Bachelor's Thesis

Construction of Relativistic Field Theories with Luttinger Fermions

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Jena, October 18, 2022

"'Ich weiß, was ich weiß. Doch nur das,
was ich nicht weiß, macht mich heiß,
weil ich's gerne besser wüsste.'"
– Daniel Dickopf in *Besserwisser*

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1 Introduction

Since Einstein formulated the Special and later the General Theory of Relativity, it has always been of great interest in physics to see if classical models and theories need to be adapted when it comes to relativistically relevant velocities. In condensed matter physics and especially in the discussion of metals, semi-metals and topological insulators, the band structure of the matter is essential for its electric properties. Interesting phenomena like non-Fermi liquids appear when the Fermi surface is reduced to a point [1, 2]. This is the case for in what is called linear band crossing where the dispersion relation in the vicinity of these Fermi points is linear in momentum. Here the short range components of electron interactions are large, while the long range components can be neglected [3]. This is well explained by a relativistic field theory of the Gross-Neveu-Yukawa type [4].

Attention is also drawn to the case of quadratic band crossing (QBC) or quadratic band touching (QBT). Here, the dispersion relation is quadratic in momentum. In the literature the fermions are sometimes referred to as *Luttinger fermions* in this situation, named after Joaquin Mazdak Luttinger for his work on so called *Luttinger liquids* [5]. A non-relativistic treatment of QBC in 2 spatial dimensions can be found in [6, 7]. The situation is rather different in 3 spatial dimensions. This problem is often treated with the use of the renormalization group [1, 8, 9]. All these models are non-relativistic. In this work we want to show that it is also possible to construct corresponding relativistically invariant versions of such models and of Luttinger fermions in general.

The thesis is structured as follows: In chapter 2 we recap the widely known formalism of the Hamiltonian and Dirac spinors. In chapter 3, we show the construction of the Hamiltonian for Luttinger fermions and find the general anti-commutator for the second-rank-tensor G. Further, we see the failure of the simplest ansatz for the spin metric. Two representations for the *Clifford algebra* in $2+1$ and later $3+1$ dimensions are chosen in chapter 4. Here we find the necessary spin metric to construct relativistically invariant actions.

2 Theoretical Foundations

2.1 Conventions

In the following work we use the so called natural units, i.e.

$$\hbar = 1 = c\,, \tag{2.1}$$

where c is the speed of light and $\hbar$ is the reduced Plank's constant. The used metric for the spacetime follows the sign convention

$$g_{\mu\nu} = \mathrm{diag}(+,-,-,-)\,. \tag{2.2}$$

If the summation is not stated explicitly, we use Einstein's summation convention where repeated indices are implicitly summed over. The underlined index, e.g. $G_{\underline{ii}}$, means that there is no summation over this index. Indices in Roman letters are meant to start counting at one, while Greek indices start at zero.

2.2 From Hamiltonian to Action

A well known way of describing a dynamical system of particles is the so called Hamiltonian function

$$H(q_i, p_j) = K + V, \tag{2.3}$$

where K stands for the kinetic term and V for the potential giving rise to a force acting on the particles. The variables q_i and p_j are the position in space and the momentum of the particles respectively. The time evolution of the system can be calculated through Hamilton's equations:

$$\dot{p}_i = \frac{\mathrm{d}p_i}{\mathrm{d}x} = -\frac{\partial H}{\partial q_i}\ ;\ \dot{q}_i = \frac{\mathrm{d}q_i}{\mathrm{d}x} = \frac{\partial H}{\partial p_i}\,, \tag{2.4}$$

where $p_i = \frac{\partial H}{\partial \dot{q}_i}$.

An equivalent description to the Hamiltonian is the Lagrangian given by

$$L(q_i, \dot{q}_i) = K - V. \tag{2.5}$$

In this description, the equations of motion are the so called Euler-Lagrange-equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0\,. \tag{2.6}$$

In the derivation of these equations time plays a special role as the principle is to keep the action $S = \int \mathrm{d}t L$ constant at all time. Thus we cannot define a covariant Lagrangian but we can use a *Lagrangian density* $\mathscr{L}$ instead, which is defined by

$$L = \int \mathrm{d}x^3 \mathscr{L}\,. \tag{2.7}$$

Now, this yields

$$S = \int \mathrm{d}t L = \int \mathrm{d}t \mathrm{d}x^3 \mathscr{L} = \int \mathrm{d}x^4 \mathscr{L}\,. \tag{2.8}$$

Instead of depending on coordinates and velocities the Lagrangian density can also depend on fields $\phi(\mathbf{x}, t) = \phi(x^\mu)$. The equations of motion now read

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_i)} \right) - \frac{\partial \mathscr{L}}{\partial \phi_i} = 0\,. \tag{2.9}$$

An example of $\mathscr{L}$ depending on Spinors Ψ can be found in equation (2.11) and the corresponding equation of motion in (2.12).

2.3 Dirac Formalism

The simplest representation of the Lorentz group, neglecting the trivial $(0,0)$ representation which corresponds to a scalar field, is the $(0,\frac{1}{2})$ and the complex conjugated $(\frac{1}{2},0)$ representation. The two-component so called *Weyl Spinors* in this representation are split into right-handed and left-handed, where right-handed spinors have chirality $+1$ and left-handed -1.

Another possible representation for the Lorentz group is described by $(0,\frac{1}{2}) \oplus (\frac{1}{2},0)$, which describes Dirac bi-spinors. In the chiral representation they can be written as

$$\Psi = \begin{pmatrix} \eta^{\dot{\alpha}} \\ \xi_\alpha \end{pmatrix},$$

where $\eta^{\dot{\alpha}}$ is the two component right-handed Weyl spinor of the $(0,\frac{1}{2})$ representation and ξ_α is the left-handed Weyl spinor from the $(\frac{1}{2},0)$ representation of the Lorentz group [10]. The product $\Psi^\dagger\Psi$ is not a Lorentz invariant. We therefore have to introduce the adjoint Dirac bi-spinor $\bar{\Psi} = \Psi^\dagger h$, where h is the spin metric that can be identified with the γ^0 matrix. The defining property for the Dirac γ matrices is the corresponding Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\, 1_{4x4} \tag{2.10}$$

for $\mu, \nu = 0,1,2,3$. Additionally one defines

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

Using the Dirac spinors $\bar{\Psi}$ and Ψ together with the γ matrices one can built new bilinear covariant quantities [11]:

- $\bar{\Psi}\gamma^5\Psi$ is a pseudoscalar.
- $\bar{\Psi}\gamma^\mu\Psi$ is a four-vector.
- $\bar{\Psi}\gamma^\mu\gamma^5\Psi$ is a pseudo four-vector.
- $\left(\bar{\Psi}\sigma^{\mu\nu}\Psi\right)$, where $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$, is an antisymmetric tensor.

Now we can write down the Dirac action

$$S_D = \int \mathrm{d}^4 x \left[i\bar{\Psi}\gamma^\mu \partial_\mu \Psi - m\bar{\Psi}\Psi \right]. \tag{2.11}$$

The first term is the kinetic term, while the second one is called mass term. From the action principle the functional derivative of the action yields the *Dirac equation*

$$0 = \frac{\delta S_D}{\delta \bar{\Psi}(x)} = (i\gamma^\mu \partial_\mu - m)\Psi. \tag{2.12}$$

It is convenient to abbreviate: $\gamma^\mu \partial_\mu =: \not{\partial}$. Hence,

$$0 = (i \not{\partial} - m)\Psi. \tag{2.13}$$

2.4 Spinor Construction

The Dirac algebra inherits a symmetry called the similarity transformations:

$$A \longrightarrow A' = SAS^{-1}. \tag{2.14}$$

Here,

$$A \epsilon GL(d_\gamma, \mathbb{C}) \qquad \text{and} \qquad S \epsilon \mathrm{SL}(d_\gamma, \mathbb{C}), \tag{2.15}$$

and d_γ is the dimension of the Clifford-algebra.

