Faculty of Physics and Astronomy

University of Heidelberg

Diploma thesis
in Physics

submitted by

Jens Hämmerling

born in Winsen/Luhe, Germany

Heidelberg
2004

GEOMETRY OF SPIN-FIELD COUPLINGS ON THE WORLDLINE

This diploma thesis has been carried out by
Jens Hämmerling
at the Institute for Theoretical Physics

under the supervision of
Dr. Holger Gies
and
Prof. Michael G. Schmidt

GEOMETRIE DER SPIN-FELDKOPPLUNG AUF DER WELTLINIE

Zusammenfassung

In dieser Arbeit leiten wir eine geometrische Darstellung der Kopplung von Spinfreiheitsgraden an Eichfelder im Weltlinienzugang zur Quantenfeldtheorie her. Dazu kombinieren wir die von der Stringtheorie inspirierte Methode des Weltlinienformalismus mit der Loop-space Formulierung von Eichtheorien. Nachdem wir die Äquivalenz zur gewöhnlichen Darstellung von Eichtheorien gezeigt haben, leiten wir eine Ortsraumdarstellung für die dem Krümmungstensor entsprechende Größe, der sogenannten Loopableitung ab. Damit stellen wir eine Verbindung zwischen dem Loop Calculus und dem gewohnten Funktionalkalkül her. Mittels dieser Darstellung ist es uns möglich einen Ableitungsoperator auf dem Raum der Holonomien zu definieren, der seine Gültigkeit auch in der Anwendung auf die Weltliniendarstellung der effektiven Wirkung behält. Dies gipfelt in einem Spin Faktor, der die Information des Spins mit der "zickzack"-Bewegung des zugehörigen Feldes in der Weltliniendarstellung verbindet. Dabei konzentrieren wir uns vornehmlich auf den Fall der Quantenelektrodynamik im äußeren Feld, in dem wir eine rein geometrische Darstellung des Pauli-Terms erhalten. Als verdeutlichendes Beispiel leiten wir die wohlbekannte Heisenberg-Euler Wirkung aus dem Zusammenspiel von Spinfaktor und Holonomie her.

GEOMETRY OF SPIN-FIELD COUPLINGS ON THE WORLDLINE

Abstract

We derive a geometric representation of couplings between spin degrees of freedom and gauge fields within the worldline approach to Quantum Field Theory. For this purpose we combine the string inspired methods of the worldline formalism with the loop-space approach to gauge theory. After we have shown the equivalence to the familiar representation of gauge theory we derive the coordinate representation of the so-called loop derivative, which corresponds to the curvature tensor in ordinary gauge theory. For this reason we associate the loop calculus with the familiar functional calculus. With the aid of the coordinate representation we are able to define a derivative operator on the space of all holonomies which can immediately be applied to the worldline representation of the effective action. This results in a spin factor that associates the information about spin with "zigzag" motion of the fluctuating field. We concentrate on the case of Quantum Electrodynamics in external fields where we obtain a purely geometric representation of the Pauli term. As an illustrative example, we rederive the well-known Heisenberg-Euler action from the interplay between spin factor and holonomy.

Contents

1 Introduction

In ordinary quantum field theory perturbative calculations are usually done in the so-called second quantized approach which involves the techniques of Feynman diagrams. But in addition to this approach to QFT there also exists a first quantized version where the fluctuations of the quantum field are represented by random trajectories of the corresponding particle in coordinate space. This approach is attributed to Richard Feynman who presented the formalism for the case of scalar QED as his first attempt at a general formulation of QFT [1]. However, this line of research did not gain much attention during the following decades, since a practicable technology for treating spinorial degrees of freedom has not been available for a long time. Decades later it was discovered that string theory reduces to field theory in the infinite string tension limit. It was shown that perturbative expansion of string theories can consist of a smaller number of Feynman diagrams, since in string perturbation theory one gets world sheet diagrams with fewer possible topologies than ordinary Feynman diagrams. There appeared analogies between string theoretic effective actions and representations of the effective action in Feynman's worldline representation. For example the Fradkin-Tseytlin path integral is a string theoretic generalization of the effective action for a particle in an external field, with no internal photon corrections [2]. The connection to field theory was made by the observation that in the infinite string tension limit these string diagrams reduce to Feynman's ordinary ones [3],[4],[5],[6].

Besides these string theoretic motivations the worldline formalism is highly interesting for its own, because it gives a first quantized approach to field theory and, connected with this fact, some very illustrative pictures about the quantum behavior of the field in different energy regimes. In practice, it provides for powerful tools to investigate field theoretic problems [7],[8],[9],[10],[11],[12]. Moreover, the worldline approach in combination with Monte-Carlo techniques offers efficient algorithms for computing quantum amplitudes numerically [13],[14],[15],[16],[17]. It is possible to interpret the quantum fluctuations of the field as clouds of loops which pick up information about the behavior of a background field from the infrared to the ultraviolet modes of the fluctuating field. For our purposes in this thesis the representation of quantum field theory in terms of the worldline is especially useful because it offers a connection to the language of loop calculus. This makes it possible to describe generic spin-field couplings in terms of geometry. More precisely we will present the loop-space formulation of gauge theory, which may be considered as more fundamental than the ordinary one, because in this approach the dynamical aspects of gauge theory are represented as kinetic properties of an equivalence class of loops. Ordinary gauge theory appears as a representation of this group of loops onto a special gauge group. For example if we map the group of loops onto the gauge group $U(1)$ we obtain the gauge theory of electrodynamics. This alternative approach to gauge theories was developed by Polyakov, Migdal and others [18],[19] but has not found much attention among the physicists community. Probably one reason for this is that it is rather difficult to quantize the classical gauge theory in the loop approach, because the space spanned by the loop states over a continuous background is far too big to provide a basis for the (seperable) Hilbert space of QFT. Other formulations of gauge theory emerged as more appropriate for the armory of quantization techniques. However, loop states are not too many or too singular in a background independent formulation and this is the key technical point on which, for instance, Loop Quantum Gravity relies [20],[21],[22],[23]. Indeed the loop approach or at least some parts of it undergo directly a revival of interest in the loop gravity approach to quantum gravity, where the states which have to be quantized are expressed in terms of loops and the holonomy becomes a quantum operator that creates loop states. In this approach to quantum gravity the point of view is that holonomies are the natural variables in a gauge theory. In this thesis we will follow this point of view, even though our focus is on quantizing the matter fields rather than the gauge field. Upon quantizing matter fields in an external gauge field in the worldline language, the gauge-field dependence occurs naturally in the form of the holonomy. However, if the matter fields carry spin, additional gauge-field dependencies arise in the form of explicit spin-field couplings. In specific examples, the spin-field coupling can be traded for a gauge-field independent spin factor which is a geometric (and sometimes topological) quantity. As a result, the gauge-field dependence is solely concentrated in the form of the fundamental holonomy.

The purpose of this thesis is to derive an expression for the spin factor in the second order formalism in contrast to the first order approach. First and second order approach refers to a difference in the representation of the functional determinant as follows. In the first order approach the one loop effective action is represented as a determinant of $\not{D}$, whereas in the second order formalism the effective action is written quadratically in $\not{D}$. In the first-order formalism, the first spin-factor representation was found by Polyakov [24], further details about the Polyakov spin factor have been studied, for instance, in [25],[26]. In this thesis we are more interested in the spin factor in the second order formalism, which has proven to be highly advantageous for both analytical as well as numerical calculations. A first step towards the notion of a spin factor in the second-order formalism has been performed by Karanikes and Ktorides [27],[28], who outlined the possibility of rewriting the spin-field coupling. However, they argued that the final result is identical to the Polyakov (first-order) spin factor, for which an ad hoc regularization seems necessary, in order to control the singularity structure.

Since in this thesis we give an exhaustive introduction into the loop calculus, the connection to the coordinate representation of the so called loop derivative and its technical treatment we are able to derive a valid representation of the geometric spin factor in the second order formalism, which is in fact different from the representation of the Polyakov spin factor in the first order formalism. We analyze in detail the singularity structure of our spin factor and develop a spin-factor calculus suitable for efficiently carrying out detailed computations. In particular the singularity structure teaches us that the random "zigzag" nature of worldlines in spacetime is an essential ingredient for the coupling between spin and an external field.

The diploma thesis is structured as follows. In the second section we introduce the notion of the effective action, which is an enormously powerful tool in quantum field theory. By computing this to one-loop order, we obtain a functional determinant which gives us an analytical expression for all one-loop Feynman diagrams. In the third chapter we develop the worldline calculus and represent the effective action to one loop order in this language. Furthermore we give some physical (as mentioned above) interpretations of the quantum behavior in the worldline approach. As an illustrative example we will calculate the Heisenberg-Euler action with these techniques. In section 4. we introduce the loop space formulation of gauge theory, giving a precise definition of holonomy, loop derivative and many of the connections to gauge theory in order to obtain the coordinate space version of the loop derivative in section 5. This provides us with a definition of the loop derivative in terms of functional calculus which is essential when dealing with path integrals. In section 6, we will introduce a loop operator on the space of holonomies which is appropriate for path integration. We show explicitly that the action of this operator on the Wilson loop (the holonomy) can be used to reformulate the familiar spin-field connection $\sigma_{\mu\nu}F_{\mu\nu}$ of the spinor-worldline path integral. After some formal manipulations of the path integral we arrive at an alternative expression for the spin-field coupling which encodes the spin field connection in a new quantity $\sigma_{\mu\nu}\omega_{\mu\nu}$. We investigate the properties of our new representation in the $F \to 0$ limit, where F indicates the field. In section 7. we show, that our new expression is also valid for an arbitrary field configuration and therefore is totally equivalent to the familiar representation of the effective action. In the last section we give an alternative derivation of the Heisenberg-Euler action [29],[30],[31], that describes the quantum-induced nonlinear corrections to Maxwell's electrodynamics [32],[33]. We conclude by outlining several direct generalizations of our formalism and possible future applications.

2 The effective Action

We consider a quantum field theory of a field ψ in the presence of an external source J. In this case a quantized field which is initially in the vacuum state need not stay in that state. All physical properties of the field can be derived from the variation of the probability amplitude for the field to remain in the vacuum state with respect to the source. This vacuum to vacuum transition amplitude is usually called $Z[J]$, the generating functional,

$$Z[J] = \int \mathcal{D}\psi \, \exp[i \int d^4x \, (\mathcal{L}[\psi] + J\psi)] = \langle 0|e^{-iHT}|0\rangle,$$

where H is the Hamiltonian. This factor $Z[J]$ is given by the sum of all vacuum to vacuum amplitudes in the presence of the current J, including the disconnected as well as the connected diagrams. The functional $W[J]$ which generates only the connected Feynman diagrams, is related to $Z[J]$ by

$$Z[J] = e^{-iW[J]}.$$

This can be understood by a heuristic argument. A general Feynman diagram that consists of N connected components will contribute to $Z[J]$ a term equal to the product of the contributions of these components, divided by the number $N!$ of permutations of vertices that merely permute all the vertices in one connected component with all the vertices in another. Therefore the sum of all graphs is

$$Z[J] = \sum_{N=0}^{\infty} \frac{1}{N!}(-iW[J])^N = \exp(-iW[J]),$$

where $W[J]$ is the sum of all connected diagrams. Another more physical interpretation of $W[J]$ is, that it is the vacuum energy in static systems multiplied by the time during we are observing the system ($\langle 0|e^{-iHT}|0\rangle = e^{-iW[J]}$). With the above definition of Z we also have an equation for defining the generating functional of all connected Feynman diagrams,

$$W[J] = i \, \ln[Z].$$

With this definition we consider now the functional derivative of $W[J]$ with respect to $J(x)$:

$$\frac{\delta}{\delta J(x)} W[J] = i\frac{\delta}{\delta J(x)} \, \ln[Z] = -\frac{\int \mathcal{D}\psi \, e^{i\int(\mathcal{L}+J\psi)}\psi(x)}{\int \mathcal{D}\psi \, e^{i\int(\mathcal{L}+J\psi)}} = -\langle 0|\psi(x)|0\rangle =: -\psi_{\text{cl}}(x).$$

This is the vacuum expectation value in presence of a classical source $J(x)$. We can interpret this amplitude as the classical field ψ_{cl}. Because it is more convenient to formulate our quantum field theory in terms of ψ_{cl} than in terms of the source $J(x)$, we carry out a Legendre transform of $W[J]$ and get,

$$\Gamma[\psi_{\text{cl}}] = -W[J] - \int d^4y J(y)\psi_{\text{cl}}(y).$$

We call this quantity the effective action. When we perform an approximation expanding about the classical action we can interprete the effective action as the classical action plus quantum corrections. It contains only the one particle irreducible (1 PI) Feyman graphs, which implies that we have to analyze fewer kinds of graphs if we decide to reconstruct the field theory in terms of a perturbative series. But the more physical advantage is that we can interpret this quantity much better than the $J(x)$ dependent generating functional. As far as the vacuum state of the theory is concerned, in fact $\Gamma[\psi_{\rm cl}]$ plays the role of an action governing the dynamics of $\psi_{\rm cl}$ in terms of equations of motion obtained by variational calculus. To motivate this let us calculate,

$$\begin{aligned} \frac{\delta}{\delta\psi_{\rm cl}(x)}\Gamma[\psi_{\rm cl}] &= -\int d^4y\ \frac{\delta J(y)}{\delta\psi_{\rm cl}(x)}\frac{\delta W[J]}{\delta J(y)} - \int d^4y\ \frac{\delta J(y)}{\delta\psi_{\rm cl}(x)}\psi_{\rm cl}(x) - J(x) \\ &= -J(x). \end{aligned}$$

If we switch-off the external source this equation becomes

$$\frac{\delta}{\delta\psi_{\rm cl}(x)}\Gamma[\psi_{\rm cl}] = 0.$$

This equation will be solved by the classical configuration of the field ψ. The procedure is totally analogous to the derivation of the equations of motion in classical physics.

From the full effective action (and nonzero J) we get the full equations of motion of the system. Since our action functional depends on $\psi_{\rm cl}$ we can say that $\Gamma[\psi_{\rm cl}]$ describes the dynamics of a weighted average of ψ over all possible fluctuations. The minimization of the effective action gives us the classical equations of motion plus quantum corrections. In the language of perturbation theory this corresponds to all tree-level amplitudes plus all loop corrections.

2.1 Loop expansion of the effective action

As the above calculation shows, the extremum of $\Gamma[\psi_{\text{cl}}]$ solving $\frac{\delta}{\delta\psi_{\text{cl}}(x)}\Gamma[\psi_{\text{cl}}] = 0$ in the absence of the source J, gives the exact vacuum state of the Quantum Field Theory. Next we want compute the effective action in a loop expansion. For this, we start from an expansion of the generating functional Z and derive the desired expression for Γ using the Legendre transform.

