathcal{B}| = \infty$. The case $|\mathcal{B}| < \infty$ is a rare exception and mostly adheres to truncations of the space of all possible action functionals, the so-called theory space $\mathcal{T}$ (see the next sentence).

[2]Note that $[\mathcal{R}_k] = 2$, cf. app. B.

In ch. 3, we will use an expansion of the propagator in terms of projectors which act upon configuration space, determine the algebraic structure of them and make a general ansatz for the regularised full propagator with a set of unknown coefficients that need to be determined. The identity $\mathbb{1} = \left(\Gamma_k^{(2)} + \mathcal{R}_k\right)\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}$ offers a system of coupled algebraic equations whose solution space yield the sought-for coefficients. Finally, we have to take care of the supertrace, i.e. we need to identify the various index species.

(4). An expansion of the RHS of eq. (2.15) in terms of elements taken from $\mathcal{B}$ - like it is done in eq. (2.16) - and a subsequent comparison with the LHS gives an infinite tower of coupled differential equations for the generalised couplings. Because of the notable complexity that this system generally reveals, one practically starts by investigating reduced systems to find approximate solutions. This translates to so-called *truncations of theory space*, in which we aim for a proper evaporation of $\mathcal{B}$ into a finite residue $\tilde{\mathcal{B}} \subset \mathcal{B}$ with $|\tilde{\mathcal{B}}| < \infty$. In this way, the infinite tower of differential equations melts down to a more or less well treatable system which can then be adressed by either analytical or numerical methods, or even a combination of those. However, sometimes also infinite dimensional truncations of theory space are considered.

Before we conclude this section, it should be made transparent that we tacitly assumed the existence of a vector space structure for theory space multiple times, though this is actually not justified. To this date, there is no convincing proposal for any generic algebraic structure upon $\mathcal{T}$. However, since theory space pertinently suggests at least aspects of a vector space by its very construction, we will adopt this conjecture like for instance in ref. [28] in what follows.

2.2

The Idea of Asymptotic Safety

Remark: for this section we mostly refer to refs. [28] & [29].

A. Beta Functions and Fixed Points

In order to approach the important notion of a fixed point by means of its definition within the FRG formalism, it is useful to introduce a specific class of functions which depend on the generalised couplings u_α. Since the LHS of the Wetterich equation (2.15) represents an action functional, and therefore an element of theory space $\mathcal{T}$, so must the RHS. Thus, whatever the supertrace on the RHS may yield, we can at least formally expand it by means of an appropriate operator basis $\mathcal{B}$. The Wetterich equation then reads [28]:

$$\sum_{\alpha\in I}\Big(\partial_t u_\alpha(k)\Big) b^\alpha\,[\Phi] = \sum_{\alpha\in I}\hat{\beta}_\alpha\Big(u(k);k\Big) b^\alpha\,[\Phi]\,. \tag{2.19}$$

The formal coefficients $\hat{\beta}_\alpha$, are the famous *generalised beta functions*, i.e the beta functions for the generalised couplings u_α. Because of the complicated structure of the RHS in eq. (2.15), the $\hat{\beta}_\alpha$ could in principle depend on all generalised couplings collected in the coupling vector $u(k) \equiv (u_1(k), u_2(k), \dots)^T \in \mathcal{T}$.

As it becomes clear from eq. (2.19), the generalised beta functions carry all the information about the *renormalisation flow*, that is, the change of the generalised couplings - and therefore of the EAA - with respect to variations of the RG time parameter t. It manifests the phenomenon which is usually dubbed as "*running of the coupling constants*", which is already known from standard QFT [31].

In concrete FRG applications, it is often beneficial to perform a transition to *dimensionless generalised couplings*. For this, let us consider any coupling u_α. Its canonical mass dimension should be $[u_\alpha] = d_\alpha$, and we use the mass scale provided by the scale parameter k to define: $\tilde{u}_\alpha := k^{-d_\alpha} u_\alpha$. In this way, we have separated the explicit scale dependence of u_α from the fundamental coupling strenght $\tilde{u}_\alpha$, which is dimensionless and depends explicitly on the RG time t. From $\partial_t u_\alpha(k) = \hat{\beta}_\alpha\Big(u(k); k\Big)$, following from eq. (2.19) for all $\alpha \in I$, we get:

$$\partial_t \tilde{u}_\alpha(t) = k^{-d_\alpha} \hat{\beta}_\alpha\Big(u(k); k\Big) - d_\alpha \tilde{u}_\alpha(t). \tag{2.20}$$

Finally, we define the **beta functions of the dimensionless generalised couplings**:

$$\tilde{\beta}_\alpha\Big(\tilde{u}(t)\Big) := \partial_t \tilde{u}_\alpha(t). \tag{2.21}$$

We should emphasise that we did not consider any scaling effects of the canonical mass dimensions d_α to this end. Nevertheless, they are actively modified and shifted by an amount that is known as the anomalous dimension as soon as interactions are turned on. Since coupling parameters are generally sensible to the RG flow and change with respect to variations of the scale parameter k, the anomalous dimension behaves accordingly. In FRG equations, the anomalous dimension appears explicitly after renormalisation of the field strength, i.e. adding another factor, the field strength renormalisation Z_k, to the definition of $\tilde{u}_\alpha$. We will keep it with this rather perfunctory discussion for now and come back to it during our analysis in ch. 3 where we give further details and a formal definition of the anomalous dimension.

Let us now clarify the meaning of fixed points[3]. Let $\tilde{u} \in \mathcal{T}$ and $\tilde{\beta} \equiv \Big(\tilde{\beta}_1, \tilde{\beta}_2, \ldots\Big)^T \in \mathcal{T}$ be the beta vector that, like the coupling vector, collects all the beta functions. The vector $\tilde{u}$ is called a **fixed point**, denoted with $\tilde{u} \equiv \tilde{u}_*$, by definition iff [28]:

$$\boxed{\tilde{\beta}\Big(\tilde{u}_*\Big) = 0.} \tag{2.22}$$

This *fixed point equation* actually reflects an inifinite tower of coupled algebraic equations, which is inherited from the character of the ERGE.

There is some kind of a domain of influence for any given fixed point $\tilde{u}_* \in \mathcal{T}$, which can formally be characterised as a subset of theory space. To describe it, we need one more concept. Let $c : \mathcal{I} \subseteq \mathbb{R} \to \mathcal{T}$ be a curve in theory space, parametrised with the RG time t. We say that c is *RG future-extendible*, if there exists an *RG future end-point* $c_\infty \in \overline{\mathcal{T}}$ in some suitable asymptotic extension of $\mathcal{T} \subset \overline{\mathcal{T}}$, such that $c(t) \to c_\infty$ in the *asymptotic RG future*, that is for $t \to \infty$. Indeed, fixed points can serve as reasonable candidates for the RG future end-points of renormalisation flow trajectories, i.e. solutions of the ERGE. Now, let us consider a fixed point $\tilde{u}_* \in \overline{\mathcal{T}}$. The set of all points $p \in \overline{\mathcal{T}}$ for which there exists an RG future-extendible (and already RG future-extended) curve c, that passes through p and is pulled into $\tilde{u}_*$ under the inverse flow, i.e. for increasing t, is referred to as the *UV critical hypersurface* of the fixed point, denoted by $\mathscr{I}_{\mathrm{UV}}\Big(\tilde{u}_*\Big)$.

In order to gain better insight about the algebraic features of the UV critical hypersurface and consequently also about the significance of fixed points in view of non-perturbative descriptions of QFTs, let us focus on the properties of the renormalisation flow in direct proximity to such a fixed point. For this, we assume the existence of a fixed point, $\tilde{u}_*$, and consider a small neighbourhood around $\tilde{u}_*$ in which we expand the beta vector:

$$\tilde{\beta}\Big(\tilde{u}(t)\Big) = \underbrace{\tilde{\beta}\Big(\tilde{u}_*\Big)}_{\triangleq 0} + \mathbf{J}_{\tilde{\beta}}\Big(\tilde{u}_*\Big)\Big(\tilde{u}(t) - \tilde{u}_*\Big) + \mathcal{o}\Big(\tilde{u}(t) - \tilde{u}_*\Big); \qquad \text{for } \tilde{u} \to \tilde{u}_*. \tag{2.23}$$

[3]For reasons of convenience, we will just write "beta functions" and "couplings", instead of, respectively, "beta functions of dimensionless generalised couplings" and "dimensionless generalised couplings" from now on.

Here, $\mathbf{J}_{\tilde{\beta}}\left(\tilde{u}_*\right)$ denotes the Jacobian matrix of the beta vector evaluated at the fixed point. It is commonly called the *stability matrix* and denoted as $\mathbf{B} \equiv \mathbf{J}_{\tilde{\beta}}\left(\tilde{u}_*\right)$. Moreover, the stability matrix satisfies an eigenvalue equation of the form $\mathbf{B}V_K = \theta_K V_K$, where the index K labels different - possibly complex - eigenvalues $\theta_K \in \mathbb{C}$ and their corresponding eigenvectors V_K. The former are also known as *critical exponents*. After a linearisation of the flow, which accounts to a truncation of eq. (2.23) after first order in the deviation $\tilde{u}(t) - \tilde{u}_*$, the remaining system of equations can straightforwardly be solved, its solution being [28]:

$$\tilde{u}(t) \simeq \tilde{u}_* + \sum_K \left(C_K \mathrm{e}^{-\theta_K t}\right) V_K. \tag{2.24}$$

Here, the C_K are constants of integration and are to be considered as free parameters of the theory.

Eigenvectors V_K that refer to eigenvalues with negative real part, $\Re[\theta_K] < 0$, are called *irrelevant* for obvious reasons: irrelevant parts within the eigenvector expansion of eq. (2.24) eventually diverge for $t \to \infty$, thus developing an unacceptable behaviour in the UV limit, i.e. that $\mathrm{e}^{-\theta_K t}$ continuously grows for increasing t. As a consequence, the eigenvector expansion in eq. (2.24) would contain at least a single divergent constituent, such that the curve $\tilde{u}$ leaves theory space for $t \to \infty$, i.e. in the asymptotic RG future. Therefore, we choose the corresponding constants of integration C_K to vanish identically. Eigenvalues θ_K which are purely imaginary, i.e. $\Re[\theta_K] = 0$, are called *marginally irrelevant* and describe oscillating contributions to the curve $\tilde{u}$ near the fixed point $\tilde{u}_*$. However, since marginally irrelevant directions V_K survive the limit $t \to \infty$ if their constants of integration C_K are nonzero, they ultimately lead to a deviation from $\tilde{u}_*$ provided we keep them in the expansion (2.24). Of course, marginally irrelevant directions do not result in blowing up the curve $\tilde{u}$ out of theory space like this would be the case for an irrelevant direction, but instead forces $\tilde{u}$ orbiting the fixed point $\tilde{u}_*$. Thus, also for the marginally irrelevant directions we require their corresponding constants of integration to vanish. The remaining C_K, for which $\Re[\theta_K] > 0$ need to be fixed by experiment. Since such a task is only attainable for a finite set of free parameters we demand $\left|\left\{C_K \mid \Re[\theta_K] > 0\right\}\right| < \infty$ for a viable fixed point to hold.

In consequence, the set of all *relevant eigenvectors* V_K, for which $\Re[\theta_K] > 0$, parametrise the space of points in $\overline{\mathcal{T}}$ which are attracted under the inverse renormalisation flow, i.e. towards the ultraviolet, thus inducing the dimension of the UV critical hypersurface:

$$\mathscr{D}_{\mathrm{UV}}\left(\tilde{u}_*\right) \equiv \dim\left[\mathscr{I}_{\mathrm{UV}}\left(\tilde{u}_*\right)\right] = \left|\left\{\theta_K \;\middle|\; \Re[\theta_K] > 0\right\}\right|. \tag{2.25}$$

The request for $\mathscr{D}_{\mathrm{UV}}\left(\tilde{u}_*\right) < \infty$ is a first example of a so-called *selection criterion*. It measures the degree of predictivity of the theory that emanates from the fixed point. In sec. 2.3, we will familiarise with another such selection criterion, namely the *global existence of fixed point actions*. To conclude this paragraph, a visualisation of $\mathscr{I}_{\mathrm{UV}}\left(\tilde{u}_*\right)$ is presented in fig. 2.3.

B. Non-perturbative Renormalisability and Asymptotic Freedom

When exploring fixed point structures of any given theoretical frame, one normally tends to establish a gross classification scheme according to which fixed points are called either *Gaussian* or *non-Gaussian*. Remembering that a fixed point is actually an action functional living in theory space, the former type of fixed points corresponds to a free field theory, i.e. that all couplings which - beside the parts that are quadratic in the fields - induce interacting terms in the operator expansion vanish at a Gaussian fixed point (GFP). In contrast, a fixed point that contains an interacting sector is referred as being a non-Gaussian fixed point (NGFP)[4].

[4]These definitions of GFPs and NGFPs can of course be formalised (although this will not be of interest for us): at a GFP, the stability matrix is diagonal with eigenvalues that coincide with the canonical mass dimensions of the couplings. This is not true for at least one eigenvalue at an NGFP [28].

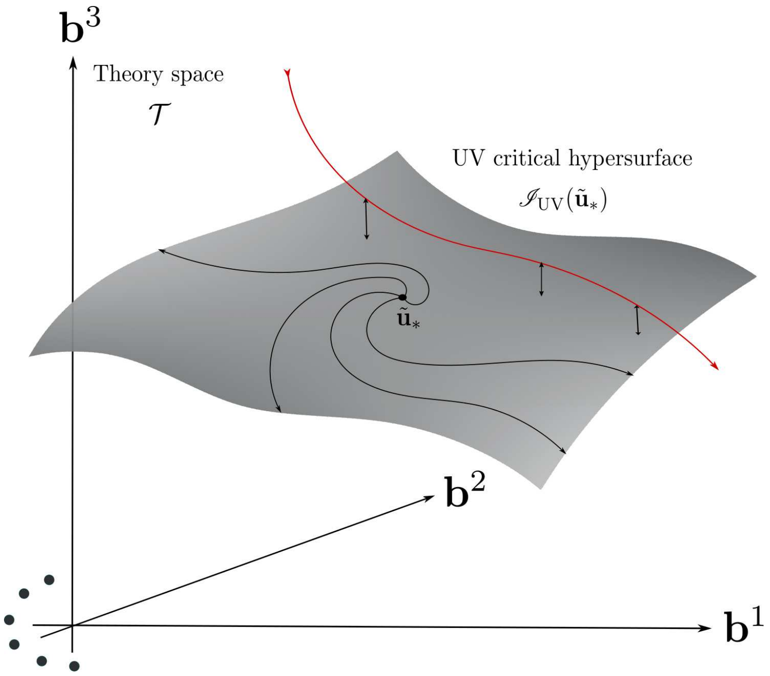

Figure 2.3: *A portrayal of the UV critical hypersurface $\mathscr{I}_{\mathrm{UV}}$ defined upon the fixed point $\tilde{u}_*$ and embedded into theory space $\mathcal{T}$. Curves that are completely built on relevant eigendirections are attracted by $\tilde{u}_*$ (black lines), whereas curves that contain the information of at least one irrelevant eigendirection miss the fixed point and eventually leave theory space (red line). The arrows that are attached to the solid lines indicate the direction of the renormalisation flow, i.e. towards the infrared.*

The strong impact of the FRG on QFT becomes visible in particular when we are confronted with the problem of perturbative non-renormalisability, like it is the case for gravity. Since perturbative QFT breaks down as soon as the couplings become considerably large as we resolve nature to shorter and shorter distance scales, we cannot expect to have success with our standard perturbative approach which ultimately leaves its domain of applicability. Instead, we would need an alternative formalism that does not rely on any diminutiveness assumptions about the couplings and therefore enables a treatment beyond the perturbative regime. In fact, the ERGE given in (2.15) can be considered as a possible instrument for this quest and serves a starting point for non-perturbative studies. Theories which may not be perturbatively renormalisable, could possibly be so from a non-perturbative perspective. An answer to this question can be found using the FRG formalism by searching for a viable fixed point in the region from which the theory receives its pathological behaving contributions. Once a suitable fixed point is singled out, we can construct well-defined theories over all energy scales by looking for solutions of the ERGE equipped with our fixed point as the "initial data" (or asymptotic data). These solutions are viewed as *asymptotically safe renormalisation flow trajectories* and represent a whole programme which is known as *asymptotic safety* [28] & [29].

In concrete calculations, one normally encounters not only a single fixed point in a given theory space, but instead a whole collection of them. Each one can be a suitable option for declaring an asymptotically safe UV behaviour. Choosing one or the other fixed point leads to qualitatively different QFTs which belong to various of so-called *universality classes* [28].

In the special case in which our theory asymptotically arrives at a GFP, we say that it is *asymptotically free*. A celebrated example of so-called *asymptotic freedom*, which was honoured by the nobel-prize in 2004, is given by QCD: it has been shown that quarks behave like freely propagating particles in the deep UV region [36]-[38].

As it was mentioned at the end of paragraph **A** of this section, the choice for a suitable fixed point is based on carefully reflected selection criteria, which are generated to a large extent upon physical admissibility conditions.

Although the word "suitable" is easily written down, justifying any specific fixed point selection in an impermeable manner is often quite laborious. We have already discussed an example of a rather popular selection criterion, that of *predictivity*, i.e. that $\mathscr{D}_{\mathrm{UV}}\big(\tilde{u}_*\big)$ is a finite (and preferably not too large) number. In the upcoming section, we will add another selection criterion that has, for instance, a significant influence on applications in statistical physics and is known under the name "global existence of fixed point actions".

2.3

Global Existence of Fixed Point Actions

A fairly natural condition on fixed points is known under the name "global existence". Let us first elucidate the precise meaning of this notion. For this, we take a fixed point (action) $\tilde{u}_* \in \overline{\mathcal{T}}$ and expand it with an appropriate operator basis $\mathcal{B}$, such that we get the following representation:

$$\tilde{u}_*\left[\Phi\right] \mathrel{\hat{=}} \sum_{\alpha\in I} \left(\tilde{u}_*\right)_\alpha b^\alpha\left[\Phi\right]. \tag{2.26}$$

We recall that b^α represents a functional of the form $b^\alpha\left[\Phi\right] = \displaystyle\int \ell^\alpha\left(\Phi\right) \mathrm{d}^d x$, where ℓ denotes an arbitrary function in the field Φ, as long as it is compatible with the symmetry constraints that were imposed beforehand. Then, eq. (2.26) becomes:

$$\tilde{u}_*\left[\Phi\right] \mathrel{\hat{=}} \int \left(\sum_{\alpha\in I} \left(\tilde{u}_*\right)_\alpha \ell^\alpha\left(\Phi\right)\right) \mathrm{d}^d x \equiv \int \mathscr{L}_*\left(\Phi\right) \mathrm{d}^d x, \tag{2.27}$$

where in the last step we have introduced the fixed point Lagrangian function, or simply *fixed function* $\mathscr{L}_* := \sum_{\alpha\in I} \left(\tilde{u}_*\right)_\alpha \ell^\alpha$. If we restrict w.l.o.g. on a single real valued field species, $\Phi \to \phi$, we have that $\phi(x) \in \mathbb{R}$ for any $x \in \mathbb{R}^d$ for which ϕ is defined. Hence, $\mathscr{L}_*$ could be considered as a partial function defined on some subset $\mathcal{S} \subseteq X \subseteq \mathbb{R}$ of the maximal domain of definition, X, which contains all values of ϕ that could possibly occur. Sometimes, $X \neq \mathbb{R}$, for instance if ϕ is constructed from a fundamentally positive quantity, like the squared norm of a vector degree of freedom in Euclidean space. Finally, we say that the fixed point (action) $\tilde{u}_*$ is **globally-existing**, or simply **global**, if the corresponding fixed function $\mathscr{L}_*$ is a total function, i.e $\mathcal{S} = X$. That means, if $\tilde{u}_*$ is a global fixed point, then $\mathscr{L}_* : X \to \mathbb{R}$ is a function free of definition singularities, i.e. points in X for which $\mathscr{L}_*$ is not defined.

In many applications which are based on derivative expansions of the action, $\mathscr{L}_*$ is often decomposed into a purely kinetic part, $\sim \left(\partial_\mu\phi\right)^2 + \ldots$, and a *fixed point potential*, $\mathscr{V}_*$, where the latter is usually assumed to contain only non-derivative expressions in the field ϕ. If we keep the explicit form of $\mathscr{V}_*$ unspecified, the renormalisation flow can be projected onto the fixed point potential by assuming a constant field amplitude, such that the fixed point equation (2.22) yields an ordinary differential equation for $\mathscr{V}_*$ [28]. In this way, another perspective on the notion of globally-existing fixed points occurs, which is to be understood as follows: within the theory of differential equations, the notions of "local" and "global" solutions are well known and can be adopted for our purposes to declare the notion of a global fixed point in an alternative manner. In this specific situation, a fixed point $\tilde{u}_*$ would be referred to as being global, if $\mathscr{V}_*$ defines a global solution of the fixed point equation of the projected renormalisation flow.

Let us now illustrate the idea of globally-existing fixed point actions by means of an explicit model.

Example. Global Wilson-Fisher Fixed Point of the Ising Model

The versatile class of $\mathrm{O}(N,\mathbb{R})$-symmetric theories is well suited for descriptions of a wide range of critical phenomena, such as the ferromagnetic phase transition in $\mathrm{O}(3,\mathbb{R})$, leading to the Heisenberg model, or the daily experienced liquid-vapor phase transition in $\mathrm{O}(1,\mathbb{R})$, which is known as the *Ising model* [39]. In a general $\mathrm{O}(N,\mathbb{R})$ theory, the degrees of freedom are collected in a field vector $\Phi \equiv (\varphi_1,\dots,\varphi_N)^T$ living in an N-dimensional flat field space. A general action functional in d spacetime dimensions which suffices the symmetry condition imposed by the $\mathrm{O}(N,\mathbb{R})$ group action is given by [40]:

$$S_{\mathrm{O}(N,\mathbb{R})}\left[\Phi\right] = \int \left(\frac{1}{2}\left(\partial_\mu \Phi^a\right)^2 + V\left(\bar{\varphi}\right)\right) \mathrm{d}^d x, \tag{2.28}$$

where $\bar{\varphi} \equiv |\Phi|$ with respect to the standard inner product on $\mathbb{R}^N$. Therein, the function V appears as a potential whose field dependence is restricted on $\mathrm{O}(N,\mathbb{R})$ invariant expressions, which are exactly the elements of the polynomial ring in the field norm $|\Phi| = \sqrt{\Phi_a\Phi^a}$ over $\mathbb{R}$.

This paragraph is devoted to give a demonstration of the concept of "globally-existing fixed points" by investigating the rather simple fixed point sector of the Ising model, i.e. $\mathrm{O}(1,\mathbb{R})$ theory - where $\mathrm{O}(1,\mathbb{R}) = \big(\{+1,-1\},\cdot\big) \cong \mathbb{Z}_2$ is the group that contains just the identity and parity inversion with the ordinary multiplication $\cdot$ acting as the group operation - and concurrently familiarise with two important analytical methods which appear as useful tools to construct global fixed point solutions. Now, our field vector carries only a single component, $\Phi \equiv \varphi$, and according to eq. (2.28) the action reads:

$$S_{\mathrm{O}(1,\mathbb{R})}\left[\varphi\right] = \int \left(\frac{1}{2}\left(\partial_\mu \varphi\right)^2 + V\left(\varphi\right)\right) \mathrm{d}^d x. \tag{2.29}$$

Let us build the effective average action from the bare action (2.29) by modifying the potential V with a scale dependence k, $V \to V_k$, which gives us the so-called *effective average potential*. Moreover, also the kinetic term acquires a flowing coupling, Z_k, which is known as the *field strength renormalisation*. However, for the sake of simplicity, we add the additional assumption that $Z_k = 1$ for all values of k. These specifications are sometimes summarised under the name *local potential approximation*, in whose spirit our ansatz for the EAA reads [28]:

$$\Gamma_k\left[\phi\right] = \int \left(\frac{1}{2}\left(\partial_\mu \phi\right)^2 + V_k\left(\phi\right)\right) \mathrm{d}^d x. \tag{2.30}$$

We note, that the argument of Γ_k does generally not coincide with the bare field φ, but should rather be interpreted, similar to the full effective description in eq. (2.5), as the vacuum expectation value of φ in presence of the scale-dependent supremum source $\tilde{J}_k$ which follows from the definition eq. (2.11). In the upcoming step, we want to set up the Wetterich equation (2.15) for the ansatz (2.30). The LHS, $\partial_t\Gamma_k$, is rather trivial, so we therefore directly turn our focus to the RHS. First, we need the second functional derivative of Γ_k, which under consideration of the elementary relation $\frac{\delta\phi(x)}{\delta\phi(y)} = \delta^{(d)}(x,y)$ gives:

$$\Gamma_k^{(2)}\left[\phi\right](x,y) \equiv \frac{\delta^2\Gamma_k\left[\phi\right]}{\delta\phi(x)\delta\phi(y)} = \left[-\partial_x^2 + V_k''\big(\phi(x)\big)\right]\delta^{(d)}(x,y), \tag{2.31}$$

in which ∂_x^2 denotes the d'Alembert operator acting on the coordinates x, and $V_k''(\phi) \equiv \frac{\partial^2 V_k(\phi)}{\partial\phi^2}$ according to our conventions. Since (2.31) represents the (x,y) entry of the operator $\Gamma_k^{(2)}\left[\phi\right]$ as it appears in the Wetterich equation and in its coordinate space representation, we can observe that $\Gamma_k^{(2)}\left[\phi\right]$ is diagonal in its coordinate indices.

Thus, the corresponding operator can be read off to:

$$\Gamma_k^{(2)}[\phi] = -\partial^2 + V_k''(\phi) \quad \xrightarrow{\mathcal{F}} \quad p^2 + V_k''(\mathcal{F}\phi)\,, \tag{2.32}$$

where we have performed the Fourier transformation $\mathcal{F}$ right after, such that $\Gamma_k^{(2)}[\phi]$ is now given in its momentum space representation. For simplicity, we set $\mathcal{F}\phi \equiv \phi$ from now on and keep in mind that ϕ is to be considered in momentum space.

Since we have set $Z_k = 1$, the $(\partial_\mu\phi)^2$-direction in the (truncated) theory space is insignificant for our analysis, because the flow in this direction is freezed by assumption. The important part belongs only to the scale-dependent effective average potential that contains purely non-derivative field contributions. Therefore, it is convenient to consider constant field configurations that rule out the uninteresting sector of the EAA by default.

Now, because of the diagonal structure of $\Gamma_k^{(2)}[\phi]$, the regularised full propagator, $G_k[\phi] = \left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)^{-1}$, is nothing but the reciprocal of its components in momentum space. Conclusively, the supertrace in (2.15) reduces to a trace-pair over coordinate and momentum space, i.e. integrals thereof, such that the insertion of the expressions above yields:

$$\Omega_d\Big[\partial_t V_k(\phi)\Big] = \frac{1}{2}\Omega_d \int \frac{\partial_t \mathcal{R}_k(p)}{p^2 + \mathcal{R}_k(p) + V_k''(\phi)} \frac{\mathrm{d}^d p}{(2\pi)^d}, \tag{2.33}$$

where Ω_d corresponds to a volume factor originating from the coordinate space integration. Since it appears on both sides, these factors simply cancel.

For the regulator we choose the optimised cutoff such as it was already introduced in eq. (2.18), $\mathcal{R}_k^{\mathrm{op}}(p) = (k^2 - p^2)\,\mathbf{1}_{[0,k^2)}(p^2)$, which is linear in p^2. In consequence, the integral on the RHS of eq. (2.33) becomes spherical in momentum space. An adequate switch to spherical coordinates yields:

$$\partial_t V_k(\phi) = \frac{1}{(4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}+1\right)} \frac{k^{d+2}}{k^2 + V_k''(\phi)}. \tag{2.34}$$

In order to obtain the fixed point equation, it is advantageous to establish dimensionless quantities first. A simple dimensional analysis reveals: $[V_k] = d$ and $[\phi] = \frac{d-2}{2}$. Hence, we define: $\tilde{V}_k\big(\tilde{\phi}\big) := k^{-d} V_k(\phi)$ with the dimensionless field $\tilde{\phi} := k^{\frac{2-d}{2}}\phi$. At a fixed point we have $\partial_t \tilde{V}_k = 0$ for $\tilde{V}_k \equiv \tilde{V}_*$. After inserting the dimensionless quantities and summarising the geometric factor by $a_d^{-1} := (4\pi)^{d/2}\,\Gamma\left(\frac{d}{2}+1\right)$, we obtain from eq. (2.34):

$$d\tilde{V}_*\big(\tilde{\phi}\big) - \frac{1}{2}(d-2)\,\tilde{\phi}\tilde{V}_*'\big(\tilde{\phi}\big) = \frac{a_d}{1 + \tilde{V}_*''\big(\tilde{\phi}\big)}. \tag{2.35}$$

For our subsequent calculation we specify on three spacetime dimensions, $d = 2+1$, where we have $a_3 = \frac{1}{6\pi^2}$. In this situation, eq. (2.35) reduces to:

$$\boxed{6\tilde{V}_*\big(\tilde{\phi}\big) - \tilde{\phi}\tilde{V}_*'\big(\tilde{\phi}\big) = \frac{1}{3\pi^2}\frac{1}{1 + \tilde{V}_*''\big(\tilde{\phi}\big)}.} \tag{2.36}$$

In view of finding a global fixed point action, our strategy provides an approach to a suitable fixed point solution by constructing it piecewise from two sides; the small-field amplitude regime and its counterpart for large field amplitudes. Thus, the treatment of the non-linear ODE (2.36) proceeds as follows[5].

[5]The following algorithm consists of three steps **(S1)**, **(S2)** and **(S3)** and will also be of relevance in ch. 3. Therefore, we pursue a more detailed and careful description in what follows.

(S1). As a first attempt, we expand the fixed point potential $\tilde{V}_*$ in terms of a formal power series with an infinitely large collection of unknown coefficients, $\{\sigma_{2n}\}_{n\in\mathbb{N}_0}$, where the numbering "$2n$" arises from the $\mathbb{Z}_2$ symmetry of $\tilde{V}_*$. As it becomes clear explicitly in the considerations below, any beta function, $\beta_{2\hat{n}}$ for $\hat{n}\in\mathbb{N}_0$, does not only depend on all previous order couplings, $\{\sigma_{2n}\}_{n=0}^{\hat{n}-1}$ together with the present order coupling $\sigma_{2\hat{n}}$, but also on the first successive one, $\sigma_{2(\hat{n}+1)}$. That means, that at the tower level $2\hat{n}$ of eq. (2.22), we gathered a system that contains one more unknown than we have equations at hand. This implies that we can choose one element of the set $\{\sigma_{2n}\}_{n=0}^{\hat{n}+1}$ as a free parameter and express all other elements in terms of it. For this, we could choose for instance σ_2, which corresponds to the first non-constant contribution in the power series expansion for $\tilde{V}_*$. Either way, an additional "exterior" information would be necessary to close the algebraic system up to $\sigma_{2(\hat{n}+1)}$. Here comes the point where we utilise a method called **small-field expansion** (SFE) [40]: for small field values $\tilde{\phi}$, it is sufficiently reasonable to truncate the power series expansion of $\tilde{V}_*$ at a certain order $2\hat{n}\geq 0$, by setting all subsequent coefficients to zero, i.e. $\sigma_{2n}\equiv 0$ for all $n>\hat{n}$. This includes in particular $\sigma_{2(\hat{n}+1)}\equiv 0$, which, after recalling that we have expressed every coefficient in terms of σ_2, yields a solution space of possible values for σ_2 and therefore all the other remaining σ_{2n} that were not be supposed to vanish. The small.field expansion ansatz is then of the form:

$$\tilde{V}_*^{\mathrm{SFE}}\big(\tilde{\phi}\big)=\sum_{n=0}^{\hat{n}}\frac{\sigma_{2n}}{(2n)!}\tilde{\phi}^{2n}. \tag{2.37}$$

Albeit the system has been successfully closed, however, since the fixed point equation ultimately sets up a non-linear algebraic system of coupled equations, we will generally find a plethora of possible solutions. To take a systematic decision on one of them, we need to identify a stabilising behaviour within the solution spaces of $\sigma_{2(\hat{n}+1)}(\sigma_2)=0$ as we go to higher orders of our truncation, $2\hat{n}\to 2(\hat{n}+1)\to\ldots$, i.e. we have to single out a specific value for σ_2 on which part of the solution space to the equation $\sigma_{2(\hat{n}+1)}(\sigma_2)=0$ converges. In this way, we solve the algebraic fixed point system approximately for arbitrary large $\hat{n}$ with increasing precision. As a result, we ensure a maximal radius of convergence for the SFE.

(S2). Once we have successfully constructed a well-defined fixed point solution for the small field amplitude regime, we can turn to large field values then. Here, the procedure works similar to what was already described in step **(S1)**, but using an expansion in negative instead of positive powers of $\tilde{\phi}$, relative to a leading power, to render the fixed point's asymptotic behaviour properly at large field amplitudes. The asymptotically dominating part of $\tilde{V}_*$, i.e. in the limit $\tilde{\phi}\to\infty$, is captured by a term $\sim\tilde{\phi}^{2N}$ for some $N\in\mathbb{Z}$ and for which a formula can be derived by means of dimensional reasoning. In d spacetime dimensions, N is given by [41]:

$$N=\frac{d}{d-2+\eta_*}, \tag{2.38}$$

in which η_* is called the *anomalous dimension* evaluated at the fixed point[6]. In this way, we can formulate the analogue of the small-field expansion by choosing once more a truncation order $2\hat{n}\geq 0$ and write down a **large-field expansion** (LFE) in a form that is suggested in [40]:

$$\tilde{V}_*^{\mathrm{LFE}}\big(\tilde{\phi}\big)=\lambda\tilde{\phi}^{2N}+\sum_{n=0}^{\hat{n}}\lambda_{2n}\tilde{\phi}^{-2n}. \tag{2.39}$$

The role of the free parameter is now assigned to λ. From here on we can proceed analogously to step **(S1)** and use the fixed point equation to express all coefficients λ_{2n} in terms of λ.

[6]As it was already explained in paragraph **A** of sec. 2.2, we will encounter the anomalous dimension again in ch. 3 together with its general definition and where we also briefly clarify its physical interpretation.

(S3). Finally, we need to determine the *radius of convergence* of both, the small-field expansion as well as the large-field expansion. For this, one generically has to use the formula expression of the *Cauchy-Hadamard theorem*, that is:

$$r_f = \frac{1}{\limsup\limits_{n\to\infty}\left(\sqrt[n]{c_n}\right)}, \tag{2.40}$$

in which c_n are the coefficients of some series expansion of a given function f, and r_f denotes its radius of convergence. Another formula, which is often considerably better to handle, is given as a consequence of the ratio test:

$$r_f = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|, \tag{2.41}$$

which is only applicable if the limit really exists. However, the radius of convergence can also readily be estimated geometrically by scanning the graphs of $\tilde{V}_*^{\mathrm{SFE}}\left(\tilde{\phi}\right)$ and $\tilde{V}_*^{\mathrm{LFE}}\left(\tilde{\phi};\lambda\right)$ for a pathological changeover as we tend to larger or smaller field values respectively. The remaining parameter λ has to be adjusted in such a way, that the interval $[r_{\mathrm{LFE}}, r_{\mathrm{SFE}}] \neq \emptyset$, in which $r_{\mathrm{SFE}} \equiv r_{\tilde{V}_*^{\mathrm{SFE}}}$ and $r_{\mathrm{LFE}} \equiv r_{\tilde{V}_*^{\mathrm{LFE}}}$. Moreover, there shall exist a field value, $\tilde{\phi}_0 \in [r_{\mathrm{LFE}}, r_{\mathrm{SFE}}]$, for which $\tilde{V}_*^{\mathrm{SFE}}\left(\tilde{\phi}_0\right) = \tilde{V}_*^{\mathrm{LFE}}\left(\tilde{\phi}_0\right)$. If these conditions are fulfilled, we eventually can weld our partial solutions at $\tilde{\phi}_0$ together and generate a global solution $\tilde{V}_*$. To improve the quality of this agglutination, we could supplementary fine tune the parameter λ, such that the total error between $\tilde{V}_*^{\mathrm{SFE}}$ and $\tilde{V}_*^{\mathrm{LFE}}$ is minimised in the overlap region, which is accounted by demanding:

$$\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\int\limits_{r_{\mathrm{LFE}}(\lambda)}^{r_{\mathrm{SFE}}}\left[\tilde{V}_*^{\mathrm{SFE}}\left(\tilde{\phi}\right) - \tilde{V}_*^{\mathrm{LFE}}\left(\tilde{\phi}\right)\right]^2 \mathrm{d}\tilde{\phi}\right)\Bigg|_{\lambda=\lambda_\star} = 0. \tag{2.42}$$

Apart from the analytical methods described above, there are a number of numerical approaches, such as for instance so-called *pseudo-spectral methods*. Though these are applied very successfully to construct global fixed point actions, the analytical algorithms do not lose their relevance and often provide a reliable starting point for the application of numerical techniques, or are already even sufficient. For a demonstration of numerical treatments of the Ising model, see [41]. An extension to various $\mathrm{O}(N,\mathbb{R})$ models can be found in [40].

Let us now begin to apply the steps **(S1)**-**(S3)** on eq. (2.36).

(S1). Small-Field Expansion

According to our instructions formulated above, we first perform a MacLaurin series expansion of the RHS of eq. (2.36) in even powers of the field $\tilde{\phi}$ to account for the $\mathbb{Z}_2$ symmetry condition on the fixed point potential and insert the small field ansatz, eq. (2.37), right after. Equating coefficients of same powers in the field amplitude yields the following relation:

$$(3-n)\,\sigma_{2n} = \frac{1}{6\pi^2}\frac{\mathrm{d}^{2n}}{\mathrm{d}\tilde{\phi}^{2n}}\left[\frac{1}{1+\tilde{V}_*''\left(\tilde{\phi}\right)}\right]\Bigg|_{\tilde{\phi}=0}, \qquad \text{for } n \geq 0. \tag{2.43}$$

The derivative structure on the RHS constitutes a total of $2n+2$ derivatives of $\tilde{V}_*$ for any $n \in \mathbb{N}_0$. Thus, evaluating for $\tilde{\phi} = 0$ at the end of our calculation do not only gives the previous coefficients, $\{\sigma_{2m}\}_{m=0}^{n}$, but in addition also their successor, σ_{2n+2}, as we expect it from the description in **(S1)**.