Maintaining this symmetry of similarity transformations while keeping the action real, as is necessary for a unitary theory, leads to the concept of Dirac conjugation. In the following, we sketch this line of argument, as it is not well known in the standard literature [12, 13].

Consider the standard Dirac algebra, say in $D = 4$,

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}. \tag{2.16}$$

The algebra is invariant under

$$\gamma_\mu \longrightarrow S\gamma_\mu S^{-1}, \ S \epsilon \mathrm{SL}(d_\gamma, \mathbb{C}), \tag{2.17}$$

where $d_\gamma = 4$ in the standard case.

Also, the Dirac equation is invariant as long as the spinors transform as

$$\Psi \longrightarrow S\Psi, \tag{2.18}$$

since

$$0 = (i \not{\partial} - m)\Psi \longrightarrow (iS \not{\partial} S^{-1} - m)S\Psi) = S\underbrace{(i \not{\partial} - m)\Psi}_{=0} = 0. \tag{2.19}$$

In order to construct an invariant action we postulate the existence of a conjugate spinor $\bar{\Psi}$ that transforms as:

$$\bar{\Psi} \longrightarrow \bar{\Psi}S^{-1}, \tag{2.20}$$

such that

$$S = \int d^4x\, \bar{\Psi}(i\,\partial\!\!\!/ - m)\Psi \tag{2.21}$$

is invariant under (2.17 & 2.18 & 2.20).

However, in order for the action to be real, $\bar{\Psi}$ must be linearly related to the adjoint spinor; we use the ansatz

$$\bar{\Psi} = \Psi^\dagger h, \tag{2.22}$$

where h can be interpreted as a spin metric. In view of (2.18), $\Psi^\dagger$ transforms as

$$\Psi^\dagger \longrightarrow \Psi^\dagger S^\dagger. \tag{2.23}$$

Since $S^\dagger \neq S^{-1}$ in general, the spin metric has to satisfy

$$S^\dagger h = hS^{-1}, \tag{2.24}$$

such that (2.20) is satisfied:

$$\bar{\Psi} \longrightarrow \bar{\Psi}S^{-1} \overset{(2.22)}{=} \Psi^\dagger h S^{-1} \overset{(2.24)}{=} \Psi^\dagger S^\dagger h. \tag{2.25}$$

In view of (2.24), h is also called a ‘hermitizer’.

Now, consider the mass term as an example,

$$\begin{aligned} S_m &= -m\int d^4x\, \bar{\Psi}\Psi \\ S_m &\overset{!}{=} S_m^* \\ &= -m\int d^4x\, \Psi^\dagger \bar{\Psi}^\dagger \\ &= -m\int d^4x\, \Psi^\dagger h^\dagger (\Psi^\dagger)^\dagger \\ &= -m\int d^4x\, \Psi^\dagger h^\dagger \Psi. \end{aligned} \tag{2.26}$$

Reality of S_m requires a hermitian spin metric

$$h^\dagger = h. \tag{2.27}$$

Performing the same consideration for the kinetic term, we get

$$\begin{aligned} S_{\text{kin}} = \int d^4x\, \bar{\Psi}\, i\,\partial\!\!\!/\, \Psi \overset{!}{=} S_{\text{kin}}^* &= -i\int d^4x\, (\partial_\mu \Psi^\dagger)\gamma^{\mu^\dagger}\bar{\Psi}^\dagger \\ &\overset{i.b.p.}{=} i\int d^4x\, \Psi^\dagger \gamma^{\mu^\dagger} h^\dagger \partial_\mu \Psi \\ &= i\int d^4x\, \Psi^\dagger \overbrace{hh^{-1}}^{\mathbb{1}} \gamma^{\mu^\dagger} h^\dagger \partial_\mu \Psi \\ &= \int d^4x\, \bar{\Psi} h^{-1}\gamma^{\mu^\dagger} h^\dagger i\partial_\mu \Psi, \end{aligned}$$

which is real, provided

$$\gamma^\mu = h^{-1}\gamma^{\mu^\dagger} h^\dagger \stackrel{(2.27)}{=} h^{-1}\gamma^{\mu^\dagger} h. \tag{2.28}$$

The last equation stresses the interpretation of a „hermitizer". In fact, one can prove that (2.27),(2.28) and a scale condition, e.g.

$$\begin{aligned} h &= h^\dagger, \\ \gamma^\mu &= h^{-1}\gamma^{\mu^\dagger} h, \\ |\det h| &= 1, \end{aligned} \tag{2.29}$$

fix the spin metric completely up to a sign [12, 14]. $|\det h| = 1$ inhibits that h introduces an artificial rescaling of conjugate spinors with respect to the original spinors.

For the choice of the metric convention $g_{\mu\nu} = \text{diag}(1,-1,-1,-1)$ and the Dirac algebra $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ with γ_0 being hermitian and γ_i being anti-hermitian, a solution to (2.29) is given by the standard choice $h = \gamma_0$.

2.5 Gauge Theories and Electrodynamics

Local or global phase changes, which keep the Lagrangian invariant, are called *gauge transformations*. They can in general be written as

$$\Psi \longrightarrow \Psi \exp\{i\alpha(\mathbf{x})\hat{A}\}, \tag{2.30}$$

where $\alpha(\mathbf{x})$ is a real function of the space coordinates and $\hat{A}$ is a unitary operator [11]. If $\alpha(\mathbf{x}) = \alpha = const.$, the gauge transformation is called global otherwise it is a local transformation. The generalization of the theory to non-Abelian groups is called *Yang-Mills-Theory*. If we choose $\hat{A}$ as one of the generators of $SU(2)$ we deal with structures similar to those of the model for the weak interaction. The same holds for the $SU(3)$ group and the strong interaction. Imposing local gauge invariance for the Lagrangian always entails the introduction of a field called *gauge field*, which comes up in an interaction term in the Lagrangian. In fact, this 'generates' the core of electrodynamics and the coupling of fermions to electromagnetic fields. In turn, one can also add this interaction term to the Lagrangian and maintain the local gauge invariance. Have a look at [11] for a good instruction how to derive the interaction term.

Here for an example, we want to focus on the Abelian version of the gauge theory and choose the generators of the $U(1)$ group for the unitary operator $\hat{A} = \mathbb{1}$. For this we take the still globally gauge invariant Lagrangian for the free Dirac particles and the electromagnetic fields

$$\mathscr{L} = i\bar{\Psi}\gamma_\mu\partial^\mu\Psi - m\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.31}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and A_μ is the four-potential of the electromagnetic field. We apply the *minimal coupling prescription*:

$$\partial^\mu \longrightarrow D^\mu = \partial^\mu + ieA^\mu. \tag{2.32}$$

Indeed, this procedure works for the Schrödinger equation and the Klein-Gordon equation as well. This yields

$$\mathscr{L} = i\bar{\Psi}\gamma_\mu D^\mu \Psi - m\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.33}$$

$$= i\bar{\Psi}\gamma_\mu \partial^\mu \Psi - -e\bar{\Psi}\gamma_\mu A^\mu \Psi - m\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.34}$$

which is indeed invariant under a local phase transformation

$$\Psi \longrightarrow \Psi e^{ie\theta(\mathbf{x})} \tag{2.35}$$

together with a simultaneous transformation of the four-potential

$$A^\mu \longrightarrow A^\mu + \partial^\mu \theta(\mathbf{x})\,, \tag{2.36}$$

as can be verified easily. These interaction terms generated with the gauge invariance condition are essential for a lot of phenomena.