Since we want to use renormalized perturbation theory, we split our bare Lagrangian into two parts. The first part contains the renormalized parameters and the second one contains the counterterms:

$$\mathcal{L} = \mathcal{L}_1 + \delta\mathcal{L}.$$

Because we want to calculate Γ as a function of ψ_{cl}, the functional $Z[J]$ however only depends on ψ_{cl} implicitly, and we have to find a relation between $J(x)$ and ψ_{cl}. As was shown above we know that to lowest order in perturbation theory this relation is the classical field equation:

$$\frac{\delta}{\delta\psi}\left(\int d^4x\ \mathcal{L}\right)\bigg|_{\psi=\psi_{cl}} + J(x) = 0.$$

Now we define $J_1(x)$ to be the function that solves this equation exactly,

$$\frac{\delta}{\delta\psi}\left(\int d^4x\ \mathcal{L}_1\right)\bigg|_{\psi=\psi_{cl}} + J_1(x) = 0. \tag{1}$$

The difference $J(x) - J_1(x) = \delta J(x)$ will be interpreted as a counterterm, so our expression for the generating functional transforms to,

$$e^{-iW[J]} = \int \mathcal{D}\psi\ e^{i\int d^4x(\mathcal{L}_1[\psi]+J_1\psi)}\ e^{i\int d^4x(\delta\mathcal{L}[\psi]+\delta J\psi)}.$$

The second exponential contains the counterterms, which we will leave aside for the moment. Now we decompose the field ψ into the classical part and one part which contains the quantum fluctuations $\psi(x) = \psi_{cl}(x) + \eta(x)$. If we expand the first exponent about ψ_{cl} it takes the form,

$$\begin{aligned}\int d^4x\ (\mathcal{L}_1 + J_1\psi) &= \int d^4x\ (\mathcal{L}_1[\psi_{\text{cl}}] + J_1\psi_{\text{cl}}) + \int d^4x\ \eta(x)\left(\frac{\delta\mathcal{L}_1}{\delta\psi} + J_1\right) \\ &+ \frac{1}{2}\int d^4x\ d^4y\ \eta(x)\eta(y)\frac{\delta^2\mathcal{L}_1}{\delta\psi(x)\delta\psi(y)} \\ &+ \frac{1}{3!}\int d^4x\ d^4y\ d^4z\ \eta(x)\eta(y)\eta(z)\frac{\delta^3\mathcal{L}_1}{\delta\psi(x)\delta\psi(y)\delta\psi(z)} + \ldots\ .\end{aligned}$$

Because of (1) the term linear in η vanishes. Therefore $\int d^4x\ (\mathcal{L}_1 + J_1\psi)$ is a Gaussian integral with correction terms. If we limit ourselves to the second order in the expansion we can write the generating functional as

$$e^{-iW[J]} = \int \mathcal{D}\eta\ \exp\left[i\left(\int d^4x\ \mathcal{L}_1[\psi_{\text{cl}}] + \frac{1}{2}\int d^4x\ d^4y\ \eta(x)\ \frac{\delta^2\mathcal{L}_1}{\delta\psi\delta\psi}\ \eta(y)\right)\right].$$

Making use of the formula $\int \mathcal{D}x\ e^{\frac{1}{2}\langle x, Ax\rangle} = (\det(-A))^{-\frac{1}{2}}$, we are guided to the expression

$$e^{-iW[J]} \;=\; \exp\left[i\int d^4x\ (\mathcal{L}_1[\psi_{\rm cl}] + J_1\psi_{\rm cl})\right]\left(\mathrm{Det}\left[-\frac{\delta^2 S_1}{\delta\psi\delta\psi}\right]\right)^{-\frac{1}{2}}.$$

This equation gives us the classical equations of motion together with the first order quantum corrections (all one-loop diagrams). In the next step of our calculation we have to consider the counterterm part of the generating functional. We expand the second exponent about $\psi = \psi_{\rm cl}$ and get,

$$(\delta\mathcal{L}[\psi_{\rm cl}] + \delta J\psi_{\rm cl}) + (\delta\mathcal{L}[\psi_{\rm cl} + \eta] - \delta\mathcal{L}[\psi_{\rm cl}] + \delta J\eta).$$

The first term is a constant which gives an additional term in $\exp\left[i\int d^4x\ (\mathcal{L}_1[\psi_{\rm cl}] + J_1\psi_{\rm cl})\right]$. If we expand the second term in η, we get counterterms which we can put in the bare Feynman diagrams. Finally we obtain an expression for the effective action valid for all one-loop quantum corrections:

$$\Gamma[\psi_{\rm cl}] = \int d^4x\ \mathcal{L}_1[\psi_{\rm cl}] + \frac{i}{2}\ \ln\ \det\left[-\frac{\delta^2 S_1}{\delta\psi\delta\psi}\right].$$

We can see what we have mentioned earlier. In this representation there is a separation of the effective action in classical and quantum terms so that we now are allowed to say that we study a theory with quantum fluctuations spread over the average, the classical value, of the field. Our derivation of the effective action made use of the bosonic path integration (we have used the bosonic Gaussian path integral formula $\int \mathcal{D}\psi\ \exp[-\frac{1}{2}\langle\psi, A\psi\rangle] = (\mathrm{Det}A)^{-\frac{1}{2}}$), but in this thesis we are focused on the fermionic theory. That means we have to do the above calculation with Grassmann quantities. This is no problem, because we only have to replace the bosonic path integration by the formula for fermions $\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ exp[-\langle\psi, A\psi\rangle] = (\mathrm{Det}A)$. Then our expression for the effective action transforms to:

$$\Gamma[\psi_{\rm cl}] = \int d^4x\ \mathcal{L}_1[\psi_{\rm cl}] - i\ \ln\ \det\left[-\frac{\delta^2 S_1}{\delta\psi\delta\bar{\psi}}\right].$$

Finally we have to do a last rewriting. Because we want to represent the quantum term of the effective action in the worldline representation, as well as using the analogies to statistical physics, we have to do a rotation to "Euclidean time" $t = -it_E$. Let us now see what happens to the action integral under the rotation to Euclidean time. Let us consider the simple 1-dimensional potential system,

$$L[q,\dot{q}] = \frac{1}{2}(\frac{dq}{dt})^2 - V(q).$$

Then

$$\begin{aligned} iS[q] &= i\int_0^{iT}(-idt_E)\left[\frac{1}{2}\left(\frac{dq}{d(-it_E)}\right)^2 - V(q)\right] \\ &= -S_E[q] \\ S_E[q] &= \int dt_E\ \left[\frac{1}{2}(\frac{dq}{dt_E})^2 + V(q)\right]. \end{aligned}$$

With the 4-dimensional analogon of this transformation the effective action changes to,

$$\Gamma[\psi_{\rm cl}] = \int d^4x \; \mathcal{L}_{E_1}[\psi_{\rm cl}] - \; \ln \det \left[-\frac{\delta^2 S_{E_1}}{\delta\psi\delta\psi} \right],$$

where all quantities are considered to be "Euclidean" unless stated otherwise. Also, in the progress of this thesis, all path integrals are meant as "Euclidean" path integrals if not explicitly mentioned otherwise.

After we have developed the formalism in a general way, we now specialize to the QED action

$$S = \int d^4x \; \left[\bar{\psi}\gamma_\mu(i\partial^\mu + eA^\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right].$$

We have to switch this expression to the Euclidean one and get

$$S_E = \int d^4x \; \left[\bar{\psi}\gamma_\mu(i\partial^\mu + eA^\mu)\psi + m\bar{\psi}\psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right],$$

where we have used that $\gamma_4 = -i\gamma_0$ and $A_4 = iA_0$ after transition to the Euclidean metric. Since we perform only the path integral over the fermion and anti-fermion field $\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \; e^{-S}$ and not the $\int \mathcal{D}A$ integration, the quantum dynamics of the electromagnetic field is not considered. The next step is to compute the effective action with the QED Lagrangian. For this we have to examine the different functional derivatives of S. Following the rules of variational calculus we get

$$\frac{\delta^2 S_E}{\delta\bar{\psi}\delta\psi} \;\; = \;\; -(i\not{D} + m),$$

with $\not{D} = \gamma_\mu D^\mu = \gamma_\mu(\partial^\mu - ieA^\mu)$, where A^μ in our case is the classical background field.

$$\frac{\delta^2 S_E}{\delta\psi\delta A_\mu} \sim \bar{\psi} = 0,$$

since we don not have a fermion background. Furthermore we get

$$\frac{\delta^2 S_E}{\delta A_\mu \delta A_\nu} \sim \text{const.}$$

When we insert this now into the quantum part of the effective action we get:

$$\Gamma_{\rm Ferm} = - \; \ln \det(i\not{D} + m).$$

Because we are working in the four-dimensional spacetime we know that $\Gamma_{\rm Ferm}$ is a Lorentz scalar. This means that $\Gamma_{\rm Ferm}$ does not depend on the sign of $i\not{D}$ and therefore we are allowed to write

$$\begin{aligned} \Gamma_{\rm Ferm} &= -\frac{1}{2} \left[\ln \det(i\not{D} + m) + \ln \det(-i\not{D} + m) \right] \\ &= -\frac{1}{2} \ln \det(\not{D}^2 + m^2). \end{aligned}$$

This is the expression for the quantum part of the effective action, we will be working with in the progress of this thesis.

3 Worldline representation of the effective action

The effective action for a gauge field that couples to a fluctuating fermion field (in the Euclidean metric) reads $\Gamma[A] = \int d^4x\mathcal{L}_{\rm cl}[A] - \ln\ \det[\frac{\delta^2 S[A]}{\delta\bar\psi\delta\psi}]$, where the first term by variation gives the classical equations of motion (in the language of perturbative quantum field theory the first part of the action will be constructed from the tree level diagrams and is therefore identical to the classical result). The second part is constructed from the one loop graphs in perturbation theory and can be viewed as the first-order quantum correction of the classical action. In this thesis we are mainly interested in the quantum regime, therefore we restrict ourselves to the second part of the effective action. We call it the fermionic effective action $\Gamma_{\rm Ferm}$.

$$\begin{aligned}\Gamma_{\rm Ferm} &= -\frac{1}{2}\ \ln\ \det(\not{D}^2 + m^2) \\ &= -\frac{1}{2}\ \mathrm{Tr}\ \ln(\not{D}^2 + m^2),\end{aligned} \tag{2}$$

where we have used the (ln det = tr ln)-identity and $D = (\partial + ieA)$. Let us modify the operator $\not{D}^2$ in order to obtain an explicit expression in terms of the spin-field coupling.

$$\begin{aligned}\not{D}^2 &= D_\mu D_\nu \gamma^\mu\gamma^\mu \\ &= D_\mu D_\nu \left[\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right] \\ &= D_\mu D_\nu[-g^{\mu\nu} - i\ \sigma^{\mu\nu}] \\ &= -D^2 - i\ \frac{1}{2}\ \sigma^{\mu\nu}[D_\mu, D_\nu] \\ &= -D^2 - \frac{e}{2}\ \sigma_{\mu\nu}F^{\mu\nu}.\end{aligned}$$

Therefore we have arrived at the quantity $\not{D}^2 = -\ D^2 - \frac{e}{2}\ \sigma_{\mu\nu}F^{\mu\nu}$, where the last term is the so-called Pauli term that carries the information about the spin coupling to the field. Inserting this expression into (2) we get

$$\Gamma_{\rm Ferm} = \frac{1}{2}\int_0^\infty \frac{dT}{T}\ \ \mathrm{Tr}\ e^{-(-D^2+m^2-\frac{e}{2}\sigma F)T},$$

since $-\mathrm{Tr}\ \ln\ (\frac{A}{B}) = \int_0^\infty \frac{dT}{T}\ \mathrm{Tr}\ (e^{-AT} - e^{-BT})$. We have suppressed the second term $\big[$which actually also appears in $\Gamma_{\rm Ferm}$ as the normalization to the free fermion theory ($\Gamma_{\rm Ferm} = -\frac{1}{2}\ \mathrm{Tr}\ \ln\ (\frac{\not{D}^2+m^2}{-\partial^2+m^2})\ )\big]$. Now we can write the functional trace as

$$\mathrm{tr}_\gamma \int d^Dx\langle x|e^{-(-D^2+m^2-\frac{e}{2}\sigma F)T}|x\rangle.$$

It is well known that we can decompose a "time" evolution operator in terms of evolution operators for infinitesimal time slices as

$$\langle q_{\rm a}|e^{-HT}|q_{\rm a}\rangle = \int\prod_{i=1}^N dq_i\langle q_{\rm a}|e^{-H\delta\tau}|q_1\rangle\langle q_1|...|q_N\rangle\langle q_N|e^{-H\delta\tau}|q_{\rm a}\rangle.$$

Let us calculate the propagator over a small segment in the path integral. That means

$$\begin{aligned}\langle q_j t_j | q_{j+1} t_{j+1}\rangle &= \langle q_j | e^{\frac{-H\delta\tau}{\hbar}} | q_{j+1}\rangle \\ &= \langle q_{j+1} | 1 - \frac{H\delta\tau}{\hbar} + O(\delta\tau^2) | q_j\rangle,\end{aligned}$$

Where $\delta\tau = t_j - t_{j+1}$. For the rest of the calculation we will put $\hbar$ to 1. Now the Hamiltonian for the "Fermionic evolution " reads

$$\begin{aligned}\hat{H} &= -D^2 + m^2 - \frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu} \\ &= -(\partial + ieA)^2 + m^2 - \frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu} \\ &= (\hat{p} + eA)^2 + m^2 - \frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu} \\ &= \hat{p}^2 + e\hat{p}A(\hat{q}) + eA(\hat{q})\hat{p} + e^2A^2(\hat{q}) + m^2 - \frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu}\end{aligned}$$

where $\hat{p} = -i\partial$.

In the next step we insert the identities $\int |p\rangle\langle p| = 1$ and $\langle q|p\rangle = e^{ipq}$ and get

$$\begin{aligned}& \int dp\ \langle q_j|\ 1 - (\hat{p}^2 + e\hat{p}A(\hat{q}) + eA(\hat{q})\hat{p} + e^2A^2(\hat{q}) + m^2 - \frac{e}{2}\,\sigma_{\mu\nu}F^{\mu\nu})\ \delta\tau|p\rangle\langle p|q_{j+1}\rangle \\ =& \int dp\ (1 - (p^2 + 2ep\bar{A}(q_j) + e^2\bar{A}^2(q_j) + m^2 - \sigma_{\mu\nu}F^{\mu\nu}(q_j))\ \delta\tau)\langle q_j|p\rangle\langle p|q_{j+1}\rangle \\ =& \int dp\ (1 - (p^2 + 2ep\bar{A}(q_j) + e^2\bar{A}^2(q_j) + m^2 - \sigma_{\mu\nu}F^{\mu\nu}(q_j)\ \delta\tau)\ e^{ip(q_j - q_{j+1})},\end{aligned}$$

Where $\bar{A}(q_j) = \frac{1}{2}(A(q_j) + A(q_{j+1}))$ and the terms proportional to A^2 have also been expressed as $\bar{A}^2$ for convenience which is possible, since it is proportional to $\delta(q_j - q_{j+1})$. Because we know that for a small space time segment the relation $\langle q_j|1 - \hat{H}\delta\tau|q_{j+1}\rangle$ is valid, we conclude that $H(q_j, p) = p^2 + 2ep\bar{A}(q_j)) + e^2\bar{A}^2(q_j) + m^2 - \sigma_{\mu\nu}F^{\mu\nu}(q_j)$. It follows that

$$\begin{aligned}& \int dp\ e^{(p^2+2ep\bar{A}(q_j))+e^2\bar{A}^2(q_j)+m^2-\sigma_{\mu\nu}\bar{F}^{\mu\nu})\delta\tau}e^{ip(q_j-q_{j+1})} \\ =& \int dp\ e^{-\delta\tau\ p^2-(2e\bar{A}(q_j)\delta\tau-i(q_j-q_{j+1}))p-(e^2\bar{A}^2(q_j)+m^2-\frac{e}{2}\sigma\bar{F}(q_j))\delta\tau} \\ =&\ e^{-(e^2\bar{A}^2(q_j)+m^2-\frac{e}{2}\sigma\bar{F}(qj))\delta\tau}\sqrt{\frac{\pi}{\delta\tau}}\ e^{\frac{[(2e\bar{A}(q_j)\delta\tau-i(q_j-q_{j+1})^2]^2}{4\delta\tau}},\end{aligned}$$

where we have used the Gaussian integral formula $\int dp\, e^{-\frac{1}{2}ap^2+Jp} = \sqrt{\frac{2\pi}{a}}\, e^{\frac{J^2}{2a}}$, with $a = 2\delta\tau$ and $J = -(2e\bar{A}(q_j)\delta\tau - i(q_j - q_{j+1})$. Let us now sum up this equation,

$$\begin{aligned}&= \sqrt{\frac{\pi}{\delta\tau}}\ e^{-(e^2\bar{A}^2(q_j)+m^2-\frac{e}{2}\sigma F(q_j))\delta\tau}\quad e^{e^2\bar{A}^2(q_j)\ \delta\tau-\frac{(q_j-q_{j+1})^2}{4\delta\tau}-ie\bar{A}(q_j)(q_j-q_{j+1})} \\ &= \sqrt{\frac{\pi}{\delta\tau}}\ e^{-m^2\delta\tau+\frac{e}{2}\sigma F(q_j)\delta\tau}e^{-ie\bar{A}(q_j)(q_j-q_{j+1})}e^{\frac{-(q_j-q_{j+1})^2}{4\delta\tau}}.\end{aligned}$$

We use the discretized version of the propagator

$$\langle q_0|e^{-HT}|q_{\mathrm{N+1}}\rangle = \int \prod_{i=1}^{N} dq_i \langle q_0|e^{-H\delta\tau}|q_1\rangle\langle q_1|...|q_N\rangle\langle q_N|e^{-H\delta\tau}|q_{\mathrm{N+1}}\rangle,$$

where we insert N+1-times the expression for the small path integral segment and therefore we get

$$\begin{aligned}\langle q_0|e^{-HT}|q_{\mathrm{N+1}}\rangle = \quad & \int \prod_{i=1}^{N} dq_i \sqrt{\frac{\pi}{\delta\tau}}^{N+1} \prod_{j=0}^{N} \exp\left(-m^2\delta\tau + \frac{e}{2}\sigma F(q_j)\delta\tau\right) \\ & \times \prod_{j=0}^{N} \exp\left(-ie\bar{A}(q_j)(q_j - q_{j+1})\right) \exp\left(\frac{-(q_j - q_{j+1})^2}{4\delta\tau}\right).\end{aligned}$$

We can conclude that

$$\prod_{j=0}^{N} e^{-m^2\delta\tau + \frac{e}{2}\sigma F(q_j)\delta\tau} = e^{-m^2T}\ \mathcal{P}e^{\frac{e}{2}\sigma\sum_{j=0}^{N} F(q_j)\delta\tau},$$

where $\mathcal{P}$ indicates the path ordering which takes care of the Dirac algebra. As well, we find

$$\prod_{j=0}^{N}\ e^{-ie\bar{A}(q_j)(q_j-q_{j+1})}\ e^{\frac{-(q_j-q_{j+1})^2}{4\delta\tau}} = e^{\sum_{j=0}^{N} -ie\bar{A}(q_j)(q_j-q_{j+1})}\ e^{\sum_{j=0}^{N}\frac{-(q_j-q_{j+1})^2}{4\delta\tau}},$$

with $(N+1)\delta\tau = T$.