Accordingly, we let σ_2 - that is the coefficient of the mass-like term $\frac{1}{2}\sigma_2\tilde{\phi}^2$ - adopt the role of a free parameter and change its symbol to $\sigma_2 \to \mu$ for better identification. Below, the first five relations are presented explicitly in terms of μ as they follow from eq. (2.43) (but skipping the trivial information $\sigma_2 = \mu$):

$$\begin{aligned}
\sigma_0 &= \frac{1}{18\pi^2}\frac{1}{1+\mu}, \\
\sigma_4 &= -12\pi^2\mu\,(1+\mu)^2\,, \\
\sigma_6 &= 72\pi^4\mu\,(1+\mu)^2\left(1+14\mu+13\mu^2\right), \\
\sigma_8 &= -25\,920\pi^6\mu^2\,(1+\mu)^4\,(1+7\mu)\,, \\
\sigma_{10} &= 103\,680\pi^8\mu^2\,(1+\mu)^5\left(2+121\mu+623\mu^2\right), \\
&\vdots
\end{aligned} \tag{2.44}$$

A truncation at order $\hat{n} \geq 0$ serves as an additional information in the spirit of the SFE, namely $\sigma_{2(\hat{n}+1)}(\mu) = 0$, which closes the algebraic system (2.44) up to $\tilde{\phi}^{2\hat{n}}$ in the power series expansion (2.37). For different truncation orders $\hat{n}$, the induced set of zeros, $\left\{\mu \,|\, \sigma_{2(\hat{n}+1)}(\mu) = 0\right\}$, is depicted in fig. 2.4. Within this chart we can clearly observe the Gaussian line of zeros at $\mu_{\mathrm{GFP}} = 0$, but in addition also a non-trivial attractor of the solution space sequence at $\mu_{\mathrm{NGFP}} \simeq -0.18606$.

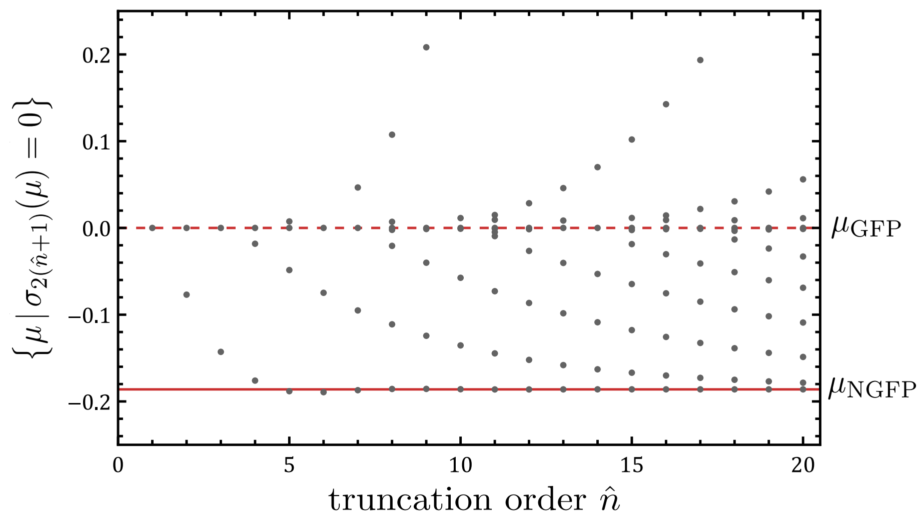

Figure 2.4: *At each truncation order $\hat{n} \geq 1$, there is a non-empty set of zeros for the first resected coefficient of the power series expansion after truncation order $\sigma_{2(\hat{n}+1)}$. Up to truncation order $\hat{n} = 20$, the solution $\mu_{\mathrm{GFP}} = 0$ is perpetually present, which only leaves the constant term $\tilde{V}_* = \sigma_0 = \frac{1}{18\pi^2}$ and corresponds to the trivial, or Gaussian, fixed point solution of eq. (2.36). In addition, several downward directed branches develop a seemingly converging behaviour at higher truncation orders. They strive to a non-trivial zero $\mu_{\mathrm{NGFP}} \simeq -0.18606$, which is expected to be approximated with increasing accuracy in direction of truncation orders beyond $\hat{n} = 20$.*

For this specific value of μ, the fixed point potential in the small-field amplitude regime $\tilde{V}_*^{\mathrm{SFE}}$ is presented in fig. 2.5, with a good perspective on the rapid change of tendency at roughly $|\tilde{\phi}| \simeq 0.5$. Hence, we obtain a graphical estimate for the radius of convergence to $r_{\mathrm{SFE}} \simeq 0.5$.

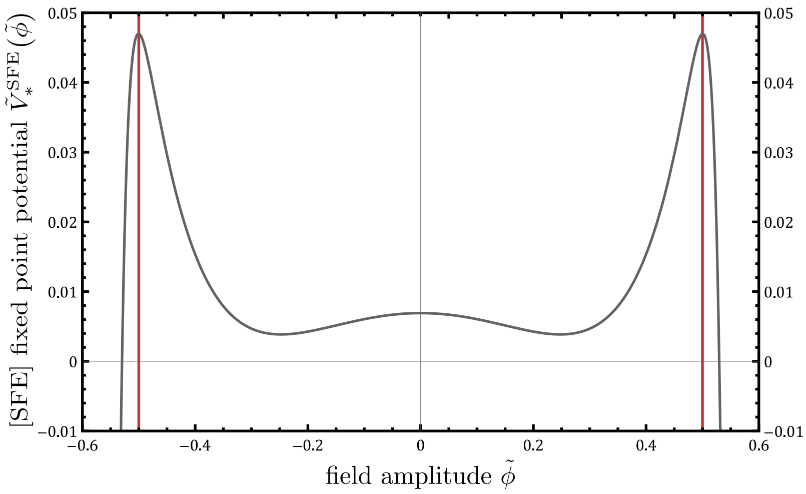

Figure 2.5: *The graph of $\tilde{V}_*^{\mathrm{SFE}}$ that contains the information of the power series expansion, eq. (2.37), up to contribution $\sim \tilde{\phi}^{40}$ for $\mu = \mu_{\mathrm{NGFP}}$. The transition to the irregular, and therefore unreliable region is markedly visible at $|\tilde{\phi}| \simeq 0.5$, which allows for a valuation of the radius of convergence as it is indicated by the red vertical lines.*

(S2). Large-Field Expansion

In order to properly describe the fixed point potential $\tilde{V}_*$ in situations where the modulus of the field amplitude becomes very large, we can refer to the ansatz (2.39) which comprises an asymptotically dominating part in the limit $|\tilde{\phi}| \to \infty$, as well as a "shrinking tail" that vanishes in the same limit process. Explicitly, we have:

$$\tilde{V}_*^{\mathrm{LFE}}\left(\tilde{\phi}\right) = \lambda\tilde{\phi}^{2N} + \sum_{n=0}^{\infty} \lambda_{2n}\tilde{\phi}^{-2n}. \tag{2.45}$$

In view of ch. 3, we already anticipate at this stage that the anomalous dimension evaluated at the NGFP, η_*, is proportional to the change of the field strength renormalisation Z_k with respect to the scale parameter k. Since we have decided on $Z_k = 1$ for all scales k, this implies that $\eta_* = 0$ within the scope of the local potential approximation. Consequently, the asymptotic power N can comfortably be inferred from eq. (2.38), yielding $N = 3$ in $d = 3$ spacetime dimensions.

It is convenient to introduce a new field variable, $\tilde{\chi} := \tilde{\phi}^{-1}$, which represents a small quantity for large field amplitudes $|\tilde{\phi}|$. As a consequence, eq. (2.36) now reads:

$$6\tilde{V}_*\left(\tilde{\chi}\right) + \tilde{\chi}\tilde{V}_*'\left(\tilde{\chi}\right) = \frac{1}{3\pi^2}\frac{1}{1 + 2\tilde{\chi}^3\tilde{V}_*'\left(\tilde{\chi}\right) + \tilde{\chi}^4\tilde{V}_*''\left(\tilde{\chi}\right)}. \tag{2.46}$$

After inserting the ansatz (2.45) for the new variable $\tilde{\chi}$ together with another MacLaurin series expansion of the RHS in even powers of $\tilde{\chi}$ afterwards, we obtain:

$$\sum_{n=0}^{\infty} 2\lambda_{2n}\left(3+n\right)\tilde{\chi}^{2n} = \frac{\tilde{\chi}^4}{3\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n)!}\left[\frac{1}{30\lambda + \tilde{\chi}^4 + \sum_{j=1}^{\infty} 2j\left(2j+1\right)\lambda_{2j}\tilde{\chi}^{2j+6}}\right]^{(2n)}_{\tilde{\chi}=0}\tilde{\chi}^{2n}. \tag{2.47}$$

Because of the generic RHS-factor $\tilde{\chi}^4$ in front of the MacLaurin series, the coefficients λ_0 and λ_2 are forced to vanish by comparison with the LHS: $\lambda_0 = \lambda_2 = 0$. For the higher order coefficients we can find an analogue of eq. (2.43) by equating prefactors of same powers in $\tilde{\chi}$ on both sides of eq. (2.47).

This results in:

$$\lambda_{2n+4} = \frac{1}{6\pi^2}\frac{1}{(2n)!(5+n)}\left[\frac{1}{30\lambda + \tilde{\chi}^4 + \sum_{j=1}^{\infty} 2j\,(2j+1)\,\lambda_{2j}\tilde{\chi}^{2j+6}}\right]^{(2n)}_{\tilde{\chi}=0}, \qquad \text{for } n \geq 0. \tag{2.48}$$

Following our argumentation in **(S2)**, we declare λ to act as a free parameter for which the first five non-vanishing relations that follow from the system (2.48) read:

$$\begin{aligned}
\lambda_4 &= \frac{1}{900\pi^2\lambda}, \\
\lambda_8 &= -\frac{1}{37\,800\pi^2\lambda^2}, \\
\lambda_{12} &= \frac{1}{1\,458\,000\pi^2\lambda^3}, \\
\lambda_{14} &= -\frac{1}{2\,430\,000\pi^4\lambda^3}, \\
\lambda_{16} &= -\frac{1}{53\,460\,000\pi^2\lambda^4}, \\
&\;\;\vdots
\end{aligned} \tag{2.49}$$

as well as $\lambda_6 = \lambda_{10} = 0$. At this point, our piecewise two-step analysis of the fixed point potential is completed and we try to appropriately combine both results in the upcoming step.

(S3). Matching the SFE and LFE

If we restrict our attention on non-negative field values, $\tilde{\phi} \geq 0$ - which is without loss of essential information because of $\mathbb{Z}_2$ symmetry - we can achieve a global solution by the formal statement:

$$\tilde{V}_*\big(\tilde{\phi}\big) = \tilde{V}_*^{\mathrm{SFE}}\big(\tilde{\phi}\big)\mathbf{1}_{[0,\tilde{\phi}_0)}\big(\tilde{\phi}\big) + \tilde{V}_*^{\mathrm{LFE}}\big(\tilde{\phi};\lambda_\star\big)\mathbf{1}_{[\tilde{\phi}_0,\infty)}\big(\tilde{\phi}\big), \tag{2.50}$$

in which $\tilde{\phi}_0$ and $\lambda_\star$ are interpreted as explained in **(S3)**. Now, we aim for an optimal choice of the remaining parameter, $\lambda \to \lambda_\star$, and accordingly deduce a concrete $\tilde{\phi}_0$. For this, we begin by extracting the information $\lambda_\star \in (2.5, 5)$ from fig. 2.6 with a corresponding radii of convergence estimated to $r_{\mathrm{LFE}}(\lambda = 5) \simeq 0.4$ and $r_{\mathrm{LFE}}(\lambda = 2.5) \simeq 0.35$ at the boundaries of the interval.

To proceed, we make a further assumption that simplifies the resolution of the optimisation equation (2.42) significantly: in case we keep the implicit λ dependence of r_{LFE}, then the evaluation of eq. (2.42) becomes challenging since we are not aware of any explicit expression for $r_{\mathrm{LFE}}(\lambda)$. The latter could, in principle, be inferred from the limit behaviour of the function sequence $|\lambda_{2n}(\lambda)/\lambda_{2(n+1)}(\lambda)|$ as $n \to \infty$. To avoid these complications, we are instead satisfied by taking the mean $[r_{\mathrm{LFE}}(2.5) - r_{\mathrm{LFE}}(5)]/2 = 0.375$ and suppose this value to refer to $r_{\mathrm{LFE}}(\lambda_\star)$, which is an adequate approximation. Then, the evaluation of (2.42) is rather straightforward and yields:

$$\lambda_\star \simeq 3.34499 \qquad \& \qquad \tilde{\phi}_0 \simeq 0.48186. \tag{2.51}$$

To conclude this example of the three dimensional Ising model, we insert the information from (2.51) in eq. (2.50) and plot the full fixed point potential that can be seen in fig. 2.7. Moreover, our results are in good agreement with [41].

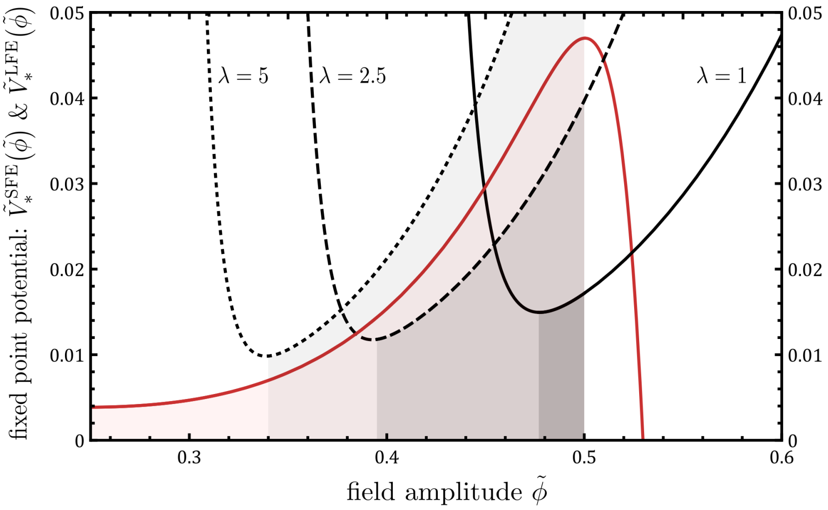

Figure 2.6: *Both, the graphs of the small-field and large-field solution are presented as red and black curves respectively. The latter are shown in three different variants according to distinct values for* λ *as indicated. The shaded areas below the large-field curves delimit the overlap regions, i.e. the closed intervals* $[r_{\mathrm{LFE}}(\lambda), r_{\mathrm{SFE}}(\lambda)]$. *The solid line coloured in black, which corresponds to* $\tilde{V}_*^{\mathrm{LFE}}\big(\tilde{\phi}; \lambda = 1\big)$, *does not show any intersection with the red solid line, but the dashed and dotted black lines enclose the sector where the optimal parameter* $\lambda_\star$ *and a point of contact,* $\tilde{\phi}_0$, *between* $\tilde{V}_*^{\mathrm{SFE}}$ *and* $\tilde{V}_*^{\mathrm{LFE}}$ *can be found.*

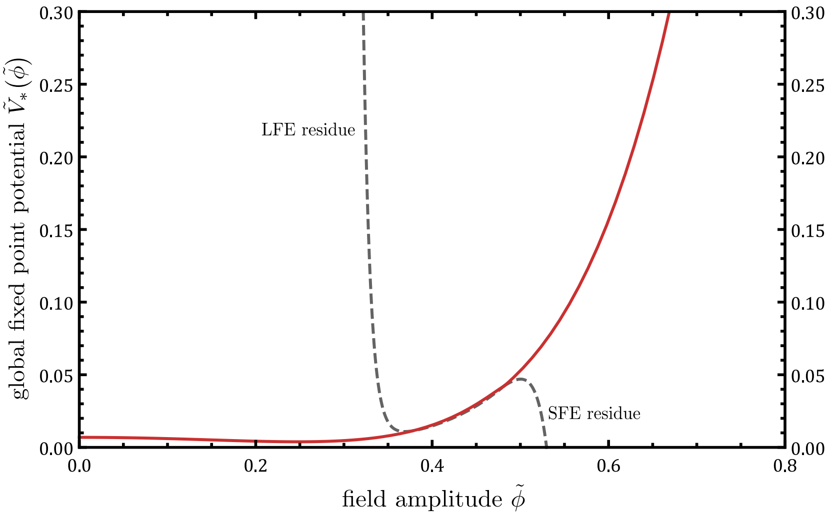

Figure 2.7: *The well-known global Wilson-Fisher fixed point potential of the three dimensional Ising model is depicted by the red solid line, whereas black dashed lines indicate the graphs according to the partial solutions of the separated analyses that were made in the context of SFE and LFE.*

2.4

Exact Renormalisation Group Methods for Gauge Theories

A. Gauge Redundancies

In this section, we will discuss the final preliminaries in prospect of ch. 3. There, we are confronted with a functional renormalisation group that entails *local symmetries* at the level of the effective action. We can think of these symmetries as being implemented by local transformation parameters. Any physical system that contains such a symmetry is referred as being *gauge invariant* under its corresponding *gauge transformation*.

A well-studied example is Maxwell's theory of electromagnetism where we have freedom to transform the four potential pointwise and continuously to any desired value without affecting Maxwell's equations and hence the observable electric and magnetic field. To be a little more concrete, let[7] $A \hat{=} (\phi, \mathbf{A})$ denote the four potential that summarises the scalar potential ϕ and the vector potential $\mathbf{A}$. Then, the electric and magnetic field, $\mathbf{E}$ and $\mathbf{B}$ respectively, can be constructed from A like follows (see for instance [51]):

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \qquad \& \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{2.52}$$

It is standard textbook knowledge that these relations are invariant under a local transformation of the form $A_\mu \mapsto A_\mu + \partial_\mu \varphi$, in which φ is supposed to be any sufficiently smooth function [51]. The special feature of this type of transformation is its local character, i.e. we transform A to a different value at each point in spacetime. Such mappings are known as *gauge transformations*. In this situation, we call the four potential A a *gauge field* and the local transformation parameter φ a *gauge function*.

Let us now gather some basic informations about how to apply the FRG in the presence of gauge symmetries. For definiteness, we shall consider a well-known instance to explain the new features that appear in the FRG implementation of gauge invariant theories, that is *Yang-Mills theory* (YM theory). A thorough discussion has already been carried out in [27]. Therefore, we will mostly stick to this reference until the end of the present section.

For a precise fomulation we begin with a collection of rather general facts about Lie groups and Lie algebras [42]. Let G be a Lie group, i.e. a differentiable manifold furnished with a group structure, such that the basic group operations are smooth. In case of Yang-Mills theory, the Lie group is usually considered to be the special unitary group $\mathrm{SU}(n)$ with $n \in \mathbb{N}$. Each Lie group can be associated with its induced Lie algebra $\mathfrak{g}$, that is the algebra defined on the vector space which is tangent to the neutral element of G together with the Lie bracket $[\cdot,\cdot]$ that serves as the vector space structure respecting multiplicative compostion. For $\mathrm{SU}(n)$, the associated Lie algebra is denoted as $\mathfrak{su}(n)$.

Now, let us take a set of basis vectors, or sometimes also called *generators*, $\{X_I\}$ for $\mathfrak{g}$ in its adjoint representation and expand any 1-form ω that has tangential components ω_μ in terms of that basis:

$$\omega = \omega_\mu \mathrm{d}x^\mu = \left(\omega_\mu^I X_I\right) \mathrm{d}x^\mu = \left(\omega_\mu^I \mathrm{d}x^\mu\right) X_I \equiv \omega^I X_I. \tag{2.53}$$

The Lie bracket relations among $\{X_I\}$ define the so-called *structure constants* $\{f^{IJK}\}$ of the Lie algebra $\mathfrak{g}$:

$$[X_I, X_J] = f_{IJ}{}^K X_K. \tag{2.54}$$

[7] At this point, we agree on more convention: the purely spatial part of any full $[(d-1)+1]$-dimensional vector v is indicated by the same, but bold-face typed letter, i.e $v \hat{=} (v_0, \mathbf{v})$. In addition, we will write all $(d-1)$-dimensional vectors in terms of bold-face letters.

After this common discussion, let us turn back to the Yang-Mills example where $G \equiv \mathrm{SU}(n)$. The classical YM action can be written in terms of the *field strength tensor*, $\mathbf{F}$, which is constructed as a 2-form upon the Lie algebra valued 1-form degrees of freedom A. In gauge theories, we identify these 1-forms with the above mentioned *gauge fields*. Either way, the *YM action* reads [27]:

$$S_{\mathrm{YM}}[A] = \int \frac{1}{4} F^I_{\mu\nu} F_I^{\mu\nu} \mathrm{d}^d x. \tag{2.55}$$

We can now take any pair of local tangential indices (μ, ν) and expand $F_{\mu\nu}$ in terms of generators from $\mathfrak{su}(n)$. According to the common formulation we get [27]:

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + g f^I{}_{JK} A^J_\mu A^K_\nu, \tag{2.56}$$

in which the upper capital latin Lie algebra indices are also known as *adjoint colour labels* and g represents a coupling parameter. If the structure constants do not vanish entirely, we would call the underlying gauge theory *non-Abelian*, which is lucid from eq. (2.54) when we interpret the Lie bracket as a commutator.

In order to define a QFT based on the YM action (2.55), we could use eq. (2.2) as an orientation and declare a gauge partition functional by:

$$\begin{aligned} \mathcal{Z}_{\mathrm{YM}}[J] &\sim \int\limits_{\mathfrak{su}(n)} \exp\left(-S_{\mathrm{YM}}[A] + \int A^I_\mu(x) J^\mu_I(x) \mathrm{d}^d x\right) [\mathcal{D}A] \\ &\equiv \int\limits_{\mathfrak{su}(n)} \exp\Big(-S_{\mathrm{YM}}[A] - S_{\mathrm{Source}}[A; J]\Big) [\mathcal{D}A], \end{aligned} \tag{2.57}$$

where $S_{\mathrm{Source}}[A; J] := -\int A^I_\mu(x) J^\mu_I(x) \mathrm{d}^d x$ denotes a source action. In eq. (2.57), we have intentionally not used an equal sign, otherwise we would commit a conceptual blunder. The reason for that is attributable to the existence of physically equivalent field configurations due to gauge invariance. More precisely, the YM action (2.55) is unchanged under the following local infinitesimal gauge transformation [27]:

$$A^I_\mu(x) \mapsto A^I_\mu(x) - \partial_\mu \alpha^I(x) + g f^I{}_{JK} \alpha^J(x) A^K_\mu(x), \tag{2.58}$$

in which α is an infinitesimal transformation parameter that is considered as a smooth function. Any continuous variation of α yields another valid transformation parameter and any finite gauge transformation can be obtained by performing a sequence of infinitesimal ones. We call the latter $\mathcal{U}_\alpha$. Gathering all possible gauge transformations of the form (2.58) within a set and equipping them with the usual composition on functions gives us the so-called *gauge group*. Moreover, we can declare an equivalence relation on $\mathfrak{su}(n)$ where two gauge fields A and $\tilde{A}$ are said to be equivalent by definition iff there exists a gauge transformation $\mathcal{U}_\alpha$ such that $\tilde{A} = \mathcal{U}_\alpha A \mathcal{U}_\alpha^{-1} - \frac{1}{g}(\nabla \mathcal{U}_\alpha)\mathcal{U}_\alpha^{-1}$. In this situation we write $A \sim_\alpha \tilde{A}$. Then, the space $\mathfrak{su}(n)/\sim_\alpha$ consists of equivalence classes that collect physically indistinguishable field configurations which are known as the *gauge orbits*.

Now, the expression $\int_{\mathfrak{su}(n)} [\mathcal{D}A]$ represents a functional integration over *all* field configurations, while we actually wish to integrate only over physically inequivalent ones. That means, we rather need an integral that accounts for the gauge orbit structure, or formally:

$$\int\limits_{\mathfrak{su}(n)} [\mathcal{D}A] \quad \longrightarrow \int\limits_{\mathfrak{su}(n)/\sim_\alpha} [\mathcal{D}A^{\mathrm{rep}}], \tag{2.59}$$

where A^{rep} denotes a representative for each gauge orbit.

In order to avoid *gauge redundancies* due to the integration over all field configurations in presence of a gauge symmetry, we can invoke an algorithm that goes back to Ludvig Dmitrievich Faddeev and Victor Nikolaevich Popov which essentially explains how the idea (2.59) can technically be realised [43]. We are going to illustrate it in the upcoming paragraph.

B. The Faddeev-Popov Procedure

Given the formal expression (2.59) we need to install a mechanism that controls the integration over gauge orbit space. Let us consider an arbitrary gauge orbit, i.e. an equivalence class $[A^{\mathrm{rep}}] \in \mathfrak{su}(n)/\sim_\alpha$ in which all elements are connected through transformations build from (2.58). As a consequence of gauge freedom, we can proceed analogously to standard methods from electrodynamics and impose an extra condition to fix a specific gauge:

$$\mathsf{G}^I[A] = 0, \tag{2.60}$$

for all index values I. If we now choose any gauge field $A \in [A^{\mathrm{rep}}]$ that does not satisfy eq. (2.60), we pick a suitable $\mathcal{U}_\alpha$ from the gauge group and transform A such that eq. (2.60) holds. The ideal scenario would be that the gauge fixing condition - which appears as a system of differential equations for the gauge transformation parameter α - is fulfilled by only a single element from each gauge orbit, however, this is generally not true, especially for non-Abelian gauge theories and refers essentially to the so-called *Gribov ambiguity* [27]. Fortunately, there are conditional solutions to that problem and in what follows we assume that the Gribov ambiguity is circumvented properly such that we can continue with the formulation of a well-defined QFT.

To incorporate the gauge fixing condition (2.60) into the path integral expression eq. (2.57), we can use a functional generalisation of the one dimensional unit relation: $1 = \int_U \delta(f)\mathrm{d}f = \int_{\mathbb{R}} |f'(x)|\delta(f(x))\,\mathrm{d}x$ where $f : \mathbb{R} \to U \subseteq \mathbb{R}$ is supposed to exhibit only a single zero. At the level of functional integrals this reads:

$$1 = \int \delta\left(\mathsf{G}^I\right)\left[\mathcal{D}\mathsf{G}^I\right] = \int_{\mathrm{SU}(n)} \delta\left(\mathsf{G}^I[A_\alpha]\right) \mathrm{Det}\left(\frac{\delta \mathsf{G}^I[A_\alpha]}{\delta\alpha^J}\right) \mathrm{d}\mu(\alpha), \tag{2.61}$$

in which μ is called the *Haar measure* that is used for the group integration over $\mathrm{SU}(n)$, or more generally, for the integration over spaces that contain group structures, and where A_α is supposed to be of the form (2.58).

The functional determinant under the integral in the last expression of (2.61) is sometimes called the *Faddeev-Popov determinant* and according to standard textbooks it has been shown that it is indeed gauge invariant, see for instance ref. [31]. Consequently, we can simplify the integral expression by making the specific choice $\alpha = 0$, i.e. $\alpha^I = 0$ for all index values I, such that the integrand becomes independent of the integration variable. Since the Haar measure is normalised, i.e $\mu(\mathrm{SU}(n)) \equiv \int_{\mathrm{SU}(n)} \mathrm{d}\mu = 1$, we obtain:

$$1 = \delta\left(\mathsf{G}^I[A]\right) \mathrm{Det}\left(\frac{\delta \mathsf{G}^I[A_\alpha]}{\delta\alpha^J}\right)\bigg|_{\alpha=0}. \tag{2.62}$$

This result can directly be inserted into (2.57) which finally replaces the $\sim$ symbol by an equal sign:

$$\mathcal{Z}_{\mathrm{YM}}[J] = \int \exp\Big(-S_{\mathrm{YM}}[A] - S_{\mathrm{Source}}[A;J]\Big)\delta\left(\mathsf{G}^I[A]\right) \mathrm{Det}\left(\frac{\delta \mathsf{G}^I[A_\alpha]}{\delta\alpha^J}\right)\bigg|_{\alpha=0} [\mathcal{D}A]. \tag{2.63}$$

It is conventional to properly enqueue also the delta functional and the Faddeev-Popov determinant in the sequence of actions under the exponential. In order to realise this for the delta functional, we can adhere to the approximation of the delta function by means of the normal distribution:

$$\delta\left(\mathsf{G}^I[A]\right) \to \lim_{\kappa\to 0} \exp\left(-\frac{1}{2\kappa}\int \mathsf{G}^I[A(x)]\,\mathsf{G}_I[A(x)]\,\mathrm{d}^d x\right) =: \mathrm{e}^{-S_{\mathrm{gf}}[A]}. \tag{2.64}$$

Here, we have introduced the **gauge-fixing action** S_{gf} that is determined by the gauge-fixing condition eq. (2.60).

For doing the exponentiation of the Faddeev-Popov determinant we recall a general representation of functional determinants in terms of path integrals over Grassmann-valued fields. In order to review it, let θ be a Grassmann-valued vector and $\mathbf{M}$ a Hermitian matrix with eigenvalues $\{\mu_i\}$. Then, the following formula proves to be correct [31], or see Proposition D.6 in app. D:

$$\int \mathrm{e}^{-\theta^\dagger \mathbf{M}\theta}\mathrm{d}\bar{\theta}\mathrm{d}\theta = \prod_i \mu_i = \det(\mathbf{M})\,. \tag{2.65}$$

The path integral analogue of relation (2.65) with $\det(\mathbf{M})$ being identified with the Faddeev-Popov determinant yields:

$$\mathrm{Det}\left(\frac{\delta \mathsf{G}^I[A_\alpha]}{\delta\alpha^J}\right)\Bigg|_{\alpha=0} = \int \exp\left[-\int \bar{c}_I(x)\left(\frac{\delta\mathsf{G}^I[A_\alpha(x)]}{\delta\alpha^J}\right)\Bigg|_{\alpha=0} c^J(x)\mathrm{d}^d x\right][\mathcal{D}\bar{c}\mathcal{D}c]\,. \tag{2.66}$$

The anticommuting real-valued fields c and $\bar{c}$ are known as *Faddeev-Popov ghosts*. Finally, we define the **ghost action** as:

$$S_{\mathrm{gh}}[c,\bar{c};A] := \int \bar{c}_I(x)\left(\frac{\delta\mathsf{G}^I[A_\alpha(x)]}{\delta\alpha^J}\right)\Bigg|_{\alpha=0} c^J(x)\mathrm{d}^d x. \tag{2.67}$$

Now we turn back to the partition functional $\mathcal{Z}_{\mathrm{YM}}$ in eq. (2.63) and use eq. (2.64) together with eqs. (2.66) & (2.67), yielding:

$$\mathcal{Z}_{\mathrm{YM}}\left[J,\Theta,\bar{\Theta}\right] = \int \mathrm{e}^{-S_{\mathrm{YM}}[A]-S_{\mathrm{Source}}\left[A;J,\Theta,\bar{\Theta}\right]-S_{\mathrm{gf}}[A]-S_{\mathrm{gh}}[c,\bar{c};A]}\,[\mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c]\,, \tag{2.68}$$

whereby we have expanded the source action by two additional parts which account for the new ghost fields. That means we now have:

$$S_{\mathrm{Source}}\left[A;J,\Theta,\bar{\Theta}\right] = -\int\left(A^I_\mu(x)J^\mu_I(x) + \bar{\Theta}_I(x)c^I(x) + \Theta^I(x)\bar{c}_I(x)\right)\mathrm{d}^d x. \tag{2.69}$$

Though we have basically finished our short review on the FRG in presence of gauge symmetries, we have not yet discussed any details on the effective action or the flow equation. However, the analysis is quite analogous to what we have already explained in sec. 2.1 and details can be found in ref. [27]. Moreover, eq. (2.15) still applies for the present case of YM theory, where we have to take care not to miss the ghost fields when computing the supertrace.

An interesting situation appears in the case of *Abelian gauge theories*, i.e. when all structure constants vanish identically. As a consequence, the ghost fields decouple from the gauge degrees of freedom and the Faddeev-Popov determinant reduces to a constant factor that can be absorbed into the path integral measure $[\mathcal{D}A]$. In this way, only the gauge-fixing action remains as the additional piece of information due to gauge invariance within the partition functional, eq. (2.68). The ghost fields are thus also irrelevant for the Wetterich equation and the renormalisation flow is determined solely by the behaviour of the effective action under variations of the gauge fields. We will come back to this discussion in ch. 3.

"Nature uses only the longest threads to weave her patterns, so that each small piece of her fabric reveals the organization of the entire tapestry."

~ Richard Phillips Feynman

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Web site: `https://www.brainyquote.com/quotes/richard_p_feynman_160463`.

3

Global Fixed Point Structures in Self-Interacting U(1) Gauge Theories

Now that we have developed a solid foundation in the course of ch. 2, we are well prepared for the core analysis of this thesis. In principle, we proceed technically quite similarly to the Ising model example presented in sec. 2.3. The structural differences primarily concern the degrees of freedom under consideration, the symmetry constraints and the restrictions imposed by the local potential approximation. More precisely, we change from scalar to bosonic spin-1 degrees of freedom, represented by vector quantities, replace the group $\mathrm{O}(1, \mathbb{R})$ by the one-dimensional unitary group U(1) which acts as the gauge group on our model, and we will refrain from the limitation $Z_k = 1$ and instead consider an RG dynamical field strength renormalisation. From a bird's eye perspective, our model can be interpreted as a renormalisation flow portrayal of a purely self-interacting massless spin-1 bosonic field without couplings to any sort of matter degrees of freedom. Incidentally, first results are due to the work of Heisenberg & Euler, who already derived the effective Lagrangian of the pure electromagnetic field up to 1-loop order in 1936, see ref. [47].

The outline of this chapter is as follows: in sec. 3.1, we discuss the basis of the theory and subsequently derive the technical setup on which we will apply the techniques that were introduced in ch. 2. In this way we try to extract relevant information about the fixed-point sector. Their demonstration and discussion takes place in sec. 3.2.

In order to improve readability, we refrain from a thorough presentation of the majority of calculations and instead merely outline their execution.

3.1

Prelude: Modelling the Theory

A. Quantum Electrodynamics Revisited and Basic Constructions

In order to classify the subject of interest, let us start with the prominent instance of QED, i.e. the quantum description of the electromagnetic interaction mediated by spin-1 bosons and acting between electrically charged spin-1/2 particles. Physically, they are commonly considered as photons and electrons respectively, whereas mathematically, the latter type of particles is formulated with elements obtained from the d-dimensional canonical representation of the spin group $\mathrm{Spin}(d-1, 1)$, which are called Dirac spinors and are usually denoted by ψ [44]. Conversely, the former particle type, that is the spin-1 boson, is described by a vector-like degree of freedom, A, that has components A_μ.

Within the Lorentz covariant formulation of classical electrodynamics, A represents the four potential, containing the scalar potential $A_0 \equiv -\phi$, and the components of the vector potential $A_i \equiv \mathbf{A}_i$.

In $d = 4$ spacetime dimensions and against the background of flat Lorentzian geometry, dynamics between spin-1/2 fermions and spin-1 bosons is encoded in the Lagrangian of QED [45]:

$$\mathscr{L}_{\mathrm{QED}}\left(A_{\mathrm{M}}, \psi\right) = -\frac{1}{4}\left(F_{\mathrm{M}}\right)_{\mu\nu}\left(F_{\mathrm{M}}\right)^{\mu\nu} + \overline{\psi}\left(\imath \not{D} - m\right)\psi. \tag{3.1}$$

Here, $\left(F_{\mathrm{M}}\right)_{\mu\nu}$ are the (covariant) components of the second degree *field strength tensor* $\mathbf{F}_{\mathrm{M}}$. Because we will move to Euclidean space later, we have decided to indicate the formulation of the field strength tensor with respect to Minkowski space by adding the subscript[1] "M", which we also apply to related quantities like, for instance, the four potential. Later on, when we pass over from Minkowski to Euclidean signature, we will ventilate our notation and drop the index "M".

Either way, the field strength tensor is related to the four potential according to:

$$\left(F_{\mathrm{M}}\right)_{\mu\nu} \triangleq \partial_\mu\left(A_{\mathrm{M}}\right)_\nu - \partial_\nu\left(A_{\mathrm{M}}\right)_\mu. \tag{3.2}$$

Furthermore, $\overline{\psi} := \psi^\dagger\gamma^0$ is referred to as the Dirac adjoint of ψ, where γ^0 denotes a member of the Dirac matrices $\{\gamma^\mu\}$. The latter are moreover used to define the Feynman dagger, which appears explicitly in eq. (3.1) as $\not{D} := \gamma^\mu \mathrm{D}_\mu$. Here, we also find the interaction link between spin-1/2 fermions and spin-1 bosons, which is incorporated within the covariant derivative, $\mathrm{D}_\mu := \partial_\mu + \imath e\left(A_{\mathrm{M}}\right)_\mu$, with e being the electric charge. Lastly, m signifies the fermion mass parameter.

An important observation on the QED Lagrangian (3.1) refers to its symmetry under *local phase transfomations*:

$$A_{\mathrm{M}}(x) \mapsto A_{\mathrm{M}}(x) + \partial\Lambda(x) \qquad \& \qquad \psi(x) \mapsto \mathrm{e}^{\imath\alpha(x)}\psi(x), \tag{3.3}$$

in which Λ and α are some sufficiently smooth functions on spacetime. Because of its local character, we recognise (3.3) as a pair of *gauge transformations* on both the bosonic and fermionic sector of $\mathscr{L}_{\mathrm{QED}}$ respectively. In this situation, the gauge group is identified with the first unitary group U(1).

With this brief recapitulation of QED, we are now in position to introduce the model of interest, starting from the Lagrangian (3.1). For the purposes of this work, we completely disregard all fermion contributions, including its interactions with the vector degrees of freedom. In compensation to that, we instead enlarge the bosonic sector by adding a generic collection of further expressions which obey - besides Lorentz invariance - the local U(1) symmetry condition imparted through (3.3) while ignoring the spinor transformation part. These modifications essentially result in a new Lagrangian of the form $\mathscr{L}_{\mathrm{M}}\left(A_{\mathrm{M}}\right) = -\frac{1}{4}\left(F_{\mathrm{M}}\right)_{\mu\nu}\left(F_{\mathrm{M}}\right)^{\mu\nu} + \mathcal{I}\left(A_{\mathrm{M}}\right)$, where the additional term $\mathcal{I}$ - which incidentally displays self-interactions among the spin-1 field - is considered to be the most general appropriate composition that can be build from the fundamental degree of freedom A_{M}.

For definiteness, we can consult the fact that an arbitrary Lorentz and local U(1) invariant quantity can always be decomposed into combinations of only two primal invariants, which read[2] [48]:

$$\mathscr{F}_{\mathrm{M}} := \frac{1}{4}\left(F_{\mathrm{M}}\right)_{\mu\nu}\left(F_{\mathrm{M}}\right)^{\mu\nu} \qquad \& \qquad \mathscr{G}_{\mathrm{M}} := \frac{1}{4}\left(F_{\mathrm{M}}\right)_{\mu\nu}\left(\star F_{\mathrm{M}}\right)^{\mu\nu}. \tag{3.4}$$

Here, $\star\mathbf{F}_{\mathrm{M}}$ is the *Hodge dual* to $\mathbf{F}_{\mathrm{M}}$ which can be constructed via the Levi-Civita tensor:

$$\left(\star F_{\mathrm{M}}\right)^{\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}\left(F_{\mathrm{M}}\right)_{\lambda\sigma}. \tag{3.5}$$

[1]Since later, the spinor part of the QED Lagrangian will not be of interest for us, we do not establish this notation for ψ or $\not{D}$, though their representation does also depend on the background manifold signature.

[2]A reasoning of this statement can be found in app. D, cf. Theorem D.8.

Let us turn back to the Lagrangian $\mathscr{L}_{\mathrm{M}}$. According to what we have just said, we can justly interpret the extra contribution $\mathcal{I}$ as a function that specifically depends only on $\mathscr{F}_{\mathrm{M}}$ and $\mathscr{G}_{\mathrm{M}}$ instead of A_{M}.