3 Hamiltonian for Luttinger Fermions

3.1 Construction

We want to construct a theory of relativistic Luttinger fermions in D spacetime dimensions. For this, we start by reviewing the Hamiltonian construction of non-relativistic Luttinger fermions in d space dimensions.

We assume:

$$H = \sum_{i,j=1}^{d} G_{ij} p_i p_j = \sum_{i=1}^{d} p_i \left(\sum_{j=1}^{d} G_{ij} p_j \right). \tag{3.1}$$

So we need to determine the coefficients of the second-rank tensor G, which is obviously symmetric. Using the assumption $H^2 = p^4 \mathbb{1}$ one can see that H^2 only contains even powers of p_i. Therefore we can do the following calculation:

$$H^2 = \left[\sum_{i=1}^{d} p_i \left(\sum_{j=1}^{d} G_{ij} p_j \right) \right]^2 \tag{3.2}$$

$$= \sum_{i=1}^{d} p_i^2 \left(\sum_{j=1}^{d} G_{ij} p_j \right)^2 + \sum_{i \neq a} p_i p_a \left(\sum_{j=1}^{d} G_{ij} p_j \right) \left(\sum_{j=1}^{d} G_{aj} p_j \right) \tag{3.3}$$

$$= \sum_{i=1}^{d} \left[\sum_{j=1}^{d} (G_{ij} p_i p_j)^2 + \underbrace{\sum_{j \neq b} p_i^2 p_j p_b G_{ij} G_{ib}}_{A} \right] + \sum_{i \neq a} p_i p_a \left(\sum_{j=1}^{d} G_{ij} p_j \right) \left(\sum_{n=1}^{d} G_{an} p_n \right). \tag{3.4}$$

Because $j \neq b$ and H^2 only consists of even powers of p_i the term A has to vanish. This means that the symmetric part of $G_{ij} G_{ib}$ vanishes, i.e.

$$G_{\underline{i}j} G_{\underline{i}b} + G_{\underline{i}b} G_{\underline{i}j} = \{G_{\underline{i}j}, G_{\underline{i}b}\} = 0 \text{ for } j \neq b. \tag{3.5}$$

The same holds for the last term. In order to get only even powers of p_i, we need to have only two non-vanishing terms: 1. $j = i, n = a$ and 2. $j = a, n = i$. The remainder has to be zero, which means, together with (3.5),

$$\{G_{\underline{i}\underline{i}}, G_{an}\} = 0 \text{ for } n \neq a. \tag{3.6}$$

The first possibility gives us the sum

$$\sum_{i<a} p_i^2 p_a^2 \{G_{ii}, G_{aa}\}, \tag{3.7}$$

whereas the second one, using the symmetry of G, leads to the sum

$$\sum_{i<a} p_i^2 p_a^2 \{G_{ia}, G_{ai}\} = \sum_{i<a} p_i^2 p_a^2 (2G_{ia}). \tag{3.8}$$

If we now combine all these leftover terms from (3.4), we get for the squared Hamiltonian

$$H^2 = \sum_{i=1}^{d} G_{ii}^2 p_i^4 + \sum_{i<j} p_i^2 p_j^2 \left(4G_{ij}^2 + \{G_{ii}, G_{jj}\}\right). \tag{3.9}$$

From this we get the third condition in order to satisfy $H^2 = p^4 \mathbb{1}$:

$$4G_{\underline{ij}}^2 + \left\{G_{\underline{ii}}, G_{\underline{jj}}\right\} = 2 \text{ for any } \underline{i} \neq \underline{j}\,. \tag{3.10}$$

So in total the assumption $H^2 = p^4$ results in three conditions (3.5),(3.6),(3.10), and the normalization $G_{\underline{ii}}^2 = 1$.

Now we want to get the last missing anti-commutator of two diagonal elements of G. If we assume that G is a traceless tensor with respect to its spacetime indices, which means $G_{ii} = 0$, then H does not depend on the invariant p^2, but only on the irreducible tensor $p^\mu p^\nu - \frac{p^2}{D} g^{\mu\nu}$. This gives rise to the property

$$0 = \left\{G_{\underline{kk}}, \sum_{i=1}^{d} G_{ii}\right\} = 2 + \sum_{i(\neq k)} \{G_{\underline{kk}}, G_{ii}\}. \tag{3.11}$$

Since the index k is arbitrary this holds true for every pair off diagonal elements. So we get

$$\left\{G_{\underline{ii}}, G_{\underline{jj}}\right\} = \frac{2}{1-d} \text{ for } i \neq j\,. \tag{3.12}$$

Combined with (3.10) we can get the squared off-diagonal elements

$$G_{ij}^2 = d/(2(d-1))\,. \tag{3.13}$$

We obtain the following four conditions for the anti-commutators:

$$(i)\, \{G_{ij}, G_{kl}\} = 0 \qquad \text{for } i \neq j, k \neq l \text{ and } (ij) \neq (kl) \tag{3.14a}$$

$$(ii)\, \{G_{\underline{ii}}, G_{kl}\} = 0 \qquad \text{for } k \neq l \tag{3.14b}$$

$$(iii)\, \{G_{\underline{ii}}, G_{\underline{ii}}\} = 2 \tag{3.14c}$$

$$(iv)\, \{G_{\underline{ii}}, G_{\underline{jj}}\} = -\frac{2}{d-1} \qquad \text{for } i \neq j\,. \tag{3.14d}$$

Since the tensor structure must respect Euclidean invariance, the right-hand side can only be spanned by the invariant tensors δ_{ij} and ϵ_{ijk}. G_{ij} is symmetric, $G_{ij} = G_{ji}$, hence the Levi-Civita symbol, anti-symmetric in all indices, drops out. Using $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, $\delta_{il}\delta_{jk}$ as a basis for ansatz, we find the general form

$$\left\{G_{ij}, G_{kl}\right\} = -\frac{2}{d-1}\delta_{ij}\delta_{kl} + \frac{d}{d-1}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,. \tag{3.15}$$

A generalization to Minkowski spacetime reads

$$\left\{G_{\mu\nu}, G_{\kappa\lambda}\right\} = -\frac{2}{D-1} g_{\mu\nu} g_{\kappa\lambda} + \frac{D}{D-1}(g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa})\,. \tag{3.16}$$

Table 1: Dimension d_γ of Clifford-algebra with d_e anti-commuting elements.

D	2	3	4	5	6
d_e	2	5	9	14	20
d_γ	2	4	16	128	1024

Now, equation (3.14a) tells us that all off-diagonal elements must anti-commute. Since $G_{\mu\nu}$ is a D dimensional symmetric matrix with respect to the $\mu\nu$ indices, it has $\frac{1}{2}D(D-1)$ off-diagonal elements.