In the limit $\lim_{N\to\infty}$ we are now lead to write

$$\begin{aligned} & \lim_{N\to\infty} e^{-m^2T}\ \mathcal{P}e^{\frac{e}{2}\sigma\sum_{j=0}^{N} F(q_j)\delta\tau} e^{\sum_{j=0}^{N} -ie\bar{A}(q_j)(q_j-q_{j+1})}\ e^{\sum_{j=0}^{N}\frac{-(q_j-q_{j+1})^2}{4\delta\tau}} \\ = \quad & \lim_{N\to\infty} e^{-m^2T}\ \mathcal{P}e^{\frac{e}{2}\sigma\sum_{j=0}^{N} F(q_j)\delta\tau} e^{\sum_{j=0}^{N}\frac{\delta\tau}{\delta\tau}(-ie\bar{A}(q_j)(q_j-q_{j+1}))}\ e^{\sum_{j=0}^{N}\frac{\delta\tau}{\delta\tau}(\frac{-(q_j-q_{j+1})^2}{4\delta\tau})} \\ = \quad & \mathcal{P}e^{\frac{e}{2}\int_0^T d\tau\ \sigma F}\ e^{-ie\int_0^T d\tau\ \dot{q}A(q)}\ e^{-\int_0^T d\tau\frac{\dot{q}^2}{4}}.\end{aligned}$$

We get a loop integral because we just consider closed paths here ($\langle q_0|e^{-HT}|q_{N+1}\rangle$ where $|q_0\rangle = |q_{N+1}\rangle$). For this reason we finally get a path integral with periodic boundary conditions. We can write the effective action as

$$\begin{aligned}\Gamma_{\mathrm{Ferm}} &= \frac{1}{2}\int_0^\infty \frac{dT}{T}\ \mathrm{Tr}\ e^{-(-D^2+m^2-\frac{e}{2}\sigma F)T} \\ &= \frac{1}{2}\int_0^\infty \frac{dT}{T}\ \mathrm{tr}_\gamma \lim_{N\to\infty}\int\prod_{i=1}^{N} dq_i \sqrt{\frac{\pi}{\delta\tau}}^{N+1} e^{-m^2T}\ \mathcal{P}e^{\frac{e}{2}\sigma\sum_{j=0}^{N} F(q_j)\delta\tau} \\ &\quad \times\ e^{\sum_{j=0}^{N} -ie\bar{A}(q_j)(q_j-q_{j+1})} e^{\sum_{j=0}^{N}\frac{-(q_j-q_{j+1})^2}{4\delta\tau}} \\ &= \frac{1}{2}\int_0^\infty \frac{dT}{T}\ e^{-m^2T}\ \mathcal{N}\int\limits_{q(0)=q(\tau)} \mathcal{D}q(\tau)\ \mathrm{tr}_\gamma\mathcal{P}\ e^{\frac{e}{2}\int_0^T d\tau\ \sigma F}\ e^{-ie\int_0^T d\tau\ \dot{q}A(q)}\ e^{-\int_0^T\frac{\dot{q}^2}{4}}, \quad (3)\end{aligned}$$

where $\mathcal{N}$ is the factor of normalization. This is the fermionic effective action transformed into a worldline path integral. This representation has found a wide range of applications, such as the computation of the one-loop N-photon amplitudes, gradient expansion of the effective action [34], higher-loop computations and β functions [35],[36],[37],[38]. The worldline representation also offers an intuitive approach for an interpretation of the quantum processes. We are allowed to examine the fermionic action as a path integral over an "effective" particle moving in a background field with propertime T, this is indicated by the kinetic term ($e^{-\int_0^T \frac{\dot{q}^2}{4}}$). In addition this term ensures that the loop cloud is heaped around a center of mass . During the motion of the particle in the backround it "scans" the field by its loop clouds. That means, for small propertimes the size of the "loop cloud" (which is meant as an ensemble of closed loops) is also small. Therefore the loop cloud picks up small-scale information about the background field. For large propertimes the loop clouds blows up and therefore we get information about the large scale behavior of the background field. There are no other constraints to the loops except continuity and closeness. This means, that the loops are allowed to be arbitrarily self-intersecting and knotty. In this way we have a colorful picture for the ultraviolet and the infrared regime of the observed quantum field fluctuations arranged by the worldline representation.

Let us now continue with fixing the normalization factor $\mathcal{N}$. This can be determined from the zero-field limit,

$$\mathcal{N} \int \mathcal{D}q(\tau)\; e^{-\int_0^T d\tau\; \frac{\dot{q}^2}{4}} \stackrel{!}{=} \langle q|e^{T\partial^2}|q\rangle \; .$$

Into the right expression we insert two times the identity. This leads us to

$$\begin{aligned}
\langle q|e^{T\partial^2}|q\rangle &= \int d^Dp \int d^D\acute{p}\; \langle q|p\rangle\langle p|e^{T\partial^2}|\acute{p}\rangle\langle\acute{p}|q\rangle \\
&= \int d^Dp \int d^D\acute{p}\; \langle q|p\rangle\langle p|\acute{p}\rangle\langle\acute{p}|q\rangle\; e^{-Tp^2} \\
&= \int d^Dp \int d^D\acute{p}\; \frac{e^{ipq}}{\sqrt{2\pi}^D}\; \delta(p-\acute{p})\; \frac{e^{-i\acute{p}q}}{\sqrt{2\pi}^D}\; e^{-Tp^2}.
\end{aligned}$$

After performing the $\acute{p}$ integral we obtain,

$$\langle q|e^{T\partial^2}|q\rangle = \int \frac{d^Dp}{2\pi^D}\; e^{-Tp^2} = \frac{1}{(4\pi T)^{\frac{D}{2}}}.$$

Therefore we find

$$\mathcal{N} = \frac{1}{(4\pi T)^{\frac{D}{2}} \int \mathcal{D}q(\tau)\; e^{-\int_0^T d\tau\; \frac{\dot{q}^2}{4}}} \; . \tag{4}$$

Inserting this into equation (3) leads us to the compact formula,

$$\Gamma_{\mathrm{Ferm}}[A] = \frac{1}{2}\,\frac{1}{(4\pi)^{\frac{D}{2}}} \int_0^\infty \frac{dT}{T^{(\frac{D}{2}+1)}}\; e^{-m^2T}\; \langle W_{\mathrm{spin}}[A]\rangle \; ,$$

where we have defined $W_{\text{spin}}[A]$ in the following way,

$$\begin{aligned} W_{\text{spin}}[A] &= \exp\left[-ie\int_0^T d\tau\, \dot{q}A(q)\right] \operatorname{tr}_\gamma \mathcal{P}\exp\left(\frac{e}{2}\int_0^T d\tau\, \sigma F\right) \\ &= \exp\left[-ie\oint dqA(q)\right] \operatorname{tr}_\gamma \mathcal{P}\exp\left(\frac{e}{2}\int_0^T d\tau\, \sigma F\right). \end{aligned}$$

We recognize in the first factor the well-known Wilson loop. This one is complemented with the spin-field coupling term (Pauli term). What we have found with this rewriting of the fermion action is, that we no longer analyze a second quantized theory, instead in our point of view we are studying a first quantized one (because the path integral has to be performed over the paths of the effective particle). Of course we are still dealing with quantum field theory, only our approach has changed.

3.1 Heisenberg-Euler action in worldline representation

In this section we calculate the (unrenormalized) Heisenberg-Euler action [29],[30],[31] with the formalism developed above. For this we have to review some technical developments as put forward in [9]. First let us consider the expectation value of the spinorial Wilson loop,

$$\langle W_{\text{spin}}[A]\rangle \;=\; \frac{\int \mathcal{D}x(\tau)\; e^{-ie\int_0^T d\tau\;\dot{x}A(x)}\;\text{tr}_\gamma\; e^{\frac{e}{2}\int_0^T d\tau\;\sigma F}\; e^{-\int_0^T d\tau \frac{\dot{x}^2}{4}}}{\int \mathcal{D}x(\tau)\; e^{-\int_0^T d\tau \frac{\dot{x}^2}{4}}}.$$

We confine ourselves to a constant field strength F, but at this point the orientation of the field is left open. Using the Schwinger-Fock gauge $A_\mu = \frac{1}{2}F_{\nu\mu}x_\nu$ the Wilson loop yields,

$$\langle W_{\text{spin}}[A]\rangle \;=\; \text{tr}_\gamma\; e^{\;\frac{e}{2}\sigma FT}\int \mathcal{D}x(\tau)\; e^{-i\;\frac{e}{2}\int_0^T d\tau\;\dot{x}(\tau)\;F_{\nu\mu}x_\nu(\tau)}\; e^{-\int_0^T d\tau\frac{\dot{x}^2(\tau)}{4}}. \tag{5}$$

The next step is to rewrite this expression in order to apply the well known Gaussian integral formulas. For this reason we consider the exponents of the exponential functions under the path integral,

$$-\frac{1}{4}\int_0^T d\tau\dot{x}^2 \;=\; -\frac{1}{2}\int_0^T d\tau\; x_\mu\;[-\frac{1}{2}\frac{d^2}{d\tau^2}\delta_{\mu\nu}]\;x_\nu,$$

where we have integrated by parts. The first term under the integral of (5) can be treated similarly,

$$-i\;\frac{e}{2}\int_0^T d\tau\;\dot{x_\mu}F_{\nu\mu}x_\nu \;=\; -\frac{1}{2}\;ie\int_0^T d\tau\;x_\mu\;[F_{\mu\nu}\frac{d}{d\tau}]\;x_\nu.$$

Finally we substitute these expressions back into the path integral and get,

$$\begin{aligned}
&\mathcal{N}\int \mathcal{D}x(\tau)\;\exp\;(-\frac{1}{2}\int_0^T d\tau\;x_\mu\;[-\frac{1}{2}\frac{d^2}{d\tau^2}\delta_{\mu\nu}+ie\;F_{\mu\nu}\frac{d}{d\tau}]\;x_\nu\;)\\
=\;&\mathcal{N}\int \mathcal{D}x(\tau)\;\exp\;(-\frac{1}{4}\int_0^T d\tau\;x_\mu\;[-\frac{d^2}{d\tau^2}\delta_{\mu\nu}+2ieF_{\mu\nu}\frac{d}{d\tau}]\;x_\nu\;)\\
=\;&\frac{\text{Det}^{-\frac{1}{2}}[-\frac{d^2}{d\tau^2}\delta_{\mu\nu}+2ieF_{\mu\nu}\frac{d}{d\tau}]}{\text{Det}^{-\frac{1}{2}}[-\frac{d^2}{d\tau^2}\delta_{\mu\nu}]},
\end{aligned}$$

where we have used the integral formula $\int \mathcal{D}x(\tau)\; e^{-\frac{1}{2}\langle x\hat{M}x\rangle} = \text{Det}^{-\frac{1}{2}}[\hat{M}]$ as well as $\mathcal{N} = \text{Det}^{\frac{1}{2}}[-\frac{d^2}{d\tau^2}\delta_{\mu\nu}]$. We have separated the factor of $\frac{1}{(4\pi T)^{\frac{D}{2}}}$ from the normalization $\mathcal{N}$, as it occurs in (4). This propertime factor will always occur with the T-integration measure in the following. In the next step of our calculation we simplify and evaluate the functional determinant,

$$\frac{\text{Det}^{-\frac{1}{2}}[-\frac{d^2}{d\tau^2}\delta_{\mu\nu}+2ieF_{\mu\nu}\frac{d}{d\tau}]}{\text{Det}^{-\frac{1}{2}}[-\frac{d^2}{d\tau^2}\delta_{\mu\nu}]} = \text{Det}^{-\frac{1}{2}}[\delta_{\mu\nu}-2ieF_{\mu\nu}(\frac{d}{d\tau})^{-1}],$$

where we have utilized the well-known identity $\mathrm{Det}(AB) = \mathrm{Det}(A)\,\mathrm{Det}(B)$. In the following we use the ($\ln\det = \mathrm{tr}\ln$)-identity and get,

$$\begin{aligned}
\mathrm{Det}^{-\frac{1}{2}}\left[1 - 2ieF(\frac{d}{d\tau})^{-1}\right] &= \exp\,(-\frac{1}{2}\ \mathrm{Tr}\ \ln[1 - 2ieF(\frac{d}{d\tau})^{-1}]) \\
&= \exp\,(-\frac{1}{2}\ \mathrm{Tr}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}[-2ieF(\frac{d}{d\tau})^{-1}]^n \\
&= \exp\,(\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2ie)^n}{n}\ \mathrm{tr}[F^n]\ \mathrm{Tr}[(\frac{d}{d\tau})^{-n}].
\end{aligned}$$

Since $\mathrm{tr}[F^n] = 0$ for n odd, we are lead to

$$\mathrm{Det}^{-\frac{1}{2}}\left[1 - 2ieF(\frac{d}{d\tau})^{-1}\right] = \exp\,(\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2ie)^{2n}}{2n}\ \mathrm{tr}[F^{2n}]\ \mathrm{Tr}[(\frac{d}{d\tau})^{-2n}].$$

Let us now calculate the functional trace. With the aid of the eigenbasis of the derivative operator $\{e^{2\pi im\frac{\tau}{T}}, m \in Z\backslash\{0\}\}$, (here we split off the zero mode $m = 0$, corresponding to a trivial shift of the center-of-mass of the worldlines)

$$\begin{aligned}
\mathrm{Tr}[(\frac{d}{d\tau})^{-2n}] &= \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty}(\frac{2\pi im}{T})^{-2n} \\
&= 2\sum_{m=1}^{\infty}(\frac{2\pi i}{T})^{-2n}\ (\frac{1}{m})^{2n} \\
&= 2(\frac{2\pi i}{T})^{-2n}\ \zeta(2n),
\end{aligned}$$

where $\zeta(2n)$ denotes the Riemannian ζ function for even numbers. We insert this into the expression for the functional determinant and get,

$$\mathrm{Det}^{-\frac{1}{2}}\left[1 - 2ieF(\frac{d}{d\tau})^{-1}\right] = \exp\left(\sum_{n=1}^{\infty}\frac{(2ie)^{2n}}{2n}\ (\frac{2\pi i}{T})^{-2n}\ \mathrm{tr}[F^{2n}]\ \zeta(2n)\right).$$

Using the identity $\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|$ which connects the ζ function with the Bernoulli numbers, as well as $\mathrm{tr}[F^{2n}] = \sum_{\alpha}(E_\alpha)^{2n}$ where E_α corresponds to the eigenvalues of the

Faraday tensor, we obtain,

$$\begin{aligned}
\mathrm{Det}^{-\frac{1}{2}}\left[1-2ieF(\frac{d}{d\tau})^{-1}\right] &= \exp\left[\sum_\alpha\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{E_\alpha eT}{\pi}\right)^{2n}\frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|\right] \\
&= \exp\left[\frac{1}{2}\sum_\alpha\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)(2n)!}\;|B_{2n}|\;(eE_\alpha T)^{2n}\right] \\
&= \exp\left[-\frac{1}{2}\sum_\alpha\;\ln\left(\frac{\sin(eTE_\alpha)}{eTE_\alpha}\right)\right] \\
&= \exp\left[-\frac{1}{2}\;\mathrm{tr}\;\ln\left(\frac{\sin(eTF)}{eTF}\right)\right] \\
&= \left[\exp\;\left(\ln\;\det\left(\frac{\sin(eTF)}{eTF}\right)\right)\right]^{-\frac{1}{2}} \\
&= \det{}^{-\frac{1}{2}}\left[\frac{\sin(eTF)}{eTF}\right].
\end{aligned}$$