Since we did not arranged any further restrictions on $\mathcal{I}$, the *unconstrained effective action*[3] of the QFT that is characterised by $\mathscr{L}_{\mathrm{M}}$, denoted as $\tilde{\Gamma}_{\mathrm{M}}$, yields:

$$\tilde{\Gamma}_{\mathrm{M}}\left[A_{\mathrm{M}}\right] = \int \mathscr{L}_{\mathrm{M}}\left(\mathscr{F}_{\mathrm{M}}, \mathscr{G}_{\mathrm{M}}\right) \mathrm{d}^4x = \int \Big[-\mathscr{F}_{\mathrm{M}} + \mathcal{I}\left(\mathscr{F}_{\mathrm{M}}, \mathscr{G}_{\mathrm{M}}\right)\Big] \mathrm{d}^4x. \tag{3.6}$$

From here on, we could begin to formulate the renormalisation flow for the effective action Γ_{M} (after gauge fixing $\tilde{\Gamma}_{\mathrm{M}} \to \Gamma_{\mathrm{M}}$) and explore in particular its fixed point sector. However, in view of the technical complexity with which we might be confronted, it is reasonable to first carry out the transition from Minkowski to Euclidean signature and to re-introduce all previous quantities that have been provided an additional subscript "M" by their Euclidean counterpart.

B. From Lorentzian to Euclidean Geometry

Let us start again with the spin-1 field Lagrangian, but this time without taking any self-interactions into account such that $\mathscr{L}_{\mathrm{M}}$ becomes purely kinetic:

$$\mathscr{L}_{\mathrm{M}} = -\mathscr{F}_{\mathrm{M}}. \tag{3.7}$$

Starting from this equation, the central demand that leads to Euclidean conditions relies on the request for an analogue expression which does not carries an overal negative sign, i.e. $\mathscr{L}_{\mathrm{M}} \to \mathscr{L} \equiv \mathscr{F}$. Here, the Euclidean equivalents $\mathscr{L}$ and $\mathscr{F}$ belong to $\mathscr{L}_{\mathrm{M}}$ and $\mathscr{F}_{\mathrm{M}}$ respectively.

How such a transition is done in detail can properly be understood by considering the explicit expression for $\mathscr{F}_{\mathrm{M}}$ in terms of the electric and magnetic field $\mathbf{E}_{\mathrm{M}}$ and $\mathbf{B}_{\mathrm{M}}$ respectively. Using the relations $(F_{\mathrm{M}})_{0i} = (\mathbf{E}_{\mathrm{M}})_i$ and $(F_{\mathrm{M}})_{ij} = \varepsilon_{ij\ell}\left(\mathbf{B}_{\mathrm{M}}\right)_\ell$, together with the antisymmetry of the field strength tensor, $(F_{\mathrm{M}})_{\mu\nu} = -\left(F_{\mathrm{M}}\right)_{\nu\mu}$, a simple calculation yields: $\mathscr{F}_{\mathrm{M}} = \frac{1}{2}\left(\mathbf{B}_{\mathrm{M}}^2 - \mathbf{E}_{\mathrm{M}}^2\right)$.

Let us now consider an Euclidean construction of the field strength tensor, denoted as $\mathbf{F}$, which again contains the components of the (Euclidean) electric and magnetic field, $\mathbf{E}$ and $\mathbf{B}$ respectively, and that is structurally equivalent to $\mathbf{F}_{\mathrm{M}}$, i.e. we still have $F_{0i} = \mathbf{E}_i$ and $F_{ij} = \varepsilon_{ij\ell}\mathbf{B}_\ell$, but where now the Euclidean metric tensor $\boldsymbol{\delta}$ is applied for raising and lowering indices instead of the Minkowski metric tensor $\boldsymbol{\eta}$. That means that we now get: $\mathscr{F} = \frac{1}{4}\delta^{\mu\lambda}\delta^{\nu\sigma}F_{\mu\nu}F_{\lambda\sigma}$, which in turn gives: $\mathscr{F} = \frac{1}{2}\left(\mathbf{B}^2 + \mathbf{E}^2\right)$.

One might now ask the question: what precisely is the connection between the Euclidean and Minkowskian quantities that were introduced in paragraph **A**? The answer can be formulated as follows: analogously to Wick rotations in coordinate space, we can consider Wick rotations in field space that only affect the "timelike" component of the gauge field A (or A_{M}, depending on where we start), that is A_0 (or $(A_{\mathrm{M}})_0$). As a consequence, the naught components of the field strength tensor, i.e. $F_{0i} = -F_{i0}$ and therefore the components of the electric feld, inherit the modifications from the Wick rotation procedure. Explicitly, this means: $\mathbf{E}_{\mathrm{M}} \mapsto \mathbf{E} = \imath\mathbf{E}_{\mathrm{M}}$ and trivially $\mathbf{B}_{\mathrm{M}} \mapsto \mathbf{B} = \mathbf{B}_{\mathrm{M}}$. Under regard of eq. (3.7) we find exactly what was initially requested:

$$\mathscr{L} = \mathscr{F} = \frac{1}{2}\left(\mathbf{B}^2 + \mathbf{E}^2\right) = \frac{1}{2}\left(\mathbf{B}_{\mathrm{M}}^2 - \mathbf{E}_{\mathrm{M}}^2\right) = \mathscr{F}_{\mathrm{M}} = -\mathscr{L}_{\mathrm{M}}. \tag{3.8}$$

Let us now summarise the new ensemble of Euclidean quantities that we have introduced during this paragraph. First of all, using the Euclidean four potential A, eq. (3.2) becomes:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.9}$$

[3] We use the term "unconstrained" to point out a special pecularity of gauge theories, which forces us to declare a gauge fixing condition to avoid redundancies due to the integration over physically gauge equivalent fields. Detailed information on this aspect were already presented in sec. 2.4.

Then, the Hodge dual to **F** reads:

$$\star F_{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma}. \tag{3.10}$$

Since the invariants given in eq. (3.4) change accordingly, we thus find:

$$\boxed{\mathscr{F} \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \& \qquad \mathscr{G} \equiv \frac{1}{4} \left(F_{\mu\nu}\right)\left(\star F^{\mu\nu}\right).} \tag{3.11}$$

These preparations are sufficient to enlarge $\mathscr{L}$ in the same manner as we did for $\mathscr{L}_\mathrm{M}$, i.e. by adding an additional contribution $\mathcal{I}$ that is considered to be a function of $\mathscr{F}$ and $\mathscr{G}$. Hence, the Minkowski version of the unconstrained effective action from eq. (3.6) becomes:

$$\tilde{\Gamma}[A] = \int \mathscr{L}(\mathscr{F},\mathscr{G})\,\mathrm{d}^d x = \int \Big[\mathscr{F} + \mathcal{I}(\mathscr{F},\mathscr{G})\Big]\mathrm{d}^d x. \tag{3.12}$$

In addition, we have decided to lift the number of spacetime dimensions from $d = 4$ to a basically unspecified number d, though the appearance of the four-dimensional Levi-Civita tensor within the definition of $\mathscr{G}$ nevertheless forces us to deviate at most infinitesimally from exactly four dimensions. That means we need to set $d = 4 - \epsilon$ for a small parameter $|\epsilon| \ll 1$.

It should also be noted that a detailed compilation of information concerning the field strength tensor, the dual field strength tensor and the invariants can be found in app. A. There, one can also look for the precise mathematical relation between **F** and $\star$**F** according to the Hodge operator. In addition to the four-dimensional Minkowskian/Euclidean formulation of **F** and $\star$**F**, corresponding representations in dimensions $d \in \mathbb{N}_0$ with $d < 4$ are presented in both Minkowski *and* Euclidean space.

Since it will be of interest in the upcoming paragraph below, let us conclude with a short look on also the second primal invariant, $\mathscr{G}$, and figure out its connection to $\mathscr{G}_\mathrm{M}$. Yet another simple calculation gives as a first result: $\mathscr{G}_\mathrm{M} = -\mathbf{E}_\mathrm{M} \cdot \mathbf{B}_\mathrm{M}$. After Wick rotation, this becomes: $-\mathbf{E}_\mathrm{M} \cdot \mathbf{B}_\mathrm{M} = \imath \mathbf{E} \cdot \mathbf{B} \equiv \imath \mathscr{G}$, i.e. that the Euclidean invariant $\mathscr{G} = -\imath \mathscr{G}_\mathrm{M}$ is purely imaginary. However, we can obtain a real valued quantity by taking the square: $\mathscr{G}^2 = -\mathscr{G}_\mathrm{M}^2 \in \mathbb{R}$.

C. The Effective Average Action and the Full Propagator

Now that we have built our foundations, we could apply the FRG procedure that has been explained in ch. 2. Specifically this means to furnish the effective action (3.12) with a scale dependence k and obtain the *unconstrained effective average action* as an interpolation between $\tilde{\Gamma}$ and an appropriate microscopic bare action. The explicit form of the latter is rather arbitrary as long as it belongs to the underlying theory space. After that, we can proceed with detailed studies of the renormalisation flow and the fixed point sector. However, before we turn to these issues, it is fairly worthwile to become more precise on the definition of theory space for the present case.

In order to uniquely specify the theory space $\mathcal{T}$, we need to fix a collection of degrees of freedom as well as a selection of imposed symmetry conditions. Recalling our discussion from paragraphs **A** and **B** of this section, the relevant degrees of freedom are represented by the invariants $\mathscr{F}$ and $\mathscr{G}$, though, in principle, the gauge field A appears as the fundamental variable. However, since we demand Lorentz and local U(1) invariance, the restriction on pure $\mathscr{F}$ and $\mathscr{G}$ dependencies is nevertheless justified, at least as long as invariants constructed from field strength derivatives can be neglected (see below). Thus, theory space is considered as the following set of action functionals $\mathcal{A}$:

$$\mathcal{T} := \Big\{\mathcal{A} \mid \mathcal{A} \mathrel{\hat{=}} \mathcal{A}[\mathscr{F},\mathscr{G}]\Big\}. \tag{3.13}$$

In terms of a derivative expansion, a generic theory space representative $\mathcal{A} \in \mathcal{T}$ appears like:

$$\mathcal{A}[\mathscr{F},\mathscr{G}] = \int \left\{ \sum_{n,m\in\mathbb{Z}} a_{nm}\mathscr{F}^n\mathscr{G}^m \right\} \equiv \int \mathscr{W}\Big(\mathscr{F},\mathscr{G}\Big)\mathrm{d}^d x, \tag{3.14}$$

which corresponds to the zeroth order expansion term. In a more general approach, also higher order derivative contributions would need to be considered in the equations, including for instance $\sim (\partial_\mu \mathscr{F})^2$ or $\sim (\partial_\mu F^{\mu\nu})^2$. In this situation, it is no longer possible to fully express a generic action functional $\mathcal{A} \in \mathcal{T}$ solely in terms of $\mathscr{F}$ and $\mathscr{G}$ and the theory space (3.13) needs to be properly adjusted. The integrand in (3.14) is then enlarged by a "kinetic function" $\mathscr{K}$ which depends on new field strength derivative invariants. In this thesis, however, we will exclusively concentrate on the homogeneous case which will be characterised further below.

It is clear, that also the effective action (3.12) shows up as an element of $\mathcal{T}$, and so does the unconstrained effective average action $\tilde{\Gamma}_k$ for all $k \in \mathbb{R}_0^+$. The unconstrained EAA possesses the same generic form like it is presented in eq. (3.14), but since it describes an RG curve in theory space, the coefficients become scale dependent, i.e. $a_{nm} \to a_{nm}(k)$. They carry all the information of the renormalisation flow and are designated as *generalised couplings* as in the language of ch. 2.

The scale attachment $\mathscr{W} \to \mathscr{W}_k$ allows us to write $\tilde{\Gamma}_k$ as follows:

$$\tilde{\Gamma}_k[A] = \int \mathscr{W}_k\Big(\mathscr{F},\mathscr{G}\Big)\mathrm{d}^d x. \tag{3.15}$$

At the end of paragraph **B** we have already seen that $\mathscr{G} \in \imath\mathbb{R}$. However, if one wants the EAA to be real-valued, this implies that some of the generalised couplings need to be purely imaginary. To see this, take for instance the term $a_{01}(k)\mathscr{G}$ in eq. (3.14). Since $\mathscr{G} \in \imath\mathbb{R}$, but $\Gamma_k \in \mathbb{R}$ for each k by request, we conclude that $a_{01} \in \imath\mathbb{R}$. Conversely, this is not true for the quadratic contribution $a_{02}\mathscr{G}^2$, because $\mathscr{G}^2$ is already a real number and thus $a_{02} \in \mathbb{R}$. Either way, to avoid collections of mixed-valued generalised couplings, one can restrict on a uniform structure in the sense that all couplings which actively drive the renormalisation flow take their values from the same algebraic field of numbers. Since $\mathscr{F}$ is a manifestly real-valued quantity, one can exclude all parts of $\mathscr{W}_k$ that contribute in odd powers of $\mathscr{G}$ and therefore appear purely imaginary.

However, there is another good reason to exclude odd powers of $\mathscr{G}$ in eq. (3.14), which relies on the fact that the Euclidean EAA must be *parity-preserving*. The invariant $\mathscr{G}$ by itself is a pseudoscalar; consider a parity transformation that can be described by a matrix $\mathcal{P} \mathrel{\hat{=}} \mathrm{diag}\,(1,-1,-1,-1)$. Now apply $\mathcal{P}$ on the field strength tensor $\mathbf{F}$, where a simple computation of the product $\mathcal{P}^T\mathbf{F}\mathcal{P}$ reveals that the electric field transforms like $\mathbf{E} \mapsto -\mathbf{E}$ and the magnetic field is unchanged, $\mathbf{B} \mapsto \mathbf{B}$. We already saw how to express $\mathscr{G}$ in terms of $\mathbf{E}$ and $\mathbf{B}$, which implies: $\mathscr{G} = \mathbf{E}\cdot\mathbf{B} \mapsto -\mathbf{E}\cdot\mathbf{B} = -\mathscr{G}$. Thus, $\mathscr{G}$ behaves odd under parity transformations and hence appears as a pseudoscalar. However, even powers of $\mathscr{G}$ precisely laminate this property and behave like parity-preserving quantities as it is required for the EAA.

According to our recent discussion, eq. (3.15) reduces to:

$$\tilde{\Gamma}_k[A] \searrow \tilde{\Gamma}_k^{\mathrm{red}}[A] = \int \mathscr{W}_k^{\mathrm{red}}\left(\mathscr{F},\mathscr{G}^2\right)\mathrm{d}^d x. \tag{3.16}$$

In order to keep the notation simple, we recycle identifiers and set $\mathscr{W}_k^{\mathrm{red}} \to \mathscr{W}_k$ as well as $\tilde{\Gamma}_k^{\mathrm{red}} \to \tilde{\Gamma}_k$. In what follows, our focus is on setting up the Wetterich equation (2.15) for the EAA (3.16).

Let us begin by deriving an expression for the regularised full propagator, $\left(\Gamma_k^{(2)}[A] + \mathcal{R}_k\right)^{-1}$, and first concentrate on the second functional derivative of $\tilde{\Gamma}_k$, putting the regulator aside for a moment. For this we have to compute:

$$\left(\tilde{\Gamma}_k^{(2)}\right)^{\mu\nu}[A](x,y) = \frac{\delta^2\tilde{\Gamma}_k[A]}{\delta A_\mu(x)\delta A_\nu(y)} = \int \frac{\delta^2\mathscr{W}_k(\mathscr{F},\mathscr{G}^2)}{\delta A_\mu(x)\delta A_\nu(y)}\mathrm{d}^d z. \tag{3.17}$$

Expanding the second functional derivative of $\mathscr{W}_k$ by using the chain rule leads to:

$$\begin{aligned}\frac{\delta^2\mathscr{W}_k\left(\mathscr{F},\mathscr{G}^2\right)}{\delta A_\mu(x)\delta A_\nu(y)} &= \frac{\partial^2\mathscr{W}_k}{\partial\mathscr{F}^2}\left(\frac{\delta\mathscr{F}(z)}{\delta A_\mu(x)}\right)\left(\frac{\delta\mathscr{F}(z)}{\delta A_\nu(y)}\right)+\frac{\partial^2\mathscr{W}_k}{\partial\left(\mathscr{G}^2\right)^2}\left(\frac{\delta\mathscr{G}^2(z)}{\delta A_\mu(x)}\right)\left(\frac{\delta\mathscr{G}^2(z)}{\delta A_\nu(y)}\right)\\ &+\frac{\partial^2\mathscr{W}_k}{\partial\mathscr{F}\partial\mathscr{G}^2}\left[\left(\frac{\delta\mathscr{F}(z)}{\delta A_\mu(x)}\right)\left(\frac{\delta\mathscr{G}^2(z)}{\delta A_\nu(y)}\right)+\left(\frac{\delta\mathscr{G}^2(z)}{\delta A_\mu(x)}\right)\left(\frac{\delta\mathscr{F}(z)}{\delta A_\nu(y)}\right)\right]\\ &+\frac{\partial\mathscr{W}_k}{\partial\mathscr{F}}\left(\frac{\delta^2\mathscr{F}(z)}{\delta A_\mu(x)\delta A_\nu(y)}\right)+\frac{\partial\mathscr{W}_k}{\partial\mathscr{G}^2}\left(\frac{\delta^2\mathscr{G}^2(z)}{\delta A_\mu(x)\delta A_\nu(y)}\right).\end{aligned} \tag{3.18}$$

To proceed, we need to take a total of four functional derivatives of the invariants (3.11). This works rather straightforwardly and can be done using standard rules of functional differentiation. Let us therefore directly jump to the results:

$$\begin{aligned}\frac{\delta\mathscr{F}(z)}{\delta A_\mu(x)} &= F^{\lambda\mu}\partial_\lambda\delta^{(d)}(z,x)\,, \qquad \frac{\delta\mathscr{G}^2(z)}{\delta A_\mu(x)} = 2\mathscr{G}\left(\star F\right)^{\lambda\mu}\partial_\lambda\delta^{(d)}(z,x)\,,\\ \frac{\delta^2\mathscr{F}(z)}{\delta A_\mu(x)\delta A_\nu(y)} &= \delta^{\mu\nu}\left[\partial_\lambda\delta^{(d)}(z,y)\right]\left[\partial^\lambda\delta^{(d)}(z,x)\right]-\left[\partial^\mu\delta^{(d)}(z,y)\right]\left[\partial^\nu\delta^{(d)}(z,x)\right],\\ \frac{\delta^2\mathscr{G}^2(z)}{\delta A_\mu(x)\delta A_\nu(y)} &= \frac{1}{4}F_{\kappa\rho}F_{\pi\tau}\left(\varepsilon^{\mu\nu\lambda\sigma}\varepsilon^{\kappa\rho\pi\tau}+2\varepsilon^{\mu\sigma\pi\tau}\varepsilon^{\nu\lambda\kappa\rho}\right)\left[\partial_\lambda\delta^{(d)}(z,y)\right]\left[\partial_\sigma\delta^{(d)}(z,x)\right].\end{aligned} \tag{3.19}$$

After insertion of (3.19) into (3.18), the integration in eq. (3.17) can be carried out piecewise; for each term we obtain an expression of the form $\sim\int\left[\partial_\lambda\delta^{(d)}(z,x)\right]\left[\partial_\sigma\delta^{(d)}(z,y)\right]\mathrm{d}^dz$, which, by means of integration by parts, can be reformulated as $\sim\int\left[\partial_\lambda\partial_\sigma\delta^{(d)}(z,x)\right]\delta^{(d)}(z,y)\mathrm{d}^dz = \partial_\lambda\partial_\sigma\delta^{(d)}(y,x)$, after having neglected additional boundary terms. However, since so far the field strength tensor is considered as a local function on spacetime, the integration by parts additionally generates derivatives of the field strength itself and therefore of the invariants $\mathscr{F}$ and $\mathscr{G}$. To avoid unnecessary complications, we follow a similar argumentation as in the context of the Wilson-Fisher example in sec. 2.3 and agree on the further assumption that the field strength derivative sector do not exert a strong influence on the flow of $\mathscr{W}_k$, as it was already suggested by the definition of $\mathcal{T}$ in (3.13) and its elements in eq. (3.14). In practice, this means that the renormalisation flow is projected onto the potential[4] $\mathscr{W}_k$. Alternatively and more accurately expressed, we *suppose a sufficiently homogeneous field strength*, $\mathbf{F}(z)\simeq$ const. for all $z\in\mathbb{R}^d$, and refer to this argument as the *homogeneity condition.*

Before we continue, it is convenient to compactify our notation, hence we define:

$$\frac{\partial\mathscr{W}_k}{\partial\mathscr{F}} =: \mathscr{W}_k' \qquad \& \qquad \frac{\partial\mathscr{W}_k}{\partial\mathscr{G}^2} =: \dot{\mathscr{W}}_k. \tag{3.20}$$

Under consideration of the homogeneity condition, the next intermediate result reads:

$$\begin{aligned}\left(\tilde{\Gamma}_k^{(2)}\right)^{\mu\nu}[A](x,y) &= \mathscr{W}_k'\left[\delta^{\mu\nu}\left(-\partial^2\right)+\partial^\mu\partial^\nu\right]\delta^{(d)}(x,y)-\Big\{\mathscr{W}_k''F^{\lambda\mu}F^{\sigma\nu}\\ &+4\mathscr{G}^2\ddot{\mathscr{W}}_k\left(\star F^{\lambda\mu}\right)\left(\star F^{\sigma\nu}\right)+2\mathscr{G}\dot{\mathscr{W}}_k'\left[F^{\lambda\mu}\left(\star F^{\sigma\nu}\right)+F^{\lambda\nu}\left(\star F^{\sigma\mu}\right)\right]\\ &+\frac{1}{4}\dot{\mathscr{W}}_kF_{\kappa\rho}F_{\pi\tau}\left(\varepsilon^{\mu\nu\lambda\sigma}\varepsilon^{\kappa\rho\pi\tau}+2\varepsilon^{\mu\sigma\pi\tau}\varepsilon^{\nu\lambda\kappa\rho}\right)\Big\}\partial_\lambda\partial_\sigma\delta^{(d)}(x,y).\end{aligned} \tag{3.21}$$

[4]It would be inadequate to identify $\mathscr{W}_k$ with a potential in the ordinary sense, since it exhibits also kinematical properties, like for instance the term $\mathscr{F}$ which displays the kinetic energy of freely propagating spin-1 bosons in Euclidean space. Anyway, for the sake of convenience we will nevertheless simply call $\mathscr{W}_k$ the "potential", or more precisely, the "effective average potential".

Instead of working with the coordinate space representation of $\tilde{\Gamma}_k^{(2)}$ which contains derivatives of delta distributions, it is conventional to perform the transition to momentum space where $\tilde{\Gamma}_k^{(2)}$ becomes a purely algebraic expression. Since the remaining local dependencies are carried by the derivative operators and the Dirac delta distributions, it is straightforward to apply a Fourier transformation, e.g. by using the common rules: $\delta^{(d)}(x) \overset{\mathcal{F}}{\mapsto} \mathbf{1}_{\mathbb{R}^d}(x)$ and $\partial_\mu f(x) \overset{\mathcal{F}}{\mapsto} -\imath p_\mu f(p)$, where $\mathcal{F}$ denotes the Fourier transformation and f is any suitable test function.

Before we turn to explicitly present the momentum space representation of $\tilde{\Gamma}_k^{(2)}$, we can make a notable simplification on the last term in eq. (3.21), which in momentum space reads:

$$\frac{1}{4}\dot{\mathscr{W}}_k F_{\kappa\rho} F_{\pi\tau}\Big(\varepsilon^{\mu\nu\lambda\sigma}\varepsilon^{\kappa\rho\pi\tau} + 2\varepsilon^{\mu\sigma\pi\tau}\varepsilon^{\nu\lambda\kappa\rho}\Big)p_\lambda p_\sigma = 2\dot{\mathscr{W}}_k \mathscr{G}\varepsilon^{\mu\nu\lambda\sigma}p_\lambda p_\sigma + 2\dot{\mathscr{W}}_k\Big(\star F^{\mu\lambda}\Big)\Big(\star F^{\nu\sigma}\Big)p_\lambda p_\sigma. \quad (3.22)$$

Here we have the totally antisymmetric Levi-Civita tensor in combination with the symmetric product $p_\lambda p_\sigma$, thus $\varepsilon^{\mu\nu\lambda\sigma}p_\lambda p_\sigma = 0$ and only the second term in (3.22) survives. Thus, from eq. (3.21) we obtain:

$$\begin{aligned}\Big(\tilde{\Gamma}_k^{(2)}\Big)^{\mu\nu}[A](p) = \mathscr{W}_k'\Big(\delta^{\mu\nu}p^2 - p^\mu p^\nu\Big) + \Big\{&\mathscr{W}_k'' F^{\mu\lambda}F^{\nu\sigma} \\ &+ 2\Big(\dot{\mathscr{W}}_k + 2\mathscr{G}^2\ddot{\mathscr{W}}_k\Big)\Big(\star F^{\mu\lambda}\Big)\Big(\star F^{\nu\sigma}\Big) \\ &+ 2\mathscr{G}\dot{\mathscr{W}}_k'\Big[F^{\mu\lambda}\Big(\star F^{\nu\sigma}\Big) + F^{\nu\lambda}\Big(\star F^{\mu\sigma}\Big)\Big]\Big\}p_\lambda p_\sigma.\end{aligned} \quad (3.23)$$

Let us now face the problem of gauge redundancies. As it was explained in sec. 2.4, the unconstrained EAA needs to be equipped with a mechanism that ensures the path integral only to integrate over physically inequivalent field configurations. For this, the regulated weight factor $\mathrm{e}^{-S-\Delta S_k}$ is enlarged by two additional actions which take care of gauge fixing and the integration over gauge orbit space. These requirements are implemented via a gauge fixing action, S_{gf}, and the Faddeev-Popov ghost action, S_{gh}, respectively. Since we are dealing with an Abelian gauge theory, the ghost fields decouple from the spin-1 bosonic degrees of freedom. In consequence, the ghost degrees of freedom do not contribute non-trivially to the renormalisation flow and hence can be dropped from our consideration.

It remains to decide for a proper gauge fixing condition. A convenient choice is the standard *Lorenz gauge*:

$$0 = \mathsf{G}(A) \equiv \partial_\mu A^\mu. \quad (3.24)$$

From this, the corresponding scale-dependent gauge fixing action can be constructed according to eq. (2.64):

$$(S_{\mathrm{gf}})_k[A] = \frac{1}{2\kappa}Z_k\int(\partial_\mu A^\mu)^2\,\mathrm{d}^d x, \quad (3.25)$$

where $\kappa \in \mathbb{R}$ is a formally necessary gauge parameter. It will drop out at some point of our calculations. The flow controlled field strength renormalisation Z_k appears as a consequence of having Z_k as the generic prefactor in front of the kinetic energy, $\mathscr{F} \sim A^2$. It further implements non-trivial flow dynamics through its k dependence.

Following the same procedure as for the unconstrained EAA, we find by a simple computation:

$$\Big(S_{\mathrm{gf}}^{(2)}\Big)^{\mu\nu}(p) = \left(\mathcal{F}\frac{\delta^2(S_{\mathrm{gf}})_k[A]}{\delta A_\mu(x)\delta A_\nu(y)}\right)(p) = \frac{1}{\kappa}Z_k p^\mu p^\nu. \quad (3.26)$$

Conclusively, the *constrained effective average action* is given as $\Gamma_k = \tilde{\Gamma}_k + (S_{\mathrm{gf}})_k$, and its second functional derivative follows from eqs. (3.23) & (3.26).

It can be written in a compactified form that uses *field space projectors*:

$$\Gamma_k^{(2)} = \zeta_k^{\perp}\mathbf{P}_{\perp} + \zeta_k^{\parallel}\mathbf{P}_{\parallel} + \zeta_k^{\vartriangle}\mathbf{P}_{\vartriangle} + \zeta_k^{\triangledown}\mathbf{P}_{\triangledown} + \zeta_k^{\lozenge}\mathbf{P}_{\lozenge}. \tag{3.27}$$

The definitions of the *projectors*, $\{\mathbf{P}_{\perp}, \mathbf{P}_{\parallel}, \mathbf{P}_{\vartriangle}, \mathbf{P}_{\triangledown}, \mathbf{P}_{\lozenge}\}$, and their coefficients, $\{\zeta_k^{\perp}, \zeta_k^{\parallel}, \zeta_k^{\vartriangle}, \zeta_k^{\triangledown}, \zeta_k^{\lozenge}\}$, are summarised in tab. 3.1 below.

Field Space Direction	Projector	Projection Coordinate
Transversal	$\mathbf{P}_{\perp} = \boldsymbol{\delta} - \frac{p\otimes p}{p^2}$	$\zeta_k^{\perp} = \mathscr{W}_k' p^2$
Longitudinal	$\mathbf{P}_{\parallel} = \frac{p\otimes p}{p^2}$	$\zeta_k^{\parallel} = \frac{1}{\kappa} Z_k p^2$
Sub-Transversal	$\mathbf{P}_{\vartriangle} = \frac{(\mathbf{F}p)\otimes(\mathbf{F}p)}{(\mathbf{F}p)^2}$	$\zeta_k^{\vartriangle} = \mathscr{W}_k'' (\mathbf{F}p)^2$
Dual Sub-Transversal	$\mathbf{P}_{\triangledown} = \frac{(\star\mathbf{F}p)\otimes(\star\mathbf{F}p)}{(\star\mathbf{F}p)^2}$	$\zeta_k^{\triangledown} = 2\left(\dot{\mathscr{W}}_k + 2\mathscr{G}^2\ddot{\mathscr{W}}_k\right)(\star\mathbf{F}p)^2$
Skew Sub-Transversal	$\mathbf{P}_{\lozenge} = \frac{(\mathbf{F}p)\otimes(\star\mathbf{F}p)+(\star\mathbf{F}p)\otimes(\mathbf{F}p)}{\mathscr{G}p^2}$	$\zeta_k^{\lozenge} = 2\mathscr{G}^2\dot{\mathscr{W}}_k' p^2$

Table 3.1: *Operator components of* $\Gamma_k^{(2)}$. *Remark: recall that* $\otimes$ *denotes the dyadic product.*

Let us briefly list some geometric properties of the projectors. Each of them acts as an operator on the A-field space and filters directional propagation informations. For instance, consider the wide class of gauge fields $A_\mu(p) = p_\mu\Lambda(p)$ for largely arbitrary functions Λ. Their transversal component of propagation is singled out by the action of $\mathbf{P}_{\perp}$, but since $A_\mu \sim p_\mu$ points in direction of motion, we basically do not expect a transversal part in this situation. Indeed, together with tab. 3.1 we find: $P_{\perp}^{\mu\nu}A_\nu(p) = \left(\delta^{\mu\nu}p_\nu - \frac{p^\mu p^\nu p_\nu}{p^2}\right)\Lambda(p) = (p^\mu - p^\mu)\Lambda(p) = 0$. By contrast, the longitudinal projection acts like an identity in this case, i.e. $P_{\parallel}^{\mu\nu}A_\nu(p) = A^\mu(p)$. This example should justify the namings "transversal" and "longitudinal" that appear in tab. 3.1.

It is trivial to verify that the transversal and longitudinal projectors add up to unity:

$$\mathbf{P}_{\perp} + \mathbf{P}_{\parallel} = \boldsymbol{\delta} \equiv \mathbb{1}. \tag{3.28}$$

Furthermore, since $(Fp)^\mu p_\mu = F^{\mu\nu}p_\mu p_\nu = 0$, because we contract an antisymmetric object, the field strength tensor $\mathbf{F}$, with the symmetric product $p_\mu p_\nu$, the actions of the remaining projectors $\{\mathbf{P}_{\vartriangle}, \mathbf{P}_{\triangledown}\mathbf{P}_{\lozenge}\}$ on $A_\mu(p) \sim p_\mu$ vanish identically. Together with eq. (3.28), it thus follows that their codomains must be a subset of the transversal portion of A-field space on which $\mathbf{P}_{\perp}$ is surjective.

Moreover, it is not difficult to show that each projector is idempotent of degree 2, so they can factually be classified as projection operators in the strict algebraic sense. The only exception to this concerns the skew sub-transversal projector $\mathbf{P}_{\lozenge}$, where $\mathbf{P}_{\lozenge}^2 \neq \mathbf{P}_{\lozenge}$ and the object rather appears as a sum of two proper projection operators, namely $\frac{(\mathbf{F}p)\otimes(\star\mathbf{F}p)}{\mathscr{G}p^2} =: \mathbf{P}_{\lozenge}^L$ and $\frac{(\star\mathbf{F}p)\otimes(\mathbf{F}p)}{\mathscr{G}p^2} =: \mathbf{P}_{\lozenge}^R$.

As it is clear from eq. (3.27), $\Gamma_k^{(2)}$ appears as a linear combination of the projectors given in tab. 3.1. Each of them is provided with a scale dependent coefficient that controls the projection coordinate while drifting along the renormalisation flow. Except for $\zeta_k^{\parallel}$, they strongly depend on the potential $\mathscr{W}_k$ and in particular on its derivatives.

Eventually, for the derivation of the full propagator we need the set of algebraic relations among the projectors under ordinary multiplication. For this purpose we introduce the *Cayley table* that includes the finite set of really projection operators, i.e. taking care of the decomposition $\mathbf{P}_\Diamond = \mathbf{P}^L_\Diamond + \mathbf{P}^R_\Diamond$ in proper sub-transversal projectors. The individual computations rely to some extent on two helpful identities:

$$\begin{aligned} F^{\mu\lambda} F^{\nu}{}_{\lambda} + (\star F)^{\mu\lambda} (\star F)^{\nu}{}_{\lambda} &= 2\mathscr{F}\delta^{\mu\nu}, \\ F^{\mu\lambda} (\star F)^{\nu}{}_{\lambda} = (\star F)^{\mu\lambda} F^{\nu}{}_{\lambda} &= \mathscr{G}\delta^{\mu\nu}. \end{aligned} \tag{3.29}$$

For a proof of them, one is invited to consult the end of app. A, cf. Proposition A.1. Finally, the Cayley table is positioned below.

$\bullet$	$\mathbf{P}_\perp$	$\mathbf{P}_\parallel$	$\mathbf{P}_\triangle$	$\mathbf{P}_\triangledown$	$\mathbf{P}^L_\Diamond$	$\mathbf{P}^R_\Diamond$
$\mathbf{P}_\perp$	$\mathbf{P}_\perp$	$\mathbf{0}$	$\mathbf{P}_\triangle$	$\mathbf{P}_\triangledown$	$\mathbf{P}^L_\Diamond$	$\mathbf{P}^R_\Diamond$
$\mathbf{P}_\parallel$	$\mathbf{0}$	$\mathbf{P}_\parallel$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathbf{P}_\triangle$	$\mathbf{P}_\triangle$	$\mathbf{0}$	$\mathbf{P}_\triangle$	$g^2\mathbf{P}^L_\Diamond$	$\mathbf{P}^L_\Diamond$	$\mathbf{P}_\triangle$
$\mathbf{P}_\triangledown$	$\mathbf{P}_\triangledown$	$\mathbf{0}$	$g^2\mathbf{P}^R_\Diamond$	$\mathbf{P}_\triangledown$	$\mathbf{P}_\triangledown$	$\mathbf{P}^R_\Diamond$
$\mathbf{P}^L_\Diamond$	$\mathbf{P}^L_\Diamond$	$\mathbf{0}$	$\mathbf{P}_\triangle$	$\mathbf{P}^L_\Diamond$	$\mathbf{P}^L_\Diamond$	$\frac{1}{g^2}\mathbf{P}_\triangle$
$\mathbf{P}^R_\Diamond$	$\mathbf{P}^R_\Diamond$	$\mathbf{0}$	$\mathbf{P}^R_\Diamond$	$\mathbf{P}_\triangledown$	$\frac{1}{g^2}\mathbf{P}_\triangledown$	$\mathbf{P}^R_\Diamond$

Table 3.2: *Multiplication relations among projection operators on A-field space. The dimensionless parameter g is defined as $g := \frac{\mathscr{G}p^2}{\sqrt{(\mathbf{F}p)^2(\star\mathbf{F}p)^2}}$.*

In order to construct the regularised full propagator, we need to implement a regularisation scheme in the form of an IR regulator as it was described in sec 2.1. For this, we adhere to eq. (2.18), but for the sake of generality without giving additional details on the shape function R until further notice. Furthermore, it has just become necessary to consider a field strength renormalisation also for the regulator itself, such that it accounts for the required mass-like damping in the deep IR regime, i.e. $\Gamma^{(2)}_k + \mathcal{R}_k = Z_k\,(p^2 + k^2) + \ldots$ for $p^2 \ll k^2$. This means, that the regulator shall reduce to $\mathcal{R}_k(p) \approx Z_k k^2$ in the IR limit to avoid an odd suppression behaviour that generates a mass k^2/Z_k instead of k^2, as it would be the case if we decide on $\lim_{p^2/k^2\to 0}\mathcal{R}_k = k^2$. This measure is also known under the name "adjusted cutoffs" and further details of it can be found in ref. [28]. With these comments, a quite generic form for $\mathcal{R}_k$ appears as a linear combination of the two disjoint projectors $\mathbf{P}_\perp$ and $\mathbf{P}_\parallel$:

$$\mathcal{R}_k(p) = Z_k p^2 R\left(\frac{p^2}{k^2}\right)\left[\mathbf{P}_\perp + \frac{1}{\kappa}\mathbf{P}_\parallel\right]. \tag{3.30}$$

Combining the eqs. (3.27) & (3.30) we obtain the *regularised full propagator*:

$$\boxed{\Gamma^{(2)}_k + \mathcal{R}_k = \left(\zeta^\perp_k + Z_k p^2 R\right)\mathbf{P}_\perp + \left(\zeta^\parallel_k + \frac{1}{\kappa} Z_k p^2 R\right)\mathbf{P}_\parallel + \zeta^\triangle_k \mathbf{P}_\triangle + \zeta^\triangledown_k \mathbf{P}_\triangledown + \zeta^\Diamond_k \mathbf{P}_\Diamond.} \tag{3.31}$$

D. The Exact Renormalisation Flow Equation

This paragraph is devoted to the derivation of an explicit formulation of the exact renormalisation group equation (ERGE). First, let us recall the Wetterich equation (2.15) and customise it to our present case:

$$\partial_t \Gamma_k [A] = \frac{1}{2} \underset{(P\cdot L)}{\mathbf{Tr}} \Big[\Big(\Gamma_k^{(2)} [A] + \mathcal{R}_k \Big)^{-1} \partial_t \mathcal{R}_k \Big]. \tag{3.32}$$

The supertrace manifests as a trace-pair; on one hand we need to take a momentum trace (P), provided that all quantities within the trace operation are also considered in their momentum space representation, otherwise we would have to integrate over coordinate space first, which we can circumvent by means of a Fourier transformation. Using the homogeneity condition, the coordinate space integral yields a volume factor Ω_d that cancels with another volume factor on the LHS of eq. (3.32), and what remains is precisely the momentum trace. On the other hand, since we are dealing with vector degrees of freedom, we additionally need to perform a trace over Lorentz indices (L).