Also, equation (3.14b) implies that all off-diagonal elements must anti-commute with all diagonal elements. There are in total D diagonal elements. However, we can always decompose $G_{\mu\nu}$ with respect to the Lorentz indices as

$$G^{\mathrm{TL}}_{\mu\nu} + \frac{1}{D} g_{\mu\nu} G\,, \tag{3.17}$$

where $G = \mathrm{tr_L}\, G_{\mu\nu} \equiv g^{\mu\nu} G_{\mu\nu}$ and $\mathrm{tr}_L\, G^{\mathrm{TL}}_{\mu\nu} = 0$ is the traceless part.

Since we intend to use $G_{\mu\nu}\partial^\nu\partial^\nu$ as a kinetic operator for the field, the trace part would simply lead to a Klein-Gordon type operator, which we would be free to add afterwards if desired. However, imposing additional symmetries of a chiral type can forbid the occurance of such a trivial term. Hence, we exclude it from now on. Concentrating on the irreducible derivative tensor structure $\partial^\mu\partial^\nu - \frac{g^{\mu\nu}}{D}\partial^2$, we consider $G_{\mu\nu}$ to be traceless for the remainder, i.e.

$$G_{\mu\nu} \equiv G^{\mathrm{TL}}_{\mu\nu}\,. \tag{3.18}$$

In this condition, only $D-1$ diagonal elements are independent. To span the space of all $G_{\mu\nu}$, we thus need

$$d_e = \frac{1}{2}D(D-1) + D - 1 = \frac{1}{2}D^2 + \frac{1}{2}D - 1 \tag{3.19}$$

anti-commuting elements. These are available in a d_γ dimensional Clifford-algebra listed in table 1.
If d_e is even, we can think of a Dirac algebra in d_e dimensions, where the dimensionality of the irreducible representation of the γ matrices is $d_\gamma = 2^{d_e/2}$.

If d_e is odd, we may use the Dirac algebra in $d_e - 1$ dimensions and include γ_* as the d_eth element, so $d_\gamma = 2^{\frac{d_e-1}{2}}$.

Now, let γ_a be the required d_γ dimensional Dirac matrices, satisfying a (Euclidean) Clifford algebra

$$\left\{\gamma_a, \gamma_b\right\} = 2\delta_{ab}\,, \tag{3.20}$$

with $a, b = 1, 2 \ldots, d_e$. Then we can span the space of all $G_{\mu\nu}$ by

$$G_{\mu\nu} = a^a_{\mu\nu}\gamma_a\,, \tag{3.21}$$

where $a^a_{\mu\nu} = a^a_{\nu\mu}$ being symmetric.

Insertion of equation (3.21) into equation (3.16) yields

$$\begin{aligned}\{G_{\mu\nu}, G_{\kappa\lambda}\} &= a^a_{\mu\nu} a^b_{\kappa\lambda} \{\gamma_a, \gamma_b\} \\ &= 2 a^a_{\mu\nu} a^a_{\kappa\lambda} \\ &= -\frac{2}{D-1} g_{\mu\nu} g_{\kappa\lambda} + \frac{D}{D-1} (g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa}). \end{aligned} \tag{3.22}$$

Contracting (3.22) with $g^{\mu\nu}$ leads us to

$$2a^a a^a_{\kappa\lambda} = -\frac{2D}{D-1} g_{\kappa\lambda} + \frac{D}{D-1}(2g_{\kappa\lambda}) = 0\,, \tag{3.23}$$

where we defined

$$a^a = g^{\mu\nu} a^a_{\mu\nu}\,. \tag{3.24}$$

Equation (3.23) reflects a trivial result as it expresses the tracelessness of $G_{\mu\nu}$, i.e. also the $a^a_{\mu\nu}$ have to be traceless 2nd rank Lorentz tensors.

Therefore, let us contract (3.22) with $g^{\mu\kappa}$:

$$\begin{aligned} 2a^{a\mu}_{\nu} a^a_{\mu\lambda} &= -\frac{2}{D-1} g_{\nu\lambda} + \frac{D}{D-1}(D g_{\nu\lambda} + g_{\nu\lambda}) \\ &= -\frac{2}{D-1} g_{\nu\lambda} + \frac{D}{D-1}(D+1) g_{\nu\lambda} \\ &= \frac{D^2 + D - 2}{D-1} g_{\nu\lambda} \\ &= \frac{(D+2)(D-1)}{D-1} g_{\nu\lambda} \\ &= (D+2) g_{\nu\lambda}\,. \end{aligned} \tag{3.25}$$

Further contraction with $g^{\nu\lambda}$ yields:

$$2a^{a\mu\nu} a^a_{\mu\nu} = D(D+2)\,. \tag{3.26}$$

Apart from the condition (3.22) and their consequences (3.23) to (3.26), there is a rather large freedom of fixing the components of $a^a_{\mu\nu}$. For a specific choice in the Euclidean case see [3]. Equation (3.21) illustrates that the $G_{\mu\nu}$ inherit a large further symmetry, namely similarity transformations of the Dirac algebra:

$$G_{\mu\nu} \longrightarrow G'_{\mu\nu} = S G_{\mu\nu} S^{-1}, \tag{3.27}$$

where

$$S \epsilon \mathrm{SL}(d_\gamma, \mathbb{C})\,. \tag{3.28}$$

The symmetry in (3.27) is the same as for Dirac spinors, which we discussed in section 2.4.

3.2 Hermiticity properties of $\mathbf{G}_{\mu\nu}$

Let us first check the hermiticity properties of the G's, say for the metric convention $g = \mathrm{diag}(1,-1,-1,-1)$. From (3.16) we get for $\kappa = \mu$, $\lambda = \nu$:

$$2\,G^2_{\underline{\mu\nu}} = -\frac{2}{D-1}\,g^2_{\underline{\mu\nu}} + \frac{D}{D-1}\left(g_{\underline{\mu\mu}}\,g_{\underline{\nu\nu}} + g_{\underline{\mu\nu}}\,g_{\underline{\nu\mu}}\right). \tag{3.29}$$

e.g.

$$\mu = \nu: \qquad G^2_{\underline{\mu\mu}} = 1 \tag{3.30a}$$

$$(\mu,\nu) = (0,i): \qquad G^2_{0,i} = -\frac{1}{2}\frac{D}{D-1} \tag{3.30b}$$

$$(\mu,\nu) = (i, j \neq i): \qquad G^2_{i,j\neq i} = \frac{1}{2}\frac{D}{D-1}\,. \tag{3.30c}$$

Equation (3.30a) shows a constraint for $D-1$ independent elements G as one of the D elements is already fixed due to tracelessness. Equation (3.30b) is a constraint for $D-1$ elements whereas the last equation (3.30c) is a constraint for $1/2(D-1)(D-2)$ elements. Summing these up

$$D-1+D-1+\frac{1}{2}(D-1)(D-2) = d_e \tag{3.31}$$

shows that this fixes all d_e anti-commuting elements to span the space of all $G_{\mu\nu}$.