Where we have used the identity $\ln\frac{\sin x}{x} = -\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}}{(2n)!}\frac{x^{2n}}{2n}$ in the second line. This equation is valid for an arbitrary constant field. Because we want to illustrate here the derivation of the Heisenberg-Euler action we now focus on a special choice for the classical field A^μ. We choose the direction of the magnetic field in the z-direction of our laboratory frame $\vec{B} = B\vec{e_z}$. This will be realized through an appropriate gauge potential $A_\mu = \frac{1}{2}B\;(0,-y,x,0)^{\mathrm{T}}$. For this field configuration the Faraday tensor becomes,

$$F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

That means we only have two nonvanishing matrix elements $F_{12} = -F_{21} = B$. Now we investigate the matrix determinant $\det^{-\frac{1}{2}}\left[\frac{\sin(eTF)}{eTF}\right]$ in this field. We diagonalize this expression with respect to the Lorentz structure. For this we have to take a look at the expansion of the $\frac{\mathrm{sin} eTF}{eTF}$-matrix.

$$\begin{aligned}
\frac{\sin(eTF)}{eTF} &= \sum_{k=0}^{\infty}(-1)^k\;\frac{(eTF)^{2k}}{(2k+1)!} \\
&= 1-\frac{(eT)^2}{3!}F^2+\frac{(eT^4)}{5!}F^4-\frac{(eT)^6}{7!}F^6+...
\end{aligned}$$

We observe that there are only even powers of F. Therefore we get,

$$F^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -B^2 & 0 & 0 \\ 0 & 0 & -B^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

With this choice we have diagonalized our operator with respect to the Lorenz structure. Summing up the series, we can represent our diagonalized matrix as,

$$\frac{\sin(eTF)}{eTF} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sinh(eTB)}{eTB} & 0 & 0 \\ 0 & 0 & \frac{\sinh(eTB)}{eTB} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

In this diagonalized form we can easily calculate the determinant

$$\det{}^{-\frac{1}{2}}\left[\frac{\sin(eTF)}{eTF}\right] = \left(\frac{\sinh(eTB)}{eTB}\right)^{-1} . \tag{6}$$

Inserting this into the expression for our effective action we get,

$$\Gamma_{\text{Ferm}}[A] = \frac{1}{2}\frac{1}{(4\pi)^2}\int_0^\infty \frac{dT}{T^3}\, e^{-m^2T}\, \text{tr}_\gamma\, e^{\frac{e}{2}\sigma FT}\left(\frac{eTB}{\sinh(eTB)}\right),$$

where we have inserted $D = 4$ for the dimension of space time. The next step of the calculation is to evaluate the γ trace. With our choice of the classical magnetic field $A_\mu = \frac{1}{2}B(0, -y, x, 0)^T$ the γ trace of the spin-field coupling transforms to,

$$\begin{aligned} \text{tr}_\gamma\, e^{\frac{e}{2}\sigma FT} &= 2\,(e^{eBT} + e^{-eBT}) \\ &= 4\cosh(eBT). \end{aligned}$$

If we insert this into our expression for the effective action we get

$$\Gamma_{\text{Ferm}}[A] = \frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\, e^{-m^2T}\,(eBT)\coth(eBT). \tag{7}$$

This is the expression for the (unrenormalized) Heisenberg-Euler Lagrangian describing the quantum-induced nonlinear corrections to Maxwell's electrodynamics. We have presented the derivation here in the worldline representation because it is a simple (and one of the few) analytical treatments of spinor QED in a background field. The other advantage of the worldline formalism is that it is relatively easy to be translated into a numerical algorithm [13],[14],[15]. Such algorithms consist in averaging over an ensemble of closed worldlines (*loop clouds*). As an advantage, we do not have to solve the problem by diagonalizing the Dirac operator, calculating the trace etc., since in general it is quite difficult if not impossible in practice to find a basis where the operator is diagonal. Doing the calculation by the worldline algorithm avoids this complicated procedure. The present approach to the worldline description has adopted the Dirac operator which enforces a spin-dependent Pauli term of the form $\sigma_{\mu\nu}F^{\mu\nu}$ into the worldline action. In the progress of this thesis we will see, that it is possible to establish a direct connection between geometrical properties of the worldline (more precisley the rate of change of the worldline expressed in terms of the so called "loop derivative ") and the conventional spin field coupling σF.

4 Loop space formulation of gauge theory and loop derivative

Let us now develop the basic elements of a loop space formulation of gauge theory [39],[40]. The gauge theory is defined on a spacetime manifold $\mathcal{M}$ with a gauge group G. In order to express this situation in the most economical way, we make use of the idea of fibre bundles. Let $\mathcal{M}$ and G define a fibre bundle $P(\mathcal{M}, G)$ where at each point of the manifold $x \in \mathcal{M}$ there is a fibre which is given by the gauge group G. Then we can interpret fields that live on the manifold (for example matter fields) as sections of the fibre bundle in the fundamental representation. The transport of these matter fields from one point of the manifold to another is performed by a Lie algebra valued one-form called the connection A. Let us begin the investigation with the definition of the notion of parallel transport in terms of holonomies.

4.1 Holonomy and parallel transport

Definition of the holonomy: Given a connection A in a group G over a manifold $\mathcal{M}$, the holonomy is defined as follows. Let a curve Φ be a continuous, piecewise smooth map from the interval $[0,1]$ into $\mathcal{M}$,

$$\begin{aligned} \Phi : [0,1] &\longrightarrow \mathcal{M} \\ s &\longmapsto x^\mu(s). \end{aligned}$$

The holonomy, or parallel transporter, $H[A,\Phi]$ of the connection A along the curve γ is the element of G defined by

$$\begin{aligned} H[A,\Phi](0) &= 1, \\ \frac{d}{ds}H[A,\Phi](s) + \dot{\Phi}^\mu(s)A_\mu\left(\Phi(s)\right)H[A,\Phi](s) &= 0 \\ H[A,\Phi] &= H[A,\Phi](1) \end{aligned}$$

where $\dot{\Phi}^\mu(s) \equiv \frac{dx^\mu(s)}{ds}$ is the tangent to the curve. The formal solution of this equation is

$$H[A,\Phi] = \mathcal{P}\mathrm{exp}\int_o^1 ds\dot{\Phi}^\mu(s)A_\mu^i\left(\Phi(s)\right)\tau_i \equiv \mathcal{P}\mathrm{exp}\int_\Phi A,$$

where τ_i is a basis in the Lie algebra of the group G and the path ordering $\mathcal{P}$ is defined by the power series expansion

$$\begin{aligned} &\mathcal{P}\mathrm{exp}\int_0^1 dsA\left(\Phi(s)\right) = \\ &\sum_{n=0}^{\infty}\int_0^1 ds_1\int_0^{s_1} ds_2\cdots\int_0^{s_{n-1}} ds_nA\left(\Phi(s_n)\right)\cdots A\left(\Phi(s_1)\right). \end{aligned}$$

The connection A is an object, that defines the notion of parallel transporting a vector from one point of the manifold to another point near the starting point. The holonomy of any curve Φ is well defined even if there are finite sets of points where Φ is non-differentiable. The reason is that we can break Φ in components where everything is differentiable and define the holonomy of Φ as the product of the holonomies of the components, which are well defined by continuity. As we have shown above we are enforced to write the parallel transporting holonomy along a curve $\Phi^x_{\mathcal{O}}$, starting at the origin $\mathcal{O}$ and ending at $x \in \mathcal{M}$, as

$$H\left(\Phi^x_{\mathcal{O}}\right) = \mathcal{P}\exp\left(\int_{\mathcal{O}}^{x} dy^\mu A_\mu(x)\right).$$

We also introduce the inverse curve $\left(\Phi^x_{\mathcal{O}}\right)^{-1} = \left(\Phi^{\mathcal{O}}_x\right)$ which runs from the point $x \in \mathcal{M}$ to the origin $\mathcal{O}$. When we consider a matter field under the gauge transformation,

$$\psi \to g\psi, \; g \in G,$$

the connection transforms as

$$A_\mu \to gA_\mu g^{-1} + g\partial_\mu g^{-1},$$

and the phase factor of the holonomy transforms as

$$H\left(\Phi^x_{\mathcal{O}}\right) \to g(x) H\left(\Phi^x_{\mathcal{O}}\right) g^{-1}(\mathcal{O}).$$

Let us now introduce an equivalence class between curves having the same starting point as well as the same end point. We define that two curves $\Phi^x_{\mathcal{O}}$ and $\Upsilon^x_{\mathcal{O}}$ are equivalent, $\Phi^x_{\mathcal{O}} \sim \Upsilon^x_{\mathcal{O}}$, if they give rise to the same phase factor :

$$H\left(\Phi^x_{\mathcal{O}}\right) = H\left(\Upsilon^x_{\mathcal{O}}\right).$$

In the following we will call the equivalence classes "paths" and denote them by,

$$\gamma^x_{\mathcal{O}} = \left[\Phi^x_{\mathcal{O}}\right].$$

We confine ourself to closed curves, starting and ending at the same point $\mathcal{O}$.

4.2 The group of loops

Now we want to consider curves on the manifold $\mathcal{M}$ which are parameterized by a value s on the manifold. Let us assume that the curves are continuous and piecewise smooth. We can regard such a curve Φ as a map in the following form

$$\Phi : [0, s_1] \cup [s_1, s_2] \cup ... \cup [s_{N-1}, 1] \to \mathcal{M}.$$

This map is smooth in each closed interval $[s_i, s_{i+1}]$ and continuous in the whole domain. For these curves, we are able to define a composition law. If there are two smooth curves

Φ_1 and Φ_2 such that the end point of Φ_1 is the same as the starting point of Φ_2 we define the curve $\Phi_1 \circ \Phi_2$ by

$$(\Phi_1 \circ \Phi_2)(s) = \begin{cases} \Phi_1(2s) & , \quad s \in [0, \frac{1}{2}] \\ \Phi_2(2(s - \frac{1}{2})) & , \quad s \in [\frac{1}{2}, 1] \end{cases}.$$

We also introduce the inverse curve by

$$\Phi^{-1}(s) = \Phi(1 - s).$$

For the calculation of the phase factors of the holonomy the parameterization of the curve is not important. Therefore we consider unparameterized curves, which we define as equivalence relation by identifying all curves, differing by orientation-preserving, differentiable reparameterizations. This means unparameterized curves are equivalence classes of parameterized curves. Closed curves start and end at the same point $\mathcal{O}$. Another object we want to introduce is the null-curve defined by

$$\mathcal{I}(s) = \mathcal{O} \ \forall s,$$

and all parameterizations. This null-curve is identical to the identity element. We denote the set off all closed curves beginning and ending at $\mathcal{O}$ as $\mathrm{L}_{\mathcal{O}}$. But this set is not a group, since for a curve $\mathcal{L} \in \mathrm{L}_{\mathcal{O}}$ the inverse curve $\mathcal{L}^{-1}$ is not the group inverse, so that $\mathcal{L} \circ \mathcal{L}^{-1} \neq \mathcal{I}$. Instead $\mathrm{L}_{\mathcal{O}}$ forms a semi-group under the composition of two curves $\mathcal{L}_1, \mathcal{L}_2 \to \mathcal{L}_1 \circ \mathcal{L}_2$. As we have mentioned above, the phase factor of the holonomy is connected with the parallel transport around a closed curve. If we have a principal fibre bundle $P(\mathcal{M}, G)$ with group G over $\mathcal{M}$ the holonomy is defined in the following way. Take a point $\hat{\mathcal{O}}$ in the fibre over $\mathcal{O}$ by using the connection $A_\mu(x)$. A closed curve $\mathcal{L}$ in $\mathcal{M}$ is lifted to a curve $\hat{\mathcal{L}}$ in the fibre bundle $P(\mathcal{M}, G)$ such that the starting point is

$$\hat{\mathcal{L}}(0) = \hat{\mathcal{O}},$$

and the end point is

$$\hat{\mathcal{L}}(1) = \hat{\mathcal{L}}(0) H(\mathcal{L}).$$

This relation defines the holonomy $H(\mathcal{L})$ as right multiplication of the group. And from the definition of the holonomy we get the following properties:

1. Composition law

$$H(\mathcal{L}_1 \circ \mathcal{L}_2) = H(\mathcal{L}_1) H(\mathcal{L}_2)$$

2. Under a gauge transformation, that means a change in the choice of the point of the fibre over $\mathcal{O}$, replacing $\hat{\mathcal{O}}$ by $\hat{\mathcal{O}}' = \hat{\mathcal{O}} g^{-1}$, the holonomy transforms as

$$H'(\mathcal{L}) = g H(\mathcal{L}) g^{-1}.$$

As we have mentioned the set of closed curves does not form a group. When we introduce a further equivalence relation which identifies all closed curves leading to the same holonomy we can "transform" $\mathrm{L}_{\mathcal{O}}$ to a group.

Definition: Let

$$H : \mathrm{L}_{\mathcal{O}} \to G$$

be the holonomy of a connection $A_\mu(x)$, defined on a bundle $P(\mathcal{M}, G)$. Two closed curves $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{L}_{\mathcal{O}}$ are equivalent, $\mathcal{L}_1 \sim \mathcal{L}_2$, if they have the same holonomy,

$$H(\mathcal{L}_1) = H(\mathcal{L}_2),$$

for every bundle $P(\mathcal{M}, G)$ and smooth connection $A_\mu(x)$.

Let us arrange that equivalence classes under the above-defined equivalence relation are called loops and are denoted by greek letters $[\mathcal{L}] = \alpha$. Under a loop we imagine not a closed curve but rather the equivalence class of all closed curves that begins and ends at the same point $\mathcal{O}$ and have the same holonomy. We also define that a closed curve $\mathcal{L}$ is called tree, if it is equivalent to the null-curve. This kind of curves does not enclose an area. From this statement follows directly, that two curves $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{L}_{\mathcal{O}}$ are equivalent if $\mathcal{L}_1 \circ \mathcal{L}_2^{-1}$ is tree. With these definitions the equivalence classes considered above form a group and we call this group the group of loops or the loop space, because the equivalence classes imply that, if $\alpha = [\mathcal{L}_1]$ and $\beta = [\mathcal{L}_2]$, then $\alpha \circ \beta = [\mathcal{L}_1 \circ \mathcal{L}_2]$, and this defines the multiplication law for loops. The equivalence relation also introduces the missing inverse element of a group that we had missed before in form of $\alpha \circ \alpha^{-1} = i$, where $i = [\mathcal{I}]$. For later purposes let us define the set of loops with base-point at $\mathcal{O}$ with $\mathcal{L}_{\mathcal{O}}$. This set forms a non-abelian group. When we regard the mapping from the group of loops $\mathcal{L}_{\mathcal{O}}$ into the gauge group G, this map defines a holonomy associated with a generalized connection $A_\mu(x)$. This relation will be investigated in more detail in the representation theory of loops.