The advantage of working with projection operators becomes markedly perceptible when we derive the inverse of (3.31). Let $\mathscr{O}$ denote an arbitrary operator, then its inverse - if it exists - is uniquely determined and satisfies the equation $\mathscr{O}\mathscr{O}^{-1} = \mathbb{1} = \mathscr{O}^{-1}\mathscr{O}$. We identify $\mathscr{O} \equiv \Gamma_k^{(2)} + \mathcal{R}_k$ and make an expansion ansatz in field space projectors $\{\mathbf{P}_\perp, \mathbf{P}_\parallel, \mathbf{P}_\vartriangle, \mathbf{P}_\triangledown, \mathbf{P}_\Diamond\}$ for $\Big(\Gamma_k^{(2)} + \mathcal{R}_k\Big)^{-1}$, which is justified as the full propagator represents yet another operator that acts on A-field space:

$$\Big(\Gamma_k^{(2)} + \mathcal{R}_k\Big)^{-1} = \xi_k^\perp \mathbf{P}_\perp + \xi_k^\parallel \mathbf{P}_\parallel + \xi_k^\vartriangle \mathbf{P}_\vartriangle + \xi_k^\triangledown \mathbf{P}_\triangledown + \xi_k^\Diamond \mathbf{P}_\Diamond. \tag{3.33}$$

The determination of the unknown coefficients $\{\xi_k^\perp, \xi_k^\parallel, \xi_k^\vartriangle, \xi_k^\triangledown, \xi_k^\Diamond\}$ proceeds as follows: first we consider $\mathbb{1} = \Big(\Gamma_k^{(2)} + \mathcal{R}_k\Big)\Big(\Gamma_k^{(2)} + \mathcal{R}_k\Big)^{-1}$ and insert (3.31) together with (3.33). Next, we make use of the Cayley table, tab. 3.2, to deal with the projector products and factorise in terms of projection operators rightafter. Then, we take account of eq. (3.28) and eventually read off the algebraic system of equations that uniquely fixes the coefficients $\{\xi_k^\perp, \xi_k^\parallel, \xi_k^\vartriangle, \xi_k^\triangledown, \xi_k^\Diamond\}$. Indeed, the whole calculation is rather tedious but elementary, since the resulting algebraic system is only weakly coupled. In order to maintain clarity we skip most of the details and directly present its solution:

$$\xi_k^\perp = \frac{1}{\zeta_k^\perp + Z_k p^2 R}, \qquad \xi_k^\parallel = \frac{1}{\zeta_k^\parallel + \frac{1}{\kappa} Z_k p^2 R},$$

$$\begin{aligned}
\xi_k^\vartriangle &= \frac{\Big(\zeta_k^\perp + Z_k p^2 R\Big)^{-1}\Big[\Big(\zeta_k^\Diamond\Big)^2 g^{-2} - \zeta_k^\vartriangle \Big(\zeta_k^\perp + \zeta_k^\triangledown + Z_k p^2 R\Big)\Big]}{\Big(\zeta_k^\perp + \zeta_k^\vartriangle + \zeta_k^\Diamond + Z_k p^2 R\Big)\Big(\zeta_k^\perp + \zeta_k^\triangledown + \zeta_k^\Diamond + Z_k p^2 R\Big) - \Big(\zeta_k^\Diamond + \zeta_k^\triangledown g^2\Big)\Big(\zeta_k^\vartriangle + \zeta_k^\Diamond g^{-2}\Big)}, \\
\xi_k^\triangledown &= \frac{\Big(\zeta_k^\perp + Z_k p^2 R\Big)^{-1}\Big[\Big(\zeta_k^\Diamond\Big)^2 g^{-2} - \zeta_k^\triangledown \Big(\zeta_k^\perp + \zeta_k^\vartriangle + Z_k p^2 R\Big)\Big]}{\Big(\zeta_k^\perp + \zeta_k^\vartriangle + \zeta_k^\Diamond + Z_k p^2 R\Big)\Big(\zeta_k^\perp + \zeta_k^\triangledown + \zeta_k^\Diamond + Z_k p^2 R\Big) - \Big(\zeta_k^\Diamond + \zeta_k^\triangledown g^2\Big)\Big(\zeta_k^\vartriangle + \zeta_k^\Diamond g^{-2}\Big)}, \\
\xi_k^\Diamond &= \frac{\Big(\zeta_k^\perp + Z_k p^2 R\Big)^{-1}\Big[\zeta_k^\vartriangle \zeta_k^\triangledown g^2 - \zeta_k^\Diamond \Big(\zeta_k^\perp + \zeta_k^\Diamond + Z_k p^2 R\Big)\Big]}{\Big(\zeta_k^\perp + \zeta_k^\vartriangle + \zeta_k^\Diamond + Z_k p^2 R\Big)\Big(\zeta_k^\perp + \zeta_k^\triangledown + \zeta_k^\Diamond + Z_k p^2 R\Big) - \Big(\zeta_k^\Diamond + \zeta_k^\triangledown g^2\Big)\Big(\zeta_k^\vartriangle + \zeta_k^\Diamond g^{-2}\Big)}.
\end{aligned} \tag{3.34}$$

Let us now take the RG time derivative of eq. (3.30):

$$\partial_t \mathcal{R}_k(p) = -Z_k p^2 \left(\eta_k R + 2\frac{p^2}{k^2} R'\right)\left[\mathbf{P}_\perp + \frac{1}{\kappa}\mathbf{P}_\parallel\right]. \tag{3.35}$$

Here, we finally re-encounter the *anomalous dimension* η_k, that by definition represents a measure for the relative change of the field strength renormalisation Z_k with respect to RG time:

$$\eta_k := -\frac{1}{Z_k}\partial_t Z_k = -\partial_t \ln\left(Z_k\right). \tag{3.36}$$

From a physical perspective, the effect that arises through the anomalous dimension becomes visible by comparing free field theories with their possible interacting extensions, for instance within a perturbation theoretical framework. In the free Gaussian case, each operator $\mathscr{O}$ changes by a factor of $a^{-\Delta}$ under a dilation $x \mapsto ax$ with $a \in \mathbb{R}$. The number Δ is called the *canonical dimension* of $\mathscr{O}$. When turning on interactions, this number could shift by an amount that refers to the anomalous dimension, i.e. $\Delta \to \Delta + \eta$, and as soon as renormalisation flow effects come into play, this shift will be scale-dependent and corresponds to what we have just defined according to eq. (3.36).

Using once more the Cayley table, tab. 3.2, the operator product of (3.33) and (3.35) can be inferred:

$$\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} \partial_t \mathcal{R}_k = -Z_k p^2 \left(\eta_k R + 2\frac{p^2}{k^2}R'\right) \left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} + \frac{1-\kappa}{\kappa}\xi_k^{\parallel}\mathbf{P}_{\parallel}\right]. \tag{3.37}$$

We are now in position to go through the various traces that we need to apply in (3.37). As it was already mentioned, from the homogeneity condition it follows that $\mathscr{W}_k$ is independent of any local coordinates, thus the LHS of the Wetterich equation (3.32) reduces to $\partial_t\Gamma_k = \Omega_d \partial_t \mathscr{W}_k$, where Ω_d denotes a volume factor that originates from the d-dimensional coordinate space integration. On the RHS of eq. (3.32), we can express all quantities in momentum space, obtaining another factor Ω_d that cancels with its counterpart on the LHS as we have recently explained. The Lorentz trace (L) is distributed among the individual projectors, whose computations can be done straightforwardly, except for $\mathbf{P}_{\Diamond}$ which requires slightly more effort. For this, we need eq. (3.29):

$$\begin{aligned}\underset{(L)}{\mathbf{Tr}}\left[\mathbf{P}_{\Diamond}\right] &= \underset{(L)}{\mathbf{Tr}}\left[\frac{(\mathbf{F}p)\otimes(\star\mathbf{F}p) + (\star\mathbf{F}p)\otimes(\mathbf{F}p)}{\mathscr{G}p^2}\right] \\ &= \frac{(Fp)^{\mu}\left(\star Fp\right)_{\mu} + \left(\star Fp\right)^{\mu}(Fp)_{\mu}}{\mathscr{G}p^2} \\ &= 2\frac{F^{\mu\lambda}\left(\star F\right)^{\nu}{}_{\lambda}\, p_{\mu}p_{\nu}}{\mathscr{G}p^2} \overset{(3.29)}{=} 2\frac{\mathscr{G}\delta^{\mu\nu}p_{\mu}p_{\nu}}{\mathscr{G}p^2} = 2.\end{aligned} \tag{3.38}$$

The remaining Lorentz traces are:

$$\underset{(L)}{\mathbf{Tr}}\left[\mathbf{P}_{\perp}\right] = d-1 \qquad \& \qquad \underset{(L)}{\mathbf{Tr}}\left[\mathbf{P}_{\parallel}\right] = \underset{(L)}{\mathbf{Tr}}\left[\mathbf{P}_{\vartriangle}\right] = \underset{(L)}{\mathbf{Tr}}\left[\mathbf{P}_{\triangledown}\right] = 1. \tag{3.39}$$

Eventually, the momentum trace (P) translates into a momentum integral that cannot be solved at this stage, unless more information on the shape function R is provided (cf. sec. 3.2).

Collecting the results from (3.37)-(3.39) and inserting them into eq. (3.32) finally yields the **exact renormalisation group equation** to leading order in a derivative expansion:

$$\boxed{\partial_t \mathscr{W}_k = -\frac{1}{2}Z_k \int p^2 \left(\eta_k R + 2\frac{p^2}{k^2}R'\right)\left((d-1)\xi_k^{\perp} + \frac{1}{\kappa}\xi_k^{\parallel} + \xi_k^{\vartriangle} + \xi_k^{\triangledown} + 2\xi_k^{\Diamond}\right)\frac{\mathrm{d}^d p}{(2\pi)^d}.} \tag{3.40}$$

Our original aim was to investigate the fixed point sector of the self-interacting locally U(1) invariant model, i.e. of the renormalisation flow which is characterised by the ERGE (3.40). For this, it is useful to perform the transition to dimensionless quantities first.

E. Dimensionless Quantities and Field Renormalisation

The goal of this paragraph is to render the functional PDE (3.40) autonomous in its scale dependence k. This mathematical condition can be achieved by switching to *dimensionless quantities*.

In what follows, we adhere to the information provided in app. B, where some basic dimensional analysis based on the model of our present case was already carried out, e.g. that $[\mathscr{F}] = [\mathscr{G}] = d$ holds in natural units where $\hbar = 1$ and $c = 1$.

The only characteristic quantity at hand defining a mass scale is given by the scale parameter k. It counts one mass unit since it represents a momentum, i.e. $[k] = 1$. Let us therefore introduce *dimensionless invariants* by multiplying $\mathscr{F}$ and $\mathscr{G}$ with an appropriate power of k, but simultaneously implement a normalisation to account for the generic field strength renormalisation factor in front of the momentum integral as presented in eq. (3.40):

$$\tilde{\mathscr{F}} := Z_k k^{-d} \mathscr{F} \qquad \& \qquad \tilde{\mathscr{G}} := Z_k k^{-d} \mathscr{G}. \tag{3.41}$$

According to these definitions, it is now possible to deduce a dimensionless field strength tensor. For this, we use the definition of $\mathscr{F}$:

$$\tilde{\mathscr{F}} \triangleq Z_k k^{-d} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) = \frac{1}{4} \left(\sqrt{Z_k} k^{-\frac{d}{2}} F_{\mu\nu}\right) \left(\sqrt{Z_k} k^{-\frac{d}{2}} F^{\mu\nu}\right) \equiv \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu}. \tag{3.42}$$

Hence we identify the *renormalised dimensionless field strength*:

$$\tilde{\mathbf{F}} = \sqrt{Z_k} k^{-\frac{d}{2}} \mathbf{F}, \tag{3.43}$$

where the same is true for the Hodge dual $\star\mathbf{F}$.

In addition to this, we define a *dimensionless version of the effective average potential* under consideration of $[\mathscr{W}_k] = d$ (cf. app. B):

$$w_k\left(\tilde{\mathscr{F}}, \tilde{\mathscr{G}}^2\right) := k^{-d} \mathscr{W}_k\left(Z_k^{-1} k^d \tilde{\mathscr{F}}, \left(Z_k^{-1} k^d \tilde{\mathscr{G}}\right)^2\right) \equiv k^{-d} \mathscr{W}_k\left(\mathscr{F}, \mathscr{G}^2\right). \tag{3.44}$$

With this, all preparations are made to translate the LHS of eq. (3.40) to its dimensionless form:

$$\begin{aligned} \partial_t \mathscr{W}_k &= k \partial_k \left(k^d w_k\right) \\ &= k\left(d k^{d-1} w_k + k^d \partial_k w_k + k^d w_k' \partial_k \tilde{\mathscr{F}} + k^d \dot{w}_k \partial_k \tilde{\mathscr{G}}^2\right) \\ &= k^d \left(d w_k + \partial_t w_k - w_k' (\eta_k + d) \tilde{\mathscr{F}} - 2\dot{w}_k (\eta_k + d) \tilde{\mathscr{G}}^2\right). \end{aligned} \tag{3.45}$$

Note that we follow the convention (3.20) once more, i.e. a prime or a dot indicates derivatives of w with respect to $\tilde{\mathscr{F}}$ or $\tilde{\mathscr{G}}^2$ respectively.

Let us now turn our attention to the RHS of eq. (3.40) and consider the collection of coefficients $\{\xi_k^{\perp}, \xi_k^{\parallel}, \xi_k^{\vartriangle}, \xi_k^{\triangledown}, \xi_k^{\Diamond}\} \equiv \Xi$. To improve readability, we will generically speak of arbitrary representatives $\xi_k \in \Xi$ in what follows. The same holds for $\zeta_k \in \Pi \equiv \{\zeta_k^{\perp}, \zeta_k^{\parallel}, \zeta_k^{\vartriangle}, \zeta_k^{\triangledown}, \zeta_k^{\Diamond}\}$. Before we continue our discussion, it is useful to derive a set of relations connecting derivatives of $\mathscr{W}_k$ with derivatives of w_k using the eqs. (3.41) & (3.44):

$$\begin{aligned} \mathscr{W}_k' &\triangleq \frac{\partial \mathscr{W}_k}{\partial \mathscr{F}} = \frac{\partial \tilde{\mathscr{F}}}{\partial \mathscr{F}} \frac{\partial}{\partial \tilde{\mathscr{F}}} \left(k^d w_k\right) = Z_k w_k' &&\longrightarrow \quad \mathscr{W}_k'' = Z_k^2 k^{-d} w_k'', \\ \dot{\mathscr{W}}_k &\triangleq \frac{\partial \mathscr{W}_k}{\partial \mathscr{G}^2} = \frac{\partial \tilde{\mathscr{G}}^2}{\partial \mathscr{G}^2} \frac{\partial}{\partial \tilde{\mathscr{G}}^2} \left(k^d w_k\right) = Z_k^2 k^{-d} \dot{w}_k &&\longrightarrow \quad \ddot{\mathscr{W}}_k = Z_k^4 k^{-3d} \ddot{w}_k, \\ \dot{\mathscr{W}}_k' &\triangleq \frac{\partial^2 \mathscr{W}_k}{\partial \mathscr{F} \partial \mathscr{G}^2} = \left(\frac{\partial \tilde{\mathscr{F}}}{\partial \mathscr{F}}\right) \left(\frac{\partial \tilde{\mathscr{G}}^2}{\partial \mathscr{G}^2}\right) \frac{\partial^2}{\partial \tilde{\mathscr{F}} \partial \tilde{\mathscr{G}}^2} \left(k^d w_k\right) = Z_k^3 k^{-2d} \dot{w}_k'. \end{aligned} \tag{3.46}$$

For definiteness and to give an illustration, let us specifically consider $\zeta_k^{\perp} \in \Pi$ and make use of the relations (3.46) (cf. also tab. 3.1):

$$\zeta_k^{\perp} \triangleq \mathscr{W}_k' p^2 = Z_k w_k' k^2 \frac{p^2}{k^2} \equiv Z_k k^2 \left(w_k' s^2 \right) \equiv Z_k k^2 \tilde{\zeta}_k^{\perp}. \tag{3.47}$$

Here, we have introduced a dimensionless parameter s that measures the momentum p in portions of the scale k, i.e. $s := p/k$. This enables us to read off the dimensionless transversal projection coordinate $\tilde{\zeta}_k^{\perp} \equiv w_k' s^2$. Since all projection operators are dimensionless per construction, each $\zeta_k \in \Pi$ must behave identically to $\zeta_k^{\perp}$ under dimensional rescaling, thus we can deduce: $\zeta_k = Z_k k^2 \tilde{\zeta}_k$ for all $\zeta_k \in \Pi$. Based on the information from tab. 3.1, we obtain the following set of dimensionless projection coordinates:

$$\begin{aligned} &\tilde{\zeta}_k^{\perp} = w_k' s^2, \qquad \tilde{\zeta}_k^{\parallel} = \frac{1}{\kappa} s^2, \\ &\tilde{\zeta}_k^{\vartriangle} = w_k'' \left(\tilde{\mathbf{F}} s \right)^2, \qquad \tilde{\zeta}_k^{\triangledown} = 2 \left(\dot{w}_k + 2\tilde{\mathscr{G}}^2 \ddot{w}_k \right) \left(\star\tilde{\mathbf{F}} s \right)^2, \qquad \tilde{\zeta}_k^{\lozenge} = 2\tilde{\mathscr{G}}^2 \dot{w}_k' s^2. \end{aligned} \tag{3.48}$$

From the observation that elements which live in Π correspond to the inverse propagator $\left(\Gamma_k^{(2)} + \mathcal{R}_k \right)$, whereas elements from Ξ belong to the full propagator, we can conclude that each $\xi_k \in \Xi$ must scale inversely to $\zeta_k \in \Pi$ under dimensional transformations $\xi_k \mapsto \tilde{\xi}_k$. That means: $\xi_k = \frac{1}{Z_k k^2} \tilde{\xi}_k$ for all $\xi_k \in \Xi$.

Let us return to the RHS of eq. (3.40) and use our recent findings:

$$\begin{aligned} &-\frac{1}{2} Z_k \int p^2 \left(\eta_k R + 2\frac{p^2}{k^2} R' \right) \left((d-1)\xi_k^{\perp} + \frac{1}{\kappa}\xi_k^{\parallel} + \xi_k^{\vartriangle} + \xi_k^{\triangledown} + 2\xi_k^{\lozenge} \right) \frac{\mathrm{d}^d p}{(2\pi)^d} \\ &\overset{(p \mapsto \frac{p}{k} = s)}{=} -\frac{1}{2} k^d \int s^2 \left(\eta_k R + 2s^2 R' \right) \left((d-1)\tilde{\xi}_k^{\perp} + \frac{1}{\kappa}\tilde{\xi}_k^{\parallel} + \tilde{\xi}_k^{\vartriangle} + \tilde{\xi}_k^{\triangledown} + \tilde{\xi}_k^{\lozenge} \right) \frac{\mathrm{d}^d s}{(2\pi)^d}. \end{aligned} \tag{3.49}$$

Note that the shape function R now depends on the variable s^2.

Equating (3.45) and (3.49) finally yields the **autonomous ERGE**:

$$\boxed{\partial_t w_k + d w_k - (\eta_k + d) \left(w_k' \tilde{\mathscr{F}} + 2\dot{w}_k \tilde{\mathscr{G}}^2 \right) = -\frac{1}{2} \int s^2 \left(\eta_k R + 2s^2 R' \right) \tilde{\Xi}_k^{(d)} \frac{\mathrm{d}^d s}{(2\pi)^d}.} \tag{3.50}$$

Here, we have introduced a compact notation[5]:

$$\tilde{\Xi}_k^{(d)} := (d-1)\tilde{\xi}_k^{\perp} + \frac{1}{\kappa}\tilde{\xi}_k^{\parallel} + \tilde{\xi}_k^{\vartriangle} + \tilde{\xi}_k^{\triangledown} + \tilde{\xi}_k^{\lozenge}. \tag{3.51}$$

Eq. (3.50) serves as the starting point for the rest of our upcoming analysis covered in sec. 3.2.

[5]Note that $\tilde{\Xi}_k^{(d)}$ do not only depend on the scale k and the number of spacetime dimensions d, but also on the dimensionless momentum measure s as well as the dimensionless invariants $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{G}}^2$.

3.2

Global Fixed Points: Results and Discourse

A. Fixed Point Equation and Truncation of Theory Space

The autonomous ERGE (3.50) describes the projected renormalisation flow of w_k, that is a family of RG trajectories within the dimensionless truncated theory space $\tilde{\mathcal{T}}_{\text{trunc}} \equiv \left\{\mathcal{A} \,|\, \mathcal{A} \doteq \mathcal{A}\left[\tilde{\mathscr{F}}, \tilde{\mathscr{G}}^2\right]\right\} \subset \tilde{\mathcal{T}} = \left\{\mathcal{A} \,|\, \mathcal{A} \doteq \mathcal{A}\left[\tilde{\mathscr{F}}, \tilde{\mathscr{G}}\right]\right\} \cong \mathcal{T}$. Each member of that family is uniquely determined once a set of well-posed boundary conditions was asserted. A careful analysis of eq. (3.50) requires information about possible RG future-end points, i.e. UV attractors for $t \to \infty$, to guarantee, among other physical conditions, a well-behaving RG flow in the asymptotic future. As it was discussed in sec. 2.2, physically reasonable candidates for such future-end points are fixed points of the RG flow, i.e. where variations of all generalised couplings with respect to RG time freezes, meaning $\partial_t w_k \to 0$ as we approach the fixed point $w_k \to w_*$. Thus, from eq. (3.50), we obtain the **fixed point equation** (FPE):

$$d w_* - (\eta_* + d)\left(w_*' \tilde{\mathscr{F}} + 2\dot{w}_* \tilde{\mathscr{G}}^2\right) = -\frac{1}{2}\int s^2 \left(\eta_* R + 2s^2 R'\right) \tilde{\Xi}_*^{(d)} \frac{\mathrm{d}^d s}{(2\pi)^d}. \tag{3.52}$$

At first glance we can make the following structural observations on eq. (3.52):

(1). Even though the fixed point equation does not appear as a functional PDE anymore, it is still a PDE in the ordinary sense. It describes the fixed point potential's dynamical behavior in the variables $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{G}}^2$.

(2). The integral on the RHS can be studied piecewise for each term of eq. (3.51). If we decide for a linear cutoff, cf. eq. (2.18), the transversal and longitudinal integrations, which correspond to $\tilde{\xi}_*^{\perp}$ and $\tilde{\xi}_*^{\parallel}$ respectively, can properly be solved analytically using properties of the hypergeometric function. However, due to the high degree of non-linearity which especially reveals in the sub-transversal portions, manifesting in rational integrands containing contributions of the form $\left(\tilde{\mathbf{F}}s\right)^2, \left(\star\tilde{\mathbf{F}}s\right)^2$ and powers of s up to forth order in both numerator and denominator, these sectors are hardly accessible by means of analytical methods.

(3). Even if we find well-suited solutions to the sub-transversal integrals, they are likely of comparable complexity as their integrands, which provokes a significant non-linear derivative structure of the fixed point potential up to second order in both variables. The resulting PDE for w_* is therefore expected to reveal a considerably complicated structure.

For these reasons, we start with a rather cautious approach which intends to truncate theory space even further. Specifically, we decide to exclude all contributions of w_* which are proportional to *any* power of $\tilde{\mathscr{G}}$, namely by supposing that all related generalised couplings vanish identically. In consequence it follows that $\dot{w}_* = \ddot{w}_* = \dot{w}_*' = 0$, which simplifies our system substantially. Moreover, the fixed point equation (3.52) reduces to a non-linear ODE which is much better to handle than a significantly non-linear PDE.

Furthermore, two dimensionless projection and inverse projection coordinates vanish in this one-dimensional situation according to (3.48): $\tilde{\zeta}_*^{\triangledown} = \tilde{\zeta}_*^{\lozenge} = 0$ and consequently $\tilde{\xi}_*^{\triangledown} = \tilde{\xi}_*^{\lozenge} = 0$.

We are left with:

$$\tilde{\xi}_*^{\perp} = \frac{1}{\tilde{\zeta}_*^{\perp} + s^2 R}, \qquad \tilde{\xi}_*^{\parallel} = \frac{1}{\tilde{\zeta}_*^{\parallel} + \frac{1}{\kappa} s^2 R}, \qquad \tilde{\xi}_*^{\Delta} = -\frac{\tilde{\zeta}_*^{\Delta}}{\left(\tilde{\zeta}_*^{\perp} + s^2 R\right)\left(\tilde{\zeta}_*^{\perp} + \tilde{\zeta}_*^{\Delta} + s^2 R\right)}. \tag{3.53}$$

Upon insertion of (3.48) in each term of (3.53) we find:

$$s^2 \tilde{\Xi}_*^{(d)} = \frac{d-1}{w_*' + R} + \frac{1}{1+R} - \frac{w_*'' \left(\tilde{\mathbf{F}} s\right)^2}{\left(w_*' + R\right)\left[\left(w_*' + R\right) s^2 + w_*'' \left(\tilde{\mathbf{F}} s\right)^2\right]}. \tag{3.54}$$

In view of integrating (3.54) by means of the fixed point equation, the only problematic structure comes through the non-linear factor $\left(\tilde{\mathbf{F}} s\right)^2$. To get a better overview, we can express it in terms of dimensionless electric and magnetic fields, $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ respectively. Assuming $d = 4$ spacetime dimensions and a vector representation $s \mathrel{\hat{=}} (s_0, \mathbf{s})^T$, we get:

$$\left(\tilde{\mathbf{F}} s\right)^2 = s_0^2 \tilde{\mathbf{E}}^2 + \tilde{\mathbf{E}}^2 \mathbf{s}^2 \cos\left(\vartheta_e\right)^2 + \tilde{\mathbf{B}}^2 \mathbf{s}^2 \sin\left(\vartheta_m\right)^2 + 2 s_0 \mathbf{s} \cdot \left(\tilde{\mathbf{E}} \times \tilde{\mathbf{B}}\right), \tag{3.55}$$

where ϑ_e and ϑ_m denote the angles enclosed between $\mathbf{s}$ and $\tilde{\mathbf{E}}$ or $\tilde{\mathbf{B}}$ respectively. To avoid severe technical difficulties and produce first concrete results, it is advantageous to restrict ourselves to a certain class of field configurations which are summarised under the term *self duality*. As the name suggests, this class relies on the assumption that $\tilde{\mathbf{F}} = \star\tilde{\mathbf{F}}$ which is equivalent to $\tilde{\mathbf{E}} = \tilde{\mathbf{B}}$. Actually, the only non-trivial additional information that we have put into our model is that both the electric and magnetic field are initially not perpendicular, because then it can be shown that there always exists a frame of reference in which $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ are parallel, provided that there is no system in which $\tilde{\mathbf{E}} \cdot \tilde{\mathbf{B}} = 0$ is true[6]. It should be noted that the idea of "self duality" only becomes meaningful in $d = 4$ spacetime dimensions, because then the field strength and dual field strength tensor are of the same degree and can therefore be mathematically compared to each other.

Either way, the key implication of self duality can be seen from the upper equation in (3.29): using $\tilde{\mathbf{F}} = \star\tilde{\mathbf{F}}$ we obtain:

$$2\tilde{\mathscr{F}} s^2 = 2\tilde{\mathscr{F}} \delta^{\mu\nu} s_\mu s_\nu = \tilde{F}^{\mu\lambda} \tilde{F}^{\nu}{}_{\lambda} s_\mu s_\nu + \left(\star\tilde{F}\right)^{\mu\lambda} \left(\star\tilde{F}\right)^{\nu}{}_{\lambda} s_\mu s_\nu = 2\left(\tilde{F} s\right)^2, \tag{3.56}$$

which leads to the elegant identity:

$$\boxed{\left(\tilde{\mathbf{F}} s\right)^2 = \tilde{\mathscr{F}} s^2.} \tag{3.57}$$

Inserting (3.57) in (3.54) results in a sequence of radial functions which can easily be integrated.

Let us now inspect the consequences on eq. (3.52) that are caused by the entirety of our recent truncations/restrictions. First of all, the RHS can be solved in three steps by splitting the integral according to eq. (3.54). For this it is useful to consider a general identity that simplifies radial integrals by performing their angular integration first. For any locally integrable function $f : \mathbb{R}^d \to U \subset \mathbb{R}$ we have:

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^d} f\left(s^2\right) \frac{\mathrm{d}^d s}{(2\pi)^d} &= v_d \int_0^\infty r^{\frac{d}{2}-1} f(r) \mathrm{d}r, \\ v_d &:= \frac{1}{2\left(4\pi\right)^{d/2} \Gamma\left(\frac{d}{2}\right)}. \end{aligned} \tag{3.58}$$

[6] For a proof of this statement, see app. D, cf. Proposition D.7.

A proof of this relation follows by a direct calculation that works for instance with d-dimensional spherical coordinates.

Using (3.58) under consideration of eq. (3.54) together with the identity (3.57) cuts the fixed point equation (3.52) down to:

$$\begin{aligned} dw_* - (\eta_* + d)\, w'_* \tilde{\mathscr{F}} = (1-d)\, v_d \int_0^\infty r^{\frac{d}{2}-1} \frac{\eta_* R(r) + 2rR'(r)}{w'_* + R(r)} \mathrm{d}r \\ - v_d \int_0^\infty r^{\frac{d}{2}-1} \frac{\eta_* R(r) + 2rR'(r)}{1 + R(r)} \mathrm{d}r \\ + v_d w''_* \tilde{\mathscr{F}} \int_0^\infty r^{\frac{d}{2}-1} \frac{\eta_* R(r) + 2rR'(r)}{\left(w'_* + R(r)\right)\left[w'_* + w''_* \tilde{\mathscr{F}} + R(r)\right]} \mathrm{d}r. \end{aligned} \tag{3.59}$$

The remaining radial integrals can be referred to as so-called *threshold functions* which commonly appear in the realm of renormalisation flow computations. They show up as a consequence of the supertrace operation that needs to be performed in order to find explicit representations of the Wetterich equation and parametrise the 1-loop integrals. Since the operators of which the supertrace needs to be taken are usually a combination of some differential operators, it can be shown that the result is expandable in an appropriate set of functions [28]. These functions refer precisely to the above mentioned threshold functions.

As it becomes visible in (3.59), the integrands share the same numerator which contains the shape function R and its derivative R'. Recalling our discussion from sec. 2.1, the shape function is constructed such that it is non-vanishing only up to values $r \simeq 1$, whereas R' is peaked around that point of transition. Consequently, the integrals in (3.59) receive their contributions primarily from the unit interval, just before the built-in suppression dominates for larger values $r > 1$ (hence the name "threshold functions").

Let us now fix a specific regularisation scheme. For reasons of convenience we choose an optimal cutoff according to eq. (2.18). The shape function is then given by:

$$\begin{aligned} R(r) \equiv R^{\mathrm{op}}(r) = \frac{1-r}{r} \mathbf{1}_{[0,1)}(r), \\ R'(r) \equiv (R^{\mathrm{op}})'(r) = -\frac{1}{r^2} \mathbf{1}_{[0,1)}(r). \end{aligned} \tag{3.60}$$

At this point, it is also convenient to introduce a pair of new abbreviations that improve readability of our formulas: $\boldsymbol{\omega}^{(1)} := 1 - w'_*$ and $\boldsymbol{\omega}^{(2)} := 1 - w'_* - w''_* \tilde{\mathscr{F}}$, so we have $\boldsymbol{\omega}^{(2)} - \boldsymbol{\omega}^{(1)} = -w''_* \tilde{\mathscr{F}}$. Insertion of (3.60) into (3.59) yields the following sequence of solutions:

(1). First the *transversal threshold function*:

$$\begin{aligned} \int_0^\infty r^{\frac{d}{2}-1} \frac{\eta_* R(r) + 2rR'(r)}{w'_* + R(r)} \mathrm{d}r = \int_0^1 r^{\frac{d}{2}-1} \frac{\eta_* (1-r) - 2}{1 - \boldsymbol{\omega}^{(1)} r} \mathrm{d}r \\ = \frac{4\eta_*}{d(d+2)} {}_2F_1\left(1, \frac{d}{2}; \frac{d}{2}+2; \boldsymbol{\omega}^{(1)}\right) - \frac{4}{d} {}_2F_1\left(1, \frac{d}{2}; \frac{d}{2}+1; \boldsymbol{\omega}^{(1)}\right). \end{aligned} \tag{3.61}$$

(2). Second the *longitudinal threshold function*:

$$\int_0^\infty r^{\frac{d}{2}-1}\frac{\eta_* R(r) + 2rR'(r)}{1+R(r)}\mathrm{d}r = \int_0^1 r^{\frac{d}{2}-1}\Big(\eta_* - 2 - \eta_* r\Big)\mathrm{d}r = -\frac{4}{d}\left(1 - \frac{\eta_*}{d+2}\right). \tag{3.62}$$

(3). Finally, the *sub-transversal threshold function*, where it is helpful to use a split of the form $\frac{1}{(1-\boldsymbol{\omega}^{(1)}r)(1-\boldsymbol{\omega}^{(2)}r)} = \frac{1}{\boldsymbol{\omega}^{(2)}-\boldsymbol{\omega}^{(1)}}\left(\frac{\boldsymbol{\omega}^{(2)}}{1-\boldsymbol{\omega}^{(2)}r} - \frac{\boldsymbol{\omega}^{(1)}}{1-\boldsymbol{\omega}^{(1)}r}\right)$ before performing the integral directly:

$$\begin{aligned}
&\int_0^\infty r^{\frac{d}{2}-1}\frac{\eta_* R(r) + 2rR'(r)}{\Big(w_*' + R(r)\Big)\left[w_*' + w_*''\tilde{\mathscr{F}} + R(r)\right]}\mathrm{d}r = \int_0^1 r^{\frac{d}{2}}\frac{\eta_*\,(1-r) - 2}{\Big(1-\boldsymbol{\omega}^{(1)}r\Big)\Big(1-\boldsymbol{\omega}^{(2)}r\Big)}\mathrm{d}r \\
&= -\frac{1}{w_*''\tilde{\mathscr{F}}}\frac{4}{d+2}\left[\boldsymbol{\omega}^{(1)}{}_2F_1\left(1, \frac{d}{2}+1; \frac{d}{2}+2; \boldsymbol{\omega}^{(1)}\right) - \boldsymbol{\omega}^{(2)}{}_2F_1\left(1, \frac{d}{2}+1; \frac{d}{2}+2; \boldsymbol{\omega}^{(2)}\right)\right] \\
&\quad + \frac{1}{w_*''\tilde{\mathscr{F}}}\frac{4\eta_*}{(d+2)(d+4)}\left[\boldsymbol{\omega}^{(1)}{}_2F_1\left(1, \frac{d}{2}+1; \frac{d}{2}+3; \boldsymbol{\omega}^{(1)}\right)\right. \\
&\qquad\qquad \left. - \boldsymbol{\omega}^{(2)}{}_2F_1\left(1, \frac{d}{2}+1; \frac{d}{2}+3; \boldsymbol{\omega}^{(2)}\right)\right].
\end{aligned} \tag{3.63}$$

The results (3.61)-(3.63) are written in terms of the *hypergeometric function* ${}_2F_1$, where we have used its Euler integral representation to rewrite the integrals. A collection of properties concerning this special function and also further representations of it can be found in app. C. It should be made transparent, that using the Euler integral formula for ${}_2F_1$ is accompanied with the extra assumption that both $\boldsymbol{\omega}^{(1)}$ and $\boldsymbol{\omega}^{(2)}$ are strictly smaller than one, such that we dont leave the *principal branch* of ${}_2F_1$. This implies $w_*' > 0$ and $w_*''\tilde{\mathscr{F}} > 0$ as two non-trivial impositions on w_* that we can consult later to evaluate the validity of SFE and LFE results.

Now that we have finished our preparations we can try to construct a globally-existing fixed point potential by combining small-field and large-field analytical methods that were described in sec. 2.3. For this, we adhere to the three-step procedure which we have formulated in the same section and whose individual steps were numbered as **(S1)**-**(S3)**. Before we begin, it just becomes necessary to now decide on a number of spacetime dimensions, such that concrete results can be extracted. Since so far we considered $d = 4 - \epsilon$, let us now take the limit $\epsilon \to 0$, i.e. $d \to 4$, and with this focus on the natural case. From eq. (3.59) and using (3.61)-(3.63) we derive the equation of interest for w_*:

$$\boxed{\begin{aligned}
w_* = &\frac{1}{128\pi^2}\left(1 - \frac{\eta_*}{6}\right) + \left(1 + \frac{\eta_*}{4}\right)w_*'\tilde{\mathscr{F}} \\
&+ \frac{3}{128\pi^2}{}_2F_1\left(1, 2; 3; \boldsymbol{\omega}^{(1)}\right) - \frac{\eta_*}{256\pi^2}{}_2F_1\left(1, 2; 4; \boldsymbol{\omega}^{(1)}\right) \\
&+ \frac{\boldsymbol{\omega}^{(1)}\eta_*}{1\,536\pi^2}{}_2F_1\left(1, 3; 5; \boldsymbol{\omega}^{(1)}\right) - \frac{\boldsymbol{\omega}^{(1)}}{192\pi^2}{}_2F_1\left(1, 3; 4; \boldsymbol{\omega}^{(1)}\right) \\
&+ \frac{\boldsymbol{\omega}^{(2)}}{192\pi^2}{}_2F_1\left(1, 3; 4; \boldsymbol{\omega}^{(2)}\right) - \frac{\boldsymbol{\omega}^{(2)}\eta_*}{1\,536\pi^2}{}_2F_1\left(1, 3; 5; \boldsymbol{\omega}^{(2)}\right).
\end{aligned}} \tag{3.64}$$

B. Fixed Point Potential: Small-Field Expansion

Basically without specifying any explicit form for w_*, we can derive a power expansion formulation of eq. (3.64) by expressing the hypergeometric function in terms of its *Gauss series representation* (cf. app. C):

$$w_* = \frac{1}{32\pi^2}\left(1-\frac{\eta_*}{6}\right)+\left(1+\frac{\eta_*}{4}\right)w_*'\tilde{\mathscr{F}}+\frac{1}{64\pi^2}\sum_{n=1}^{\infty}\frac{6+2n-\eta_*}{(n+2)(n+3)}\left[\left(\boldsymbol{\omega}^{(1)}\right)^n+\frac{1}{2}\left(\boldsymbol{\omega}^{(2)}\right)^n\right]. \quad (3.65)$$

Focusing on small field amplitudes, i.e. small $\tilde{\mathscr{F}}$ values, a power series expansion of w_* is justly feasible:

$$w_*^{\mathrm{SFE}}\left(\tilde{\mathscr{F}}\right)=\sum_{J=0}^{\infty}\frac{\sigma_J}{J!}\tilde{\mathscr{F}}^J=\sigma_0+\tilde{\mathscr{F}}+\frac{1}{2}\sigma_2^2\tilde{\mathscr{F}}^2+\ldots \quad (3.66)$$

It is important to note that $\sigma_1 = 1$. That is because of the field normalisation established in eq. (3.41). Alternatively, we could have also omitted this renormalisation procedure and instead just defined a dimensionless quantity, e.g. $\hat{\tilde{\mathscr{F}}} := k^{-d}\mathscr{F}$, but then the field strength renormalisation Z_k would still appear explicitly in our equations and in this case we would have had identified the kinematic coefficient σ_1 with Z_k. In contrast, introducing a field normalisation as in eq. (3.41) we do not lose any information, but rather re-adjust the flow effects which originate from Z_k to a non-trivial coupling of its relative RG time change through the anomalous dimension η_k.