Using the representation (3.21) of the $G_{\mu\nu}$ in terms of a Euclidean Dirac algebra, the results of (3.30) also hold for the coefficients $a^a_{\mu\nu}$ analogously.

$$\begin{aligned} &\sum_a (a^a_{\underline{\mu\mu}})^2 = 1, \\ &\sum_a (a^a_{ij})^2 = \frac{1}{2}\frac{D}{D-1}, \qquad i \neq j, \\ &\sum_a (a^a_{0i})^2 = -\frac{1}{2}\frac{D}{D-1}. \end{aligned} \tag{3.32}$$

The simplest choice satisfying the sign constraints of (3.32) would be

$$a^a_{\underline{\mu\mu}}\,\epsilon\,\mathbb{R}, \qquad a^a_{ij}\,\epsilon\,\mathbb{R}, \qquad a^a_{0i}\,\epsilon\, i\mathbb{R}. \tag{3.33}$$

In this case, $G_{\underline{\mu\mu}}$ and G_{ij} would be hermitian, and G_{0i} would be anti-hermitian.

3.3 Simplest Choice

Let us see how far we get with assumption (3.33). Following a similar reasoning as for the Dirac spinors, reality of the mass term implies (c.f. (2.26) & (2.27)):

$$h = h^\dagger. \tag{3.34}$$

Using the constructed kinetic term of the Hamiltonian in chapter 3.1, we can now take a closer look at the kinetic term for which we make the ansatz:

$$S_{\text{kin}} = \int \mathrm{d}^4x \, \bar{\Psi} G_{\mu\nu} (i\partial^\mu)(i\partial^\nu) \Psi \tag{3.35}$$

$$S^*_{\text{kin}} = \int \mathrm{d}^4x \left[(-i\partial^\mu)(-i\partial^\nu) \Psi^\dagger \right] G^\dagger_{\mu\nu} \bar{\Psi}^\dagger \tag{3.36}$$

$$\overset{i.b.p.}{=} \int \mathrm{d}^4x \, \Psi^\dagger G^\dagger_{\mu\nu} (i\partial^\mu)(i\partial^\nu) \bar{\Psi}^\dagger \tag{3.37}$$

$$= \int \mathrm{d}^4x \, \Psi^\dagger h h^{-1} G^\dagger_{\mu\nu} (i\partial^\mu)(i\partial^\nu) h^\dagger \Psi \tag{3.38}$$

$$= \int \mathrm{d}^4x \, \bar{\Psi} h^{-1} G^\dagger_{\mu\nu} h^\dagger (i\partial^\mu)(i\partial^\nu)) \Psi . \tag{3.39}$$

From the reality of $S^*_{\text{kin}} = S_{\text{kin}}$ we can conclude:

$$G_{\mu\nu} = h^{-1} G^\dagger_{\mu\nu} h^\dagger , \tag{3.40}$$

which again underlines the reason for the name ‘hermitizer’ for the spin metric h.

Based on assumption (3.33) we get for the anti-hermitian matrices G_{0i} in (3.40)

$$\begin{aligned} G_{0i} &= h^{-1} G^\dagger_{0i} h^\dagger = -h^{-1} G_{0i} h \\ \Rightarrow h G_{0i} &= -G_{0i} h^\dagger = -G_{0i} h \\ \Rightarrow \{h, G_{0i}\} &= 0 . \end{aligned} \tag{3.41}$$

Any choice of $h = \gamma^{\bar{a}}$ for any „$\bar{a}$“ together with $G_{0i} = i\gamma^b$ with $b \neq \bar{a}$ would satisfy (3.41).

Now we turn to the hermitian matrices $G_{\underline{\mu\mu}}$ or G_{ij}:

$$\begin{aligned} G_{ij} &= h^{-1} G^\dagger_{ij} h^\dagger = h^{-1} G_{ij} h^\dagger \\ \Rightarrow h G_{ij} &= G_{ij} h \\ \Rightarrow \left[h, G_{ij}\right] &= 0 , \end{aligned} \tag{3.42a}$$

and similarly for

$$\left[h, G_{\underline{\mu\mu}}\right] = 0 . \tag{3.42b}$$

Assuming that there is a choice $h = \gamma^{\bar{a}}$, „$\bar{a}$ “ fixed, then (3.42a) implies

$$\gamma^{\bar{a}} \gamma^b a^b_{ij} \overset{!}{=} \gamma^b \gamma^{\bar{a}} a^b_{ij} = -\gamma^{\bar{a}} \gamma^b a^b_{ij} + 2\delta^{\bar{a}b} a^b_{ij} \tag{3.43}$$

$$\Rightarrow 2\gamma^{\bar{a}} \gamma^b a^b_{ij} = \delta^{\bar{a}b} a^b_{ij} \tag{3.44}$$

$$\Rightarrow a^b_{ij} = 0 \quad \text{for } b \neq \bar{a} . \tag{3.45}$$

This contradicts the condition (3.13). Therefore the simplest assumption $h = \gamma^{\bar{a}}$ fails in general.

4 Representations

4.1 In $2+1$ Dimensions

Before we turn back to the problem of the spin metric, let us study more concretely the construction of an explicit representation of the G´s satisfying the Minkowski space constraints (3.32) and of course the defining algebra (3.16).

For simplicity, we begin with $D = 3$ and use a Euclidean Dirac algebra $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$, $a, b = 1, \ldots, 5$, $\gamma^a = (\gamma^a)^\dagger$, as building blocks. The following choice is a representation of the defining algebra:

$$\begin{aligned} &G_{00} = \gamma^4, && G_{11} = \frac{1}{2}\gamma^4 + \frac{\sqrt{3}}{2}\gamma^5, \quad G_{22} = \frac{1}{2}\gamma^4 + \frac{\sqrt{3}}{2}\gamma^5 \\ &G_{0j} = i\frac{\sqrt{3}}{2}\gamma^j,\ j = 1,2 && G_{12} = \frac{\sqrt{3}}{2}\gamma^3. \end{aligned} \tag{4.1}$$

This is a relativistic generalization of the representation found in [3]. It is straight forward to see that (4.1) satisfies the normalizing constraints (3.32) and the tracelessness condition

$$g^{\mu\nu} G_{\mu\nu} = G_{00} - G_{11} - G_{22} = 0\,. \tag{4.2a}$$

For the remaining check of the algebra (3.16) we note that

$$\begin{aligned} &\left\{G_{\underline{\mu\mu}}, G_{\nu i}\right\} = 0 \text{ for } \nu \neq i \\ &\{G_{0i}, G_{12}\} = 0\,, \end{aligned} \tag{4.2b}$$

both as it should be when using the right hand side of (3.16). We see that this representation satisfies the hermiticity properties

$$G_{0i} = -G_{0i}^\dagger\,, \qquad G_{\underline{\mu\mu}} = G_{\underline{\mu\mu}}^\dagger\,, \qquad G_{12} = G_{12}^\dagger\,, \tag{4.3}$$

as it corresponds to the simplest choice (3.33).

Now, using SL$(4,\mathbb{C})$ spin base transformations, we can reach any other representation. In turn any other representation can be transformed into (4.1).

In chapter 3.3 we have seen that the easiest assumption $h = \gamma^{\bar{a}}$, $\bar{a}$ fixed, fails. This failure implies in particular that we do not know how to define the conjugate spinor $\bar{\Psi}$ for which we need the spin metric h.

Let us propose a different construction. For this, we define

$$\tilde{\Psi}(t, \vec{x}) = \Psi^\dagger(t, -\vec{x})\,. \tag{4.4}$$

Note that in $2+1$ dimensions, this corresponds to a charge conjugation and a rotation of the spatial plane by π. In $3+1$ dimensions this would correspond to a charge conjugation and a parity transformation.