4.3 The loop derivative

In this section we want to introduce the notion of a derivative operator on the loop space. Let us consider a function of loop variables $F(\gamma)$, where $\gamma \in \mathcal{L}_{\mathcal{O}}$. We want to investigate the variation of $F(\gamma)$ when the loop $\gamma \in \mathcal{L}_{\mathcal{O}}$ which is fixed at $\mathcal{O}$ is varied by the addition of an infinitesimal loop $\delta\gamma$ attached to a point x. This means that the two loops are connected by a path $\pi_{\mathcal{O}}^x$, but we are also free to choose the point x on the loop γ. The variation of the loop functional is determined by,

$$\delta F = F(\pi_{\mathcal{O}}^x \circ \delta\gamma \circ \pi_x^{\mathcal{O}} \circ \gamma) - F(\gamma).$$

Let us now explain in more detail how we realize the small variation of γ. A possible choice is to generate the infinitesimal loop $\delta\gamma$ as follows,

$$\delta\gamma = \delta u \ \delta v \ \delta\bar{u} \ \delta\bar{v},$$

where $\delta u^\mu = \epsilon_1 u^\mu$ and $\delta v^\nu = \epsilon_2 v^\nu$ are infinitesimal vectors and their inverse is defined by $\delta\bar{u} \equiv -\delta u$. The change generated by the infinitesimal loop depends only on the vectors $\delta u^\mu = \epsilon_1 u^\mu$ and $\delta v^\nu = \epsilon_2 v^\nu$, so we get the expansion

$$\begin{aligned} F(\pi^x_{\mathcal{O}} \circ \delta\gamma \circ \pi^{\mathcal{O}}_x \circ \gamma) &= \Big[1 + \epsilon_1 u^\mu Q_\mu(\pi^x_{\mathcal{O}}) + \epsilon_2 v^\nu P_\nu(\pi^x_{\mathcal{O}}) \\ &\quad + \frac{1}{2}\epsilon_1\epsilon_2(u^\mu v^\nu + v^\mu u^\nu) S_{\mu\nu}(\pi^x_{\mathcal{O}}) \\ &\quad + \frac{1}{2}\epsilon_1\epsilon_2(u^\mu v^\nu - v^\mu u^\nu)\Omega_{\mu\nu}(\pi^x_{\mathcal{O}}) + O({\epsilon_i}^2)\Big] F(\gamma). \end{aligned}$$

Q, P, S and Ω are differential operators on the space of loop functions. If $\epsilon_1 = 0, \epsilon_2 = 0$ or u^μ, v^ν are co-linear, the infinitesimal path $\delta\gamma$ is a tree. Since we have defined the loops as equivalence classes of closed curves with the same holonomy, the loop is not changed, because a tree does not contribute to the holonomy. Therefore it is evident that the loop functional does not change, if our infinitesimal loop $\delta\gamma$ is a tree. We are forced to conclude that a number of terms on the right-hand side of the above equation have to vanish in general : $Q_\mu(\pi^x_{\mathcal{O}}) = P_\mu(\pi^x_{\mathcal{O}}) = S_{\mu\nu}(\pi^x_{\mathcal{O}}) = 0$. Since for co-linear vectors u^μ, v^ν we have $u^\mu v^\nu - v^\mu u^\nu = 0$, the term $\Omega_{\mu\nu}(\pi^x_{\mathcal{O}})$ need not vanish. To summerize we find for the variation of the loop function under an infinitesimal change of loop

$$F(\pi^x_{\mathcal{O}} \circ \delta\gamma \circ \pi^{\mathcal{O}}_x \circ \gamma) = \left(1 + \frac{1}{2} s^{\mu\nu}\Omega_{\mu\nu}(\pi^x_{\mathcal{O}})\right) F(\gamma),$$

where we have defined the area enclosed by the infinitesimal loop $\delta\gamma$ by

$$s^{\mu\nu} = \epsilon_1\epsilon_2(u^\mu v^\nu - v^\mu u^\nu),$$

with $\epsilon_1\epsilon_2$ infinitesimal. The quantity $\Omega_{\mu\nu}(\pi^x_{\mathcal{O}})$ defines the loop derivative.

4.4 Loop deformation operator, Mandelstam derivative and the Ricci identity

We introduce a loop-dependent operator $U(\alpha)$ on the space of functions of loops that generates a finite deformation in the argument of a loop functional,

$$U(\alpha)F(\gamma) \equiv F(\alpha \circ \gamma).$$

The inverse and the composition law of this operator are defined as follows:

$$U^{-1}(\alpha) = U(\alpha^{-1})$$

and

$$U(\alpha)U(\beta)F(\gamma) = U(\alpha \circ \beta)F(\gamma).$$

We consider now a path which arises as a deformation of

$$\gamma \to \pi_{\mathcal{O}}^{x} \circ \delta\gamma \circ \pi_{x}^{\mathcal{O}} \circ \gamma.$$

Such a valid deformation is given by

$$(\alpha \circ \pi_{\mathcal{O}}^{x}) \circ \delta\gamma \circ (\pi_{x}^{\mathcal{O}} \circ \alpha^{-1}) \circ \gamma,$$

where $\pi_{\mathcal{O}}^{x}$ is deformed to $\alpha \circ \pi_{\mathcal{O}}^{x}$ and α is a loop, but $\alpha \circ \pi_{\mathcal{O}}^{x}$ is still a path. We can express the loop derivative of this deformed path in the following way

$$\left[1 + \frac{1}{2} s^{\mu\nu} \Omega_{\mu\nu}(\alpha \circ \pi_{\mathcal{O}}^{x})\right] F(\gamma) = F((\alpha \circ \pi_{\mathcal{O}}^{x}) \circ \delta\gamma \circ (\pi_{x}^{\mathcal{O}} \circ \alpha^{-1}) \circ \gamma). \tag{8}$$

This expression can be obtained in a different way by using the operator $U(\alpha)$ defined above:

$$\begin{aligned} U(\alpha) F(\pi_{\mathcal{O}}^{x} \circ \delta\gamma \circ \pi_{x}^{\mathcal{O}} \circ \alpha^{-1} \circ \gamma) &= U(\alpha) \left(1 + \frac{1}{2} s^{\mu\nu} \Omega_{\mu\nu}(\pi_{\mathcal{O}}^{x})\right) F(\alpha^{-1} \circ \gamma) \\ &= U(\alpha) \left(1 + \frac{1}{2} s^{\mu\nu} \Omega_{\mu\nu}(\pi_{\mathcal{O}}^{x})\right) U(\alpha)^{-1} F(\gamma). \end{aligned}$$

Comparing this expression with (8) we are lead to conclude that

$$\Omega_{\mu\nu}(\alpha \circ \pi_{\mathcal{O}}^{x}) = U(\alpha) \Omega_{\mu\nu}(\pi_{\mathcal{O}}^{x}) U^{-1}(\alpha).$$

This relation gives us the transformation property of the loop derivative under finite deformation of its path dependence. In the progress of this thesis we will use this identity to obtain the gauge invariance of the curvature tensor in the language of loops.

The next quantity to be introduced is the point derivative which is also known as the Mandelstam derivative of loops. Let us consider a function of an open path $F(\pi_{\mathcal{O}}^{x})$, a local chart at the point x of the manifold $\mathcal{M}$ and a vector u^{μ} in that chart. We define the Mandelstam derivative by a change in the path function, if the path is extended from point x to $x + \epsilon u^{\mu}$:

$$F(\pi_{\mathcal{O}}^{x} \circ \delta u) = (1 + \epsilon u^{\mu} D_{\mu}(x)) F(\pi_{\mathcal{O}}^{x}), \ \delta u = \epsilon u^{\mu}.$$

We denote the new path by $\pi_{\mathcal{O}}^{x+\epsilon u}$. After this technical remarks we turn to the problem of the Ricci identity. Let us consider an infinitesimal loop $\delta\gamma$, which is connected to an open path $\pi_{\mathcal{O}}^{x}$. We can realize this loop as $\delta\gamma = \delta u \ \delta v \ \delta\bar{u} \ \delta\bar{v}$. If we now use the definition of the loop derivative, we get

$$\begin{aligned} F(\pi_{\mathcal{O}}^{x} \circ \delta\gamma) &= F(\pi_{\mathcal{O}}^{x} \circ \delta\gamma \circ \pi_{x}^{\mathcal{O}} \circ \pi_{\mathcal{O}}^{x}) \\ &= \left(1 + \frac{1}{2} s^{\mu\nu}(x) \ \Omega_{\mu\nu}(\pi_{\mathcal{O}}^{x})\right) F(\pi_{\mathcal{O}}^{x}). \end{aligned}$$

It is also possible to express the functional F in terms of the Mandelstam derivative in the following way

$$\begin{aligned} F(\pi^x_{\mathcal{O}} \circ \delta\gamma) &\equiv F(\pi^x_{\mathcal{O}} \circ \delta u \circ \delta v \circ \delta\bar{u} \circ \bar{v}) \\ &= (1 + \epsilon_1 u^\mu D_\mu(x))(1 + \epsilon_2 v^\nu D_\nu(x + \epsilon_1 u)) \\ &\quad (1 - \epsilon_1 u^\kappa D_\kappa(x + \epsilon_1, u + \epsilon_2 v))(1 - \epsilon_2 v^\lambda D_\lambda(x + \epsilon_2 v))F(\pi^x_{\mathcal{O}}). \end{aligned}$$

If we expand this to first order in ϵ_1 and ϵ_2, we observe that terms linear in ϵ_1 or ϵ_2 vanish. We also recognize that the leading-order term is proportional to $\epsilon_1\epsilon_2$, so we get

$$\begin{aligned} F(\pi^x_{\mathcal{O}} \circ \delta\gamma) &\approx \big(1 + \epsilon_1\epsilon_2 \left(u^\mu v^\nu D_\mu(x) D_\nu(x) - u^\mu v^\lambda D_\mu(x) D_\lambda(x)\right. \\ &\quad \left.-v^\nu u^\kappa D_\nu(x) D_\kappa(x) + u^\kappa v^\lambda D_\kappa(x) D_\lambda(x)\right)\big)\, F(\pi^x_{\mathcal{O}}) \\ &= (1 + \epsilon_1\epsilon_2 u^\mu v^\nu \left[D_\mu(x), D_\nu(x)\right])\, F(\pi^x_{\mathcal{O}}) \\ &= (1 + \frac{1}{2}\, s^{\mu\nu}\left[D_\mu, D_\nu\right])F(\pi^x_{\mathcal{O}}), \end{aligned}$$

where $s^{\mu\nu} = \epsilon_1\epsilon_2(u^\mu v^\nu - u^\nu v^\mu)$ is the infinitesimal area element. If we now compare both expressions for $F(\pi^x_{\mathcal{O}} \circ \delta\gamma)$ we are lead to the expression

$$\Omega_{\mu\nu} F(\pi^x_{\mathcal{O}}) = \left[D_\mu(x), D_\nu(x)\right] F(\pi^x_{\mathcal{O}}).$$

This is the analogon of

$$[D_\mu, D_\nu] = F_{\mu\nu},$$

which is familiar from gauge theories. Later we will show that $\Omega_{\mu\nu}$ is indeed related to the curvature 2-form in gauge theories.

4.5 Connection derivative and its relation to the loop derivative

As we have seen in the last section, there exists a similarity between the loop derivative $\Omega_{\mu\nu}$ and the curvature in gauge theories. Now we introduce a derivative which corresponds to the gauge connection. We consider a loop functional $F(\pi^x_{\mathcal{O}} \circ \delta u \circ \pi^{\mathcal{O}}_{x+\delta u} \circ \gamma)$ where $\pi^x_{\mathcal{O}} \circ \delta u \circ \pi^{\mathcal{O}}_{x+\delta u}$ means an infinitesimal loop, starting at the origin $\mathcal{O}$ (as the loop γ), going to a point x, then shifted by an infinitesimal displacement $\delta u = \epsilon u$ and running back to the origin. If $F(\pi^x_{\mathcal{O}} \circ \delta u \circ \pi^{\mathcal{O}}_{x+\delta u} \circ \gamma)$ allows an expansion in terms of the infinitesimal displacement δu of the form

$$F(\pi^x_{\mathcal{O}} \circ \delta u \circ \pi^{\mathcal{O}}_{x+\delta u} \circ \gamma) = (1 + \epsilon u^\mu \Gamma_\mu(\pi^x_{\mathcal{O}}))F(\gamma),$$

we say that $F(\gamma)$ is connection differentiable and the quantity $\Gamma_\mu(\pi^x_{\mathcal{O}}) \equiv \Gamma^\pi_\mu(x)$ is called the connection derivative. The deformation of the loop γ which was performed above could have been generated by the application of successive loop derivatives. So any function which is loop differentiatble should also be connection differentiatable. We now investigate the relation between these two derivatives which will be similar to the relation between connection and curvature in gauge theory. We consider an infinitesimal loop $\delta\gamma$

$$\delta\gamma = \pi^x_{\mathcal{O}} \circ \delta u \; \delta v \; \delta\bar{u} \; \delta\bar{v} \circ \pi^{\mathcal{O}}_x . \tag{9}$$

Another choice for representing this loop is

$$\begin{aligned} \delta\gamma \;\; &\equiv \;\; \pi^x_{\mathcal{O}} \circ \delta u \circ \pi^{\mathcal{O}}_{x+\epsilon_1 u} \circ \pi^{x+\epsilon_1 u}_{\mathcal{O}} \circ \delta v \circ \pi^{\mathcal{O}}_{x+\epsilon_1 u+\epsilon_2 v} \\ & \circ \pi^{x+\epsilon_1 u+\epsilon_2 v}_{\mathcal{O}} \circ \delta\bar{u} \circ \pi^{\mathcal{O}}_{x+\epsilon_2 v} \circ \pi^{x+\epsilon_2 v}_{\mathcal{O}} \circ \delta\bar{v} \circ \pi^{\mathcal{O}}_x . \end{aligned} \tag{10}$$

If we now apply the definition of the loop derivative to $F(\delta\gamma \circ \gamma)$ with $\delta\gamma$ given in form of equation (9) and the definition of the connection derivative of the function $F(\delta\gamma \circ \gamma)$ with the infinitesimal loop $\delta\gamma$ given in the form (10) we obtain

$$\begin{aligned} (1 + \epsilon_1\epsilon_2 u^\mu v^\nu \Omega_{\mu\nu}(\pi^x_{\mathcal{O}}))F(\gamma) \;\; &= \;\; (1 + \epsilon_1 u^\mu \Gamma^\pi_\mu(x))(1 + \epsilon_2 v^\nu \Gamma^\pi_\nu(x + \epsilon_1 u)) \\ & (1 - \epsilon_1 u^\kappa \Gamma^\pi_\kappa(x + \epsilon_1 u + \epsilon_2 v)) \\ & (1 - \epsilon_2 v^\lambda \Gamma^\pi_\lambda(x + \epsilon_2 v))F(\gamma). \end{aligned}$$

If we expand all terms on the right-hand side to first order in $\epsilon_1\epsilon_2$ we get,

$$\begin{aligned} & \left[1 + \epsilon_1 u^\mu \Gamma^\pi_\mu(x)\epsilon_2 v^\nu \Gamma^\pi_\nu(x) + \epsilon_2 v^\nu \epsilon_1 u^\mu \partial_\mu \Gamma^\pi_\nu(x) - \epsilon_2\epsilon_1 v^\nu u^\kappa \Gamma^\pi_\nu(x)\Gamma^\pi_\kappa(x) \right. \\ & \left. -\epsilon_1 u^\kappa \epsilon_2 v^\nu \partial_\nu \Gamma^\pi_\kappa(x) + ...\right] F(\gamma) \\ = \;\; & \left[1 + \epsilon_1\epsilon_2 u^\mu v^\nu \left(\partial_\mu \Gamma^\pi_\nu(x) - \partial_\nu \Gamma^\pi_\mu(x) + [\Gamma^\pi_\mu(x), \Gamma^\pi_\nu(x)]\right)\right] F(\gamma). \end{aligned}$$

Comparing this expression with $(1 + \epsilon_1\epsilon_2 u^\mu v^\nu \Omega_{\mu\nu}(\pi^x_{\mathcal{O}}))F(\gamma)$ we arrive at the following relation

$$\Omega_{\mu\nu}(\pi^x_{\mathcal{O}}) = \partial_\mu \Gamma^\pi_\nu(x) - \partial_\nu \Gamma^\pi_\mu(x) + [\Gamma^\pi_\mu(x), \Gamma^\pi_\nu(x)].$$

We recognize the structural equivalence to the relation between the connection $A_\mu(x)$ and the curvature $F_{\mu\nu}(x)$ in gauge theory. Another point which we only want to mention here without a proof is that there exists an analogue of the gauge dependence of the connection in a gauge theory in terms of the connection derivative in loop space corresponding to a path dependence. This culminates in the conclusion that

$$U(\gamma) = \mathcal{P}\left(\int_\gamma dy^\mu \; \Gamma^\pi_\mu(y)\right)$$

is an operator representation of the loop γ in the space of loop functions. And from this identity we obtain the following property of the operator $U(\gamma)$,

$$U(\gamma_1)U(\gamma_2) = U(\gamma_1 \circ \gamma_2),$$

which is easy to verify with the definition of $U(\gamma)$.