We can now give concrete formulas for $\boldsymbol{\omega}^{(1)}$ and $\boldsymbol{\omega}^{(2)}$ using eq. (3.66):

$$\boldsymbol{\omega}^{(1)}\left(\tilde{\mathscr{F}}\right)=-\sum_{J=1}^{\infty}\frac{\sigma_{J+1}}{J!}\tilde{\mathscr{F}}^J \qquad \& \qquad \boldsymbol{\omega}^{(2)}\left(\tilde{\mathscr{F}}\right)=-\sum_{J=1}^{\infty}\frac{\sigma_{J+1}}{J!}(J+1)\tilde{\mathscr{F}}^J. \quad (3.67)$$

Let us adapt the strategy from the Wilson-Fisher example in sec. 2.3 and assign $\sigma_2 \to \mu$ as a free parameter, such that all other coefficients σ_J with $J \neq 2$ can be expressed in terms of μ. An order by order scan of the system (3.65) reveals a sequence of prefactors similar to (2.44). Here, we present its first few coefficients (skipping over the trivial cases $\sigma_1 = 1$ and $\sigma_2 \equiv \mu$):

$$\begin{aligned}
\sigma_0(\mu) &= \frac{1}{32\pi^2}\left(1-\frac{4}{3}\frac{\mu}{\mu+96\pi^2}\right),\\
\sigma_3(\mu) &= \frac{\mu}{800\pi^2}\left(30\,720\pi^4+3\,040\pi^2\mu+3\mu^2\right),\\
\sigma_4(\mu) &= \frac{\mu}{5}\left(12\,288\pi^4+2\,880\pi^2\mu+\frac{2\,074}{15}\mu^2+\frac{71}{240\pi^2}\mu^3+\frac{3}{12\,800\pi^4}\mu^4\right),\\
&\;\;\vdots
\end{aligned} \quad (3.68)$$

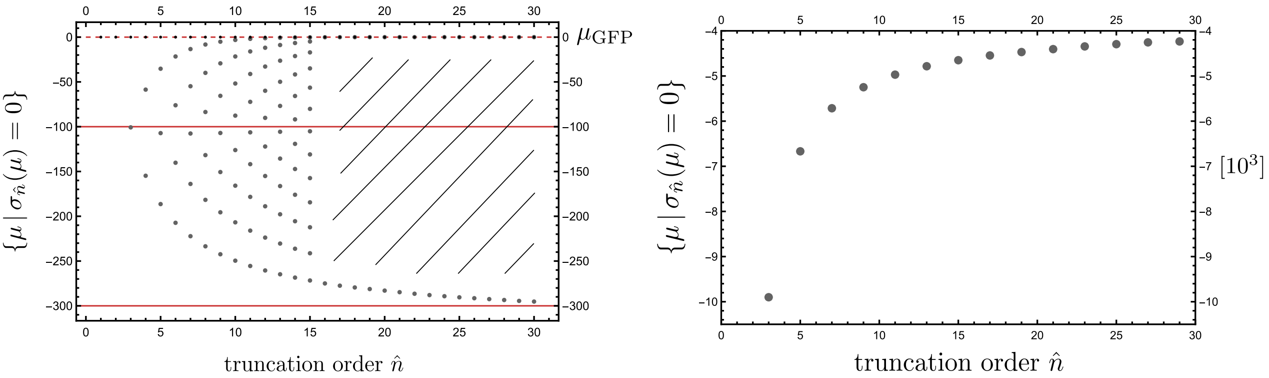
In addition, the anomalous dimension satisfies:

$$\eta_*(\mu)=\frac{8\mu}{96\pi^2+\mu}. \quad (3.69)$$

There are basically two options open to us for further investigation. On the one hand, we can adopt the "Wilson-Fisher perspective" according to sec. 2.3 and prescribe a truncation order $\hat{n}$ to the system (3.68) which renders it uniquely solvable. The extra input $\sigma_{\hat{n}}(\mu) = 0$ yields a set of zeros $\{\mu \,|\, \sigma_{\hat{n}}(\mu) = 0\}$ whose evolution with respect to $\hat{n}$ needs to be sampled for a stabilising behaviour as one tends to larger truncation orders. In this sense we follow the SFE procedure from the Ising model example in sec. 2.3. Once a value for μ has been found, the anomalous dimension is determined by eq. (3.69).

Alternatively, we can also specify the anomalous dimension η_* initially, in order to fix the free parameter μ by means of the inverse relation of eq. (3.69). All further coefficients σ_J follow from the system (3.68). However, we will first concentrate on the Wilson-Fisher perspective as it is more familiar to us.

A visualisation of the set $\{\mu \,|\, \sigma_{\hat{n}}(\mu) = 0\} \equiv M_{\hat{n}}$ for different truncation orders $\hat{n}$ is given in fig. 3.1.



(a) **(b)**

Figure 3.1: *Plots that illustrate the evolution of $M_{\hat{n}}$ with respect to the truncation order $\hat{n}$, depending on the system (3.68). Part **(a)** shows a fork-shaped structure that consists of several bifurcations, initiating around $\mu \simeq -100$. The shaded region contains further solutions of $\sigma_{\hat{n}}(\mu) = 0$ for $\hat{n} > 15$, but because of computational limitations it is omitted, except for the outermost branches which are displayed until truncation order $\hat{n} = 30$. Besides the Gaussian fixed point at $\mu_{\mathrm{GFP}} = 0$, indicated by the red dashed line, we can identify another stabilising behaviour, approximately approaching $\mu \simeq -300$ which could indicate a non-Gaussian fixed point. In addition, part **(b)** portrays another, separated branch of zeros which only continues at odd truncation orders.*

As it can be seen from part **(a)** of fig. 3.1, the cardinality of $M_{\hat{n}}$ increases as we go to larger truncation orders, which is lucid at least up to order $\hat{n} = 15$. Structurally, the formation of two-pronged forks initiates for every odd truncation order beginning from $\hat{n} = 3$, each with an upper and lower branch. New forks bifurcate approximately at the red solid middle line , i.e. for $\mu \simeq -100$. For computational capacity reasons, the evolution from $\hat{n} = 15$ onwards is only traced for the branches of the outermost fork up to and including $\hat{n} = 30$. This has been done by means of a bisection procedure, whereby the shaded area certainly contains further zeros extending the inner forks. Unfortunately, it is difficult to judge from our numerical perspective whether the branches of the outermost fork actually converge. For a better illustration, a logarithmic representation of the upper and lower branches is therefore presented in fig. 3.2. Here, it becomes visible that the upper branch arranges around a linear, falling behaviour. Supposing that this behaviour is maintained as $\hat{n}$ increases, this branch converges towards the Gaussian fixed point at $\mu_{\mathrm{GFP}} = 0$. The lower branch, on the other hand, is much more difficult to assess using the information from fig. 3.2. For this reason, the absolute ratios of successive branch points are plotted in fig. 3.3 (lower course). We can infer that the ratio $|\frac{\mu_n}{\mathrm{succ}(\mu_n)}|$ presumably approaches the constant value 1, such that convergence of the lower branch becomes likely. Hence, the limit value for the lower branch - if it really exists - can graphically be estimated and is marked as a red solid line at $\mu \simeq -300$ in part **(a)** of fig. 3.3.

When comparing successive forks, it becomes noticeable that the successor branches approach on their predecessor branches for increasing truncation order on both sides of the bifurcation line at $\mu \simeq -100$. In order to formalise this statement, let $f_{\hat{n}}$ denote the fork that bifurcates at order $\hat{n}$. Thus, the outermost fork is f_3 and its successor f_5, or more generally: $\mathrm{succ}\,(f_{\hat{n}}) = f_{\hat{n}+2}$. Let us further identify the upper and lower branch of $f_{\hat{n}}$ with $f^{\uparrow}_{\hat{n}}$ and $f^{\downarrow}_{\hat{n}}$ respectively. Note that $f_{\hat{n}}$ describes a set of certain solutions to $\sigma_n(\mu) = 0$, where $\mathbb{N} \ni n \geq \hat{n}$.

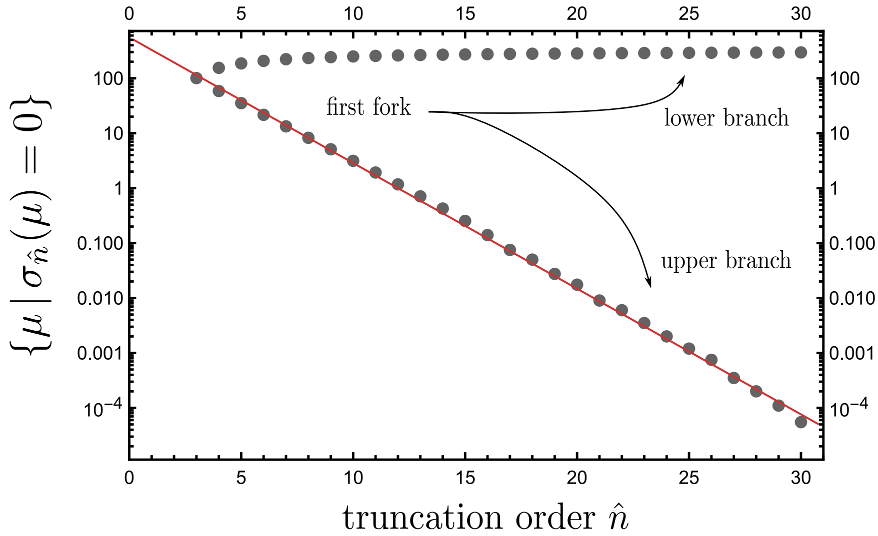


Figure 3.2: *Logarithmic plot of $M_{\hat{n}}$ that includes only the outermost fork according to part* ***(a)*** *of fig. 3.1. The upper branch indicates a linearly decreasing behaviour that ultimately converges onto the Gaussian fixed point, provided the falling course is maintained for higher truncation orders. Variations of the lower branch are hard to resolve within this illustration, instead cf. fig. 3.3.*

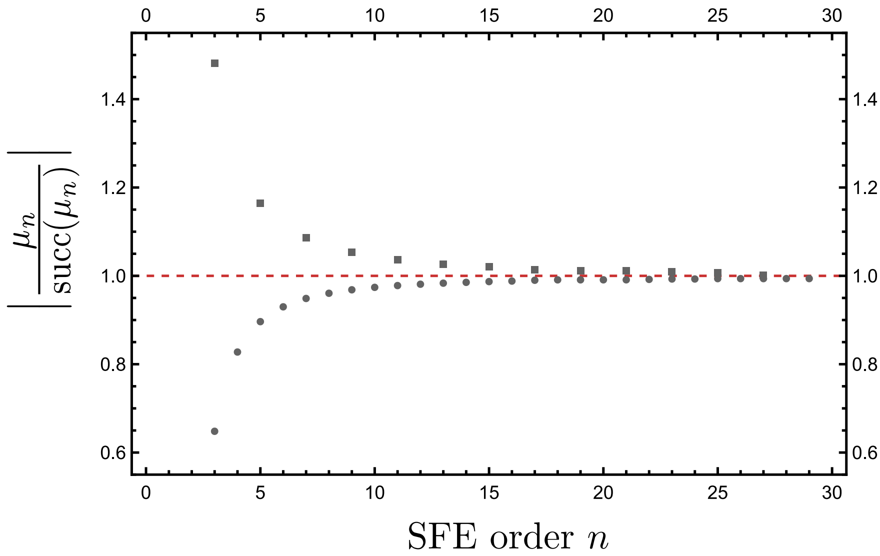


Figure 3.3: *Ratios of consecutive solutions μ_n, i.e. $\sigma_n(\mu_n) = 0$, for different SFE orders n. Dotted points correspond to the lower branch of the outermost fork in part* ***(a)*** *of fig. 3.1, whereas boxes correspond to the separated branch which is displayed in part* ***(b)*** *of the same figure. Both courses apparently approach the value 1 as one tends to increasing SFE orders.*

Therefore it is meaningful to consider $f^{\uparrow}_{\hat{n}} \ni f^{\uparrow}_{\hat{n}}(n) := \mu^{\uparrow}_n[f_{\hat{n}}]$ for which $\sigma_n(\mu^{\uparrow}_n[f_{\hat{n}}]) = 0$, and analogously for $f^{\downarrow}_{\hat{n}}(n)$. In other words, $\mu^{\uparrow}_n[f_{\hat{n}}]$ denotes the point that one arrives at, when following the upper branch of $f_{\hat{n}}$ until order n, and the same holds for $\mu^{\downarrow}_n[f_{\hat{n}}]$.

The last preparation that we need concerns *restrictions*. The set $f_{\hat{n}}|_N$ with $N > \hat{n}$ is defined as the proper subset of $f_{\hat{n}}$ that contains all points of the fork that bifurcates at order $\hat{n}$, except for those at orders $\hat{n} \leq n < N$. Analogous definitions also hold for the branches $f^{\uparrow}_{\hat{n}}$ and $f^{\downarrow}_{\hat{n}}$.

Finally, we can formulate the following *SFE-conjecture*:

Conjecture (SFE). In the situation of fig. 3.1 part **(a)**, it is true that:

$$\forall \hat{n} \in \mathbb{N} \setminus 2\mathbb{N},\ \hat{n} \geq 3: \ \operatorname{succ}(f_{\hat{n}})|_N = f_{\hat{n}}|_N \quad \text{for } N \to \infty.$$

In this case we write $\operatorname{succ}(f_{\hat{n}}) \to f_{\hat{n}}$.

If we suppose the validity of this conjecture, then from $f_7 \to f_5$ and $f_5 \to f_3$ it follows $f_7 \to f_3$ by means of transitivity. More generally we can say, that $f_{\hat{n}} \to f_3$ for all $\hat{n} \in \mathbb{N} \setminus 2\mathbb{N}$ with $\hat{n} > 3$. Consequently, the fork structure in part **(a)** of fig. 3.1 admits only two stabilising solutions at all, namely $\mu = \mu_{\text{GFP}} = 0$ and $\mu \simeq -300$. However, this assumption cannot be conclusively substantiated within the framework of this thesis due to the limited computational and temporal capacities. A final and unambiguous clarification is in any case only possible by analytical means, which, however, is expected to require a great amount of effort due to the considerable complexity of the problem. Numerically, on the other hand, one can only collect further indications in favor or against the above mentioned conjecture by continuing the fork structure from part **(a)** of fig. 3.1 to truncation orders beyond $\hat{n} = 30$.

There is yet another, separated branch which is presented in part **(b)** of fig. 3.1. The peculiarity here is that there is a solution only for every odd truncation order starting at $\hat{n} = 3$. Nevertheless, we include also this branch in our considerations and first analyse its convergence behaviour. Fig. 3.3 again serves for this purpose - now focusing on the upper course - in which the ratios of successive branch points are shown. Once again, a tendency for $|\frac{\mu_n}{\operatorname{succ}(\mu_n)}| \to 1$ apparently occurs, which makes convergence of this branch seem possible. The limit value can be determined graphically and is estimated at $\mu \simeq -4200$.

In order to summarise our findings, we note that, first of all, the Wilson-Fisher perspective reveals a trivial fixed point solution, that is for $\mu = \mu_{\text{GFP}} = 0$ which implies $\eta_* = 0$ and the only non-vanishing coefficient, besides $\sigma_1 = 1$, is $\sigma_0 = \frac{1}{32\pi^2}$. The dimensionless Gaussian fixed point potential is therefore given by: $w_*^{\text{GFP}} = \frac{1}{32\pi^2} + \tilde{\mathscr{F}}$. In addition, we seemingly have two non-trivial fixed point solutions available; one at $\mu \simeq -300 \equiv \mu_*^{(1)}$, and another at $\mu \simeq -4200 \equiv \mu_*^{(2)}$. The corresponding anomalous dimensions follow according to eq. (3.69):

$$\eta_* \left(\mu_*^{(1)}\right) = -3.7067 \qquad \& \qquad \eta_* \left(\mu_*^{(2)}\right) = 10.3305. \tag{3.70}$$

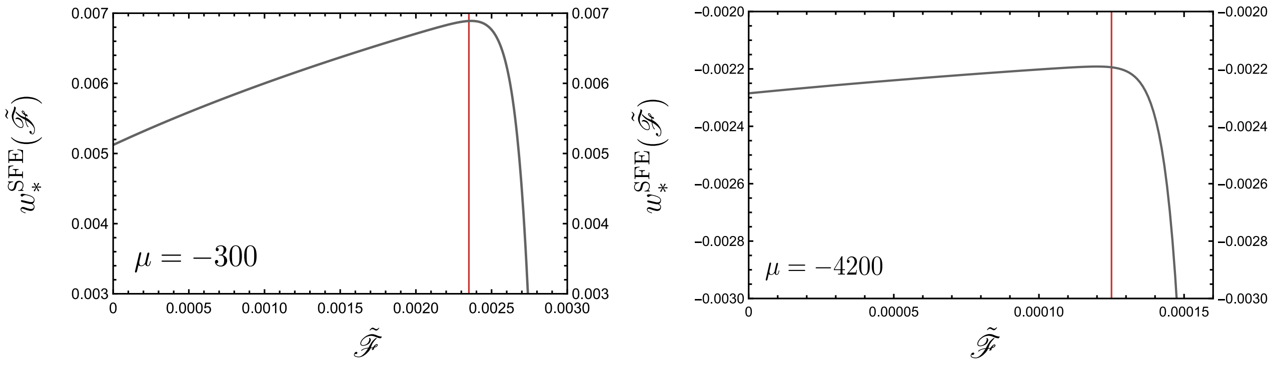
The fact that η_* becomes negative at $\mu = \mu_*^{(1)}$ is basically without any inconsistency, since by definition, eq. (3.36), this just implies that $\partial_t |Z_k| > 0$ at the fixed point. However, at $\mu = \mu_*^{(2)}$, the anomalous dimension η_* gets exceedingly large and should therefore be treated with care.

If we insert $\mu = \mu_*^{(1)}$ and $\mu = \mu_*^{(2)}$ in (3.68) and (3.69), we can graphically illustrate and contrast the fixed point potentials w_*^{SFE} for small field amplitudes according to eq. (3.66). This is done in fig. 3.4, from which the radii of convergence, $r_{\text{SFE}}(\mu)$, can be read off to:

$$r_{\text{SFE}} \left(\mu_*^{(1)}\right) \simeq 0.00235 \qquad \& \qquad r_{\text{SFE}} \left(\mu_*^{(2)}\right) \simeq 0.000125. \tag{3.71}$$

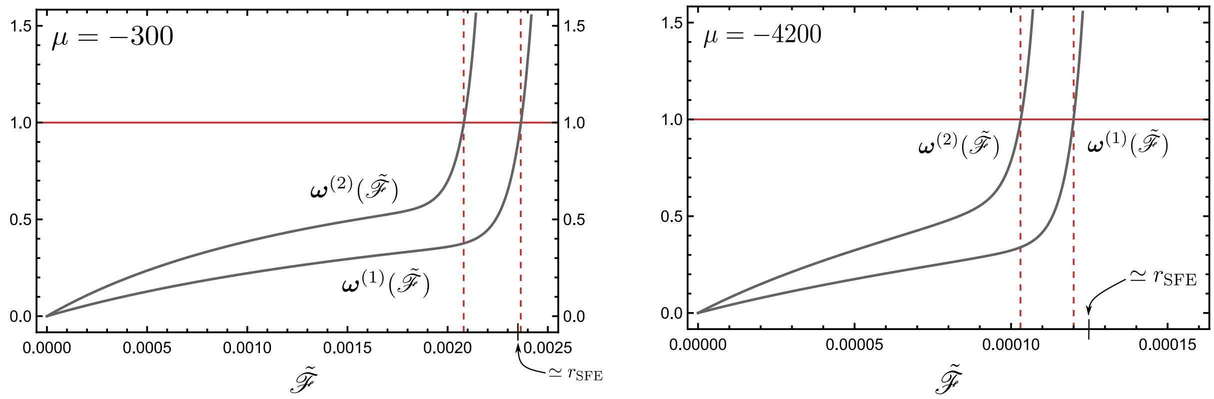
Let us now recall the conditions $\boldsymbol{\omega}^{(1)} < 1$ & $\boldsymbol{\omega}^{(2)} < 1$, ensure the applicability of Euler's integral representation of the hypergeometric function within its principal branch. In order to verify that both conditions hold within the radii of convergence given in (3.71), let us take a look at fig. 3.5. As it becomes visible, the situation $\mu = \mu_*^{(1)}$ with $r_{\text{SFE}}(\mu_*^{(1)}) \simeq 0.00235$ is compatible with $\boldsymbol{\omega}^{(1)} < 1$, but not with $\boldsymbol{\omega}^{(2)} < 1$, since $\boldsymbol{\omega}^{(2)} \left(r_{\text{SFE}}(\mu_*^{(1)})\right) > 1$. In contrast, the case $\mu = \mu_*^{(2)}$ with $r_{\text{SFE}}(\mu_*^{(2)}) \simeq 0.000125$ even does neither account for one nor the other condition. Thus, we need to restrict both solutions further on their *radii of validity*, $r_{\text{SFE}}^{\text{val}}$, which are given by:

$$r_{\text{SFE}}^{\text{val}} \left(\mu_*^{(1)}\right) \simeq 0.00209 \qquad \& \qquad r_{\text{SFE}}^{\text{val}} \left(\mu_*^{(2)}\right) \simeq 0.000103. \tag{3.72}$$



(a) **(b)**

Figure 3.4: *Graphs of the fixed point potential w_*^{SFE}, according to eq. (3.66) for $\mu = \mu_*^{(1)}$ (part **(a)**) and $\mu = \mu_*^{(2)}$ (part **(b)**). Both plots show w_*^{SFE} as a function of $\tilde{\mathscr{F}}$ in the small-field amplitude regime. The estimated radii of convergence are indicated by red solid vertical lines.*



(a) **(b)**

Figure 3.5: *Plots of $\boldsymbol{\omega}^{(1)}$ and $\boldsymbol{\omega}^{(2)}$ as functions of $\tilde{\mathscr{F}}$ according to (3.67) for $\mu = \mu_*^{(1)}$ (part **(a)**) and $\mu = \mu_*^{(2)}$ (part **(b)**). The conditions $\boldsymbol{\omega}^{(1)} < 1$ & $\boldsymbol{\omega}^{(2)} < 1$ determine the radius of validity of the SFE. Its boundaries are marked by vertical dashed red lines. Within the radii of convergence, $r_{\mathrm{SFE}}(\mu_*^{(1)})$ & $r_{\mathrm{SFE}}(\mu_*^{(2)})$, at least one condition is violated in some region. Thus, the radius of validity supplants the radius of convergence as the relevant parameter that evaluates the domain of applicability of the SFE.*

With these results, let us stop at this point and switch from the Wilson-Fisher perspective to an alternative point of view that makes use of the inverse relation of eq. (3.69); instead of truncating the system (3.68) and subsequently solving it for the parameter μ, we can initially specify a value for η_* and compute μ from eq. (3.69), i.e.:

$$\mu\left(\eta_*\right) = 96\pi^2 \frac{\eta_*}{8 - \eta_*}. \tag{3.73}$$

In principle, any value $\eta_* \neq 8$ can be inserted into eq. (3.73), but one might ask wether this approach is even meaningful and worth a deeper illumination at this point of the analysis, since we actually already found concrete results from the Wilson-Fisher perspective. Because the ansatz (3.73) is conceptually not different from the Wilson-Fisher perspective, we do not expect to find new results or stabilisation patterns, which means that the answer to the question above is basically negative, *unless* there exists a preferred choice for η_*.

In fact, the anomalous dimension of the pure photon field, η_{ph}, can serve as such a choice. It has been derived to first order in perturbation theory in ref. [45]:

$$\eta_{\text{ph}} = \frac{2\alpha}{3\pi} \simeq 0.00155, \tag{3.74}$$

in which $\alpha \simeq \frac{1}{137}$ denotes the *fine-structure constant*. It should be noted that in ref. [45], one assumes the QED Lagrangian (3.1) with massless leptons, i.e. $m \to 0$. Therefore we should be aware of the fact, that η_{ph}, as it appears in the form (3.74), includes only effects which arise from the kinetic term $\mathscr{F}$ that contributes solely to the photon Lagrangian. Nevertheless, let us consider $\eta_* \equiv \pm\eta_{\text{ph}}$ and evaluate its consequences on the fixed point potential. Firstly, from eq. (3.73) it follows:

$$\mu_\pm \equiv \mu\left(\pm\eta_{\text{ph}}\right) = 96\pi^2 \frac{(\pm\eta_{\text{ph}})}{8 \mp \eta_{\text{ph}}} \simeq \begin{cases} 0.1836 & + \\ -0.1835 & - \end{cases}. \tag{3.75}$$

The fixed point potential w_*^{SFE} for both options $\mu_\pm$ is depicted in fig. 3.6 for various truncation orders.

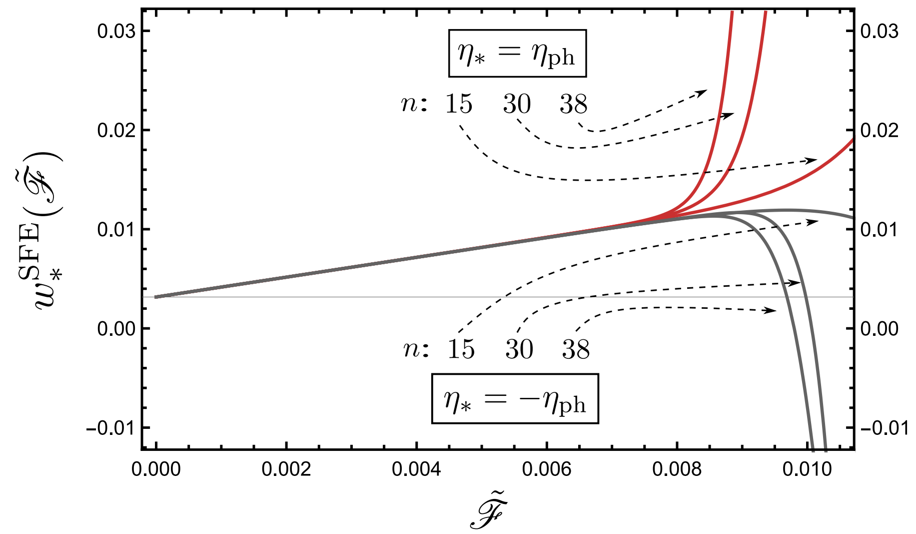


Figure 3.6: *Plots of the fixed point potential w_*^{SFE} for small field amplitudes $\tilde{\mathscr{F}}$ and for different SFE truncation orders n under initial specification of the anomalous dimension η_*. Red solid lines indicate graphs of w_*^{SFE} for $\eta_* = +\eta_{\text{ph}}$, whereas black solid lines correspond to $\eta_* = -\eta_{\text{ph}}$. As one chooses larger SFE truncation orders n, the transition from the domain of convergence to the ill-behaving region becomes progressively sharper in both cases.*

The radii of convergence can approximately and collectively be found in the interval $[0.0075, 0.01]$. The radii of validity, corresponding to the conditions $\boldsymbol{\omega}^{(1)} < 1$ & $\boldsymbol{\omega}^{(2)} < 1$, can be inferred from fig. 3.7. Here we can see that these conditions are not violated even beyond $\tilde{\mathscr{F}} = 0.01$, such that the radii of convergence do not need to be restricted as it was the case in the Wilson-Fisher perspective. A more precise determination of $r_{\text{SFE}}(\mu_\pm)$ follows from the ratio test, cf. eq. (2.41). For instance, the first few elements of the sequence $(n+1)\left|\frac{\sigma_n}{\sigma_{n+1}}\right|$ with $n \in \mathbb{N}$, assuming $\mu = \mu_+$, are presented up to order $n = 39$ in fig. 3.8. We find $r_{\text{SFE}}(\mu_+) \simeq 0.00718$ for $\eta_* = \eta_{\text{ph}}$, provided the course in fig. 3.8 stabilises for higher truncation orders beyond $n = 39$. Comparing this result with (3.71) shows that $r_{\text{SFE}}(\mu_+) > r_{\text{SFE}}(\mu_*^{(1)}) > r_{\text{SFE}}(\mu_*^{(2)})$, i.e. a (considerably) larger domain of convergence than that which the Wilson-Fisher perspective was able to reveal.

However, these findings could actually be anticipated, which can be seen as follows; the anomalous dimension η_* vanishes in the limit $\mu \to 0$, as it is clear from eq. (3.69).

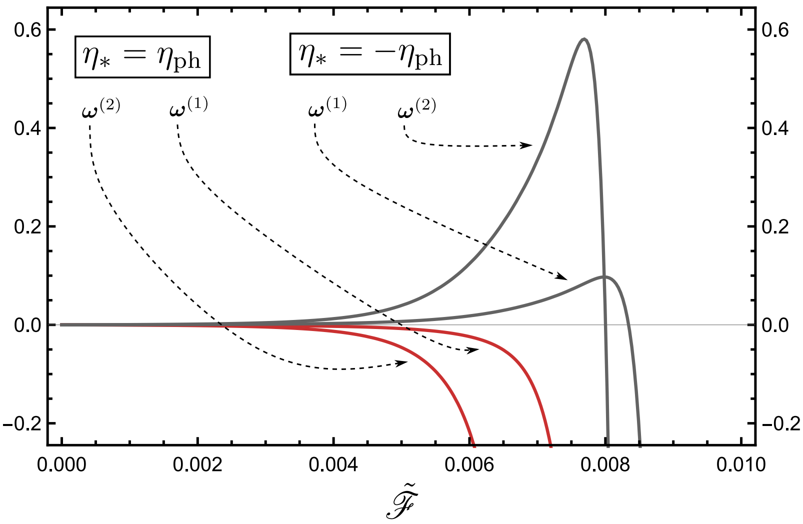


Figure 3.7: *Plots of the quantities $\boldsymbol{\omega}^{(1)}$ and $\boldsymbol{\omega}^{(2)}$ for $\eta_* = \pm\eta_{\text{ph}}$ according to (3.67). Red solid lines correspond to $\eta_* = +\eta_{\text{ph}}$ and black solid lines to $\eta_* = -\eta_{\text{ph}}$. The conditions $\boldsymbol{\omega}^{(1)} < 1$ & $\boldsymbol{\omega}^{(2)} < 1$ hold throughout the plotted range.*

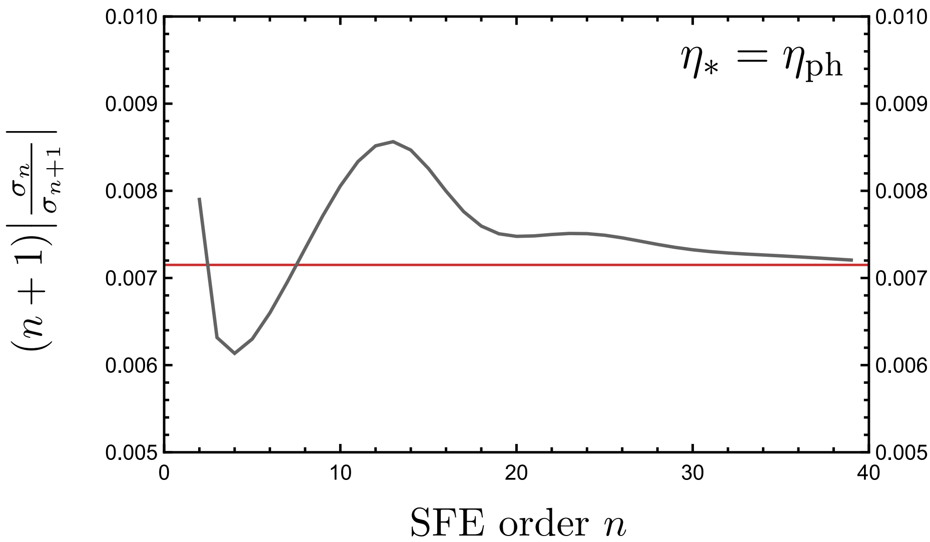


Figure 3.8: *Continuous course of the ratio test sequence that serves as an estimation aid for the radius of convergence r_{SFE} of w_*^{SFE}, assuming $\eta_* = +\eta_{\text{ph}}$. Sequence elements are shown up to order $n = 39$. The red solid line estimates the suspected limit value which corresponds to r_{SFE}.*

Hence, as $\mu \to 0$ we simultaneously approach the Gaussian fixed point, where the solution of the fixed point equation becomes exact, namely $w_*^{\text{GFP}} = \frac{1}{32\pi^2} + \tilde{\mathscr{F}}$, supposing an optimised regularisation $R = R^{\text{op}}$. In this sense, we basically expect an increasing radius of convergence as we tend closer to the GFP, which happens precisely for decreasing values of μ. We conclude, that the observed increase of r_{SFE} is more a consequence of choosing smaller values for μ rather than a “natural” choice for the anomalous dimension.

With this, let us finish our studies of the small-field regime and turn to large field amplitudes now.

C. Fixed Point Potential: Large-Field Expansion

The large-field sector is describable by an ansatz of the form (2.39), which adapted to our model reads:

$$w_*^{\mathrm{LFE}}\left(\tilde{\mathscr{F}}\right) = \lambda \tilde{\mathscr{F}}^N + \sum_{n=0}^{\infty} \lambda_n \tilde{\mathscr{F}}^{-n} = \lambda \tilde{\mathscr{F}}^N + \lambda_0 + \frac{\lambda_1}{\tilde{\mathscr{F}}} + \frac{\lambda_2}{\tilde{\mathscr{F}}^2} + \dots . \tag{3.76}$$

Therefore, the LFE exhibits a set of parameters $\{\lambda_n\}_{n\in\mathbb{N}_0} \cup \{\lambda\}$ yet to be determined, for which the fixed point equation can be applied. Similar to the Ising model example from sec. 2.3, the "*asymptotic parameter*" λ could turn out to be a free parameter of the LFE, which can be fixed by including an additional condition that demands an optimised transition between the SFE and LFE domains, cf. eq. (2.42). Furthermore, in the limit of arbitrarily large field amplitudes, the LFE ansatz is either dominated by the zeroth order parameter λ_0, or the leading term $\lambda\tilde{\mathscr{F}}^N$. According to eq. (2.38), the former happens to be the case for $N < 0$, which is equivalent to $\eta_* < 2 - d$, where d denotes the number of spacetime dimensions. Since the anomalous dimension, evaluated at a (viable) fixed point, usually turns out to be a small and positive number, this situation is rather the exception for $d \geq 2$. Commonly, the inequality $\eta_* > 2-d$, or equivalently $N > 0$, holds instead for $d \geq 2$. Then, the LFE fixed point potential follows a power law with N being the corresponding "*asymptotic exponent*".

For reasons of convenience it is useful to introduce a new variable $\chi := \tilde{\mathscr{F}}^{-1}$ as we also did for the LFE concerning the Ising model in sec. 2.3. In this way, we effectively switch from the large-field to a small-field sector since $\chi \to 0$ as $\tilde{\mathscr{F}}$ accepts large values in the realm of the LFE.

During our previous investigations, we have often encountered the first and second derivative of w_* with respect to its argument, i.e. $\tilde{\mathscr{F}}$. These derivatives now need to be expressed in terms of the variable χ, for which we use the chain rule. For the derivative operator, this means: $\frac{\mathrm{d}}{\mathrm{d}\tilde{\mathscr{F}}} = \frac{\mathrm{d}\chi}{\mathrm{d}\tilde{\mathscr{F}}}\frac{\mathrm{d}}{\mathrm{d}\chi} \triangleq -\chi^2 \frac{\mathrm{d}}{\mathrm{d}\chi}$. Assuming $w_* \equiv w_*^{\mathrm{LFE}}$, it thus follows:

$$\begin{aligned} w_*^{\mathrm{LFE}}(\chi) &= \lambda \chi^{-N} + \sum_{n=0}^{\infty} \lambda_n \chi^n, \\ \left(w_*^{\mathrm{LFE}}\right)'\left(\tilde{\mathscr{F}}(\chi)\right) &= -\chi^2 \left(w_*^{\mathrm{LFE}}\right)'(\chi) = \lambda N \chi^{1-N} - \sum_{n=1}^{\infty} n\lambda_n \chi^{n+1}, \\ \left(w_*^{\mathrm{LFE}}\right)''\left(\tilde{\mathscr{F}}(\chi)\right) &= 2\chi^3 \left(w_*^{\mathrm{LFE}}\right)'(\chi) + \chi^4 \left(w_*^{\mathrm{LFE}}\right)''(\chi) \\ &= \lambda N (N-1)\, \chi^{2-N} + \sum_{n=1}^{\infty} n(n+1)\lambda_n \chi^{n+2}. \end{aligned} \tag{3.77}$$

With these relations, we are in position to express the important quantities $\boldsymbol{\omega}^{(1)}$ and $\boldsymbol{\omega}^{(2)}$ as functions of χ:

$$\begin{aligned} \boldsymbol{\omega}^{(1)}(\chi) &= 1 - \lambda N \chi^{1-N} + \sum_{n=1}^{\infty} n\lambda_n \chi^{n+1}, \\ \boldsymbol{\omega}^{(2)}(\chi) &= 1 - \lambda N^2 \chi^{1-N} - \sum_{n=1}^{\infty} n^2 \lambda_n \chi^{n+1}. \end{aligned} \tag{3.78}$$

Now, let us leave some further comments on the "asymptotic exponent" N. First of all, its value is determined by eq. (2.38). In $d = 4$ spacetime dimensions and provided that the anomalous dimension at a fixed point is known, we find:

$$N(\eta_*) = \frac{4}{2 + \eta_*}. \tag{3.79}$$

From the SFE results we already have several, explicit options for η_* at hand. The "standard procedure", which is supposed to be the Wilson-Fisher perspective as it does not depend on a rather randomly and externally prescribed value for the anomalous dimension η_*, yielded two stabilising values which are given in (3.70). Because $\eta_*\left(\mu_*^{(2)}\right) \simeq 10$ is extraordinarily large and hence not expected to represent a viable fixed point, we will concentrate on the alternative option: $\eta_*\left(\mu_*^{(1)}\right) \simeq -3.7067$, such that N becomes:

$$N \simeq -2.3437. \tag{3.80}$$

It follows that the "asymptotic exponent" for our present U(1) gauge theory appears as a real-valued decimal number, which explicitly differs from the $d = 3$ dimensional Ising model where $N = 3$ was found to be an integer as a consequence of the local potential approximation. As soon as we allow for a scale-dependent field strength renormalisation, this generally changes as eq. (3.79) demonstrates. Indeed, since $N < 0$ and η_* is comparatively large, we basically deviate significantly from customary, reliable results, but regarding the lack of better alternatives, we nonetheless continue with the present configuration.

The first step consists in finding a suitable starting point for the LFE, i.e. a formulation of the fixed point equation that is mathematically compatible with the LFE ansatz (3.76). What this means becomes clear after we have enumerated the available representations of the fixed point equation and recall the conditions leading from one to another representation. First, we refer to eq. (3.59) as the *integral representation* of the FPE, for which we actually should decide on our first trial, because the only preceding assumptions at this point are pure $\tilde{\mathscr{F}}$ dependencies of the fixed point potential and self-dual field configurations, which are both independent of any specific field amplitude sector. If we instead aim to begin with the *hypergeometric representation* of the FPE, eq. (3.64), we need to argue that the Euler integral formulation of the hypergeometric function is reconcilable with the LFE, i.e. that $\boldsymbol{\omega}^{(1)} < 1$ and $\boldsymbol{\omega}^{(2)} < 1$ hold simultaneously. This is then automatically consistent with the *Gauss representation* of the FPE, which is given by eq. (3.65) (cf. app. C). In the following, we are going to present an analysis for the integral and the Gauss representation, beginning with the former.