Now, let us study

$$\tilde{S}_{\text{kin}} = \int \mathrm{d}^3x\, \tilde{\Psi}(t, \vec{x}) G_{\mu\nu}(-i\partial^\mu)(-i\partial^\nu)\Psi(t, \vec{x})\,. \tag{4.5}$$

For the action to be real, we need to proof $\tilde{S}^*_{\text{kin}} = \tilde{S}_{\text{kin}}$:

$$\begin{aligned}
\tilde{S}^*_{\text{kin}} &= \int \mathrm{d}^3x\,(i\partial^\mu)(i\partial^\nu)\Psi^\dagger(t,\vec{x})G^\dagger_{\mu\nu}\Psi(t,-\vec{x}) \\
&\overset{i.b.p.}{=} \int \mathrm{d}^3x\,\Psi^\dagger(t,\vec{x})G^\dagger_{\mu\nu}(-i\partial^\mu)(-i\partial^\nu)\Psi(t,-\vec{x}) \\
&= \int \mathrm{d}^3x\,\Psi^\dagger(t,\vec{x})\Big[G^\dagger_{00}(-i\partial^0)(-i\partial^0) + 2G^\dagger_{0i}(-i\partial^0)(-i\partial^i) \\
&\qquad\qquad + G^\dagger_{ij}(-i\partial^i)(-i\partial^j)\Big]\Psi(t,-\vec{x}) \\
&\overset{(4.3)}{=} \int \mathrm{d}^3x\,\Psi^\dagger(t,\vec{x})\Big[G_{00}(-i\partial^0)(-i\partial^0) - 2G_{0i}(-i\partial^0)(-i\partial^i) \\
&\qquad\qquad + G_{ij}(-i\partial^i)(-i\partial^j)\Big]\Psi(t,-\vec{x}).
\end{aligned} \tag{4.6}$$

In a last step, we now substitute the integration variable

$$\vec{x} \longrightarrow -\vec{x}, \tag{4.7}$$

i.e.

$$\int \mathrm{d}^3x = \int \mathrm{d}t\mathrm{d}^2x \overset{\vec{x}\to-\vec{x}}{=} \int \mathrm{d}t\mathrm{d}^2x = \int \mathrm{d}^3x, \tag{4.8}$$

but

$$(-i\partial^0)(-i\partial^j) \longrightarrow -(-i\partial^0)(-i\partial^j); \tag{4.9}$$

all other terms keep their sign. With this substitution we get

$$\begin{aligned}
\tilde{S}^*_{\text{kin}} &= \int \mathrm{d}^3x\,\Psi^\dagger(t,-\vec{x})\Big[G_{00}(-i\partial^0)(-i\partial^0) + 2G_{0i}(-i\partial^0)(-i\partial^i) \\
&\qquad\qquad + G_{ij}(-i\partial^i)(-i\partial^j)\Big]\Psi(t,\vec{x})
\end{aligned}\tag{4.10}$$

$$\equiv \int \mathrm{d}^3x\,\tilde{\Psi}(t,\vec{x})G_{\mu\nu}(-i\partial^\mu)(-i\partial^\nu)\Psi(t,\vec{x}) \tag{4.11}$$

$$= \tilde{S}_{\text{kin}}. \tag{4.12}$$

Thus, the action $\tilde{S}_{\text{kin}}$ is real.

The ansatz (4.4) shows that we need the rotated spinor. Since the rotation is part of the Lorentz group, there should be a corresponding $\mathrm{Sl}(d_e,\mathbb{C})$ element that 'does the job' of describing this rotation.

In fact, there is one, but even without explicitly constructing this rotation in $\mathrm{Sl}(d_e,\mathbb{C})$, the spin metric can be found by realizing that it has to correspond to a rotation in the $1-2$ plane. The desired spin metric is indeed given by

$$h := i\gamma^1\gamma^2. \tag{4.13}$$

Let us verify its necessary properties:

$$h^\dagger = (i\gamma^1\gamma^2)^\dagger = -i\gamma^{2^\dagger}\gamma^{1^\dagger} = -i\gamma^2\gamma^1 = i\gamma^1\gamma^2 = h. \tag{4.14}$$

And using the representation (4.1),

$$hG_{0i} = i\frac{\sqrt{3}}{2}i\gamma^1\gamma^2\gamma^i = -i\frac{\sqrt{3}}{2}\gamma^i i\gamma^1\gamma^2 = -G_{0i}h \Rightarrow \{h, G_{0i}\} = 0 \tag{4.15}$$

satisfies (3.41).

Finally, note that $G_{\underline{\mu\mu}}$ and G_{ij} are $\sim \gamma^a$, $a = 3,4,5$, such that

$$h\gamma^a = i\gamma^1\gamma^2\gamma^a = -i\gamma^1\gamma^a\gamma^2 = \gamma^a i\gamma^1\gamma^2 = \gamma^a h \Rightarrow \left[h, G_{ij}\right] = 0 = \left[h, G_{\underline{\mu\mu}}\right] \tag{4.16}$$

satisfies (3.42a) and (3.42b).

Therefore, we have found the required conjugate spinor

$$\bar{\Psi} = \Psi^\dagger h, \qquad\qquad h = i\gamma^1\gamma^2, \tag{4.17}$$

such that

$$S_{\text{kin}} = \int \mathrm{d}^3x\,\bar{\Psi}G_{\mu\nu}(i\partial^\mu)(i\partial^\nu)\Psi \tag{4.18}$$

is the desired Lorentz invariant action for $D = 2+1$!

4.2 In $3+1$ Dimensions

Let us now turn to the $3+1$ dimensional case. As said before, the ansatz (4.4) in $3+1$ dimensions describes a parity transformation which is not part of the space the generators γ^a, $a = 1,\ldots,9$, span. The substitution $\vec{x} \longrightarrow -\vec{x}$ is similar as to before:

$$\int \mathrm{d}^4x = \int \mathrm{d}t\mathrm{d}^3x \overset{\vec{x}\to-\vec{x}}{=} \int \mathrm{d}t\mathrm{d}^3x = \int \mathrm{d}^4x. \tag{4.19}$$

In detail, from equation (4.6) we get:

$$\begin{aligned}\tilde{S}^*_{\text{kin}} = \int \mathrm{d}^4x\,\Psi^\dagger(t,\vec{x})\Big[&G_{00}(-i\partial^0)(-i\partial^0) - 2G_{0i}(-i\partial^0)(-i\partial^i)\\ &+G_{ij}(-i\partial^i)(-i\partial^j)\Big]\Psi(t,-\vec{x}),\end{aligned} \tag{4.20}$$

implementing the substitution $\vec{x} \to -\vec{x}$ yields

$$\begin{aligned}\tilde{S}^*_{\text{kin}} &= \int \mathrm{d}^4x\,\Psi^\dagger(t,-\vec{x})\Big[G^\dagger_{00}(-i\partial^0)(-i\partial^0) + 2G^\dagger_{0i}(-i\partial^0)(-i\partial^i)\\ &\qquad +G^\dagger_{ij}(-i\partial^i)(-i\partial^j)\Big]\Psi(t,\vec{x})\\ &\equiv \int \mathrm{d}^4x\,\tilde{\Psi}(t,\vec{x})G_{\mu\nu}(-i\partial^\mu)(-i\partial^\nu)\Psi(t,\vec{x})\\ &= \tilde{S}_{\text{kin}}.\end{aligned} \tag{4.21}$$