4.6 Representation of the group of loops

All the results of the above discussions are general relations in the abstract loop space. We have recognized many formal analogies between the loop calculus and familiar gauge theory. The next step in our investigation is to study the general relations in a particular representation in terms of a gauge group. We will see that all kinematic structures of gauge theories will naturally emerge if we map the group of loops onto the gauge group. More detailed this means that gauge theories arise as representations of the group of loops. In a certain sense we can consider the group of loop as a more fundamental object than the gauge group. Let us consider a homomorphism of the group of loops onto a gauge group,

$$\mathcal{H} : \mathcal{L}_{\mathcal{O}} \to G.$$

This homomorphism defines a mapping from a loop γ onto an element of the gauge group $H(\gamma) \in G$,

$$H : \gamma \to H(\gamma).$$

Because $H(\gamma) \in G$ is a representation of the group of loops, it has to satisfy the composition law of the loop group $\mathcal{L}_{\mathcal{O}}$,

$$H(\gamma_1)H(\gamma_2) = H(\gamma_1 \circ \gamma_2).$$

If we now compare the above equation with the definition of the loop operator $U(\gamma_1)F(\gamma_2) = F(\gamma_1 \circ \gamma_2)$, which is valid for any loop function $F(\gamma)$ and hence in particular for $F(\gamma) = H(\gamma)$, we conclude that $H(\gamma)$ is a representation of $U(\gamma)$ in the gauge group.

Let us now derive the well-known local objects which are connected with the kinematic structure of gauge theory in the loop language. We assume that the representation $H(\gamma) \in G$ is loop differentiable. Consider an infinitesimal loop

$$\delta\gamma = \pi_{\mathcal{O}}^{x} \circ \delta u \circ \pi_{x+\epsilon u}^{\mathcal{O}}$$

which is attached to a given loop γ. If $H(\gamma)$ is the representation of the group of loops we get

$$H(\pi_{\mathcal{O}}^{x} \circ \delta u \circ \pi_{x+\epsilon u}^{\mathcal{O}} \circ \gamma) = H(\pi_{\mathcal{O}}^{x} \circ \delta u \circ \pi_{x+\epsilon u}^{\mathcal{O}})H(\gamma).$$

If we now use the definition of the connection derivative we have

$$H(\pi_{\mathcal{O}}^{x} \circ \delta u \circ \pi_{x+\epsilon u}^{\mathcal{O}} \circ \gamma) = (1 + \epsilon u^{\mu} \Gamma_{\mu}^{\pi}(x))H(\gamma).$$

Because the infinitesimal loop $\delta\gamma$ is close to the identity (the identity loop) and $H(\gamma)$ is a continuous differential representation that is localized in the gauge group, we must have the expansion

$$H(\pi_{\mathcal{O}}^{x} \circ \delta u \circ \pi_{x+\epsilon u}^{\mathcal{O}}) = (1 + \epsilon u^{\mu} A_{\mu}^{\pi}(x))H(\gamma),$$

where

$$A_\mu(x) = A_\mu^a T^a \in \mathcal{G}$$

is some element of the algebra $\mathcal{G}$ of the gauge group. Using the definition of the connection derivative we are lead (with support of the above equation) to

$$\Gamma_\mu^\pi(x)H(\gamma) = A_\mu(x)H(\gamma).$$

This relation in fact implies that the connection derivative is related to the connection of the gauge group. With an analogous discussion one shows that the loop derivative is indeed related to the curvature:

$$\Omega_{\mu\nu}(\pi_{\mathcal{O}}^x)H(\gamma) = F_{\mu\nu}(x)H(\gamma).$$

Also from the derived expression above,

$$\Omega_{\mu\nu}(\pi_{\mathcal{O}}^x) = \partial_\mu \Gamma_\nu^\pi(x) - \partial_\nu \Gamma_\mu^\pi(x) + [\Gamma_\mu^\pi(x), \Gamma_\nu^\pi(x)],$$

we find the representation of the curvature in terms of the gauge potential,

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)].$$

The path $\pi_{\mathcal{O}}^x$ in the above discussion was arbitrary. But as we have mentioned earlier, changing the path description changes the loop derivative. To illustrate this in more detail let us consider the transformation

$$\pi_{\mathcal{O}}^x \to \chi_{\mathcal{O}}^x = \alpha \circ \pi_{\mathcal{O}}^x, \quad \alpha = \chi_{\mathcal{O}}^x \circ \pi_x^{\mathcal{O}}.$$

Under this transformation of the path the loop derivative transforms as follows,

$$\Omega_{\mu\nu}(\alpha \circ \pi_{\mathcal{O}}^x) = U(\alpha)\Omega_{\mu\nu}(\pi_{\mathcal{O}}^x)U^{-1}(\alpha).$$

Where we have used the results from the investigation of the loop deformation operator. From this relation we find the transformation law of the curvature under a change of path

$$F'_{\mu\nu}(x) = H(x)F_{\mu\nu}(x)H^{-1}(x),$$

where we have used

$$H(x) \equiv H(\chi_{\mathcal{O}}^x \circ \pi_x^{\mathcal{O}}).$$

Here we recognize the usual transformation law of the curvature tensor under gauge transformation. When we investigate the behaviour of the connection derivative under the above change of the path $\pi_{\mathcal{O}}^x$ it is possible to show the validity of the following expression,

$$\Gamma_\mu^\chi(x) = U(x)\Gamma_\mu^\pi(x)U^{-1}(x) + U(x)\partial_\mu U^{-1},$$

with

$$U(x) \equiv U\left(\chi^x_{\mathcal{O}} \circ \pi^{\mathcal{O}}_x\right).$$

From this relation and our discussion of the representation theory we are lead to the transformation law of the connection under gauge transformation,

$$A'_\mu(x) = H(x)A_\mu(x)H^{-1}(x) + H(x)\partial_\mu H^{-1}.$$

By the definition of the loop deformation operator $U(\alpha)$ and the group composition law we have

$$U(\alpha)H(\gamma) = H(\alpha \circ \gamma) = H(\alpha)H(\gamma).$$

Because we have mentioned earlier that it is possible to show that

$$U(\gamma) = \mathcal{P}\exp\left(\int_\gamma dy^\mu \Gamma^\pi_\mu(y)\right),$$

we observe that

$$U(\alpha)H(\gamma) = \mathcal{P}\exp\left(\oint dy^\mu \Gamma^\pi_\mu(y)\right)H(\gamma).$$

If we now use the above derived relation $\Gamma^\pi_\mu(x)H(\gamma) = A_\mu(x)H(\gamma)$ we find

$$H(\alpha) = \mathcal{P}\exp\left(\oint dy^\mu A_\mu(y)\right).$$

We immediately recognize the familiar representation of the holonomy in terms of the gauge potential. The above investigations show that gauge theories arise as representations of the group of loops. All the quantities like connection and curvature can be expressed in terms of loops. Furthermore the kinematic properties of gauge theories do not depend on the choice of the gauge group to represent the group of loops. Or more detailed, the corresponding generators of the group of loops are connected with curvature and connection in a more abstract sense. Only when we investigate a particular representation of the group of loops in terms of a gauge group these abstract quantities can be interpreted in the usual sense of gauge theory. In this sense it is allowed to say that the loop representation of gauge theory is more fundamental than the ordinary one.

5 Representation of the loop derivative in terms of ordinary functional calculus

We have introduced the loop derivative in the last section in an abstract way. But because we want to calculate with this new quantity in Quantum Field Theory we want to represent the loop derivative in a more practical manner. For our purposes the most useful representation for the loop derivative is the representation in terms of functional derivatives which are commonly used in the QFT community. As we have used the worldline approach to quantum field theory we have taken a first quantization approach. We have described the quantum properties of the field with an effective particle moving on closed curves (loops) in the background field. Due to this we can reproduce the spin-field connection upon considering the rate of change of the Wilson loop. In this approach the Wilson loop is an element of the so called loop space (holonomy group) and therefore we have to define a derivative operator in this space. Because we now search an expression of the loop derivative in coordinate space we abstain from the rather abstract discussions of the past section [19],[18]. Let us consider a closed line in coordinate space. This curve can be described in terms of periodic functions, namely

$$\mathcal{C}: \; x_\mu = \mathcal{C}_\mu(\tau) = \mathcal{C}_\mu(\tau + T),$$

where τ is just a parameter, but in our approach it is the propertime of the particle moving along the loops. We consider the loops as a class of periodic functions and define each loop as a point in the loop space. We do not care about smoothness of the loops, more important for our purposes is continuity. There is indeed the possibility that the loop intersects itself. This happens at all points where

$$\mathcal{C}_\mu(\tau_1) = \mathcal{C}_\mu(\tau_2),$$

with $\tau_1 \neq \tau_2$. Of course there is the possibility of more than one intersection.

The next step is to consider infinitesimal variations of a loop. Instead of the loop $\mathcal{C}$ we study a product of loops $\mathcal{C} \circ \tilde{\mathcal{C}}$. If we choose a basis in coordinate space the loops are represented as

$$\mathcal{C}: \; x_\mu = \mathcal{C}_\mu(\tau), \; \mathcal{C}_\mu(0) = \mathcal{C}_\mu(T) = x,$$

and for the other loop

$$\tilde{\mathcal{C}}: \; x_\mu = \tilde{\mathcal{C}}_\mu(\tau), \; \tilde{\mathcal{C}}_\mu(0) = \tilde{\mathcal{C}}_\mu(\tilde{T}) = x.$$

Our goal is to understand how we can represent the loop derivative in coordinate space. Therefore we have to investigate variations of a loop. This is realized if we attach an infinitesimally small loop to a "macroscopic" one. The action of this procedure is that we have deformed the original loop by a little "perturbance". This reads

$$\mathcal{C} \circ \tilde{\mathcal{C}}: \; x_\mu = \mathcal{C}_\mu(\tau), \; \forall \; 0 \leq \tau \leq T,$$

$$\mathcal{C} \circ \tilde{\mathcal{C}}: \; x_\mu = \tilde{\mathcal{C}}_\mu(\tau - T), \; \forall \; T \leq \tau \leq T + \tilde{T}.$$

Therefore the connected loop $\mathcal{C} \circ \tilde{\mathcal{C}}$ begins as the unperturbed loop $\mathcal{C}$, continues as $\tilde{\mathcal{C}}$ at the common point x, circulates along the infinitesimal loop and then returns to the same point x. We can now compare the unperturbed with the perturbed loop. This is the basis for the definition of a derivative. The product of the two loops is a periodic function with two periods. We define the loop derivative as the leading part of the variation of the functional:

$$F(\mathcal{C} \circ \tilde{\mathcal{C}}) - F(\mathcal{C}) \to s_{\mu\nu}(\tilde{\mathcal{C}}) \; \frac{\delta F(\mathcal{C})}{\delta s_{\mu\nu}(x)}.$$

The first-order term

$$\int\limits_{\tilde{\mathcal{C}}} dy_\gamma = 0,$$

vanishes for our closed loops. The area element $s_{\mu\nu}$ is defined as

$$s_{\mu\nu}(\tilde{\mathcal{C}}) = \frac{1}{2} \int\limits_{\tilde{\mathcal{C}}} y_\mu \; dy_\nu,$$

the second order invariant. All other higher invariants also vanish because we are dealing only with infinitesimally deformed loops. Therefore in terms of functional calculus we can write

$$\delta F(\mathcal{C}) = \delta s_{\mu\nu}(\tilde{\mathcal{C}}) \; \frac{\delta F(\mathcal{C})}{\delta s_{\mu\nu}(x)}.$$

We mention that the area element is an antisymmetric tensor

$$\delta s_{\mu\nu} + \delta s_{\nu\mu} = \frac{1}{2} \int\limits_{\tilde{\mathcal{C}}} (y_\mu dy_\nu + y_\nu dy_\mu) = \frac{1}{2} \int\limits_{\tilde{\mathcal{C}}} d(y_\mu y_\nu) = 0.$$

If we imagine $\tilde{\mathcal{C}}$ as a little plaquette in various $\mu\nu$ planes we are able to calculate all components of the area derivative. All the above discussions are equivalent to the comments about the loop calculus in the last section. But as mentioned above we need a derivative in terms of ordinary functional derivatives operating on the coordinate space representation of the loops, because its easier to work with such expressions. Therefore we have to make a connection between the area derivative and our familiar functional calculus. Considering the above definition of the area derivative we want to find the functional derivative of both sides with respect to a point $x_\mu = \tilde{\mathcal{C}}_\mu(\tau)$ belonging to the infinitesemal loop $\tilde{\mathcal{C}}$. We differentiate $s_{\mu\nu}(\tilde{\mathcal{C}})$ and get

$$\delta s_{\mu\nu}(\mathcal{C}) = \frac{1}{2} \int \delta\mathcal{C}_\mu(\tau) \; \dot{\mathcal{C}}_\nu(\tau) \; d\tau + \frac{1}{2} \int \mathcal{C}_\mu(\tau) \; \delta\dot{\mathcal{C}}_\nu(\tau) \; d\tau. \tag{11}$$

Where we have used that

$$s_{\mu\nu}(\tilde{\mathcal{C}}) = \frac{1}{2} \int\limits_{\tilde{C}} y_\mu \; dy_\nu = \frac{1}{2} \int \mathcal{C}_\mu(\tau) \; \dot{\mathcal{C}}_\nu \; d\tau.$$

After we have integrated the second term in (11) by parts we get

$$\delta s_{\mu\nu}(\tilde{\mathcal{C}}) = \frac{1}{2}\int \delta\mathcal{C}_\mu(\tau)\ \dot{\mathcal{C}}_\nu(\tau)\ d\tau - \frac{1}{2}\int \dot{\mathcal{C}}_\nu(\tau)\ \delta\mathcal{C}_\mu(\tau)\ d\tau.$$

With this expression we obtain a functional derivative with respect to the loop $\mathcal{C}$

$$\frac{\delta s_{\mu\nu}(\tilde{\mathcal{C}})}{\delta\tilde{\mathcal{C}}_\alpha(\tau)} = \frac{1}{2}\ (\delta_{\mu\alpha}\ \dot{\mathcal{C}}_\nu(\tau) - \delta_{\nu\alpha}\ \dot{\mathcal{C}}_\mu(\tau)).$$

Therefore we find

$$\begin{aligned}\frac{\delta F(\mathcal{C})}{\delta\mathcal{C}_\alpha(\tau)} &= \frac{\delta s_{\mu\nu}(\tilde{\mathcal{C}})}{\delta\tilde{\mathcal{C}}_\alpha(\tau)}\ \frac{\delta F(\mathcal{C})}{\delta s_{\mu\nu}(\tau)} \\ &= \frac{\delta F}{\delta s_{\alpha\beta}(\tau)}\ \dot{\mathcal{C}}_\beta(\tau).\end{aligned}$$

We have found a functional derivative of F with respect to the loop $\mathcal{C}$. However for our cases it is more useful to get an expression for the increment in terms of $\frac{\delta}{\delta s_{\alpha\beta}}$. Therefore we have to perform one more functional derivative and get

$$\begin{aligned}\frac{\delta^2 F(\mathcal{C})}{\delta\mathcal{C}_\alpha(\tau)\ \delta\mathcal{C}_\beta(\tau')} &= \left(\frac{\delta}{\delta\mathcal{C}_\beta(\tau')}\ \frac{\delta F}{\delta s_{\alpha\gamma}(\tau)}\right)\dot{\mathcal{C}}_\gamma(\tau) + \left(\frac{\delta\dot{\mathcal{C}}_\gamma(\tau)}{\delta\mathcal{C}_\beta(\tau')}\right)\ \frac{\delta F}{\delta s_{\alpha\beta}(\tau)} \\ &= \left(\frac{\delta}{\delta s_{\alpha\gamma}(\tau)}\ \frac{\delta F}{\delta\mathcal{C}_\beta(\tau')}\right)\ \dot{\mathcal{C}}_\gamma(\tau) + \left(\frac{\delta\dot{\mathcal{C}}_\gamma(\tau)}{\delta\mathcal{C}_\beta(\tau')}\right)\ \frac{\delta F}{\delta s_{\alpha\beta}(\tau)} \\ &= \left(\frac{\delta}{\delta s_{\alpha\gamma}(\tau)}\ \frac{\delta F}{\delta s_{\beta\eta}(\tau')}\ \dot{\mathcal{C}}_\eta(\tau')\right)\ \dot{\mathcal{C}}_\gamma(\tau) + \left(\frac{\delta\dot{\mathcal{C}}_\gamma(\tau)}{\delta\mathcal{C}_\beta(\tau')}\right)\ \frac{\delta F}{\delta s_{\alpha\beta}(\tau)} \\ &= \left(\frac{\delta}{\delta s_{\alpha\gamma}(\tau)}\ \frac{\delta F}{\delta s_{\beta\eta}(\tau')}\right)\ \dot{\mathcal{C}}_\eta(\tau')\ \dot{\mathcal{C}}_\gamma(\tau) + \left(\frac{\delta\dot{\mathcal{C}}_\gamma(\tau)}{\delta\mathcal{C}_\beta(\tau')}\right)\ \frac{\delta F}{\delta s_{\alpha\beta}(\tau)}.\end{aligned}$$