In eq. (3.59), we again agree on the optimised regulator $R = R^{\mathrm{op}}$ and reuse the intermediate results in (3.60)-(3.63) for $d = 4$ spacetime dimensions. After rearranging terms, the fixed point equation reads:

$$\begin{aligned} w_*(\chi) = {} & \frac{1}{128\pi^2}\left(1-\frac{\eta_*}{6}\right) + \left(1+\frac{\eta_*}{4}\right)\frac{w_*'\left(\tilde{\mathscr{F}}(\chi)\right)}{\chi} - \frac{3}{128\pi^2}\int_0^1 r\,\frac{\eta_*(1-r)-2}{1-\boldsymbol{\omega}^{(1)}(\chi)r}\mathrm{d}r \\ & + \frac{\boldsymbol{\omega}^{(1)}(\chi)}{128\pi^2}\int_0^1 r^2\frac{\eta_*(1-r)-2}{1-\boldsymbol{\omega}^{(1)}(\chi)r}\mathrm{d}r - \frac{\boldsymbol{\omega}^{(2)}(\chi)}{128\pi^2}\int_0^1 r^2\frac{\eta_*(1-r)-2}{1-\boldsymbol{\omega}^{(2)}(\chi)r}\mathrm{d}r. \end{aligned} \tag{3.81}$$

The next step is to compare coefficients of equal powers in χ on both sides of eq. (3.81) which yields a set of relations among the unknown parameters $\{\lambda_n\}_{n\in\mathbb{N}_0} \cup \{\lambda\}$. For this, we need to express the integral terms on the RHS of eq. (3.81) in powers of χ, which is feasible by means of a MacLaurin series expansion under the integrals for the terms $\frac{1}{1-\boldsymbol{\omega}^{(1)}(\chi)r}$ and $\frac{1}{1-\boldsymbol{\omega}^{(2)}(\chi)r}$, both considered as functions of χ, where r now plays the role of a parameter. In order to gain first insights, let us start with the simplest case and read off the first relation arising from eq. (3.81) by setting $w_* \equiv w_*^{\mathrm{LFE}}$ following (3.76) and equating the coefficients of the constant contributions on both sides.

Using (3.78) we note that $\boldsymbol{\omega}^{(1)}(0) = 1 = \boldsymbol{\omega}^{(2)}(0)$ and find:

$$
\begin{aligned}
128\pi^2\lambda_0 = 1 - \frac{\eta_*}{6} - 3\int_0^1 r\frac{\eta_*(1-r)-2}{1-\boldsymbol{\omega}^{(1)}(0)r}\mathrm{d}r + \int_0^1 r^2\frac{\eta_*(1-r)-2}{1-\boldsymbol{\omega}^{(1)}(0)r}\mathrm{d}r \\
- \int_0^1 r^2\frac{\eta_*(1-r)-2}{1-\boldsymbol{\omega}^{(2)}(0)r}\mathrm{d}r \qquad (3.82) \\
= 1 - \frac{\eta_*}{6} - 3\int_0^1 r\left(\eta_* - \frac{2}{1-r}\right)\mathrm{d}r = -5 - \frac{5}{3}\eta_* - 6\lim_{\Lambda\to 0^+}\ln(\Lambda)\,.
\end{aligned}
$$

Here, the new quantity Λ - which essentially corresponds to a sharp momentum cutoff - arises from taking the integral over $\frac{r}{1-r}$ with respect to r. Hence, it turns out that a (logarithmic) *divergence* already appears at the level of the constant LFE term; $\lambda_0 \to \infty$ as soon as the UV momentum modes, defined with respect to the fixed point momentum scale, are fully integrated out. As a consequence, the LFE fixed point potential w_*^{LFE} diverges logarithmically at zeroth order and thus for all possible values of χ. Therefore, it fails to be a reasonable candidate for the large field amplitude sector of a global fixed point action, since it is obviously not combinable with the SFE results.

One could continue with this process and, for instance, collect coefficients that correspond to χ^{-N} and to linear order in χ on both sides of eq. (3.81). Since $\frac{\mathrm{d}}{\mathrm{d}\chi}\left[\frac{1}{1-\boldsymbol{\omega}^{(1)}(\chi)r}\right]\Big|_{\chi=0} = 0 = \frac{\mathrm{d}}{\mathrm{d}\chi}\left[\frac{1}{1-\boldsymbol{\omega}^{(2)}(\chi)r}\right]\Big|_{\chi=0}$, the following relations hold:

$$
\lambda = \left(1+\frac{\eta_*}{4}\right)N\lambda \qquad \& \qquad \lambda_1 = -\left(1+\frac{\eta_*}{4}\right)\lambda_1. \tag{3.83}
$$

Because $N \neq 0$ and $\eta_* \neq -4$, this implies that $\lambda = 0 = \lambda_1$, i.e. λ does not appear as a free parameter, like it was the case for the Ising model, but instead is forced to vanish due to the structure of the fixed point equation. We refrain from continuing with higher order terms and directly conclude, that a proper LFE of the form (3.76) (or of a similar kind) does not seem to be constructable under our present conditions. If we relax the assumption $N < 0$, but instead suppose a positive $N > 1$ (which implies $\eta_* \in (-2,2)$), $\boldsymbol{\omega}^{(1)}(\chi)$ and $\boldsymbol{\omega}^{(2)}(\chi)$ diverge at $\chi = 0$. Thus, all integral expressions in (3.82) are totally suppressed and we get a finite $\lambda_0 = \frac{1}{128\pi^2}\left(1-\frac{\eta_*}{6}\right)$. However, we are not aware of any such appropriate η_* that has emerged from the SFE Wilson-Fisher perspective. Therefore, our conclusion remains unchanged: the integral representation of the FPE does not seem to allow for a well-behaving LFE of the fixed point potential.

From the previous considerations it is highly expected that the LFE also fails when performing an anlogous analysis that starts from the Gauss series representation (3.65), since here the set of conditions includes at least all aspects that already led to the integral representation of the FPE which we just have discussed. Nevertheless, for reasons of completeness we now give a more detailed explanation of this statement and show that λ_0 diverges exactly as before. We start by collecting all constant contributions on the RHS of eq. (3.65) for $w_* \equiv w_*^{\mathrm{LFE}}$. Here we note, that the Gauss series contributes a constant term that arises from the expression $\left(\boldsymbol{\omega}^{(1)}\right)^n + \frac{1}{2}\left(\boldsymbol{\omega}^{(2)}\right)^n$ for each summation index value $n \in \mathbb{N}$. Both, $\boldsymbol{\omega}^{(1)}(\chi)$ and $\boldsymbol{\omega}^{(2)}(\chi)$, can be written as $1-\chi f_1(\chi)$ and $1-\chi f_2(\chi)$ respectively, with some functions f_1 and f_2 that can, in principle, be read off from (3.78).

Thus, we can separate the constant term explicitly by means of the *binomial theorem*:

$$
\begin{aligned}
\left(\boldsymbol{\omega}^{(1)}(\chi)\right)^n + \frac{1}{2}\left(\boldsymbol{\omega}^{(2)}(\chi)\right)^n &= \left(1-\chi f_1(\chi)\right)^n + \frac{1}{2}\left(1-\chi f_2(\chi)\right)^n \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k f_1(\chi)^k \chi^k + \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}(-1)^k f_2(\chi)^k \chi^k \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\left[f_1(\chi)^k + \frac{1}{2}f_2(\chi)^k\right]\chi^k \\
&= \binom{n}{0}\left[1+\frac{1}{2}\right] + \ldots = \frac{3}{2} + \ldots .
\end{aligned}
\tag{3.84}
$$

Therefore we arrive at:

$$
128\pi^2\lambda_0 = 4\left(1-\frac{\eta_*}{6}\right) + 2\frac{3}{2}\sum_{n=1}^{\infty}\frac{6+2n-\eta_*}{(n+2)(n+3)} = 10 - \frac{5}{3}\eta_* + 6\sum_{n=1}^{\infty}\frac{n}{(n+2)(n+3)}. \tag{3.85}
$$

For the remaining series we find:

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{n}{(n+2)(n+3)} &= \sum_{n=1}^{\infty}\frac{n}{n+2} - \sum_{n=1}^{\infty}\frac{n}{n+3} = \frac{1}{3} + \sum_{n=2}^{\infty}\frac{n}{n+2} - \sum_{n=2}^{\infty}\frac{n-1}{n+2} \\
&= \frac{1}{3} + \left(\sum_{n=2}^{\infty}\frac{n}{n+2} - \sum_{n=2}^{\infty}\frac{n}{n+2}\right) + \sum_{n=2}^{\infty}\frac{1}{n+2} \\
&= \sum_{n=1}^{\infty}\frac{1}{n} - \frac{3}{2} = \lim_{n\to\infty} H_n - \frac{3}{2} = \infty,
\end{aligned}
\tag{3.86}
$$

where H_n denotes the n-th partial sum of the (divergent) *harmonic series*. This shows, that also by means of the Gauss series representation, the LFE fixed point potential diverges at zeroth order, like it was initially predicted.

There is no misconception in nature,
but know, that misconception is within you.

~ Leonardo da Vinci

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[last visited November 26, 2022].

4

Review: Final Remarks & Outlook

The aim of this work was to investigate the *global existence of fixed point actions* - which often serves as an important criterion of viable fixed points - for a theory technically adjacent to gravity. Taking tensor structures and local symmetries into account, the choice fell on a *generic, but self-interacting U(1) gauge theory*, the details of which were provided by an in-depth explanation that is part of ch. 3. The structural proximity to gravitational theory should reveal further indications to the *asymptotic safety hypothesis*, the final evaluation of which is a central concern of the *asymptotic safety programme*.

This final chapter now serves the orderly disclosure as well as evaluation of all findings that have emerged from this work. Furthermore, central problems are enumerated and the additional conditions generated by them are critically evaluated. Finally, we finish our considerations with a classification of the results produced in the context of this work, measured against the above formulated goal, and subsequently provide open issues as well as unresolved questions for a potentially consecutive investigation.

A. Chronology: from the initial to the final theory

We begin with a non-judgemental and chronologically structured presentation of the key steps leading from the initially formulated to the finally investigated U(1) gauge theory. In other words, this paragraph can be understood as a minimal summary of sec. 3.1 and paragraph **A** of sec. 3.2. In parallel, we provide an overview of the assumptions/conditions induced by each step, with the aim of increasing the level of transparency.

(A1). The Lagrangian of the initial theory was motivated by the locally U(1) invariant Lagrangian of QED, cf. eq. (3.1); the quantum field theory of the electromagnetic interaction. The FRG suggests to start with the most general expression of the effective action Γ, or rather the scale-dependent EAA Γ_k, that is compatible with a set of imposed symmetry conditions and then solve the Wetterich equation to determine the interesting flow trajectories within the corresponding theory space $\mathcal{T}$. For this, one often uses a derivative expansion of the underlying effective average Lagrangian, which is defined by $\Gamma_k \sim \int \mathscr{L}_k$, and restricts the theory space further if it becomes necessary in order to obtain explicit results, e.g. to regulate technical barriers. In our case, we have decided to *exclude all matter degrees of freedom* from our consideration ($\psi \to 0$) and only keep the massless, bosonic degrees of freedom, which physically could correspond to photons for instance. Furthermore, the bosonic sector of $\mathscr{L}_{\text{QED}}$ was enlarged by a maximal collection of locally U(1) invariant expressions that were built from the fundamental invariants $\mathscr{F}_{\text{M}}$ and $\mathscr{G}_{\text{M}}$, defined against a four dimensional Minkowskian background. A schematic summary of this step is presented at the top of the next page.

$$\mathscr{L}_{\mathrm{QED}}\left(A_{\mathrm{M}},\psi\right) = -\mathscr{F}_{\mathrm{M}} - \bar{\psi}\left(\imath\not{D} - m\right)\psi$$

$\swarrow$ enlarged $\searrow$ $\qquad$ $\searrow$ excluded $\swarrow$

$$\boxed{\mathscr{L}_{\mathrm{M}}\left(\mathscr{F}_{\mathrm{M}},\mathscr{G}_{\mathrm{M}}\right)}$$

However, from a more physical point of view, one can also interpret the resulting non-linear theory as being generated by fluctuations of matter degrees of freedom, which is precisely the inverse process of matter created by sufficiently strong electromagnetic fields [47].

(A2). For an application of the FRG formalism, the gauge-fixed effective action $\Gamma_{\mathrm{M}} = \int \mathscr{L}_{\mathrm{M}} \mathrm{d}^4x + (\Gamma_{\mathrm{gf}})_{\mathrm{M}}$, which is additionally equipped with a gauge-fixing action $(\Gamma_{\mathrm{gf}})_{\mathrm{M}}$ that accounts for gauge redundancies (cf. sec. 2.4), was modified with a continuous momentum scale dependence k, i.e. $\Gamma_{\mathrm{M}} \to (\Gamma_{\mathrm{M}})_k$, such that $\mathscr{L}_{\mathrm{M}} \to (\mathscr{L}_{\mathrm{M}})_k$ and $(\Gamma_{\mathrm{gf}})_{\mathrm{M}} \to (\Gamma_{\mathrm{gf},k})_{\mathrm{M}}$. Moreover, we have *generalised the number of spacetime dimensions from 4 to d*, but under the restriction $d = 4 - \epsilon$ with $|\epsilon| \ll 1$.

In preparation for upcoming calculations we performed a *Wick rotation in coordinate and field space*, such that the Minkowskian background geometry was replaced by a Euclidean background. The Euclidean counterparts of all previously introduced quantities are essentially designated as before, but omitting the subscript “M”, e.g. $\Gamma_{\mathrm{M}} \to \Gamma$, or $\mathscr{F}_{\mathrm{M}} \to \mathscr{F}$. One of the key aspects of the field space Wick rotation is given by the relation between the purely kinetic Minkowskian and Euclidean Lagrangian: $-\left(\mathscr{L}^{\mathrm{kin}}_{\mathrm{M}}\right)_k = \mathscr{L}^{\mathrm{kin}}_k$, where $\left(\mathscr{L}^{\mathrm{kin}}_{\mathrm{M}}\right)_k \sim \mathscr{L}^{\mathrm{kin}}_{\mathrm{M}} \equiv -\mathscr{F}_{\mathrm{M}}$.

In the next step, we have implemented the first far-reaching assumption into our model, which we referred to as the *homogeneity condition*. This means, that the field strength tensor is supposed to be constant throughout spacetime. It does not imply that also the fundamental degree of freedom, i.e. the gauge field A, is constant as well, but rather is such, that the combination $\partial_\mu A_\nu - \partial_\nu A_\mu$ vanishes for all pairs of indices (μ,ν) when taking further derivatives. According to the definitions of $\mathscr{F}$ and $\mathscr{G}$, the homogeneity condition implies constancy also for these invariants. Another direct consequence of the homogeneity condition manifests in the fact, that all derivative terms, for example by means of a derivative expansion of $\mathscr{L}_k$, vanish identically. The remnant of this process consists of all non-derivative terms, which we have summarised in an *effective average potential* $\mathscr{W}_k$, i.e. $\mathscr{L}_k \to \mathscr{W}_k$. An illustration of this step can be found below.

$$\Gamma_{\mathrm{M}}\left[A_{\mathrm{M}}\right]$$

$$\downarrow$$

$$\boxed{\mathrm{d}^4x \to \mathrm{d}^dx} + \boxed{\text{Wick rotation}} + \boxed{\Gamma \to \Gamma_k} + \boxed{\text{homogeneity}}$$

$$\Downarrow$$

$$\Gamma_k\left[A\right] = \int \mathscr{W}_k\left(\mathscr{F},\mathscr{G}\right)\mathrm{d}^dx + \Gamma_{\mathrm{gf},k}\left[A\right]$$

(A3). From the EAA $\Gamma_k \hat{=} \Gamma_k[A]$, we were able to give an explicit expression for the Wetterich equation (2.15), especially for its RHS where we were calculating the regularised full propagator $\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1}$ and have performed the trace operation $\mathbf{Tr}$. The regulator $\mathcal{R}_k$ was supposed to be expressible by a *general combination of the perpendicular and parallel projector*: $\mathcal{R}_k \sim Z_k R \left[\mathbf{P}_\perp + \kappa^{-1}\mathbf{P}_\parallel\right]$, in which R denotes a shape function not specified in more detail at this point, and κ is a gauge-fixing parameter. We point out, that in contrast to the Ising model, where we have established a local potential approximation, we instead refrained from this restriction in the locally U(1) invariant theory by *allowing for a non-constant field strength renormalisation*, i.e. $\partial_k Z_k \neq 0$ in general. In this way, the anomalous dimension η_k becomes a non-trivial flow-dependent quantity with in general non-vanishing fixed point values.

The list of further assumptions made during the derivation of the flow equation are limited to firstly a choice of a gauge condition needed for the gauge-fixing action $\Gamma_{\mathrm{gf},k}$, and secondly a limitation of theory space $\mathcal{T}$ by *excluding odd powers of the invariant* $\mathscr{G}$. The latter has already been justified in ch. 3 and will be taken up again in the following paragraph. The gauge-fixing was realised by the technically easy to handle *Lorenz gauge condition*; $\partial_\mu A^\mu = 0$. Finally, the explicit scale dependencies of the flow equation (3.40) were eliminated as a consequence of *introducing dimensionless quantities*, e.g. $\mathscr{F} \to \tilde{\mathscr{F}}, \mathscr{G} \to \tilde{\mathscr{G}}, \mathscr{W}_k \to w_k$ etc., which has ultimately led to the autonomous ERGE (3.50).

(A4). The interest of this work is primarily directed towards fixed point structures. For this reason, the fixed point equation (3.52) was derived starting from the autonomous ERGE (3.50) and using the fixed point condition; $\partial_t w_k = 0$ at a fixed point. Due to high technical requirements needed to fully solve the FPE, we had to restrict the structure of the FPE considerably in order to obtain first results in view of the elementary scope of this work. For this, the *PDE character of the FPE was disintegrated by the assumption of sole* $\tilde{\mathscr{F}}$ *dependencies of the dimensionless fixed point effective average potential* w_*, i.e. $\dot{w}_* = \ddot{w}_* = \dot{w}'_* = 0$. In consequence, the FPE has reduced to an ODE for $w_* \hat{=} w_*\left(\tilde{\mathscr{F}}\right)$. Lastly, we have *resolved the remaining angular dependencies* $\sim \left(\tilde{\mathbf{F}}s\right)^2$ appearing in eq. (3.52) under consideration of (3.54). This was done by *restricting to self-dual field configurations*, for which $\tilde{\mathbf{F}} = \star\tilde{\mathbf{F}}$ holds, implying $\left(\tilde{\mathbf{F}}s\right)^2 \sim s^2$. As a consequence, the integral representation (3.59) of the FPE was obtained, which has recently served as our starting point for the LFE, cf. sec. 3.2. Under *one more assumption, namely that* $w'_*\left(\tilde{\mathscr{F}}\right) > 0 \wedge w'_* > -w''_*\left(\tilde{\mathscr{F}}\right)\tilde{\mathscr{F}}$ *is true for all (relevant)* $\tilde{\mathscr{F}}$, the FPE could be transferred to a more compact formulation by means of Euler's integral representation of the hypergeometric function and its corresponding Gauss series. This was then used to investigate small field amplitudes in the realm of the SFE.

B. Evaluation and alternative paths

In this paragraph, we will take a critical eye on some of the assumptions listed in paragraph **A**, which were formerly introduced in the course of ch. 3. We aim to give qualitative judgements and evaluations on their significance and restrictive potential. In individual cases, it is worthwhile to examine whether alternative ways exist that can restore the degree of generality while weighing up the increased effort. Thus, there will also be further comments on this aspect at appropriate points.

(B1). *Exclusion of matter*: In step **(A1)**, we have explained by which aspects the research object of this work, which is the self-interacting locally U(1) invariant theory, was inspired. The origin of its construction is allocated to QED, but with the matter sector being totally suppressed. It is already this initial assumption, which should be classified as a harsh restriction.

All results based on this premise, in particular regarding the global existence of fixed point actions, can shift significantly as soon as matter degrees of freedom are incorporated in the theory.

The effect of such inclusions in FRG approaches to gravity is an issue of current research and is being phenomenologically and quantitatively investigated, for example in ref. [48]. In the results of this publication, the requirement for the existence of a suitable fixed point in the scope of the Einstein-Hilbert truncation restricts the underlying matter model in the number of its different particle species. A similar effect could also occur for isolated systems of massless bosons, as in the case of this work. Thus, conversely to Einstein-Hilbert gravity, global existence of fixed point actions might only be realised when different sorts of matter are taken into account. Progress in this direction can for instance be achieved by re-including Dirac particles (fermionic matter) ψ into the theory. For instance, one can start to consider purely self-interacting contributions, e.g. an action of the form

$$S_{\mathrm{mat},k}\left[\psi,\bar{\psi}\right] \sim Z_{\mathrm{mat},k} \int \bar{\psi}(x)\left(\imath \not{D} - m\right)\psi(x)\mathrm{d}^d x, \tag{4.1}$$

in which $Z_{\mathrm{mat},k}$ denotes the scale-dependent spinor field strength renormalisation. Further contributions can describe also interactions between vector bosons and spin-1/2 fermions etc., as long as one takes care ensuring local U(1) invariance for every additional term which is added to the full effective action. In the context of this work, however, such an analysis was deliberately dispensed, firstly to reduce the technical complications and secondly to get a first access to a predominantly still little studied theory.

(B2). *Homogeneity*: Our theory ignores all field strength kinematical contributions, e.g. including terms of the form $F_{\mu\nu}\Box^n F^{\mu\nu}$ for $n \in \mathbb{N}$ and where $\Box \equiv \partial_\mu \partial^\mu$ means the d'Alembert operator. For definiteness, let us gather all possible locally U(1) invariant combinations that contain derivatives of the field strength tensor within the symbol $\mathscr{K}$. Then, the full Lagrangian would read; $\mathscr{L} = \mathscr{W} + \mathscr{K}$, where $\mathscr{W}$ denotes the effective potential that was considered throughout ch. 3 (or more precisely, its scale-dependent modification $\mathscr{W}_k$). However, since we are interested in globally-existing fixed point actions, i.e. fixed point action functionals which are defined for any value of the field amplitude, it suffices to consider the Lagrangian as a function of the field amplitude as the independent variable and choose $F_{\mu\nu} = \text{const.}$ for all pairs of indices (μ, ν), rather than describing its variations with respect to an extra dependency on spacetime points; $\mathbf{F} \mathrel{\hat{=}} \mathbf{F}(x)$. In other words, we are only interested in the values of the field amplitude, and not in its fluctuations over spacetime, implying that the homogeneity condition can essentially be understood as a priority alignment as it automatically neglects the irrelevant kinetic sector; $\mathscr{K} \to 0$. Therefore, the homogeneity condition can ultimately be seen as a mild assumption for our purposes.

(B3). *Elimination of $\tilde{\mathscr{G}}$ dependencies*: Restricting a fixed point potential w_* which depends on both field variables $\left(\tilde{\mathscr{F}}, \tilde{\mathscr{G}}\right)$ to a single field dependency in $\tilde{\mathscr{F}}$ is indeed an approximation. It was originally established in order to render the fixed point equation as an ODE instead of working with a much more complicated PDE that contains derivatives of w_* up to second order in both field variables. In this way, an analytical treatment became more accessible. However, further investigations that refrain from this constraint and repeat sec. 3.2 starting from the full FPE (3.52) would be a fruitful extension to the current status of this work. Since the FPE consequently turns to a PDE again, SFE and LFE approaches accordingly need to be adjusted for both field arguments of the fixed point potential, i.e. that one has to combine individual small- and large-field expansions for each argument. Although this clearly leads to a modification of the equations, it is unclear whether this is sufficient to establish a working LFE for the $\tilde{\mathscr{F}}$ sector. We will come back to this in paragraph **D**, where we refer to this as an open issue.

(B4). *Self-duality*: We have seen that self-duality is a powerful tool to simplify structures of relevant equations like the FPE; for example avoiding complicated angular dependencies over which we would need to integrate otherwise. Oppositely, the problem that comes with self-dual field configurations is the indistinguishability between the invariant quantities $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{G}}$. That is, because the electric and magnetic field are now equivalent, $\tilde{\mathbf{E}} = \tilde{\mathbf{B}}$, and thus $\tilde{\mathscr{F}} = \frac{1}{2}\left(\tilde{\mathbf{E}}^2 + \tilde{\mathbf{B}}^2\right) = \tilde{\mathbf{B}}^2 = \tilde{\mathbf{E}} \cdot \tilde{\mathbf{B}} = \tilde{\mathscr{G}}$. However, there are several alternative options which provide for both; simplifying structures while restoring the distinguishability of $\tilde{\mathscr{F}}$ and $\tilde{\mathscr{G}}$. For example, in the situation of eq. (3.55), one can consider pure magnetic field configurations, i.e $\tilde{\mathbf{E}} = \mathbf{0}$ while keeping $\tilde{\mathbf{B}}$ arbitrary. As a consequence, one arrives at $\left(\tilde{\mathbf{F}}s\right)^2 = \tilde{\mathbf{B}}^2\mathbf{s}^2\sin(\vartheta)^2$, where ϑ denotes the angle enclosed by the magnetic field $\tilde{\mathbf{B}}$ and the dimensionless spatial momentum $\mathbf{s}$. Indeed, angular terms obviously remain, but are much more easy to handle than working with the full expression given in (3.55).

Though not presented here, first calculations show that the structure of the FPE for either purely magnetic or self-dual field configurations basically hardly differ from each other. Furthermore, numeric SFE results for the former case seem to deviate from self-duality only by individual factors for the coefficients, but which do not entail substantial changes in how w_*^{SFE} behaves when plotted against the field amplitude $\tilde{\mathscr{F}}$. At this point, it is still unclear whether this is also true for the LFE.

C. Conclusion

The declared aim of this work was to contribute further indications that serve for the clarification of the asymptotic safety hypothesis, which involves the search for a non-trivial fixed point of the gravitational renormalisation flow. Our focus was thereby placed on an important property of fixed point actions which is known as global existence and provides a criterion for sifting out physically viable fixed points. However, we refrained from conducting a direct FRG analysis for gravitational systems and chose an implicit strategy instead. In particular, we have decided in favour of a self-interacting locally U(1) invariant theory, which is structurally related to central features of gravity, such as a dynamical metric tensor and diffeomorphism invariance, for which corresponding counterparts are given by a dynamical field strength tensor and U(1) gauge symmetry. Our model takes into account all locally U(1) invariant terms of zeroth order with respect to a derivative expansion of the underlying effective Lagrangian.

In other words, the initially described objective now specifies on scanning our theory for globally-existing fixed point actions, for the progress of which the relevant quantities of the Wetterich equation were calculated and the flow equation - and with this also the fixed point equation - were derived in a momentum-scale-autonomous formulation.

For the explicit construction of a globally-existing fixed point action, proven analytical methods were used, in particular small- and large-field expansions while taking into account the conditions listed in paragraph **A** and subsequently evaluated in paragraph **B** of this chapter. Following this procedure, it provides for a piecewise composition of the sought-for fixed point potential connecting solutions of the FPE for small and large field amplitudes.

It was shown that, in the scope of the SFE, non-trivial fixed point solutions emerge continuously from the trivial fixed point at $w_*^{\mathrm{GFP}} = \frac{1}{32\pi^2} + \tilde{\mathscr{F}}$. However, these solutions do not find a connection to partner solutions for large field amplitudes within the limit of their radius of convergence. This manifests in a divergent large field behaviour of w_* in the realm of the LFE, which thus prevents the construction of a globally-existing fixed point potential.

In conclusion, it is still unclear whether these hurdles can be removed by softening the previously met conditions under which we deduced all our results. However, due to the structural similarities of the restricted system considered here to the general situation, the expectation for this does not turn out in favour of global existence. It remains the assumption that the latter might only be achieved if additional matter degrees of freedom are taken into account.

D. Open Issues

Let us finish with a short list of unresolved problems which could provide for consecutive questionings.

As often noted before, it is unknown to what extent the results presented here differ from the results that follow from the full fixed point equation. An analysis in this direction requires the application of methods from the mathematical theory of partial differential equations and their solutions.

Another interesting issue that we have mentioned several times before is the effect of matter particles such as scalar or spinor fields as additional degrees of freedom.

Furthermore, the question of a meaningful expansion of the fixed point potential for large field amplitudes can alternatively be pursued by means of a flow linearisation around the fixed point under consideration. Here, a so far completely neglected detailed investigation of the stability matrix and its eigenvalues - the critical exponents - can yield a fruitful extension to the previous results.

Correspondingly to the last mentioned point, further insights about the fixed point sector, but also of the renomalisation flow as a whole, can be inferred from considerations of finite dimensional truncations of the underlying theory space $\mathcal{T}$. Enlarging these truncations step-by-step can provide important information on fixed point truncation artifacts and help to distinguish them from possibly viable fixed points. In this way, also a variety of regulator functions can be tested and compared with each other, which yields a thriving contrast to the structure of this thesis that relies exclusively on the optimised regulator.

Appendix

A

Field Strength in Minkowskian and Euclidean Geometry

The field strength tensor $\mathbf{F}$ is mainly known from electrodynamics where it is formed from the four potential A. However, its construction extends over a much more general scope. For instance, besides the field strength tensor from Maxwell's theory, there is an additional field strength in Yang-Mills theory which differs from the usual definition especially in the case of non-Abelian gauge theories when structure constants do not vanish in general (see our discussion from sec. 2.4). In this thesis, however, we are primarily concerned with Abelian gauge theories with vector-like gauge fields as the fundamental degrees of freedom and where we do not have to worry about non-trivial structure constants. Furthermore, our considerations will initially take place in d dimensions, though at some points this will be specified in what follows.

The simplest construction rule that applies to the components of the field strength tensor in the Abelian case reads [51]:

$$F_{ab} := \partial_a A_b - \partial_b A_a. \tag{A.1}$$

In order to calculate the invariants $\mathscr{F}$ and $\mathscr{G}$, which belong to the key objects of ch. 3 where they serve as the functional arguments for the EAA, we do not only need the field strength tensor itself, but in addition the dual field strength tensor, $\star\mathbf{F}$, which is the so-called *Hodge dual* to $\mathbf{F}$. Based on the definition (A.1), this is obtained by applying the *Hodge star operator*, $\star$, to $\mathbf{F}$ and is likewise designated by $\star\mathbf{F}$.

In order to summarise some details on the Hodge star operator, let us consider a (pseudo-)Riemannian manifold $\mathcal{M} \equiv (X, \mathbf{g})$, that consists of a set X together with a metric tensor $\mathbf{g}$, and let us further consider an open subset $U \subseteq \mathcal{M}$. Moreover, let $\{e^a\}_{a=1}^d$ be a local frame of 1-forms for U. Then, we can expand any p-form $\boldsymbol{\alpha}_p$ in terms of that frame [54]:

$$\boldsymbol{\alpha}_p \mathrel{\hat{=}} \frac{1}{p!} \alpha_{i_1 \ldots i_p} e^{i_1} \wedge \cdots \wedge e^{i_p}. \tag{A.2}$$

Here, the $\alpha_{i_1 \ldots i_p}$ are smooth component functions for $\boldsymbol{\alpha}_p$ with respect to the frame $\{e^a\}_{a=1}^d$. Finally, the *Hodge dual* to $\boldsymbol{\alpha}_p$ is given by the following definition [54]:

$$\star\boldsymbol{\alpha}_p := \frac{1}{p!(d-p)!} \sqrt{|\det(\mathbf{g})|} \alpha_{i_1 \ldots i_p} g^{i_1 j_1} \ldots g^{i_p j_p} \varepsilon_{j_1 \ldots j_p j_{p+1} \ldots j_d} e^{j_{p+1}} \wedge \cdots \wedge e^{j_d}. \tag{A.3}$$

Again, the components g_{ab} are defined with respect to the local frame $\{e^a\}_{a=1}^d$, i.e. $\mathbf{g} \mathrel{\hat{=}} g_{ab} e^a \wedge e^b$. By common conventions, upper indices g^{ab} indicate components of the inverse of $\mathbf{g}$. From eq. (A.3) we can observe that $\star\boldsymbol{\alpha}_p$ is a $(d-p)$-form, but we refrain from indicating this by an adjusted subscript, instead the $\star$ symbol gives the information about the dual character.

In this Appendix, we give concrete representations for $\mathbf{F}$ and $\star\mathbf{F}$ according to the eqs. (A.1)-(A.3) in $d = 2, 3$ and $d = 4$ dimensions and for both Minkowskian and Euclidean geometry.

The case $d = 1$, i.e. spacetime consists of only a single time direction and no spatial degrees of freedom at all, is trivial since $\mathbf{F} \equiv \mathbf{0}$. This becomes immediately lucid from eq. (A.1).

We further assume the metric signature $\sigma(\boldsymbol{\eta}) = (3,1)$ for the Minkowski metric tensor $\boldsymbol{\eta}$, and $\sigma(\boldsymbol{\delta}) = (4,0)$ for the Euclidean metric tensor $\boldsymbol{\delta}$. At the end of this Appendix, we eventually give a direct derivation of the identities (3.29) in $d = 4$ dimensions.

A. Field Strength in Minkowski Space

In this paragraph, we have $\mathbf{g} \equiv \boldsymbol{\eta}$ and $|\det(\boldsymbol{\eta})| = 1$ in any number of spacetime dimensions. Let us now go through the various options for d.

$\lfloor d = 2 \rfloor$ The field strength tensor can be represented by a 2×2-matrix with only a single independent component. As in the familiar case of four spacetime dimensions, we interpret this component as the electric (scalar) field, whereas a magnetic field does not exist.

There are two generally non-vanishing entries $F_{01} \equiv -E$ and $F_{10} = -F_{01} = E$, i.e. in covariant indices we have:

$$(\mathbf{F}_{ab}) \mathrel{\hat{=}} E \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A.4}$$

or alternatively in contravariant indices:

$$\left(\mathbf{F}^{ab}\right) = \left(\boldsymbol{\eta}^{ac}\boldsymbol{\eta}^{bd}\mathbf{F}_{cd}\right) \mathrel{\hat{=}} E \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.5}$$

From eqs. (A.4) & (A.5) we can compute the quantity $\mathscr{F}$:

$$\mathscr{F} \triangleq \frac{1}{4} F_{ab} F^{ab} = -\frac{1}{2} E^2. \tag{A.6}$$

For the dual field strength we can use eq. (A.3) with $d = 2 = p$:

$$\begin{aligned} \star\mathbf{F} &= \frac{1}{2!(2-2)!} \sqrt{|\det(\boldsymbol{\eta})|} F_{ab} \eta^{ac} \eta^{bd} \varepsilon_{cd} \\ &= \frac{1}{2} F_{01} \eta^{00} \eta^{11} \varepsilon_{01} + \frac{1}{2} F_{10} \eta^{11} \eta^{00} \varepsilon_{10} \\ &= -E. \end{aligned} \tag{A.7}$$

In the last step we have considered $\varepsilon_{01} = -1$ according to our conventions (i.e. from $\varepsilon^{01} = 1$ it follows that $\varepsilon_{01} = \eta_{0a}\eta_{1b}\varepsilon^{ab} = \eta_{00}\eta_{11}\varepsilon^{01} = -1$).

$\lfloor d = 3 \rfloor$ In this situation, the information carried by the field strength tensor can be arranged within a 3×3-matrix, such that the electric field counts two independent components, i.e. $\mathbf{E} \mathrel{\hat{=}} (E_1, E_2)^T$. Moreover, also a magnetic (scalar) field, B, is now present. In summary, we obtain in covariant indices:

$$(\mathbf{F}_{ab}) \mathrel{\hat{=}} \begin{pmatrix} 0 & -E_1 & -E_2 \\ E_1 & 0 & B \\ E_2 & -B & 0 \end{pmatrix}, \tag{A.8}$$

or in contravariant indices:

$$\left(\mathbf{F}^{ab}\right) = \left(\boldsymbol{\eta}^{ac}\boldsymbol{\eta}^{bd}\mathbf{F}_{cd}\right) \mathrel{\hat{=}} \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & B \\ -E_2 & -B & 0 \end{pmatrix}. \tag{A.9}$$

As we have done in the two-dimensional case, the quantity $\mathscr{F}$ is obtained by a simple computation which uses eqs. (A.8) & (A.9):

$$\mathscr{F} \triangleq \frac{1}{4} F_{ab} F^{ab} = \frac{1}{2}\left(B^2 - \mathbf{E}^2\right). \tag{A.10}$$

Finally, the dual field strength 1-form follows from eq. (A.3) with $d = 3$ and $p = 2$:

$$\begin{aligned}
\star\mathbf{F} &= \frac{1}{2!(3-2)!}\sqrt{|\det(\boldsymbol{\eta})|}F_{bc}\eta^{bm}\eta^{cn}\varepsilon_{mna}e^a \\
&= \frac{1}{2}F_{bc}\eta^{bm}\eta^{cn}\varepsilon_{mna}e^a \\
&= F_{12}\varepsilon_{120}e^0 - F_{02}\varepsilon_{021}e^1 - F_{01}\varepsilon_{012}e^2 \\
&= \left(F_{12}e^0 + F_{02}e^1 - F_{01}e^2\right)\varepsilon_{012} \\
&= -Be^0 + E_2 e^1 - E_1 e^2.
\end{aligned} \tag{A.11}$$

From this result, we can read off a vector representation for $\star\mathbf{F}$ by choosing a Cartesian coordinate frame, first in contravariant indices and then also in covariant indices by using the Minkowski metric:

$$(\star\mathbf{F}^a) \mathrel{\hat{=}} \begin{pmatrix} -B \\ E_2 \\ -E_1 \end{pmatrix} \qquad \& \qquad (\star\mathbf{F}_a) \mathrel{\hat{=}} \begin{pmatrix} B \\ E_2 \\ -E_1 \end{pmatrix}. \tag{A.12}$$

$\lfloor d = 4 \rfloor$ We conclude this paragraph with the natural case of three spatial plus one temporal dimension, in which both the electric and magnetic field appear as vector fields. That means respectively $\mathbf{E} \mathrel{\hat{=}} (E_1, E_2, E_3)^T$ and $\mathbf{B} \mathrel{\hat{=}} (B_1, B_2, B_3)^T$. The field strength tensor is explicitly represented by a 4×4-matrix:

$$\begin{aligned}
(\mathbf{F}_{ab}) &\mathrel{\hat{=}} \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}, \\
\left(\mathbf{F}^{ab}\right) = \left(\boldsymbol{\eta}^{ac}\boldsymbol{\eta}^{bd}\mathbf{F}_{cd}\right) &\mathrel{\hat{=}} \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}.
\end{aligned} \tag{A.13}$$

Again, a direct calculation yields:

$$\mathscr{F} = \frac{1}{4} F_{ab} F^{ab} = \frac{1}{2}\left(\mathbf{B}^2 - \mathbf{E}^2\right). \tag{A.14}$$

We could obtain the Hodge dual to $\mathbf{F}$ in the same manner as for $d = 2$ and $d = 3$ spacetime dimensions, but instead we take a shortcut and use the results that can be found in the standard electrodynamics literature, e.g. [51].