It is easy to prove that there is no way to describe the parity transformation in terms of γ matrices. In other words, its is not possible to find the spin metric h satisfying the condition (3.40) in the irreducible representation of the Clifford algebra (3.20). If we move away from the irreducible representation of the $\gamma^a, a = 1,\dots,9$, with the dimensionality $d_\gamma = 2^{d_e/2} = 16$ to a reducible representation with $\gamma^a, a = 1,\dots,11$, and $d_\gamma = 32$ then we have two more anti-commuting matrices which we can use to build the spin metric h. In this case it is always possible to find h, which is hermitian and satisfies the commutators (3.42a) and (3.42b) as well as the anti-commutator (3.41) such that the action

$$S = S_m + S_{\text{kin}} \tag{4.22}$$

$$= -m\int \mathrm{d}^4x\,\bar{\Psi}\Psi + \int \mathrm{d}^4x\,\bar{\Psi}G_{\mu\nu}(i\partial^\mu)(i\partial^\nu)\Psi \tag{4.23}$$

is real and

$$\bar{\Psi} = \Psi^\dagger h \tag{4.24}$$

is the conjugated spinor.

Let us see how this is done in an explicit example. We can try to construct the $G_{\mu\nu}$ similarly to the one in $2+1$ dimensions:

$$G_{0i} = i a_{\underline{i}}\gamma^{\underline{i}} \qquad \text{where } a_i \epsilon \mathbb{R} \text{ and } i = 1,2,3\,, \tag{4.25}$$

with a_i to be determined. We also assume

$$G_{12} = a_4\gamma^4 \qquad G_{23} = a_5\gamma^5 \qquad G_{13} = a_6\gamma^6\,, \tag{4.26}$$

where $a_{4,5,6}\,\epsilon\,\mathbb{R}$. We may set

$$G_{00} = \gamma^7 \tag{4.27}$$

and construct G_{11}, G_{22}, G_{33} from a suitable linear combination of $\gamma^7,\gamma^8,\gamma^9$ with real prefactors such that the $G_{\mu\nu}$ algebra (3.16) and tracelessness is satisfied. This choice fullfills the hermiticity constraints (3.33). Now, we need to find the spin metric h such that (3.42a),(3.42b), and (3.41) are satisfied. The naive choices $h = \gamma^{10}$ as well as $h = \gamma^{11}$ do not comply with the commutators $[h, G_{ij}] = 0 = \left[h, G_{\underline{\mu\mu}}\right]$.

Instead, we suggest

$$h = \gamma^1\gamma^2\gamma^3\gamma^{10}. \tag{4.28}$$

First, let us check the hermiticity of h:

$$h^\dagger = \gamma^{10^\dagger}\gamma^{3^\dagger}\gamma^{2^\dagger}\gamma^{1^\dagger} = \gamma^{10}\gamma^3\gamma^2\gamma^1 = -\gamma^1\gamma^{10}\gamma^3\gamma^2 = -\gamma^1\gamma^2\gamma^{10}\gamma^3 = \gamma^1\gamma^2\gamma^3\gamma^{10} = h\,. \tag{4.29}$$

Because

$$\{h,\gamma^1\} = \gamma^1\gamma^2\gamma^3\gamma^{10}\gamma^1 + \gamma^1\gamma^1\gamma^2\gamma^3\gamma^{10} = 0 \tag{4.30}$$

and similarly for $\{h,\gamma^2\}$ and $\{h,\gamma^3\}$, the anti-commutator

$$\{h, G_{0i}\} = 0 \tag{4.31}$$

is satisfied. To check the commutators we see that

$$[h,\gamma^a] = \gamma^1\gamma^2\gamma^3\gamma^{10}\gamma^a - \gamma^a\gamma^1\gamma^2\gamma^3\gamma^{10} \tag{4.32}$$
$$= \gamma^a\gamma^1\gamma^2\gamma^3\gamma^{10} - \gamma^a\gamma^1\gamma^2\gamma^3\gamma^{10} \tag{4.33}$$
$$= 0 \tag{4.34}$$

for $a = 4,5,6,7,8,9$. Therefore, we get

$$[h, G_{ij}] = 0 = \left[h, G_{\underline{\mu\mu}}\right] \tag{4.35}$$

as required. This means that $h = \gamma^1\gamma^2\gamma^3\gamma^{10}$ is indeed a spin metric for the chosen construction of the G's with the following coefficients:

$$G_{0i} = i\frac{\sqrt{2}}{3}\gamma^i, \tag{4.36}$$
$$G_{12} = \frac{\sqrt{2}}{3}\gamma^4, \quad G_{23} = \frac{\sqrt{2}}{3}\gamma^5, \quad G_{13} = \frac{\sqrt{2}}{3}\gamma^6, \tag{4.37}$$
$$G_{00} = \gamma^7, \tag{4.38}$$
$$G_{11} = \frac{1}{3}\gamma^7 + \frac{2\sqrt{2}}{3}\gamma^8, \tag{4.39}$$
$$G_{22} = \frac{1}{3}\gamma^7 - \frac{\sqrt{2}}{3}\gamma^8 - \sqrt{\frac{2}{3}}\gamma^9, \tag{4.40}$$
$$G_{33} = \frac{1}{3}\gamma^7 - \frac{\sqrt{2}}{3}\gamma^8 + \sqrt{\frac{2}{3}}\gamma^9. \tag{4.41}$$

With this we have proven that it is possible to find the spin metric h and built the conjugated spinor

$$\bar{\Psi} = \Psi^\dagger h, \tag{4.42}$$

such that

$$S_{\text{kin}} = \int \mathrm{d}^4x \bar{\Psi} G_{\mu\nu}(i\partial^\mu)(i\partial^\nu)\Psi \tag{4.43}$$

is real.

5 Coupling to Electromagnetic-Fields

Now, we want to investigate the behaviour of the new kinetic operator when coupled to electromagnetic fields. For this we perform the *minimal coupling prescription* and obtain

$$\mathscr{L} = \bar{\Psi} G_{\mu\nu}(iD^{\mu})(iD^{\nu})\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{5.1}$$

Replacing $D^{\mu} = \partial^{\mu} + ieA^{\mu}$ yields

$$\begin{aligned}\mathscr{L} = \bar{\Psi} G_{\mu\nu}(i\partial^{\mu})(i\partial^{\nu})\Psi - \bar{\Psi} G_{\mu\nu}(i\partial^{\mu})(eA^{\nu})\Psi - \bar{\Psi} G_{\mu\nu}(eA^{\mu})(i\partial^{\nu})\Psi \\ + \bar{\Psi} G_{\mu\nu}(eA^{\mu})(eA^{\nu})\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.\end{aligned} \tag{5.2}$$

The Lagrangian indeed comprises a local gauge invariance for the simultaneous transformations

$$\begin{aligned}\Psi &\longrightarrow \Psi' = e^{ie\theta(\mathbf{x})}\Psi, \\ A^{\mu} &\longrightarrow A^{\mu} - \partial^{\mu}\theta(\mathbf{x}).\end{aligned} \tag{5.3}$$

From these, the following transformations

$$\begin{aligned}\bar{\Psi} &\longrightarrow \bar{\Psi}' = \bar{\Psi} e^{-ie\theta(\mathbf{x})}, \\ D^{\mu} &\longrightarrow D'^{\mu} = \partial^{\mu} + ieA^{\mu} - ie\partial^{\mu}\theta(\mathbf{x}) = e^{ie\theta(\mathbf{x})} D e^{-ie\theta(\mathbf{x})}\end{aligned}$$

arise trivially.