In this expression we are only interested in the second, the antisymmetric part, because our goal is to define a derivative operator in terms of the area element $s_{\alpha\beta}$. As we have shown, this one is antisymmetric, so we get

$$\left(\frac{\delta^2 F}{\delta\mathcal{C}_\alpha(\tau)\ \delta\mathcal{C}_\beta(\tau')}\right)_{\text{antisym}} = \dot{\delta}(\tau-\tau')\ \frac{\delta F}{\delta s_{\alpha\beta}(\tau)}\ .$$

Integrating over the small interval $(\tau-\tau')$ with the proper weight $(\tau-\tau')$ we arrive at the expression

$$\frac{\delta F}{\delta s_{\alpha\beta}(\tau)} = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\ \rho\ \frac{\delta^2 F}{\delta\mathcal{C}_\alpha(\tau+\frac{\rho}{2})\ \delta\mathcal{C}_\beta(\tau-\frac{\rho}{2})}\ .$$

We conclude that the expression for the derivative operator is

$$\frac{\delta}{\delta s_{\alpha\beta}(\tau)} = \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} d\rho \ \rho \ \frac{\delta^2}{\delta \mathcal{C}_\alpha(\tau + \frac{\rho}{2}) \ \delta \mathcal{C}_\beta(\tau - \frac{\rho}{2})}.$$

Therefore we have an expression in terms of ordinary functional derivatives for the loop (area) derivative. This involves on the technical side the great advantage that we can operate with well known mathematical quantities. On the other hand, with this mathematical operator, we get a different insight into the dynamics of gauge fields, because we are now able to study for instance the Wilson loop $e^{ie \oint A_\mu dx_\mu}$. As mentioned in the last section the Wilson loop is an object of the holonomy group and the holonomoy is the parallel transport matrix along a closed curve. And indeed we will calculate explicitly that, with our notion of loop (area) derivative along a curve, it is possible to reproduce the Faraday tensor. Beyond this we identify a geometrical expression for the spin-field coupling. The idea that holonomies are the natural variables in a gauge theory can be clarified by a heuristic argument. We can understand for example electromagnetic phenomena in terms of "lines of force". Two key ideas underlie this intuition. First, the relevant physical variables fill up space. This notion is indeed the origin of field theory. Second, the relevant variables do not refer to what happens at a point, but rather refer to the relation between different points connected by a line. The mathematical quantity that expresses this idea is the holonomy of the gauge potential along the line. And in the approach followed in this thesis the dynamics of the system is coupled to the Wilson loop.

6 Loop derivative representation of the worldline path integral

In this section we develop an alternative approach to the fermionic worldline integral. In terms of the directly developed methods of "loop-calculus" it is possible to formulate the worldline integral in a more geometrical context. In this approach all the dynamical entities, even the gauge field quantities like connection and curvature appear in a natural manner. The goal of this section is to develop a worldline expression in terms of the area derivative and to show that the obtained quantity is equivalent to the standart result of the worldline expression. We want to show first that the loop derivative applied to the Wilson loop gives us the familiar result for the worldline integral. The first step in this calculation is to show that the action of the loop (area) derivative on the Wilson loop (the holonomy) indeed gives us the spin field connection. That means we have to show that the following relation is valid

$$\begin{aligned}\langle W_{\text{spin}}[A]\rangle &= \int \mathcal{D}x(\tau)\ e^{ie\oint dxA}\ \text{tr}_\gamma \mathcal{P}e^{\frac{e}{2}\int d\tau\sigma F}\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} \\ &= \int \mathcal{D}x(\tau)\left[\text{tr}_\gamma\ \mathcal{P}e^{-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}}\ e^{ie\oint dxA}\right]\ e^{-\int d\tau\frac{\dot{x}^2(\tau)}{4}},\end{aligned}$$

where $\frac{\delta}{\delta s(\tau)}$ is our operator of the loop derivative and all $d\tau$ integrals are meant as $\int_0^T d\tau$ in the progress of this thesis if not explicitly mentioned otherwise. In coordinate space the above expression takes the form

$$\frac{\delta}{\delta s_{\mu\nu}(\tau)} = \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\ \frac{\delta^2}{\delta x_\mu(\tau+\frac{\rho}{2})\ \delta x_\nu(\tau-\frac{\rho}{2})}.$$

Therefore we have to prove that the action of the exponentiated loop derivative on the Wilson loop reproduces the spin coupling term in the path integral. In formulas

$$\mathcal{P}e^{-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}}\ e^{ie\oint dxA} \quad = \quad e^{ie\oint dxA}\ \mathcal{P}e^{\frac{e}{2}\int d\tau\sigma F}.$$

To obtain this identity we expand the eponential operator in terms of $\frac{\delta}{\delta s(\tau)}$ and get

$$\left(1+\left(-\frac{i}{2}\right)\int d\tau\sigma\frac{\delta}{\delta s(\tau)}+...\right)\exp\left[ie\int d\tau'\ \dot{x}_\mu(\tau')A^\mu(x(\tau'))\right].$$

6.1 First-order calculation

At first we restrict ourselves to the action of the leading order term, the linear one, in order to gain some experience with the calculational techniques,

$$\left(-\frac{i}{2}\int d\tau\ \sigma_{\mu\nu}\frac{\delta}{\delta s_{\mu\nu}(\tau)}\right)\ \exp\left[ie\int d\tau'\ \dot{x}_\mu(\tau')A^\mu(x(\tau'))\right].$$

Let us investigate the action of one of the functional derivatives, namely $\frac{\delta}{\delta x_\nu(\tau-\frac{\rho}{2})}$. Applied to the Wilson loop we obtain the following relation

$$
\begin{aligned}
& \frac{\delta}{\delta x_\nu(\tau-\frac{\rho}{2})} \exp\left[ie\int d\tau'\ \dot{x}_\mu(\tau')A^\mu(x(\tau'))\right] = \frac{\delta}{\delta x_\nu(\tau-\frac{\rho}{2})} \exp\,[K] \\
= \quad & \exp\,[K]\ ie\int d\tau' \left[\left(\frac{\delta \dot{x}_\mu(\tau')}{\delta x_\nu(\tau-\frac{\rho}{2})}\right) A^\mu(x(\tau')) + \dot{x}_\mu(\tau')\ \left(\frac{\delta A^\mu(x(\tau'))}{\delta x_\nu(\tau-\frac{\rho}{2})}\right)\right] \\
= \quad & \exp\,[K]\ ie\int d\tau'\ \left[\ \dot{\delta}\left(\tau' - \left[\tau-\frac{\rho}{2}\right]\right) A^\nu(x(\tau')) + \dot{x}_\mu(\tau')\ \frac{\partial A^\mu(x(\tau'))}{\partial x_\nu}\ \delta\left(\tau'-\left[\tau-\frac{\rho}{2}\right]\right)\right] \\
= \quad & \exp\,[K]\ ie\int d\tau'\ \left[\ -\delta\left(\tau' - \left[\tau-\frac{\rho}{2}\right]\right) \dot{A}^\nu(x(\tau')) + \dot{x}_\mu(\tau')\ \frac{\partial A^\mu(x(\tau'))}{\partial x_\nu}\ \delta\left(\tau'-\left[\tau-\frac{\rho}{2}\right]\right)\right].
\end{aligned}
$$

In the last line we have integrated the first term by parts. Now we perform the τ' integral and get

$$
\begin{aligned}
& \exp\,[K]\ ie\left[\dot{x}_\mu\ \frac{\partial A^\mu(x(\tau-\frac{\rho}{2}))}{\partial x_\nu} - \dot{A}^\nu\left(x\left(\tau-\frac{\rho}{2}\right)\right)\right] \\
= \quad & \exp\,[K]\ ie\left[\frac{dx^\mu}{d\tau}\ \frac{\partial A_\mu(x(\tau-\frac{\rho}{2}))}{\partial x_\nu} - \frac{dx_\mu}{d\tau}\ \frac{\partial A_\nu(x(\tau-\frac{\rho}{2}))}{\partial x_\mu}\right] \\
= \quad & \exp\,[K]\ (-ie)\ \dot{x}_\mu F^{\mu\nu}(\tau-\frac{\rho}{2}),
\end{aligned}
$$

where the propertime argument always refers to all τ-dependent quantities in each term unless stated otherwise. The next step is to perform the second derivative:

$$
\begin{aligned}
& \frac{\delta}{\delta x_\mu(\tau+\frac{\rho}{2})}\ \left(-\exp\,[K]\ ie\ \dot{x}_\alpha F^{\alpha\nu}(\tau-\frac{\rho}{2})\right) \\
= \quad & \left(\frac{-\delta\ \exp\,[K]}{\delta x_\mu(\tau+\frac{\rho}{2})}\right)\ ie\ \dot{x}_\alpha(\tau-\frac{\rho}{2})\ F^{\alpha\nu} -\ \exp\,[K]\ ie\ \left(\frac{\delta[\dot{x}_\alpha(\tau-\frac{\rho}{2})\ F^{\alpha\nu}(x(\tau-\frac{\rho}{2}))]}{\delta x_\mu(\tau+\frac{\rho}{2})}\right) \\
= \quad & \exp[K](ie)^2\ \dot{x}_\beta F^{\beta\mu}(x(\tau+\frac{\rho}{2}))\ \dot{x}_\alpha F^{\alpha\nu}(x(\tau-\frac{\rho}{2})) \\
& -\exp[K]\ ie\left[\frac{\delta \dot{x}_\alpha(\tau-\frac{\rho}{2})}{\delta x_\mu(\tau+\frac{\rho}{2})}\ F^{\alpha\nu}\ +\dot{x}_\alpha\ \frac{\delta F^{\alpha\nu}(x(\tau-\frac{\rho}{2}))}{\delta x_\mu(\tau+\frac{\rho}{2})}\right] \\
= \quad & \mathcal{A} - \exp[K]ie\left[\dot{\delta}(-\rho)\ \delta_\alpha{}^\mu F^{\alpha\nu}(x(\tau-\frac{\rho}{2})) + \dot{x}_\alpha\frac{\partial F^{\alpha\nu}}{\partial x_\mu}\ \delta(-\rho)\right],
\end{aligned}
$$

where we have put $\mathcal{A} = \exp[K]\ (ie)^2\ \dot{x}_\beta F^{\beta\mu}(x(\tau+\frac{\rho}{2}))\dot{x}_\alpha F^{\alpha\nu}(x(\tau-\frac{\rho}{2}))$. Then we get

$$
\frac{\delta^2\ \exp[K]}{\delta x_\nu(\tau-\frac{\rho}{2})\delta x_\mu(\tau+\frac{\rho}{2})}\ = \mathcal{A} - (ie)\ \exp[K]\left[\dot{\delta}(\rho)\ F^{\mu\nu}(x(\tau-\frac{\rho}{2})) + \left(\dot{x}_\alpha\frac{\partial F^{\alpha\nu}}{\partial x_\mu}(\tau-\frac{\rho}{2})\right)\delta(\rho)\right].
$$

Next we perform the integral over ρ. For this, we have to calculate

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int\limits_{-\epsilon}^{\epsilon} d\rho\rho\, \frac{\delta^2\, \exp[K]}{\delta x_\mu(\tau+\frac{\rho}{2})\, \delta x_\nu(\tau-\frac{\rho}{2})} \\
= \quad &\lim_{\epsilon\to 0}\int\limits_{-\epsilon}^{\epsilon} d\rho\rho\, \left(\mathcal{A}-(ie)\,\exp[K]\left[\dot\delta(\rho)\; F^{\mu\nu}(x(\tau-\frac{\rho}{2}))+\dot x_\alpha\frac{\partial F^{\alpha\nu}}{\partial x_\mu}(\tau-\frac{\rho}{2})\;\delta(\rho)\right]\right).
\end{aligned}$$

For the term $\sim\delta(\rho)$, we can carry out the ρ integral and obtain

$$\lim_{\epsilon\to 0}\int\limits_{-\epsilon}^{\epsilon} d\rho\rho\,\left(-(ie)\,\exp[K]\;\dot x_\alpha\frac{\partial F^{\alpha\nu}}{\partial x_\mu}\;\delta(\rho)\right)=0.$$

Assuming that $\dot x F$ is sufficiently smooth immediately leads us to

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\;\mathcal{A}=0. \tag{12}$$

Actually, we will see later (cf. sect. with Green's functions 6.2) that this assumption could be too naive. Nevertheless, (12) does indeed hold also for Wick-contracted worldline products as will become clear in the next section. So we only have to deal with the expression

$$(-ie)\,\exp[K]\;\lim_{\epsilon\to 0}\int\limits_{-\epsilon}^{\epsilon} d\rho\;\dot\delta(\rho)\left(\rho\;F^{\mu\nu}(x(\tau-\frac{\rho}{2})\right).$$

After integration by parts we get

$$(ie)\,\exp[K]\;\int\limits_{-\epsilon}^{\epsilon} d\rho\left[\frac{d\rho}{d\rho}\;F^{\mu\nu}(x(\tau-\frac{\rho}{2}))+\rho\;\frac{dF^{\mu\nu}}{d\rho}\right]\delta(\rho)=(ie)\;F^{\mu\nu}(x(\tau))\;\exp[K].$$

To summarize, we finally obtain to first order in the spin-factor expansion

$$\begin{aligned}
-\frac{i}{2}\int d\tau\;\sigma_{\mu\nu}\;\frac{\delta\exp[K]}{\delta s_{\mu\nu}(\tau)} \quad &= \quad -\frac{i}{2}\left(\int d\tau\;\sigma_{\mu\nu}\;(ie)\;F^{\mu\nu}(x(\tau))\right)\;\exp[K] \\
&= \quad \left(\frac{e}{2}\int d\tau\;\sigma_{\mu\nu}F^{\mu\nu}\right)\exp[K],
\end{aligned}$$

which in itself is an exact result. This is indeed the first-order term of the general expression $\mathcal{P}\exp\;[\frac{e}{2}\int d\tau\sigma F]\;\exp[ie\int d\tau\;\dot x A]$. Therefore we have shown that the action of the area derivative gives us (at least in first order approximation) the familiar result of the worldline representation. This also shows that the action of the area derivative to the holonomy (the

Wilson loop) indeed produces the field-strength tensor. So we get, with our definition of the derivative operator, automatically the spin-field connection from the action on the Wilson loop. Most importantly we observe that in the ϵ limit only terms survive, if they contain a derivative of the delta function. This fact is essential for any non-zero spin-field coupling contribution and will be frequently used in the progress of the thesis. To be precise, any less singular term vanishes in the limit $\epsilon \to 0$, whereas more singular terms, e.g.$\sim \ddot{\delta}(\rho)$ would be ill-defined. As will become obvious in the following sections, more singular terms can indeed occur in various places, but we will be able to demonstrate that these ill-defined singularities do always cancel in the total sum of all terms. As a rule of general validity, it will turn out that we have to watch out for only those terms $\sim \dot{\delta}(\rho)$, as we have just learnt from this first-order calculation. Note that this structure arises from the contraction of terms containing one $\ddot{x}$ and one $\dot{x}$. The next step in our calulation for demonstrating the equivalence between the two approaches is to enter the next orders in the expansion of $\mathcal{P}\exp[-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}]$. We have to show that the resulting expression together with the Wilson loop indeed develop the exponential series for exp $[\frac{e}{2}\int d\tau\sigma F]\exp[ie\int d\tau\ \dot{x}A]$. Because we need to take care of the fact that worldline monomials are subject to Wick contractions in this demonstration, we first have to analyze the worldline propagators and their derivatives.