In contravariant as well as covariant indices we find:

$$
\begin{aligned}
(\star\mathbf{F}_{ab}) &\mathrel{\hat{=}} \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & -E_3 & E_2 \\ B_2 & E_3 & 0 & -E_1 \\ B_3 & -E_2 & E_1 & 0 \end{pmatrix}, \\
\left(\star\mathbf{F}^{ab}\right) &\mathrel{\hat{=}} \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & E_3 & 0 & -E_1 \\ -B_3 & -E_2 & E_1 & 0 \end{pmatrix}.
\end{aligned}
\tag{A.15}
$$

A special feature which is exclusive to the present case of four spacetime dimensions relies on the fact that the dual field strength tensor has the same degree as the field strength tensor, because $d - p = 2 = p$. Accordingly, we can construct an analog to $\mathscr{F}$ by combining $\mathbf{F}$ and $\star\mathbf{F}$ to:

$$
\mathscr{G} \triangleq \frac{1}{4} F_{ab}\left(\star F^{ab}\right) = -\mathbf{E}\cdot\mathbf{B}. \tag{A.16}
$$

B. Field Strength in Euclidean Space

This paragraph is basically a repetition of the analysis from paragraph **A**, but with the difference that we now assume an Euclidean background and thus have $\mathbf{g} \equiv \boldsymbol{\delta}$. It is lucid that $|\det(\boldsymbol{\delta})| = 1$.

Before we begin to list the relevant information in different dimensions we should emphasise that the electric and magnetic field need to be considered with respect to the Euclidean background. From a physical perspective , they must be interpreted accordingly, that is by means of a *Wick rotation* which connects Minkowskian and Euclidean geometry. More precisely, the electric field in Euclidean space is related to its counterpart in Minkowski space by a factor of $\imath$, whereas the magnetic field coincides in both situations. However, the matrix representations for $\mathbf{F}$ in different dimensions are structurally analogous to the Minkowskian case.

Lastly, it is notable, though obvious, that there is no distinction in covariant and contravariant indices in Euclidean space.

$\lfloor d = 2 \rfloor$ Let us begin with two dimensions. The field strength tensor is essentially given by eq. (A.5), although the quantity E is now the Euclidean electric scalar field:

$$
(\mathbf{F}_{ab}) = \left(\mathbf{F}^{ab}\right) \mathrel{\hat{=}} E\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.17}
$$

For $\mathscr{F}$ it follows:

$$
\mathscr{F} \triangleq \frac{1}{4} F_{ab}F^{ab} = \frac{1}{2}E^2. \tag{A.18}
$$

Finally, eq. (A.3) yields:

$$
\begin{aligned}
\star\mathbf{F} &= \frac{1}{2!(2-2)!}\sqrt{|\det(\boldsymbol{\delta})|}\, F_{ab}\delta^{ac}\delta^{bd}\varepsilon_{cd} \\
&= \frac{1}{2}F^{cd}\varepsilon_{cd} = F^{01}\varepsilon_{01} \\
&= E.
\end{aligned}
\tag{A.19}
$$

$\lfloor d=3 \rfloor$ The field strength tensor can be copied from eq. (A.9):

$$(\mathbf{F}_{ab}) = \left(\mathbf{F}^{ab}\right) = \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & B \\ -E_2 & -B & 0 \end{pmatrix}. \tag{A.20}$$

Hence, for $\mathscr{F}$ we find:

$$\mathscr{F} \triangleq \frac{1}{4} F_{ab} F^{ab} = \frac{1}{2}\left(B^2 + \mathbf{E}^2\right), \tag{A.21}$$

where $\mathbf{E} \mathrel{\hat{=}} (E_1, E_2)^T$. Next, we calculate the dual field strength tensor:

$$\begin{aligned} \star\mathbf{F} &= \frac{1}{2!(3-2)!}\sqrt{|\det(\boldsymbol{\delta})|} F_{ab}\delta^{ac}\delta^{bd}\varepsilon_{cda}e^a \\ &= \frac{1}{2}F^{cd}\varepsilon_{cda}e^a \\ &= F^{12}\varepsilon_{120}e^0 + F^{02}\varepsilon_{021}e^1 + F^{01}\varepsilon_{012}e^2 \\ &= Be^0 - E_2 e^1 + E_1 e^2. \end{aligned} \tag{A.22}$$

In Cartesian coordinates the dual field strength can be represented in vector format:

$$(\star\mathbf{F}_a) = (\star\mathbf{F}^a) \mathrel{\hat{=}} \begin{pmatrix} B \\ -E_2 \\ E_1 \end{pmatrix}. \tag{A.23}$$

$\lfloor d=4 \rfloor$ In the four-dimensional case, we can adopt $\mathbf{F}$ from eq. (A.13):

$$(\mathbf{F}_{ab}) = \left(\mathbf{F}^{ab}\right) \mathrel{\hat{=}} \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{A.24}$$

Thus we obtain:

$$\mathscr{F} \triangleq \frac{1}{4} F_{ab} F^{ab} = \frac{1}{2}\left(\mathbf{B}^2 + \mathbf{E}^2\right), \tag{A.25}$$

in which $\mathbf{B} \mathrel{\hat{=}} (B_1, B_2, B_3)^T$ and $\mathbf{E} \mathrel{\hat{=}} (E_1, E_2, E_3)^T$ are the magnetic and electric field respectively. The dual field strength 2-form is given by:

$$\begin{aligned} \star\mathbf{F} &= \frac{1}{2!(4-2)!}\sqrt{|\det(\boldsymbol{\delta})|} F_{ab}\delta^{ac}\delta^{bd}\varepsilon_{cdmn}e^m \wedge e^n \\ &= \frac{1}{4}F^{cd}\varepsilon_{cdmn}e^m \wedge e^n \\ &= \frac{1}{2}\Big(F^{23}\varepsilon_{2301}e^0 \wedge e^1 + F^{13}\varepsilon_{1302}e^0 \wedge e^2 + F^{12}\varepsilon_{1203}e^0 \wedge e^3 \\ &\quad + F^{23}\varepsilon_{2310}e^1 \wedge e^0 + F^{13}\varepsilon_{1320}e^2 \wedge e^0 + F^{12}\varepsilon_{1230}e^3 \wedge e^0 \\ &\quad + F^{03}\varepsilon_{0312}e^1 \wedge e^2 + F^{02}\varepsilon_{0213}e^1 \wedge e^3 + F^{01}\varepsilon_{0123}e^2 \wedge e^3 \\ &\quad + F^{03}\varepsilon_{0321}e^2 \wedge e^1 + F^{02}\varepsilon_{0231}e^3 \wedge e^1 + F^{01}\varepsilon_{0132}e^3 \wedge e^2\Big) \end{aligned} \tag{A.26}$$

$$= \frac{1}{2}B_1\left(e^0\wedge e^1 - e^1\wedge e^0\right) + \frac{1}{2}B_2\left(e^0\wedge e^2 - e^2\wedge e^0\right)$$
$$+ \frac{1}{2}B_3\left(e^0\wedge e^3 - e^3\wedge e^0\right) + \frac{1}{2}E_3\left(e^1\wedge e^2 - e^2\wedge e^1\right)$$
$$- \frac{1}{2}E_2\left(e^1\wedge e^3 - e^3\wedge e^1\right) + \frac{1}{2}E_1\left(e^2\wedge e^3 - e^3\wedge e^2\right).$$

From this result we can infer the matrix representation of $\star\mathbf{F}$ in a local Cartesian coordinate frame:

$$(\star\mathbf{F}_{ab}) = \left(\star\mathbf{F}^{ab}\right) \mathrel{\hat{=}} \begin{pmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & E_3 & -E_2 \\ -B_2 & -E_3 & 0 & E_1 \\ -B_3 & E_2 & -E_1 & 0 \end{pmatrix}. \tag{A.27}$$

In $d = 4$ spacetime dimensions we can once again exclusively compute the quantity $\mathscr{G}$ using eq. (A.16):

$$\mathscr{G} \triangleq \frac{1}{4}F_{ab}\left(\star F^{ab}\right) = \mathbf{E}\cdot\mathbf{B}. \tag{A.28}$$

C. Algebraic Identities

During sec. 3.1 we make use of two algebraic identities which relate $\mathbf{F}$ and $\star\mathbf{F}$ with $\mathscr{F}$ and $\mathscr{G}$ in $d = 4$ spacetime dimensions. Now that we have concrete matrix representations for $\mathbf{F}$ and $\star\mathbf{F}$ in both Minkowskian and Euclidean space, and under consideration of eqs. (A.14) & (A.16) together with eqs. (A.25) & (A.28), it is an elementary exercise to prove these identities by direct matrix multiplication. But before we do so, let us first formulate them.

Proposition A.1. Let $\mathcal{M} \equiv (X, \mathbf{g})$ be a four-dimensional (pseudo-)Riemannian manifold that consists of a set of points X and a metric tensor $\mathbf{g}$. Moreover, let $\mathbf{F}$ be the field strength 2-form and $\star\mathbf{F}$ its Hodge dual. Then, the following relations hold:

(*a*) For $\mathbf{g} \equiv \boldsymbol{\eta}$ it is:

(*i*) $F^{\mu\lambda}F^{\nu}{}_{\lambda} - (\star F)^{\mu\lambda}(\star F)^{\nu}{}_{\lambda} = 2\mathscr{F}\eta^{\mu\nu}$

(*ii*) $F^{\mu\lambda}(\star F)^{\nu}{}_{\lambda} = (\star F)^{\mu\lambda}F^{\nu}{}_{\lambda} = \mathscr{G}\eta^{\mu\nu}$

(*b*) For $\mathbf{g} \equiv \boldsymbol{\delta}$ it is:

(*i*) $F^{\mu\lambda}F^{\nu}{}_{\lambda} + (\star F)^{\mu\lambda}(\star F)^{\nu}{}_{\lambda} = 2\mathscr{F}\delta^{\mu\nu}$

(*ii*) $F^{\mu\lambda}(\star F)^{\nu}{}_{\lambda} = (\star F)^{\mu\lambda}F^{\nu}{}_{\lambda} = \mathscr{G}\delta^{\mu\nu}$

Proof. We begin with case (a) no. (i) and consider the LHS:

$$F^{\mu\lambda}F^{\nu}{}_{\lambda} - (\star F)^{\mu\lambda}(\star F)^{\nu}{}_{\lambda} = F^{\mu\lambda}\eta^{\nu\sigma}F_{\sigma\lambda} - (\star F)^{\mu\lambda}\eta^{\nu\sigma}(\star F)_{\sigma\lambda}$$
$$= F^{\mu\lambda}F^T_{\lambda\sigma}\eta^{\sigma\nu} - (\star F)^{\mu\lambda}(\star F)^T_{\lambda\sigma}\eta^{\sigma\nu} = \left[\left(\mathbf{F}^\bullet\mathbf{F}_\bullet^T - (\star\mathbf{F}^\bullet)(\star\mathbf{F}_\bullet)^T\right)\boldsymbol{\eta}\right]^{\mu\nu},$$

in which $\mathbf{F}^\bullet$ and $\mathbf{F}_\bullet$ indicate the field strength tensor in contravariant and covariant indices respectively, where the same is true also for the dual field strength tensor. Using eqs. (A.13) & (A.15), the matrix products can be calculated explicitly in a local Cartesian frame by elementary algebraic principles. The result reads:

$$\mathbf{F}^\bullet\mathbf{F}_\bullet^T - (\star\mathbf{F}^\bullet)(\star\mathbf{F}_\bullet)^T = \left(\mathbf{B}^2 - \mathbf{E}^2\right)\mathbb{1} = 2\mathscr{F}\mathbb{1}.$$

In the last line we have used eq. (A.14). Consequently, we arrive at:

$$\left[\mathbf{F}^\bullet \mathbf{F}_\bullet^T - (\star\mathbf{F}^\bullet)(\star\mathbf{F}_\bullet)^T\right]\boldsymbol{\eta} = 2\mathscr{F}\boldsymbol{\eta}.$$

Case (b) no. (i) works similar; here, we do not have to distinguish between covariant and contravariant indices since we work on an Euclidean background, i.e. $\mathbf{F}^\bullet = \mathbf{F}_\bullet \equiv \mathbf{F}$ and analogously for the dual field strength. In this way, a similar calculation as for case (a) no. (i) above gives:

$$F^{\mu\lambda}F^{\nu}{}_{\lambda} + (\star F)^{\mu\lambda}(\star F)^{\nu}{}_{\lambda} = \left[\left(\mathbf{F}\mathbf{F}^T + (\star\mathbf{F})(\star\mathbf{F})^T\right)\boldsymbol{\delta}\right]^{\mu\nu}.$$

Again, we can perform the matrix multiplication explicitly using eqs.(A.24) & (A.27). Together with eq. (A.21) the result is:

$$\left(\mathbf{F}\mathbf{F}^T + (\star\mathbf{F})(\star\mathbf{F})^T\right)\boldsymbol{\delta} = \left(\mathbf{B}^2 + \mathbf{E}^2\right)\boldsymbol{\delta} = 2\mathscr{F}\boldsymbol{\delta}.$$

Now let us concentrate on case (a) no. (ii). First, a simple restructuring of the LHS yields:

$$F^{\mu\lambda}(\star F)^{\nu}{}_{\lambda} = F^{\mu\lambda}(\star F)^T_{\lambda\sigma}\eta^{\sigma\nu} = \left[\mathbf{F}^\bullet(\star\mathbf{F}_\bullet)^T\boldsymbol{\eta}\right]^{\mu\nu}.$$

With eqs. (A.13) & (A.15) we find:

$$\mathbf{F}^\bullet(\star\mathbf{F}_\bullet)^T\boldsymbol{\eta} = -\mathbf{E}\cdot\mathbf{B}\boldsymbol{\eta} = \mathscr{G}\boldsymbol{\eta}.$$

To perform the last step we have used eq. (A.16). Since $\boldsymbol{\eta}$ is symmetric, so must $\mathbf{F}^\bullet(\star\mathbf{F}_\bullet)^T\boldsymbol{\eta}$ and hence:

$$\mathbf{F}^\bullet(\star\mathbf{F}_\bullet)^T\boldsymbol{\eta} = \left[\mathbf{F}^\bullet(\star\mathbf{F}_\bullet)^T\boldsymbol{\eta}\right]^T = \boldsymbol{\eta}(\star\mathbf{F}_\bullet)(\mathbf{F}^\bullet)^T = \boldsymbol{\eta}\Big(\boldsymbol{\eta}(\star\mathbf{F}^\bullet)\boldsymbol{\eta}\Big)\Big(\boldsymbol{\eta}\mathbf{F}_\bullet\boldsymbol{\eta}\Big)^T = (\star\mathbf{F}^\bullet)\mathbf{F}_\bullet^T\boldsymbol{\eta}.$$

This can be expressed in components as:

$$\left[\mathbf{F}^\bullet(\star\mathbf{F}_\bullet)^T\boldsymbol{\eta}\right]^{\mu\nu} = \left[(\star\mathbf{F}^\bullet)\mathbf{F}_\bullet^T\boldsymbol{\eta}\right]^{\mu\nu} = (\star F)^{\mu\lambda}F^T_{\lambda\sigma}\eta^{\sigma\nu} = (\star F)^{\mu\lambda}F^{\nu}{}_{\lambda},$$

which proves the first equality of case (a) no. (ii), and with $\mathbf{F}^\bullet(\star\mathbf{F}_\bullet)^T\boldsymbol{\eta} = \mathscr{G}\boldsymbol{\eta}$ we also have successfully showed the whole equality sequence. The proof for case (b) no. (ii) works analogous and is thus not presented here explicitly.

□

B

Dimensional Analysis

In search of fixed points for a given renormalisation flow within the FRG framework, it is necessary to find suitable solutions to the fixed point equation (2.22). In order to simplify calculations, it often might be beneficial to perform a transition to *dimensionless generalised couplings*. In this way, the ERGE becomes autonomous and the fixed point equation takes a concise form, which we have already extensively discussed in sec. 2.2. A successful accomplishment of this step requires a careful dimensional analysis of the underlying field operators beforehand. Since the focus of this work aims on vector particles which are described by a gauge field A and a set of two U(1) group action invariant quantities, $\mathscr{F}$ and $\mathscr{G}$, we deduce a collection of basic information about the mass units of these and closely related objects in the course of this Appendix.

Before we begin, it is worth recalling our conventions, according to which we work in natural units where both the speed of light c and the reduced Planck constant $\hbar$ acquire the numerical value 1 and are therefore considered to be dimensionless.

Actions. Now that $\hbar$ carries the same physical unit as an action, this implies that all action functionals are dimensionless in natural units, e.g. the bare action S, the effective action Γ, or the effective average action Γ_k:

$$[S] = [\Gamma] = [\Gamma_k] = 0. \tag{B.1}$$

The same is indeed also true for the gauge-fixing action S_{gf} as well as the cutoff action ΔS_k.

Fields. Let us consider a general theory, characterised by an action S, that in principle can contain several degrees of freedom which we summarise in a collective field $\Phi \mathrel{\hat{=}} (\phi_1, \phi_2, \dots)$ defined with respect to coordinate space. It is rather common to assume that each theory exhibits a kinetic contribution of the generic form $\sim (\partial_\mu \Phi^n)(\partial^\mu \Phi_n)$, where summation over the index n is understood. Recalling that action functionals are dimensionless, it follows:

$$0 = [S] = \left[(\partial_\mu \Phi^n)(\partial^\mu \Phi_n)\, \mathrm{d}^d x\right] = \left[\partial^2\right] + \left[\mathrm{d}^d x\right] + \left[\Phi^2\right] = 2 - d + 2[\Phi], \tag{B.2}$$

where we have used the fact that $[x] = -1$, which directly follows from choosing natural units[1]. Hence $\left[\mathrm{d}^d x\right] = -d$ and $[\partial] = 1$. Solving for $[\Phi]$ yields:

$$[\Phi] = \frac{d}{2} - 1. \tag{B.3}$$

However, instead of integrating over coordinate space, we can alternatively switch to momentum space by means of a Fourier transformation; $\Phi \mapsto \tilde{\Phi}$; $\tilde{\Phi}(p) \triangleq \int \Phi(x) \mathrm{e}^{-\imath p \cdot x} \mathrm{d}^d x$, $\partial_\mu \mapsto -\imath p_\mu$.

[1] Since $c = 1$, temporal and spatial distances are measured by equal units. The same is true for energies and masses according to $E = mc^2$, as well as for inverse energies and temporal distances because of $\hbar = 1$. It follows, that lengths and times each carry one inverse mass unit.

Here, we agree on an exception to our conventions and explicitly indicate the momentum space representation of Φ by an extra symbol $\tilde{\Phi}$. From eq. (B.3) we get:

$$\left[\tilde{\Phi}\right] = \left[\Phi \mathrm{d}^d x\right] = [\Phi] + \left[\mathrm{d}^d x\right] = \frac{d}{2} - 1 - d = -\frac{d}{2} - 1. \tag{B.4}$$

Combining the eqs. (B.3) & (B.4), we find a useful formula:

$$\left[\tilde{\Phi}\right] = -\left[\Phi\right] - 2. \tag{B.5}$$

We see, that dynamical field variables do in general not agree in different representations in terms of their units. For example, in $d = 4$ spacetime dimensions, we have $[\Phi] = 1 \neq -3 = \left[\tilde{\Phi}\right]$.

Regulator. A key object of the FRG formalism is clearly the regulator function $\mathcal{R}_k$, as it combines Wilson's idea of the renormalisation group with the concept of the effective action. In order to determine the number of mass units for the regulator function, we adhere to eq. (2.9) where we first have mentioned the regulator in relation to the cutoff action ΔS_k. Since $[\Delta S_k] = 0$, the same must hold for the integral expression on the RHS. Though (2.9) is formulated in terms of a scalar field theory, the structure in which ΔS_k appears is quite common and usually also used for a wide spectrum of theories. Therefore, supposing a cutoff action as presented in eq. (2.9), we can directly infer that[2]:

$$0 = [\Delta S_k] = \left[\tilde{\Phi}\mathcal{R}_k\tilde{\Phi}\mathrm{d}^d p\right] = 2\left[\tilde{\Phi}\right] + \left[\mathrm{d}^d p\right] + [\mathcal{R}_k] = -d - 2 + d + [\mathcal{R}_k]\,, \tag{B.6}$$

in which eq. (B.5) was used. Solving for $[\mathcal{R}_k]$ gives:

$$[\mathcal{R}_k] = 2. \tag{B.7}$$

Hence it useful to define a dimensionless regulator by factoring out a squared momentum, $\mathcal{R}_k \sim p^2 R$, which ultimately introduces the shape function R with $[R] = 0$ per construction.

Potential. In ch. 3 we primarily work with the effective average potential $\mathscr{W}_k$, whose integral over some spacetime volume essentially leads to the effective average action. What is more, not only $\mathscr{W}_k$, but basically all Lagrangian-like objects that yield action-like quantities by means of their spacetime integration must carry the same number of mass units as the number of considered spacetime dimensions. This is because the integration measure, $\mathrm{d}^d x$, brings $-d$ mass units, which we have to compensate to render the action dimensionless. In particular we have:

$$[\mathscr{W}_k] = d. \tag{B.8}$$

Invariants. As it was mentioned in ch. 3, $\mathscr{F}$ represents the (Euclidean) Lagrangian for freely propagating spin-1 bosons, thus $[\mathscr{F}] = [\mathscr{W}_k]$. In addition, $\mathscr{F}$ is constructed from the total contraction of the field strength tensor, $\mathbf{F}$, with itself. Since the Levi-Civita tensor is dimensionless, $\mathbf{F}$ and its Hodge dual, $\star\mathbf{F}$, must agree on their units. Hence, we can pithily state that $\mathscr{G} \sim \mathbf{F}^2 \sim \mathscr{F}$ in terms of units. In summary, our findings are:

$$[\mathscr{F}] = d = [\mathscr{G}]\,. \tag{B.9}$$

Field strength und dual field strength. From eq. (B.9) and the fact that, for instance, $\mathscr{F} \sim \mathbf{F}^2$, we immediately find the number of mass units for the field strength and dual field strength tensor:

$$[\mathbf{F}] = \frac{d}{2} = [\star\mathbf{F}]\,. \tag{B.10}$$

Gauge field. If we go back one more step starting from the field strength tensor, we finally arrive at the fundamental degree of freedom, i.e. the gauge field A. As it is well known, it is

[2]Here we note, that momenta are defined as products of mass and velocity. Since $c = 1$, it immediately follows: $[p] = [\text{mass}] = 1$.

related to the field strength according to $\mathbf{F} \sim \partial A$. Using eq. (B.10) and recalling that $[\partial] = 1$, we find:

$$[A] = \frac{d}{2} - 1, \tag{B.11}$$

which coincides with eq. (B.3).

Generalised couplings. The effective average potential $\mathscr{W}_k$ can be interpreted as the zeroth order contribution of a derivative expansion of the full effective average Lagrangian $\mathscr{L}_k$, that, besides non-derivative terms, can also contain derivative structures applied on the field strength tensor $\mathbf{F}$. Since $\mathscr{W}_k$ only depends on the invariants $\mathscr{F}$ and $\mathscr{G}$, we can suppose the following formulation:

$$\mathscr{W}_k\left(\mathscr{F}, \mathscr{G}\right) = \sum_{(n,m)\in\mathbb{Z}^2} u_{nm}(k)\mathscr{F}^n\mathscr{G}^m, \tag{B.12}$$

where u_{nm} denotes the generalised (scale-dependent) coupling belonging to the pair of indices (n, m). We are interested in its unit. For this we make use of eqs. (B.8) & (B.9), from which we find for all $(n, m) \in \mathbb{Z}^2$:

$$[\mathscr{W}_k] = [u_{nm}\mathscr{F}^n\mathscr{G}^m] \quad \Rightarrow \quad d = n\,[\mathscr{F}] + m\,[\mathscr{G}] + [u_{nm}] = (n+m)d + [u_{nm}]\,. \tag{B.13}$$

This finally leads to:

$$[u_{nm}] = (1 - n - m)\,d. \tag{B.14}$$

Potential derivatives. Another set of useful relations concern different derivative structures over $\mathscr{W}_k$. They can be obtained by combining the eqs. (B.8) & (B.9) and read:

$$\begin{aligned} [\mathscr{W}_k'] &= \left[\frac{\partial\mathscr{W}_k}{\partial\mathscr{F}}\right] = d - d = 0, & [\mathscr{W}_k''] &= \left[\frac{\partial^2\mathscr{W}_k}{\partial\mathscr{F}^2}\right] = d - 2d = -d, \\ \left[\dot{\mathscr{W}}_k\right] &= \left[\frac{\partial\mathscr{W}_k}{\partial\mathscr{G}^2}\right] = d - 2d = -d, & \left[\ddot{\mathscr{W}}_k\right] &= \left[\frac{\partial^2\mathscr{W}_k}{\partial\left(\mathscr{G}^2\right)^2}\right] = d - 4d = -3d. \end{aligned} \tag{B.15}$$

Moreover, for a mixed derivative we have:

$$\left[\dot{\mathscr{W}}_k'\right] = \left[\frac{\partial^2\mathscr{W}_k}{\partial\mathscr{F}\partial\mathscr{G}^2}\right] = d - d - 2d = -2d. \tag{B.16}$$

Field strength renormalisation. Let us turn to another quantity that first gains relevance during ch. 3. There, we are often confronted with the field strength renormalisation Z_k. As the name suggests, Z_k can be understood as a normalisation factor for field operators that respects dimensional settings. This means that field operators exhibit the same unit as before renormalisation, hence Z_k must be dimensionless:

$$[Z_k] = 0. \tag{B.17}$$

Anomalous dimension. Finally, from the definition of the anomalous dimension, $\eta_k \triangleq -\partial_t \ln\left(Z_k\right)$, and after recalling that the RG time parameter is dimensionless per construction, we deduce:

$$[\eta_k] = 0. \tag{B.18}$$

This completes our list of basic information about the dimensional settings of this thesis.

C

The Hypergeometric Function

During sec. 3.2 we are confronted with so-called *threshold functions* which are build upon the choice of an optimised regulator. The former are structurally closely related to a special mathematical function that is known as the *hypergeometric function*. In order to prepare for an appropriate treatment, let us collect basic facts about this function and list some identites for it in what follows.

Differential equation. The hypergeometric function serves to express solutions for second-order ordinary differential equations that exhibit at most three regular singular points [53]. Differential equations of this type can always brought into a generic form which is known as the *hypergeometric differential equation*. A solution to this equation, that is a complex function $w : G \to U \subseteq \mathbb{C}$ which is defined for some region $G \subseteq \mathbb{C}$, satisfies [53]:

$$z\,(1-z)\,\frac{\mathrm{d}^2 w(z)}{\mathrm{d}z^2} + \Big[c - (a+b+1)\,z\Big]\frac{\mathrm{d}w(z)}{\mathrm{d}z} - ab w(z) = 0, \qquad \text{for all } z \in G. \tag{C.1}$$

There exists two fundamental solutions to eq. (C.1). One of them is the hypergeometric function itself, which is often denoted as ${}_2F_1\,(a, b; c; z)$ with complex valued parameters $a, b \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Gauss series. A useful series representation of ${}_2F_1$, properly defined on the unit disk where $|z| < 1$ holds, is provided by the so-called *Gauss series* [49]:

$$\boxed{{}_2F_1\,(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n}. \tag{C.2}$$

Here, Γ denotes the Gamma function, which can be explicitly calculated for instance using Euler's integral formula:

$$\Gamma\,(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t, \tag{C.3}$$

for all $z \in \mathbb{C}$ with $\Re\,[z] > 0$.

Principal branch. In many applications and formulas regarding the hypergeometric function ${}_2F_1$, one presupposes their validity only on the *principal branch*, which is the complex subset $\{z \in \mathbb{C}\,|\,|\arg\,(1-z)\,| \leq \pi\}$. It is topologically obtained by cutting the interval $(1, \infty) \subset \mathbb{R}$ from the complex plane. Like it was said before, a majority of the identities for the hypergeometric function are only valid in the region of the principal branch. This circumstance should be taken with care during explicit computations that include different representations of the hypergeometric function.

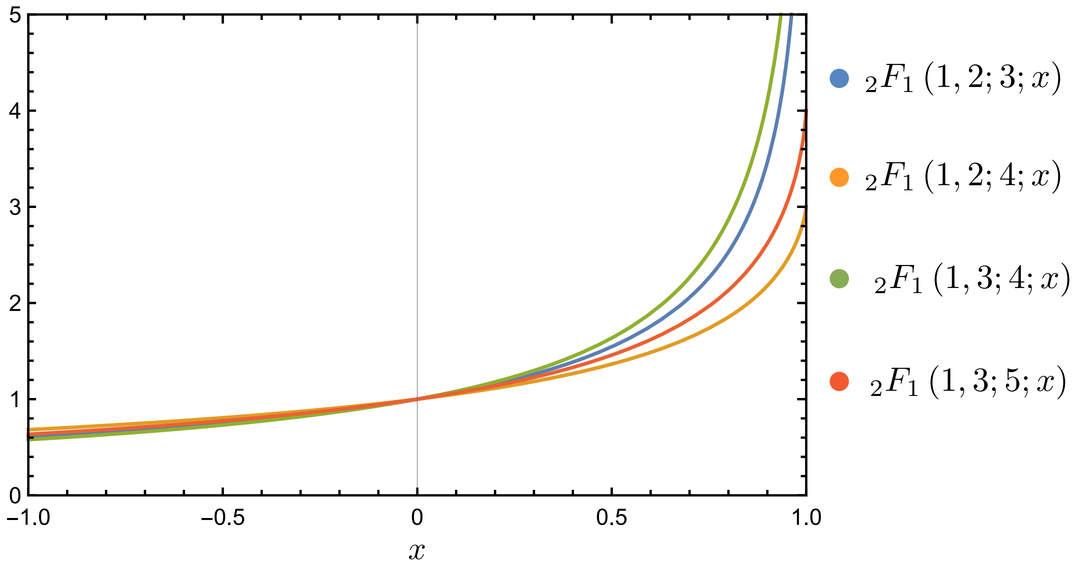

Figure C.1: *Four examples that show concrete realisations of the hypergeometric function over the symmetric unit interval on* $\mathbb{R}$*. The chosen parameter values are motivated through the fixed point equation derived in sec. 3.2, where here we have set the number of spacetime dimensions to four. In this context, the real variable* x *is either referred to* $1-w'_*$ *or* $1-w'_*-w''_*\tilde{\mathscr{F}}$*. As it can be seen from eq. (C.5), the blue and green curve suggest to develop an irregular behaviour as* $x\to 1$*, because* $c-a-b=0$ *holds in both cases and* $\Gamma(0)$ *is ill-defined.*

Integral representation. Beside the Gauss series representation, we can express the hypergeometric function by its *Euler type integral representation* which is of particular interest for our purposes. It is valid on the principal branch excluding the branch point $z=1$ and given by the following formula [53]:

$$\mathrm{B}\,(b,c-b)\,{}_2F_1\,(a,b;c;z)=\int_0^1 \frac{r^{b-1}\,(1-r)^{c-b-1}}{(1-zr)^{a}}\mathrm{d}r,\quad \text{for } \Re\,(c)>\Re\,(b)>0. \tag{C.4}$$

Here, B denotes the Euler beta function. It is defined by the following integral:

$$B(u,v)=\int_0^1 t^{u-1}(1-t)^{v-1}\mathrm{d}t, \tag{C.5}$$

for all $u,v\in\mathbb{C}$ with $\Re[u],\Re[v]>0$.

Beta function. The central result regarding the theory of Euler's beta function is its famous relation to the Gamma function, given by:

$$\mathrm{B}\,(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},\quad \text{for } u,v\in\mathbb{C} \text{ with } \Re[u],\Re[v]>0. \tag{C.6}$$

Conclusively, a selection of sample plots for some triples (a,b,c) are depicted in fig. C.1. They correspond to solutions of threshold functions with which we are dealing during the fixed point analysis in sec. 3.2 (cf. eq. (3.64)).

D

Proof & Computation History

During chs. 2 & 3, we have stated some results but deliberately omitted corresponding reasonings due to their rather large extent. This happened with the intention to prevent undesired disruptions of the text flow. The purpose of this Appendix is now to catch up on these technical details and provide proofs of individual results. We will referencing each calculation to its corresponding equation number or position in the main text.

Lemma D.1 (*Field expectation value*, eq. (2.5)).
For a scalar field theory, with field operators $\hat{\varphi}$ and effective action $\Gamma \mathrel{\hat{=}} \Gamma[\phi]$, the following relation holds:

$$\phi(x) = \left[\frac{1}{\mathcal{Z}[J]}\frac{\delta \mathcal{Z}[J]}{\delta J(x)}\right]\Bigg|_{J=\tilde{J}}, \tag{D.1}$$

where $\tilde{J}$ denotes the sorce function which fulfills the supremum condition for Γ.

Proof. First, we recall the definition of the effective action from eq. (2.4):

$$\Gamma[\phi] \triangleq \sup_J \left(\int \phi(x)J(x)\mathrm{d}^d x - \mathcal{W}[J]\right),$$

where $\mathcal{W}$ is defined in eq. (2.2). When evaluating the LHS of eq. (D.1) at the source $J = \tilde{J}$ for which the supremum is accepted, we arrive at an extremum of the corresponding expression by means of its least upper bound when considered as a functional of the source J. That means, the (functional) derivative of the supremum argument with respect to $J(x)$ must vanish when evaluated at $J = \tilde{J}$:

$$\begin{aligned} 0 &= \left[\frac{\delta}{\delta J(x)}\left(\int \phi(y)J(y)\mathrm{d}^d y - \mathcal{W}[J]\right)\right]\Bigg|_{J=\tilde{J}} \\ &= \left[\int \phi(y)\frac{\delta J(y)}{\delta J(x)}\mathrm{d}^d y - \frac{\delta \mathcal{W}[J]}{\delta J(x)}\right]\Bigg|_{J=\tilde{J}} \\ &= \underbrace{\int \phi(y)\delta^{(d)}(x,y)\mathrm{d}^d y}_{=\phi(x)} - \left[\frac{\delta \mathcal{W}[J]}{\delta J(x)}\right]\Bigg|_{J=\tilde{J}}. \end{aligned}$$

Rearranging terms and using $\mathcal{W} = \ln(\mathcal{Z})$ finally yields:

$$\phi(x) = \left[\frac{\delta \mathcal{W}[J]}{\delta J(x)}\right]\Bigg|_{J=\tilde{J}} \triangleq \left[\frac{1}{\mathcal{Z}[J]}\frac{\delta \mathcal{Z}[J]}{\delta J(x)}\right]\Bigg|_{J=\tilde{J}}$$

Side note: according to eq. (2.3), we can declare the 1-point correlation function, i.e. the vacuum expectation value $\langle\cdot\rangle_{\text{vac}}$ of the field operator $\hat{\varphi}$ in presence of a source term:

$$\langle\hat{\varphi}_{\tilde{J}}(x)\rangle_{\text{vac}} := \left[\frac{1}{\mathcal{Z}[J]}\frac{\delta\mathcal{Z}[J]}{\delta J(x)}\right]\Bigg|_{J=\tilde{J}} \equiv \phi(x).$$

This reproduces eq. (2.5) in total.

□

Lemma D.2 (*Regularisation operator*, eq. (2.9)).
In the framework of the functional renormalisation group, let $\mathcal{R}_k$ be the regulator which implements the infrared momentum mode suppression and let ΔS_k denote the cutoff action built upon $\mathcal{R}_k$. Furthermore, let φ be the underlying dynamical field variable. The operator representation of $\mathcal{R}_k$ can be written by means of ΔS_k as:

$$\Delta S_k[\varphi] = \frac{1}{2}\int \varphi(x)\mathcal{R}_k\left(-\partial^2\right)\varphi(x)\mathrm{d}^d x, \tag{D.2}$$

and is equivalent to eq. (2.9). Here, $\partial^2 \equiv \square$ means the d'Alembert operator.

Proof. Since we know, that the cutoff action ΔS_k in its form of eq. (2.9) is already well-constructed, we start with eq. (D.2) and show that eq. (2.9) follows directly from it. Using the Fourier decomposition of φ and noting that $\mathrm{e}^{\imath p\cdot x}$ is an eigenvector of $-\partial^2$ with the corresponding eigenvalue being p^2, we get:

$$\begin{aligned}\int \varphi(x)\mathcal{R}_k\left(-\partial^2\right)\varphi(x)\mathrm{d}^d x &= \int \varphi(x)\mathcal{R}_k\left(-\partial^2\right)\left(\int \varphi(p)\mathrm{e}^{\imath p\cdot x}\frac{\mathrm{d}^d p}{(2\pi)^d}\right)\mathrm{d}^d x\\ &= \int\int \varphi(x)\underbrace{\left(\mathcal{R}_k\left(-\partial^2\right)\mathrm{e}^{\imath p\cdot x}\right)}_{=\mathcal{R}_k(p^2)\mathrm{e}^{\imath p\cdot x}}\varphi(p)\mathrm{d}^d x\frac{\mathrm{d}^d p}{(2\pi)^d}\\ &= \int \underbrace{\left(\int \varphi(x)\mathrm{e}^{\imath p\cdot x}\mathrm{d}^d x\right)}_{=\varphi(-p)}\mathcal{R}_k(p^2)\varphi(p)\frac{\mathrm{d}^d p}{(2\pi)^d}\\ &\equiv \int \varphi(-p)\mathcal{R}_k(p)\varphi(p)\frac{\mathrm{d}^d p}{(2\pi)^d}\\ &\equiv 2\Delta S_k[\varphi].\end{aligned}$$

This proves that eq. (2.9) is indeed equivalent to eq. (D.2).

□

Theorem D.3 (*Quantum equations of motion*, sec. 2.1; par. **A**).
The effective action Γ satisfies the so-called quantum equations of motion:

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = \tilde{J}(x). \tag{D.3}$$

Here, $\tilde{J}$ denotes the source function which matches the supremum condition on the LHS of eq. (2.4).

Proof. We use the supremum definition of the effective action, evaluate at $J = \tilde{J}$ and perform a functional derivative with respect to ϕ:

$$\begin{aligned}
\frac{\delta\Gamma[\phi]}{\delta\phi(x)} &\triangleq \frac{\delta}{\delta\phi(x)}\left(\int \phi(y)\tilde{J}(y)\mathrm{d}^d y - \mathcal{W}\left[\tilde{J}\right]\right) \\
&= \int \frac{\delta}{\delta\phi(x)}\left(\phi(y)\tilde{J}(y)\right)\mathrm{d}^d y - \frac{\delta\mathcal{W}\left[\tilde{J}\right]}{\delta\phi(x)} \\
&= \int \left(\frac{\delta\phi(y)}{\delta\phi(x)}\right)\tilde{J}(y)\mathrm{d}^d y + \int \phi(y)\left(\frac{\delta\tilde{J}(y)}{\delta\phi(x)}\right)\mathrm{d}^d y - \int \frac{\delta\mathcal{W}\left[\tilde{J}\right]}{\delta\tilde{J}(y)}\frac{\delta\tilde{J}(y)}{\delta\phi(x)}\mathrm{d}^d y,
\end{aligned}$$

where we took into account that $\tilde{J} \,\hat{=}\, \tilde{J}(\phi)$ implicitly depends on the variable ϕ, since it is the supremum of a ϕ dependent expression. Furthermore, we have used the chain rule of functional calculus for the last term $\delta\mathcal{W}\left[\tilde{J}\right]/\delta\phi(x)$ in the last line. According to Lemma D.1, we can replace $\delta\mathcal{W}\left[\tilde{J}\right]/\delta\tilde{J}(y)$ by the field expectation value $\phi(y)$ and obtain:

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = \int \tilde{J}(y)\delta^{(d)}(x,y)\mathrm{d}^d y + \underbrace{\int \phi(y)\left(\frac{\delta\tilde{J}(y)}{\delta\phi(x)}\right)\mathrm{d}^d y - \int \phi(y)\left(\frac{\delta\tilde{J}(y)}{\delta\phi(x)}\right)\mathrm{d}^d y}_{=0} = \tilde{J}(x).$$

This finishes the proof.