To verify the local gauge invariance, we observe

$$\begin{aligned}F_{\mu\nu}F^{\mu\nu} &\longrightarrow F_{\mu\nu}F^{\mu\nu}, \\ \bar{\Psi} G_{\mu\nu}(iD^{\mu})(iD^{\nu})\Psi &\longrightarrow \bar{\Psi} e^{-ie\theta(\mathbf{x})} G_{\mu\nu} e^{ie\theta(\mathbf{x})}(iD^{\mu}) e^{-ie\theta(\mathbf{x})} e^{ie\theta(\mathbf{x})}(iD^{\nu}) e^{-ie\theta(\mathbf{x})} e^{ie\theta(\mathbf{x})}\Psi \\ &= \bar{\Psi} G_{\mu\nu}(iD^{\mu})(iD^{\nu})\Psi.\end{aligned}$$

Therefore, the Lagrangian is invariant under the local gauge transformation (5.3). The Lagrangian shows the kinetic terms for the fermions as well as for the electromagnetic fields, but also shows three interaction terms.

6 Conclusion

This thesis discusses the construction of relativistic field theories with Luttinger fermions.

After reviewing the Dirac formalism and the corresponding Dirac equation we took into account how the Dirac conjugate spinor $\bar{\Psi}$ is built to keep the action for fermionic theories real. We also reminded ourselves about gauge theories and the coupling of electro-magnetic fields to fermions.

In the next chapter we started off by constructing the Hamiltonian for Luttinger fermions and lifting the resulting anti-commutator to the general Minkowski spacetime. Realizing that the simplest choice for the coefficients $a^a_{\mu\nu}$ of the γ matrices and the simplest ansatz for the Dirac conjugate leads to a contradiction, we saw a different way to solve the problem and find the spin metric in $2+1$ dimensions. For $3+1$ dimensions we had to move to a reducible representation of the Clifford algebra.

Furthermore, we were able to show that it is possible to find a spin metric in $3+1$ dimensions and built a Dirac-like action with the new found kinetic operator:

$$G_{\mu\nu}(i\partial^\mu)(i\partial^\nu). \tag{6.1}$$

We found that the spin metric h has to satisfy the following conditions:

$$h = h^\dagger, \tag{6.2}$$

$$G_{\mu\nu} = h^{-1} G^\dagger_{\mu\nu} h, \tag{6.3}$$

$$|\det h| = 1. \tag{6.4}$$

In the last section we showed that the new action when coupled to a gauge field comprises the same gauge invariance as known for the Dirac action and the electro-magnetic fields.

In further research it would be interesting to treat this relativistic field theory with the methods of the renormalization group and look out for quantum critical fix points. Another way to go further could be to see if this construction is possible in higher dimensions as well.

Bibliography

[1] Lucile Savary, Eun Gook Moon, and Leon Balents. "New Type of Quantum Criticality in the Pyrochlore Iridates". In: *Physical Review D* (2014), p. 041027.

[2] Igor F. Herbut and Lukas Janssen. "Topological Mott Insulator in Three-Dimensional Systems with Quadratic Band Touching". In: *Phys. Rev. Lett.* 113 (10 Sept. 2014), p. 106401. URL: https://link.aps.org/doi/10.1103/PhysRevLett.113.106401.

[3] Lukas Janssen and Igor F. Herbut. "Nematic quantum criticality in three-dimensional fermi system with quadratic band touching". In: *Physical Review B* 92.4 (2015).

[4] Francesco Parisen Toldin, Martin Hohenadler, Fakher F. Assaad, and Igor F. Herbut. "Fermionic quantum criticality in honeycomb and π-flux Hubbard models: Finite-size scaling of renormalization-group-invariant observables from quantum Monte Carlo". In: *Physical Review B* 91.16 (Apr. 2015). URL: https://doi.org/10.1103%2Fphysrevb.91.165108.

[5] J. M. Luttinger. "An Exactly Soluble Model of a Many-Fermion System". In: *Journal of Mathematical Physics* 4.9 (1963), pp. 1154–1162. eprint: https://doi.org/10.1063/1.1704046. URL: https://doi.org/10.1063/1.1704046.

[6] Kai Sun, Hong Yao, Eduardo Fradkin, and Steven A. Kivelson. "Topological Insulators and Nematic Phases from Spontaneous Symmetry Breaking in 2D Fermi Systems with a Quadratic Band Crossing". In: *Phys. Rev. Lett.* 103 (4 July 2009), p. 046811. URL: https://link.aps.org/doi/10.1103/PhysRevLett.103.046811.

[7] Y. D. Chong, Xiao-Gang Wen, and Marin Solja či ć. "Effective theory of quadratic degeneracies". In: *Phys. Rev. B* 77 (23 June 2008), p. 235125. URL: https://link.aps.org/doi/10.1103/PhysRevB.77.235125.

[8] Vladimir Cvetkovic, Robert E. Throckmorton, and Oskar Vafek. "Electronic multicriticality in bilayer graphene". In: *Phys. Rev. B* 86 (7 Aug. 2012), p. 075467. URL: https://link.aps.org/doi/10.1103/PhysRevB.86.075467.

[9] Santanu Dey and Joseph Maciejko. "Quantum-critical electrodynamics of Luttinger fermions". In: *Physical Review B* 106.3 (July 2022). URL: https://doi.org/10.1103%2Fphysrevb.106.035140.

[10] Peter Woit. *Quantum Theory, Groups and Representations - An Introduction.* Berlin, Heidelberg: Springer, 2017. Chap. 41.

[11] Alessandro De Angelis and Mário Pimenta. *Introduction to Particle and Astroparticle Physics - Multimessenger Astronomy and its Particle Physics Foundations.* Berlin, Heidelberg: Springer, 2018.

[12] Holger Gies and Stefan Lippoldt. "Fermions in gravity with local spin-base invariance". In: *Physical Review D* 89.6 (Mar. 2014). URL: https://doi.org/10.1103%2Fphysrevd.89.064040.

[13] H. Arthur Weldon. “Fermions without vierbeins in curved space-time”. In: *Physical Review D* 63.10 (Apr. 2001). URL: https://doi.org/10.1103%2Fphysrevd.63.104010.

[14] Stefan Lippoldt. “Spin-base invariance of fermions in arbitrary dimensions”. In: *Phys. Rev. D* 91 (10 May 2015), p. 104006. URL: https://link.aps.org/doi/10.1103/PhysRevD.91.104006.

Acknowledgment

Here, I want to say thank you to the people who supported me throughout the time I worked for this thesis.
I want to thank my supervisor Prof. Dr. Holger Gies for the time he took for me out of his busy schedule, for all the answers to my questions and making my plans of a year abroad possible. Also I would like to thank Prof. Dr. Stefan Flörchinger for providing the second opinion.
Thank you Adrian Minnich and Elias Wahl for proofreading my thesis and for your helpful support.
Special thanks go to Marlen Hund for always having a friendly ear for my thoughts and troubles.
And lastly I want to thank my family and friends who supported me in any way.

Statement of authorship

I declare that I completed this thesis on my own and that information which has been directly or indirectly taken from other sources has been noted as such. Neither this nor a similar work has been presented to an examination committee.

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Johannes Laufkötter