□

Corollary D.4 (*Regulated quantum equations of motion*).
Let Γ_k be the smooth interpolation between a bare action S and an effective action Γ, for which a regulator $\mathcal{R}_k$ is chosen. Then, the quantum equations of motion (D.2) extend to:

$$\frac{\delta\Gamma_k[\phi]}{\delta\phi(x)} = \tilde{J}(x) - (\mathcal{R}_k\phi)(x). \tag{D.4}$$

Proof. We start with the definition of the effective average action Γ_k, given by eq. (2.11):

$$\Gamma_k[\phi] \triangleq \sup_J\left(\int \phi(x)J(x)\mathrm{d}^d x - \mathcal{W}_k[J]\right) - \Delta S_k[\phi],$$

in which ΔS_k denotes the cutoff action from eq. (2.9), and $\mathcal{W}_k$ is the scale dependent Schwinger functional, defined with respect to eq. (2.8) by $\mathcal{W}_k \equiv \ln(\mathcal{Z}_k)$. With Theorem D.3, we find:

$$\frac{\delta\Gamma_k[\phi]}{\delta\phi(x)} = \underbrace{\frac{\delta}{\delta\phi(x)}\left(\int \phi(y)\tilde{J}(y)\mathrm{d}^d y - \mathcal{W}_k\left[\tilde{J}\right]\right)}_{=\tilde{J}(x)} - \frac{\delta\Delta S_k[\phi]}{\delta\phi(x)},$$

where we have used the scale dependent version of Lemma D.1, which follows by an analogous proof[1] and states that $\phi(x) = \delta\mathcal{W}_k\left[\tilde{J}\right]/\delta\tilde{J}(x)$. Let us now concentrate on the functional derivative of the cutoff action with respect to ϕ. Under consideration of Lemma D.2 it follows:

$$\begin{aligned}
\frac{\delta\Delta S_k[\phi]}{\delta\phi(x)} &\triangleq \frac{1}{2}\frac{\delta}{\delta\phi(x)}\int \phi(y)\mathcal{R}_k\left(-\partial^2\right)\phi(y)\mathrm{d}^d y \\
&= \frac{1}{2}\left[\int \frac{\delta\phi(y)}{\delta\phi(x)}\mathcal{R}_k\left(-\partial^2\right)\phi(y)\mathrm{d}^d y + \int \phi(y)\mathcal{R}_k\left(-\partial^2\right)\frac{\delta\phi(y)}{\delta\phi(x)}\mathrm{d}^d y\right].
\end{aligned}$$

[1] The only new aspect appears by taking the functional derivative of $\Delta S_k[\phi]$ with respect to the source function J (not the "supremum source" $\tilde{J}$!), which obviously vanishes.

Now we use that $\delta\phi(y)/\delta\phi(x) = \delta^{(d)}(x,y)$, upon which the first integral directly collapses on $y = x$. The second integral is more subtle, since the operator $\mathcal{R}_k(-\partial^2)$ acts on the Dirac distribution. Nevertheless, we can circumvent any problems in this context by using integration by parts to transfer the (derivative) operator $\mathcal{R}_k(-\partial^2)$ in front of $\phi(y)$ and then solve the integral straightforwardly. For this we need the assumption of considering only situations where boundary terms can be neglected, what is, however, often done. In total, we have to perform two integration by parts, because $\mathcal{R}_k(-\partial^2)$ is considered as a second order differential operator. As an intermediate result, we find:

$$\int \phi(y)\mathcal{R}_k\left(-\partial^2\right)\frac{\delta\phi(y)}{\delta\phi(x)}\mathrm{d}^d y = \int \delta^{(d)}(x,y)\mathcal{R}_k\left(-\partial^2\right)\phi(y)\mathrm{d}^d y = \mathcal{R}_k\left(-\partial^2\right)\phi(x).$$

Altogether we arrive at:

$$\frac{\delta\Delta S_k[\phi]}{\delta\phi(x)} = \frac{1}{2}\left[\mathcal{R}_k\left(-\partial^2\right)\phi(x) + \mathcal{R}_k\left(-\partial^2\right)\phi(x)\right] = \mathcal{R}_k\left(-\partial^2\right)\phi(x) \equiv \left(\mathcal{R}_k\phi\right)(x).$$

Going back to the functional differentiation of Γ_k we finally obtain:

$$\frac{\delta\Gamma_k[\phi]}{\delta\phi(x)} = \tilde{J}(x) - \frac{\delta\Delta S_k[\phi]}{\delta\phi(x)} = \tilde{J}(x) - \left(\mathcal{R}_k\phi\right)(x),$$

which completes the proof.

□

Theorem D.5 (*Wetterich equation*, eq. (2.15)).
Let S be the bare action of a scalar field theory with φ being the field variable. Let Γ_k denote the effective average action that corresponds to S and a given regulator function $\mathcal{R}_k$. Then, Γ_k satisfies the Wetterich flow equation:

$$\partial_t\Gamma_k[\phi] = \frac{1}{2}\mathbf{Tr}\left[\left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)^{-1}\partial_t\mathcal{R}_k\right]. \tag{D.5}$$

Here, $t \equiv \ln(k/k_0)$ with k_0 being an arbitrary reference scale, ϕ means the vacuum expectation value of φ in presence of the source function $\tilde{J}$ that satisfies the supremum condition for Γ_k, and $\Gamma_k^{(2)}$ is a shortcut for the second functional derivative of Γ_k with respect to its argument ϕ.

Proof. We start with the scale-dependent version of Lemma D.1, which we recently used to prove Corollary D.4 (cf. footnote on the previous page) and take an additional functional derivative:

$$\frac{\delta\phi(x)}{\delta\tilde{J}(y)} = \frac{\delta}{\delta\tilde{J}(y)}\left(\frac{\delta\mathcal{W}\left[\tilde{J}\right]}{\delta\tilde{J}(x)}\right) = \frac{\delta^2\mathcal{W}\left[\tilde{J}\right]}{\delta\tilde{J}(x)\delta\tilde{J}(y)} \equiv G_k(x-y).$$

In the last step, we have introduced the scale dependent connected propagator G_k (cf. paragraph **B** of sec. 2.1). On the other hand, we can also take on more derivative of the source $\tilde{J}$ with respect to the field ϕ using Corollary D.4:

$$\frac{\delta\tilde{J}(x)}{\delta\phi(y)} = \frac{\delta}{\delta\phi(y)}\left(\frac{\delta\Gamma_k[\phi]}{\delta\phi(x)} + \left(\mathcal{R}_k\phi\right)(x)\right) = \underbrace{\frac{\delta^2\Gamma_k[\phi]}{\delta\phi(x)\delta\phi(y)}}_{\equiv\Gamma_k^{(2)}[\phi](x,y)} + \frac{\delta}{\delta\phi(y)}\mathcal{R}_k\left(-\partial^2\right)\phi(x)$$

$$= \Gamma_k^{(2)}[\phi](x,y) + \mathcal{R}_k\left(-\partial^2\right)\delta^{(d)}(x,y) \equiv \Gamma_k^{(2)}[\phi](x,y) + \mathcal{R}_k(x,y).$$

Here we have defined a shortcut; $\mathcal{R}_k(x,y) \equiv \mathcal{R}_k\left(-\partial^2\right)\delta^{(d)}(x,y)$. With these results we can now derive eq. (2.13):

$$\delta^{(d)}(x,y) = \frac{\delta\phi(x)}{\delta\phi(y)} = \int \frac{\delta\phi(x)}{\delta\tilde{J}(u)}\frac{\delta\tilde{J}(u)}{\delta\phi(y)}\mathrm{d}^d u = \int G_k(x-u)\left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)(u,y)\mathrm{d}^d u.$$

The last expression represents the (x,y) component of the operator product between G_k and $\Gamma_k^{(2)}[\phi] + \mathcal{R}_k$, which is normalised and non-vanishing only for $x=y$. In operator notation, this implies that $G_k \cdot \left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)$ must be equal to unity. We can express this by:

$$G_k = \left(\Gamma_k^{(2)}[\phi] + \mathcal{R}_k\right)^{-1}.$$

Before we continue, it is important to note that the "supremum source" $\tilde{J}$ does not only exhibit an implicit ϕ dependence, but since it chosen in such a way, that the scale-dependent expression $\int \phi(x)J(x)\mathrm{d}^d x - \mathcal{W}_k[J]$ evaluated for $J = \tilde{J}$ approaches its supremum, there is also an implicit k dependence for $\tilde{J}$. We can make this clear by writing $\tilde{J} \mathrel{\hat{=}} \tilde{J}_k[\phi]$, though we will not adhere to this notation and use $\tilde{J}$ further on, but keeping this aspect in mind. Conclusively, the t derivative of Γ_k follows from its definition, eq. (2.11), and reads:

$$\begin{aligned}
\partial_t\Gamma_k[\phi] &\triangleq \partial_t\left(\int \phi(x)\tilde{J}(x)\mathrm{d}^d x - \mathcal{W}_k\left[\tilde{J}\right] - \Delta S_k[\phi]\right) \\
&= \int \phi(x)\left(\partial_t\tilde{J}(x)\right)\mathrm{d}^d x - \partial_t\left(\mathcal{W}_k\left[\tilde{J}\right]\right) - \partial_t\Delta S_k[\phi] \\
&= \int \phi(x)\left(\partial_t\tilde{J}(x)\right)\mathrm{d}^d x - \partial_t\mathcal{W}_k\left[\tilde{J}\right] - \int \frac{\delta\mathcal{W}_k\left[\tilde{J}\right]}{\delta\tilde{J}(x)}\left(\partial_t\tilde{J}(x)\right)\mathrm{d}^d x - \partial_t\Delta S_k[\phi].
\end{aligned}$$

In the last line, we considered to use the chain rule when differentiating $\mathcal{W}_k\left[\tilde{J}\right]$ with respect to t, since $\mathcal{W}_k\left[\tilde{J}\right]$ depends explicitly on k per construction (indicated by the k subscript), but in addition depends also implicitly on k via its argument $\tilde{J}$. With the scale-dependent extension of Lemma D.1, we can replace $\delta\mathcal{W}_k\left[\tilde{J}\right]/\delta\tilde{J}(x)$ with $\phi(x)$ and thus the first and third term in our last result cancel with each other. The remaining expression reads:

$$\partial_t\Gamma_k[\phi] = -\partial_t\mathcal{W}_k\left[\tilde{J}\right] - \partial_t\Delta S_k[\phi].$$

Now we need to evaluate the explicit t derivative of the scale-dependent Schwinger functional. For this, we refer to eq. (2.8) and use the definition of $\mathcal{W}_k$ by means of the scale-dependent partition functional:

$$\begin{aligned}
\partial_t\mathcal{W}_k\left[\tilde{J}\right] &= \partial_t\ln\left(\mathcal{Z}_k\left[\tilde{J}\right]\right) = \frac{1}{\mathcal{Z}_k\left[\tilde{J}\right]}\partial_t\mathcal{Z}_k\left[\tilde{J}\right] \\
&= \frac{1}{\mathcal{Z}_k\left[\tilde{J}\right]}\partial_t\int \exp\left(-S[\varphi] - \Delta S_k[\varphi] + \int \varphi(x)\tilde{J}(x)\mathrm{d}^d x\right)[\mathcal{D}\varphi] \\
&= -\frac{1}{\mathcal{Z}_k\left[\tilde{J}\right]}\int \mathrm{e}^{-S[\varphi]-\Delta S_k[\varphi]+\int\varphi(x)\tilde{J}(x)\mathrm{d}^d x}\partial_t\Delta S_k[\varphi][\mathcal{D}\varphi] \\
&= -\frac{1}{2}\int\left(\frac{1}{\mathcal{Z}_k\left[\tilde{J}\right]}\int \varphi(-p)\varphi(p)\mathrm{e}^{-S[\varphi]-\Delta S_k[\varphi]+\int\varphi(x)\tilde{J}(x)\mathrm{d}^d x}[\mathcal{D}\varphi]\right)\partial_t\mathcal{R}_k(p)\frac{\mathrm{d}^d p}{(2\pi)^d}.
\end{aligned}$$

In the last step, we have used eq. (2.9) in order to express ΔS_k in terms of the regulator function $\mathcal{R}_k$. The term in paranthesis is per definition precisely the vacuum expectation value of the operator product $\hat{\varphi}(-p)\hat{\varphi}(p)$ in presence of the source $\tilde{J}$, i.e:

$$\frac{1}{\mathcal{Z}_k\left[\tilde{J}\right]}\int \varphi(-p)\varphi(p)\mathrm{e}^{-S[\varphi]-\Delta S_k[\varphi]+\int \varphi(x)\tilde{J}(x)\mathrm{d}^dx}\left[\mathcal{D}\varphi\right] \equiv \langle\hat{\varphi}(-p)\hat{\varphi}(p)\rangle_{\tilde{J}}.$$

In order to proceed, we first compute the second functional derivative of the scale dependent Schwinger functional with respect to its argument:

$$\begin{aligned}
\frac{\delta^2\mathcal{W}_k\left[J\right]}{\delta J(x)\delta J(y)} &= \frac{\delta}{\delta J(y)}\left(\frac{\delta \ln\left(\mathcal{Z}_k\left[J\right]\right)}{\delta J(x)}\right) = \frac{\delta}{\delta J(y)}\left(\frac{1}{\mathcal{Z}_k\left[J\right]}\frac{\delta\mathcal{Z}_k\left[J\right]}{\delta J(x)}\right)\\
&= -\frac{1}{\mathcal{Z}_k\left[J\right]^2}\frac{\delta\mathcal{Z}_k\left[J\right]}{\delta J(x)}\frac{\delta\mathcal{Z}_k\left[J\right]}{\delta J(y)} + \frac{1}{\mathcal{Z}_k\left[J\right]}\frac{\delta^2\mathcal{Z}_k\left[J\right]}{\delta J(x)\delta J(y)}\\
&= \frac{1}{\mathcal{Z}_k\left[J\right]}\frac{\delta^2\mathcal{Z}_k\left[J\right]}{\delta J(x)\delta J(y)} - \left(\frac{1}{\mathcal{Z}_k\left[J\right]}\frac{\delta\mathcal{Z}_k\left[J\right]}{\delta J(x)}\right)\left(\frac{1}{\mathcal{Z}_k\left[J\right]}\frac{\delta\mathcal{Z}_k\left[J\right]}{\delta J(y)}\right)\\
&\equiv \langle\hat{\varphi}(x)\hat{\varphi}(y)\rangle_J - \langle\hat{\varphi}(x)\rangle_J\langle\hat{\varphi}(y)\rangle_J\\
&= \langle\hat{\varphi}(x)\hat{\varphi}(y)\rangle_J - \phi(x)\phi(y),
\end{aligned}$$

where Lemma D.1 was applied in the last step. It follows:

$$\langle\hat{\varphi}(-p)\hat{\varphi}(p)\rangle_{\tilde{J}} = \underbrace{\frac{\delta^2\mathcal{W}_k\left[\tilde{J}\right]}{\delta\tilde{J}(-p)\delta\tilde{J}(p)}}_{\equiv G_k(p)} + \phi(-p)\phi(p) = G_k(p) + \phi(-p)\phi(p).$$

Turning back to our calculation of $\partial_t\mathcal{W}_k\left[\tilde{J}\right]$ we arrive at:

$$\begin{aligned}
\partial_t\mathcal{W}_k\left[\tilde{J}\right] &= -\frac{1}{2}\int\Big(G_k(p) + \phi(-p)\phi(p)\Big)\partial_t\mathcal{R}_k(p)\frac{\mathrm{d}^dp}{(2\pi)^d}\\
&= -\frac{1}{2}\int\partial_t\mathcal{R}_k(p)G_k(p)\frac{\mathrm{d}^dp}{(2\pi)^d} - \partial_t\underbrace{\left(\frac{1}{2}\int\phi(-p)\mathcal{R}_k(p)\phi(p)\frac{\mathrm{d}^dp}{(2\pi)^d}\right)}_{\triangleq\Delta S_k[\phi]}\\
&= -\frac{1}{2}\int\partial_t\mathcal{R}_k(p)G_k(p)\frac{\mathrm{d}^dp}{(2\pi)^d} - \partial_t\Delta S_k\left[\phi\right].
\end{aligned}$$

Inserting this expression in our last intermediate result of $\partial_t\Gamma_k\left[\phi\right]$ we finally obtain:

$$\begin{aligned}
\partial_t\Gamma_k\left[\phi\right] &= -\partial_t\mathcal{W}_k\left[\tilde{J}\right] - \partial_t\Delta S_k\left[\phi\right]\\
&= \frac{1}{2}\int\partial_t\mathcal{R}_k(p)G_k(p)\frac{\mathrm{d}^dp}{(2\pi)^d} + \partial_t\Delta S_k\left[\phi\right] - \partial_t\Delta S_k\left[\phi\right]\\
&= \frac{1}{2}\int\left(\Gamma_k^{(2)}\left[\phi\right] + \mathcal{R}_k\right)^{-1}(p)\partial_t\mathcal{R}_k(p)\frac{\mathrm{d}^dp}{(2\pi)^d}.
\end{aligned}$$

Here, we now integrate (or sum) over all diagonal values of the operator $\left(\Gamma_k^{(2)}\left[\phi\right]+\mathcal{R}_k\right)^{-1}\partial_t\mathcal{R}_k$ in its momentum space representation, which is equivalent to a continuous version of the trace operator. In this sense, we accordingly abbreviate the last expression and obtain the Wetterich equation (for a scalar field theory):

$$\partial_t\Gamma_k\left[\phi\right]=\frac{1}{2}\mathbf{Tr}\left[\left(\Gamma_k^{(2)}\left[\phi\right]+\mathcal{R}_k\right)^{-1}\partial_t\mathcal{R}_k\right].$$

With this, we have shown what was claimed.

□

Proposition D.6 (*Integral-determinant relation*, eq. (2.65)).
Let V be an n-dimensional complex vector space and $\Lambda\left(V\right)$ be its exterior algebra with respect to a basis $\mathcal{B}$ of V. Let further θ be a vector with Grassmann-valued components $\theta_k\in\Lambda\left(V\right)$, where $k\in\{1,\dots,N\}$, then the determinant of a Hermitian $N\times N$ matrix $\mathbf{M}$ with eigenvalues $\{\mu_k\}_{k=1}^N$ can be represented by the following integral relation:

$$\det\left(\mathbf{M}\right)=\int \mathrm{e}^{-\theta^\dagger\mathbf{M}\theta}\mathrm{d}\bar{\theta}\mathrm{d}\theta. \tag{D.6}$$

Proof. Let us begin to calculate an ordinary one-dimensional Gaussian integral including a complex constant $\mu\in\mathbb{C}$:

$$\int\limits_{\Lambda(V)^2}\mathrm{e}^{-\bar{\theta}\mu\theta}\mathrm{d}\bar{\theta}\mathrm{d}\theta.$$

As usual, we can expand the exponential function in terms of a power series:

$$\mathrm{e}^{-\bar{\theta}\mu\theta}=\sum_{n\in\mathbb{N}_0}(-1)^n\frac{\left(\bar{\theta}\mu\theta\right)^n}{n!}=1-\mu\bar{\theta}\theta+\frac{1}{2}\underbrace{\bar{\theta}\mu\theta\bar{\theta}\mu\theta}_{=-\bar{\theta}^2\theta^2=0}+\dots,$$

where we have used the characteristic anticommutation property of Grassmann numbers; $\theta^2=0$ for all $\theta\in\Lambda\left(V\right)$. Since all higher order terms include at least two factors θ and $\bar{\theta}$, the whole power series expansion terminates after first order. This means:

$$\mathrm{e}^{-\bar{\theta}\mu\theta}=1-\mu\bar{\theta}\theta.$$

Recalling the basic rules of integral calculus over exterior algebras, in particular $\int_{\Lambda(V)}\mathrm{d}\theta=0$ and $\int_{\Lambda(V)}\theta\mathrm{d}\theta=1$, the Gaussian integral computes to:

$$\int\limits_{\Lambda(V)^2}\mathrm{e}^{-\bar{\theta}\mu\theta}\mathrm{d}\bar{\theta}\mathrm{d}\theta=\int\limits_{\Lambda(V)^2}\left(1-\mu\bar{\theta}\theta\right)\mathrm{d}\bar{\theta}\mathrm{d}\theta=\underbrace{\int\limits_{\Lambda(V)}\left(\int\limits_{\Lambda(V)}\mathrm{d}\bar{\theta}\right)\mathrm{d}\theta}_{=0}-\int\limits_{\Lambda(V)}\left(\int\limits_{\Lambda(V)}\bar{\theta}\theta\mathrm{d}\bar{\theta}\right)\mathrm{d}\theta$$

$$=\mu\int\limits_{\Lambda(V)}\theta\underbrace{\left(\int\limits_{\Lambda(V)}\bar{\theta}\mathrm{d}\bar{\theta}\right)}_{=1}\mathrm{d}\theta=\mu\underbrace{\int\limits_{\Lambda(V)}\theta\mathrm{d}\theta}_{=1}=\mu.$$

Consider now a Hermitian $N\times N$ matrix $\mathbf{M}$ for which there exists a unitary matrix $\mathbf{U}$, such that $\mathbf{D}\equiv\mathbf{U}^\dagger\mathbf{M}\mathbf{U}$ is a diagonal matrix with $D_{kk}=\mu_k$. Here, $\mathbf{U}^\dagger\equiv\bar{\mathbf{U}}^T$ denotes the Hermitian conjugate of $\mathbf{U}$. From these observations we deduce:

$$-\theta^\dagger\mathbf{M}\theta=-\theta^\dagger\mathbf{U}\mathbf{D}\mathbf{U}^\dagger\theta=-\left(\mathbf{U}^\dagger\theta\right)^\dagger\mathbf{D}\left(\mathbf{U}^\dagger\theta\right).$$

Define a new variable of integration $\tilde{\theta} := \mathbf{U}^\dagger \theta$. Since $\mathbf{U}$ describes a unitary transformation, the integration measure does not change. Thus:

$$\int \mathrm{e}^{-\theta^\dagger \mathbf{M}\theta} \mathrm{d}\bar{\theta}\mathrm{d}\theta = \int \mathrm{e}^{-\left(\mathbf{U}^\dagger\theta\right)^\dagger \mathbf{D}\left(\mathbf{U}^\dagger\theta\right)} \mathrm{d}\bar{\theta}\mathrm{d}\theta = \int \mathrm{e}^{-\theta^\dagger \mathbf{D}\theta} \mathrm{d}\bar{\theta}\mathrm{d}\theta$$

$$= \int \mathrm{e}^{-\sum_{i=1}^N \bar{\theta}_i \mu_i \theta_i} \left(\prod_{j=1}^N \mathrm{d}\bar{\theta}_j \mathrm{d}\theta_j\right) = \prod_{i=1}^N \int \mathrm{e}^{-\bar{\theta}_i \mu_i \theta_i} \mathrm{d}\bar{\theta}_i \mathrm{d}\theta_i$$

$$= \prod_{i=1}^N \mu_i = \det\left(\mathbf{D}\right) = \det\left(\mathbf{U}^\dagger \mathbf{M}\mathbf{U}\right) = \det\left(\mathbf{U}\mathbf{U}^\dagger \mathbf{M}\right) = \det\left(\mathbf{M}\right).$$

In the penultimate step we took advantage of the fact, that the factors of the matrix product within a determinant can be arbitrarily shifted. This finishes the proof.

□

Proposition D.7 (*Parallel alignment of electric and magnetic fields*, sec. 3.2; par. **A**). Let $(\mathbf{E}, \mathbf{B})$ be a pair of an electric and magnetic field with respect to a frame of reference Σ, considered in $d = 4$ dimensional Minkowski space. If $\mathbf{E} \cdot \mathbf{B} \neq 0$, then there exists a Lorentz transformation $\Sigma \to \tilde{\Sigma}$ with a new frame of reference $\tilde{\Sigma}$ in which the transformed fields $\left(\tilde{\mathbf{E}}, \tilde{\mathbf{B}}\right)$ are parallel.

Proof. The electric and magnetic field, $\mathbf{E}$ and $\mathbf{B}$ respectively, are covariantly summarised in the field strength tensor $\mathbf{F}$. Its contravariant matrix representation in $d = 4$ spacetime dimensions is given in (A.13):

$$\left(\mathbf{F}_{\mu\nu}\right) \mathrel{\hat{=}} \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}.$$

We consider a general Lorentz boost with velocity vector $\mathbf{v} \in \Sigma$, and define the dimensionless transformation parameter[2] $\boldsymbol{\beta} := \mathbf{v}/c$, where c is the speed of light (during this proof, we will exceptionally work in SI units). In a Cartesian coordinate frame, a general Lorentz boost $\mathscr{B}$ can be represented in matrix form as:

$$\mathscr{B}\left(\boldsymbol{\beta}\right) \mathrel{\hat{=}} \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1 + (\gamma - 1)\frac{\beta_x^2}{\beta^2} & (\gamma - 1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma - 1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma - 1)\frac{\beta_y\beta_x}{\beta^2} & 1 + (\gamma - 1)\frac{\beta_y^2}{\beta^2} & (\gamma - 1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma - 1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma - 1)\frac{\beta_z\beta_y}{\beta^2} & 1 + (\gamma - 1)\frac{\beta_z^2}{\beta^2} \end{pmatrix},$$

where $\boldsymbol{\beta} \mathrel{\hat{=}} \left(\beta_x, \beta_y, \beta_z\right)^T$ and $\gamma \equiv \frac{1}{1-\beta^2}$ denotes the Lorentz factor. This matrix can, for instance, be obtained by performing three individual Lorentz boosts along each axis of a Cartesian coordinate frame, i.e. multiplying three special Lorentz boost matrices; $\mathscr{B}\left(\boldsymbol{\beta}\right) = \mathscr{B}_z\left(\beta_z\right)\mathscr{B}_y\left(\beta_y\right)\mathscr{B}_x\left(\beta_x\right)$. The field strength tensor in $\tilde{\Sigma}$ can now be computed via an ordinary matrix product:

$$\tilde{\mathbf{F}} = \mathscr{B}^T \mathbf{F} \mathscr{B}.$$

[2]Note that $\boldsymbol{\beta}$ should not be confused with the beta vector introduced in sec. 2.2.

From the result of this calculation, we can read off the transformed fields $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$:

$$\tilde{\mathbf{E}} = \gamma\left(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}\right) - \frac{\gamma^2}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta},$$

$$\tilde{\mathbf{B}} = \gamma\left(\mathbf{B} - \boldsymbol{\beta}\times\mathbf{E}\right) - \frac{\gamma^2}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{B}\right)\boldsymbol{\beta}.$$

Since we exclude the case $\mathbf{E}\cdot\mathbf{B} = 0$, both field vectors are not vanishing. Therefore, we can consider $\tilde{\mathbf{E}}\times\tilde{\mathbf{B}} \overset{!}{=} 0$ as a necessary and sufficient condition for showing that two vectors are parallel. Explicitly, this means:

$$\begin{aligned}
\mathbf{0} &= \tilde{\mathbf{E}}\times\tilde{\mathbf{B}} \\
&= \left[\gamma\left(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}\right) - \frac{\gamma^2}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta}\right]\times\left[\gamma\left(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E}\right) - \frac{\gamma^2}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{B}\right)\boldsymbol{\beta}\right] \\
&= \gamma\Bigg[\mathbf{E}\times\mathbf{B} - \mathbf{E}\times\left(\boldsymbol{\beta}\times\mathbf{E}\right) - \frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{B}\right)\left(\mathbf{E}\times\boldsymbol{\beta}\right) + \left(\boldsymbol{\beta}\times\mathbf{B}\right)\times\mathbf{B} \\
&\qquad - \left(\boldsymbol{\beta}\times\mathbf{B}\right)\times\left(\boldsymbol{\beta}\times\mathbf{E}\right) - \frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{B}\right)\left(\boldsymbol{\beta}\times\mathbf{B}\right)\times\boldsymbol{\beta} - \frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\left(\boldsymbol{\beta}\times\mathbf{B}\right) \\
&\qquad + \frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\boldsymbol{\beta}\times\left(\boldsymbol{\beta}\times\mathbf{E}\right) + \left(\frac{\gamma}{\gamma+1}\right)^2\left(\boldsymbol{\beta}\cdot\mathbf{E}\right)\left(\boldsymbol{\beta}\cdot\mathbf{B}\right)\left(\boldsymbol{\beta}\times\boldsymbol{\beta}\right)\Bigg].
\end{aligned}$$

Obviously, the last term $\sim \boldsymbol{\beta}\times\boldsymbol{\beta}$ vanishes. For the numerous double cross products we can use the Grassmann identity; $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}\,(\mathbf{a}\cdot\mathbf{b})$, or $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - \mathbf{a}\,(\mathbf{b}\cdot\mathbf{c})$ for some vectors $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^3$. In addition, many terms contain an inner product; either $\boldsymbol{\beta}\cdot\mathbf{E}$ or $\boldsymbol{\beta}\cdot\mathbf{B}$. Therefore, we can highly simplify the last intermediate result by supposing an ansatz of the form $\boldsymbol{\beta} = \alpha\mathbf{E}\times\mathbf{B}$, where $\beta\equiv|\boldsymbol{\beta}|$, such that $\boldsymbol{\beta}$ is perpendicular to both $\mathbf{E}$ and $\mathbf{B}$. In this way, all terms $\sim\boldsymbol{\beta}\cdot\mathbf{E}$ and $\sim\boldsymbol{\beta}\cdot\mathbf{B}$ vanish identically. The proportionality factor α is yet to be determined. Dividing by $\gamma\neq 0$ on both sides of our last expression we obtain:

$$\begin{aligned}
\mathbf{0} &= \mathbf{E}\times\mathbf{B} - \boldsymbol{\beta}\left(\mathbf{E}\cdot\mathbf{E}\right) + \mathbf{E}\underbrace{\left(\mathbf{E}\cdot\boldsymbol{\beta}\right)}_{=0} + \mathbf{B}\underbrace{\left(\boldsymbol{\beta}\cdot\mathbf{B}\right)}_{=0} - \boldsymbol{\beta}\left(\mathbf{B}\cdot\mathbf{B}\right) - \left(\boldsymbol{\beta}\times\mathbf{B}\right)\times\left(\boldsymbol{\beta}\times\mathbf{E}\right) \\
&= \mathbf{E}\times\mathbf{B} - \boldsymbol{\beta}\left(E^2+B^2\right) - \alpha^2\Big[\left(\mathbf{E}\times\mathbf{B}\right)\times\mathbf{B}\Big]\times\Big[\left(\mathbf{E}\times\mathbf{B}\right)\times\mathbf{E}\Big] \\
&= \mathbf{E}\times\mathbf{B} - \boldsymbol{\beta}\left(E^2+B^2\right) - \alpha^2\Big[\mathbf{B}\left(\mathbf{E}\cdot\mathbf{B}\right) - \mathbf{E}\left(\mathbf{B}\cdot\mathbf{B}\right)\Big]\times\Big[\mathbf{B}\left(\mathbf{E}\cdot\mathbf{E}\right) - \mathbf{E}\left(\mathbf{B}\cdot\mathbf{E}\right)\Big] \\
&= \mathbf{E}\times\mathbf{B} - \boldsymbol{\beta}\left(E^2+B^2\right) - \alpha^2\Big[-\left(\mathbf{E}\cdot\mathbf{B}\right)^2\mathbf{B}\times\mathbf{E} - E^2B^2\mathbf{E}\times\mathbf{B}\Big] \\
&= \Big[1 - \alpha\left(E^2+B^2\right) - \alpha^2\left(\left(\mathbf{E}\cdot\mathbf{B}\right)^2 - E^2B^2\right)\Big]\mathbf{E}\times\mathbf{B}.
\end{aligned}$$

As we have argued above, neither of the field vectors $\mathbf{E}$ or $\mathbf{B}$ vanish. If $\mathbf{E}\parallel\mathbf{B}$, then $\mathbf{E}\times\mathbf{B} = \mathbf{0}$ and there is nothing to do. The sought for Lorentz transformation is simply the identity. Thus, let us concentrate on the situtation where $\mathbf{E}$ and $\mathbf{B}$ are not parallel (and also not perpendicular by assumption).

Then, the prefactor of the our last intermediate result must vanish, which yields a quadratic equation for α. It reads:

$$0 = 1 - \alpha\left(E^2 + B^2\right) + \alpha^2\left(E^2B^2 - (\mathbf{E}\cdot\mathbf{B})^2\right)$$

$$= 1 + \left[\alpha\sqrt{E^2B^2 - (\mathbf{E}\cdot\mathbf{B})^2} - \frac{E^2+B^2}{2\sqrt{E^2B^2 - (\mathbf{E}\cdot\mathbf{B})^2}}\right]^2 - \left(\frac{E^2+B^2}{2\sqrt{E^2B^2 - (\mathbf{E}\cdot\mathbf{B})^2}}\right)^2 .$$

Here we note that $E^2B^2 \geq (\mathbf{E}\cdot\mathbf{B})^2 = E^2B^2\cos(\vartheta)^2$, with the angle ϑ enclosed by $\mathbf{E}$ and $\mathbf{B}$. The two solutions for α are:

$$\alpha = \frac{1}{|\mathbf{E}\times\mathbf{B}|^2}\left[\frac{E^2+B^2}{2} \pm \sqrt{\left(\frac{E^2-B^2}{2}\right)^2 + (\mathbf{E}\cdot\mathbf{B})^2}\right],$$

where we have compactified the root expression as follows: $E^2B^2-(\mathbf{E}\cdot\mathbf{B})^2 = E^2B^2\left(1-\cos(\vartheta)^2\right) = E^2B^2\sin(\vartheta)^2 = |\mathbf{E}\times\mathbf{B}|^2$. It obvious that $\alpha \in \mathbb{R}$ for both signs and hence $\boldsymbol{\beta} = \alpha\mathbf{E}\times\mathbf{B}$ a valid transformation parameter that exists for all non-perpendicular and initially non-parallel field configurations. The corresponding proper Lorentz transformation is given by $\mathscr{B}(\alpha\mathbf{E}\times\mathbf{B})$. This is what we intended to show.

□

Remark: Even if $\mathbf{E}$ and $\mathbf{B}$ are perpendicular and hence $\mathbf{E}\cdot\mathbf{B} = 0$, α would still be a real number and describes a proper Lorentz transformation by means of $\boldsymbol{\beta} = \alpha\mathbf{E}\times\mathbf{B}$. However, the assumption that $\mathbf{E}\cdot\mathbf{B} \neq 0$ is nevertheless necessary. One can see this, by calculating the inner product of the transformed fields, i.e. $\tilde{\mathbf{E}}\cdot\tilde{\mathbf{B}}$. After some elementary algebra, one finds that the inner product is indeed a Lorentz invariant: $\tilde{\mathbf{E}}\cdot\tilde{\mathbf{B}} = \mathbf{E}\cdot\mathbf{B}$. Thus, if both fields are perpendicular to each other in one frame of reference, then they are perpendicular in all frames. Therefore, there would be no possibility to make them appear parallel by means of a proper Lorentz transformation.

Theorem D.8 (*Basis of local U(1) invariants*, sec. 3.1; par. **A**).
In the situation of ch. 3, the set $\{\mathscr{F},\mathscr{G}\}$ forms a basis in the space of all locally U(1)- and Lorentz invariant non-derivative scalar quantities.

Sketch of a proof. Let us consider an arbitrary locally U(1)- and Lorentz invariant scalar quantity $\mathcal{S} \mathrel{\hat{=}} \mathcal{S}(\mathbf{X})$. Within the tuple $\mathbf{X}$ we collect all possible objects from which we can construct manifestly U(1)- and Lorentz invariant expressions. The first task is to characterise the tuple $\mathbf{X}$ and enumerate the individual objects of which it consists. One can be convinced that the only such objects are the field strength tensor $\mathbf{F}$, the totally antisymmetric Levi-Civita tensor $\boldsymbol{\varepsilon}$, and the metric tensor $\mathbf{g}$, whereby $\mathbf{g}$ is chosen to be either the Minkowski- or the Euclidean metric. Instead of the Levi-Civita tensor, we can also include the dual field strength tensor $\star\mathbf{F}$, which is a manifestly locally U(1) invariant combination of $\boldsymbol{\varepsilon}$ and $\mathbf{F}$. It follows:

$$\mathcal{S}(\mathbf{X}) = \mathcal{S}(\mathbf{g}, \mathbf{F}, \star\mathbf{F}) .$$

Since $\mathcal{S}$ is supposed to be a Lorentz scalar, it must be constructed from terms which are proportional to a full contraction of $\mathbf{g}, \mathbf{F}$ and $\star\mathbf{F}$. However, the metric $\mathbf{g}$ has the effect of shifting index positions, so full contractions can be reduced between $\mathbf{F}$ and $\star\mathbf{F}$ only.

Now, three different situations can occur:

(i) Full contractions of **F** with itself either yield 0 ($\sim F^{\mu}{}_{\mu}$), or, per construction, factors of $\mathscr{F}$ ($\sim F_{\mu\nu}F^{\mu\nu}$).

(ii) Full contractions between **F** and $\star$**F** lead to factors $\mathscr{G}$, either by definition ($\sim F_{\mu\nu}\star F^{\mu\nu}$), or by employing Proposition A.1 of app. A ($\sim F^{\mu\lambda}\star F^{\nu}{}_{\lambda} = \mathscr{G}\delta^{\mu\nu}$).

(iii) Full contractions of $\star$**F** with itself can be expressed in terms of contractions of **F** with itself plus a term proportional to $\mathscr{F}$, using again Proposition A.1 of app. A.

Since in all three cases we are able to reduce arbitrary contractions on factors of $\mathscr{F}$ and $\mathscr{G}$, the scalar $\mathcal{S}$ depends only on these invariants:

$$\mathcal{S} \mathrel{\hat{=}} \mathcal{S}\left(\mathscr{F}, \mathscr{G}\right).$$

Except for some more detailed argumentations on the characterisation of **X** and some formal statements on the cases (i)-(iii), this essentially completes the proof.

□

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Eidesstattliche Erklärung

Ich erkläre hiermit, dass die vorliegende Masterarbeit mit dem Titel "Wilsonian Perspective on Generic Self-Interacting U(1) Gauge Theories" von mir eigenständig und ohne die Verwendung nicht verzeichneter Hilfsmittel verfasst wurde.

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Affidavit

I declare that this Master's thesis entitled "Wilsonian Perspective on Generic Self-Interacting U(1) Gauge Theories" was written by me independently and without the use of unrecorded resources.

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Weimar, den 28.11.2022

Julian Schirrmeister