6.2 Worldline Green's functions

To obtain the worldline Green's function we have to calulate the quantity $\langle\tau_1|\ \mathcal{G}\ |\tau_2\rangle = \mathcal{G}(\tau_1,\tau_2)$ defined by $\langle\ x_\mu(\tau_1)\ x_\nu(\tau_2)\rangle = -\delta_{\mu\nu}\ \mathcal{G}(\tau_1,\tau_2)$. We extract the worldline Green's function from the generating functional of worldline correlators (not to be confused with the Schwinger functional)

$$
\begin{aligned}
Z[J] &= \frac{\int \mathcal{D}x(\tau)\ e^{-\frac{1}{2}\int d\tau\ x_\mu(\tau)\left[-\frac{1}{2}\frac{d^2}{d\tau^2}\ \delta_{\mu\nu}\right]x_\nu(\tau)+x_\alpha(\tau)J^\alpha(\tau)}}{\int \mathcal{D}x(\tau)\ e^{-S}} \\
&= \exp\left(\frac{1}{2}\int d\tau\ J_\mu(\tau_1)\left[-\frac{1}{2}\frac{d^2}{d\tau^2}\ \delta_{\mu\nu}\right]^{-1} J_\nu(\tau_2)\right),
\end{aligned}
$$

where we have used the formula

$$
\frac{\int \mathcal{D}x\ e^{-\frac{1}{2}\langle x,A\ x\rangle+\langle J,x\rangle}}{\int \mathcal{D}x\ e^{-\frac{1}{2}\langle x,A\ x\rangle}} = e^{\frac{1}{2}\langle J,A^{-1}J\rangle}.
$$

It is easy to derive the Green's function from the generating functional. We can do so by differentiating $Z[J]$ twice with respect to J:

$$
\begin{aligned}
\frac{\delta^2 Z[J]}{\delta J_\mu(\tau_1)\ \delta J_\nu(\tau_2)} &= \frac{\int \mathcal{D}x(\tau)\ x_\mu(\tau)x_\nu(\tau)\ e^{-S}}{\int \mathcal{D}x(\tau)\ e^{-S}} = \langle x_\mu(\tau_1)x_\nu(\tau_2)\rangle \\
&= \left[-\frac{1}{2}\frac{d^2}{d\tau^2}\ \delta_{\mu\nu}\right]^{-1}_{(\tau_1,\tau_2)} = -\delta_{\mu\nu}\ 2\left[\frac{d}{d\tau}\right]^{-2}_{(\tau_1,\tau_2)} \\
&= -\delta_{\mu\nu}\ \mathcal{G}(\tau_1,\tau_2).
\end{aligned}
$$

Let us now calculate the quantity $\mathcal{G}$,

$$
\mathcal{G}\to\mathcal{G}(\tau_1,\tau_2) = \langle\tau_1|\ \mathcal{G}\ |\tau_2\rangle.
$$

For this we consider again the eigenfunctions of the derivative operator on the circle with circumference T, $\{e^{2\pi i n\frac{\tau}{T}},\ n\in Z\backslash\{0\}\}$. Here we exclude the zero mode $n=0$ which simply corresponds to a global center-of-mass shift of the worldline. We can write

$$
\begin{aligned}
2\ \langle\tau_1|\left[\frac{d}{d\tau}\right]^{-2}|\tau_2\rangle &= 2\sum_n \langle\tau_1|n\rangle\left[\frac{d}{d\tau}\right]^{-2}\langle n|\tau_2\rangle \\
&= 2\sum_n \frac{1}{\sqrt{T}}\ e^{-2\pi i n\frac{\tau_1}{T}}\left[\frac{d}{d\tau}\right]^{-2}\frac{1}{\sqrt{T}}\ e^{2\pi i n\frac{\tau_2}{T}}.
\end{aligned}
$$

Because we have the eigenfunctions of the derivative operator, it is easy to calculate the eigenvalues of the inverse derivative operator, because we know that $\mathcal{A}\ |n\rangle = a_n\ |n\rangle \to$

$\mathcal{A}^{-1}|n\rangle = a_n{}^{-1}|n\rangle$. Therefore we apply the derivative operator to the eigenfunctions and invert the corresponding eigenvalues.

$$\left[\frac{d}{d\tau}\right]^2 \frac{1}{\sqrt{T}}\ e^{2\pi in\frac{\tau}{T}} = \left(\frac{2\pi in}{T}\right)^2 \frac{1}{\sqrt{T}}\ e^{2\pi in\frac{\tau}{T}}.$$

Using the above mentioned conclusion about diagonal operators, we get

$$\left[\frac{d}{d\tau}\right]^{-2} \frac{1}{\sqrt{T}}\ e^{2\pi in\frac{\tau}{T}} = \left(\frac{T}{2\pi in}\right)^2 \frac{1}{\sqrt{T}}\ e^{2\pi in\frac{\tau}{T}}.$$

Inserting this into the definition for the propagator we get

$$\begin{aligned}
\mathcal{G}(\tau_1,\tau_2) &= 2\sum_n \frac{1}{\sqrt{T}}\ e^{-2\pi in\frac{\tau_1}{T}} \left(\frac{T}{2\pi in}\right)^2 \frac{1}{\sqrt{T}}\ e^{2\pi in\frac{\tau_2}{T}} \\
&= 2T \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi in\frac{(\tau_2-\tau_1)}{T}}}{(2\pi in)^2} \\
&= |\tau_2-\tau_1| - \frac{(\tau_2-\tau_1)^2}{T}.
\end{aligned}$$

Here we have suppressed a constant term that does not contribute to physical amplitudes. The full worldline propagator reads

$$\begin{aligned}
\langle x_\mu(\tau_1)x_\nu(\tau_2)\rangle &= -\delta_{\mu\nu}\ \mathcal{G}(\tau_1,\tau_2) \\
&= -\delta_{\mu\nu}\ |\tau_2-\tau_1| + \delta_{\mu\nu}\ \frac{(\tau_2-\tau_1)^2}{T}.
\end{aligned}$$

This quantity describes the propagation of a particle in four-dimensional space on the worldline. Like the use of propagators in conventional quantum field theory, the worldline propagator pictures the quantum dynamics of the effective particle (which is as mentioned above just another formulation for our effective action, the quantum theory of the field under study). We also need derivatives of the Green's function with respect to τ. In the following, we list a number of derivates of the worldline Green's function $\mathcal{G}_{\mu\nu} = \langle x_\mu(\tau_1)x_\nu(\tau_2)\rangle$:

$$\langle \dot{x}_\mu(\tau_1)x_\nu(\tau_2)\rangle = \left[-\mathrm{sign}(\tau_1-\tau_2) + \frac{2}{T}\ (\tau_1-\tau_2)\right]\delta_{\mu\nu},$$

$$\langle \ddot{x}_\mu(\tau_1)x_\nu(\tau_2)\rangle = \left[-2\ \delta(\tau_1-\tau_2) + \frac{2}{T}\ \right]\delta_{\mu\nu},$$

$$\langle \dot{x}_\mu(\tau_1)\dot{x}_\nu(\tau_2)\rangle = \left[2\ \delta(\tau_1-\tau_2) - \frac{2}{T}\ \tau_2\right]\delta_{\mu\nu},$$

$$\langle \ddot{x}_\mu(\tau_1)\dot{x}_\nu(\tau_2)\rangle \;=\; 2\,\dot{\delta}(\tau_1-\tau_2)\,\delta_{\mu\nu}, \tag{13}$$

and finally

$$\langle \ddot{x}_\mu(\tau_1)\ddot{x}_\nu(\tau_2)\rangle \;=\; -2\,\ddot{\delta}(\tau_1-\tau_2)\,\delta_{\mu\nu}.$$

Because the worldline Green's function and their derivatives are symmetric under permutation of the indices $\langle x_\mu(\tau_1)x_\nu(\tau_2)\rangle = \langle x_\nu(\tau_1)x_\mu(\tau_2)\rangle$, other possible contractions can be deduced from this list. As we have observed in the last section it is important to get terms which involve a derivative of the delta function for "surviving" the ϵ limit. Upon close inspection of the above-calculated identities, we recognize that only the term (13) will give us such a mathematical structure. This observation will be important later. The higher order derivatives are not important, since induced selfcontractions will ensure that all these terms do vanish.

6.3 Complete loop derivative representation

We have already performed the calculation to first order in the loop derivative in subsection 6.1. The next challenge will be to prove that $\mathcal{P}\exp[-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}]\ \exp[ie\int d\tau\dot{x}A] = \mathcal{P}\exp[\frac{e}{2}\int d\tau\sigma F]\exp[ie\int d\tau\dot{x}A]$ is valid to all orders of the expansion. For the following calculations, the path ordering does not play a role. Path ordering takes care of the proper order of the Dirac matrices along the path. Since the following arguments do not rely on the properties of the Dirac algebra, we simply suppress the path ordering symbol and reinstate it, when it comes to the final and complete relations that we intend to prove. Let us start by expanding the derivative operator

$$\begin{aligned}
&\exp\left[-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}\right]\ \exp\left[ie\int d\tau\dot{x}A\right] \\
= &\left(1+\left(-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}\right)+\frac{1}{2}\left(-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}\right)^2+...\right)\exp\left[ie\int d\tau\dot{x}A\right].
\end{aligned}$$

We calculated the first term in the subsection 6.1 and got

$$-\frac{i}{2}\int d\tau\ \sigma_{\mu\nu}\ \frac{\delta\exp[K]}{\delta s_{\mu\nu}(\tau)} = \left(\frac{e}{2}\int d\tau\ \sigma_{\mu\nu}F^{\mu\nu}\right)\exp[K],$$

where we have chosen $K=\left[ie\int d\tau\dot{x}A\right]$. Let us now concentrate on the second-order term which is highly instructive to be studied in detail:

$$\begin{aligned}
&\frac{1}{2}\left(-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}\right)^2\exp[K] \\
= &\frac{1}{2}\left(-\frac{i}{2}\int d\tau''\sigma\frac{\delta}{\delta s(\tau'')}\right)\left(-\frac{i}{2}\int d\tau'\sigma\frac{\delta}{\delta s(\tau')}\right)\exp\left[ie\int d\tau\dot{x}A\right] \\
= &\frac{1}{2}\left(-\frac{i}{2}\int d\tau''\sigma\frac{\delta}{\delta s(\tau'')}\right)\left[\left(\frac{e}{2}\int d\tau'\ \sigma_{\mu\nu}F^{\mu\nu}(x(\tau'))\right)\exp[K]\right] \\
= &\frac{e}{4}\left(-\frac{i}{2}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}d\rho\rho\int d\tau' d\tau''\ \sigma_{\mu\nu}\sigma_{\kappa\lambda}\ \frac{\delta^2\left(F^{\mu\nu}(x(\tau'))\ \exp[K]\right)}{\delta x_\kappa(\tau''-\frac{\rho}{2})\ \delta x_\lambda(\tau''+\frac{\rho}{2})}\right).
\end{aligned}$$

We have to evaluate the action of the derivative operator on the product of the field-strength tensor and the Wilson loop. In this mathematical process we get additional terms to our desired result. In the progress of the calculation we will show that these counterterms do not at all contribute to our worldline path integral, leading to the expected result, namely our well known expression for the expectation value of the Wilson loop.

$$
\begin{aligned}
& \frac{1}{2}\left(-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}\right)^2 \exp[K] \\
= \; & \frac{e}{4}\left(-\frac{i}{2}\lim_{\epsilon\to 0}\int\limits_{-\epsilon}^{\epsilon} d\rho\rho \int d\tau' d\tau'' \; \sigma_{\mu\nu}\sigma_{\kappa\lambda}\;\frac{\delta}{\delta x_\kappa(\tau-\frac{\rho}{2})}\right) \\
& \times\left[\frac{\delta F^{\mu\nu}(x(\tau'))}{\delta x_\lambda(\tau+\frac{\rho}{2})}\exp[K] + F^{\mu\nu}(x(\tau'))\frac{\delta \exp[K]}{\delta x_\lambda(\tau+\frac{\rho}{2})}\right] \\
= \; & \frac{e}{4}\left(-\frac{i}{2}\lim_{\epsilon\to 0}\int\limits_{-\epsilon}^{\epsilon} d\rho\rho \int d\tau' d\tau'' \; \sigma_{\mu\nu}\sigma_{\kappa\lambda}\right) \\
& \times\Bigg(\frac{\delta\left[\frac{\partial F^{\mu\nu}(x(\tau'))}{\partial x_\lambda}\;\delta\left(\tau'-\left[\tau+\frac{\rho}{2}\right]\right)\right]}{\delta x_\kappa(\tau-\frac{\rho}{2})}\exp[K] + \frac{\partial F_{\mu\nu}(x(\tau'))}{\partial x_\lambda}\;\delta\left(\tau'-\left[\tau+\frac{\rho}{2}\right]\right) \\
& \times\frac{\delta\exp[K]}{\delta x_\kappa(\tau-\frac{\rho}{2})} + \frac{\delta F^{\mu\nu}(x(\tau'))}{\delta x_\kappa(\tau-\frac{\rho}{2})}\frac{\delta \exp[K]}{\delta x_\lambda(\tau+\frac{\rho}{2})} + F^{\mu\nu}(x(\tau'))\frac{\delta^2\exp[K]}{\delta x_\lambda(\tau+\frac{\rho}{2})\delta x_\kappa(\tau-\frac{\rho}{2})}\Bigg) \\
= \; & \frac{e}{4}\left(-\frac{i}{2}\lim_{\epsilon\to 0}\int\limits_{-\epsilon}^{\epsilon} d\rho\rho \int d\tau' d\tau'' \; \sigma_{\mu\nu}\sigma_{\kappa\lambda}\right) \\
& \Bigg(\left[\left(\frac{\delta}{\delta x_\kappa(\tau-\frac{\rho}{2})}\frac{\partial F^{\mu\nu}(x(\tau'))}{\partial x_\lambda}\right)\delta\left(\tau'-\left[\tau+\frac{\rho}{2}\right]\right)\right]\exp[K] \\
& + \left[\frac{\partial F^{\mu\nu}(x(\tau'))}{\partial x_\lambda}\;\delta\left(\tau'-\left[\tau+\frac{\rho}{2}\right]\right)\right]\frac{\delta\exp[K]}{\delta x_\kappa(\tau-\frac{\rho}{2})} \\
& + \left[\frac{\partial F^{\mu\nu}(x(\tau'))}{\partial x_\kappa}\;\delta\left(\tau'-\left[\tau-\frac{\rho}{2}\right]\right)\right]\frac{\delta\exp[K]}{\delta x_\lambda(\tau+\frac{\rho}{2})} \\
& + F^{\mu\nu}(x(\tau'))\;\frac{\delta^2\exp[K]}{\delta x_\lambda(\tau+\frac{\rho}{2})\delta x_\kappa(\tau-\frac{\rho}{2})}\Bigg).
\end{aligned}
$$

We can see that the last term gives us the desired result for the second order in the expansion of the exponential function. It remains to be shown that the additional terms do not contribute to the worldline path integral. First let us have a more precisely look at the first term of the sum. We want to determine the limit of the integral

$$
\begin{aligned}
& \lim_{\epsilon\to 0}\int\limits_{-\epsilon}^{\epsilon} d\rho\rho \int d\tau' d\tau'' \;\left(\frac{\delta}{\delta x_\kappa(\tau-\frac{\rho}{2})}\frac{\partial F^{\mu\nu}(x(\tau'))}{\partial x_\lambda}\right)\delta\left(\tau'-\left[\tau+\frac{\rho}{2}\right]\right)\exp[K] \\
= & \lim_{\epsilon\to 0}\int\limits_{-\epsilon}^{\epsilon} d\rho\rho \int d\tau' d\tau'' \;\left[\partial_\kappa\partial_\lambda F^{\mu\nu}(\tau')\delta\left(\tau'-\left[\tau-\frac{\rho}{2}\right]\right)\right]\delta\left(\tau'-\left[\tau+\frac{\rho}{2}\right]\right)\exp[K],
\end{aligned}
$$

where we have supressed the σ matrices and the constant coefficients. Performing the τ'-integral we get

$$\int d\tau \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} d\rho\rho\, \partial_\kappa\partial_\lambda F^{\mu\nu}(\tau+\frac{\rho}{2})\, \exp[K]\, \delta\left(\left[\tau+\frac{\rho}{2}\right]-\left[\tau-\frac{\rho}{2}\right]\right)$$

$$= \int d\tau\, \partial_\kappa\partial_\lambda F^{\mu\nu}\, \exp[K] \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\, \delta(\rho) = 0.$$

The first problematic term is defused. But there still remain the other unwanted terms. We make use of the singularity structure of derivatives of the worldline propagators. We showed that we only get a $\dot{\delta}$ structure, if we contract $\dot{x}$ and $\ddot{x}$ with corresponding arguments. Let us show next that these contractions are absent. Therefore the other two "disturbing" terms also do not contribute to our worldline expression. For this reason we expand the derivative of the Faraday tensor into a Taylor series,

$$\begin{aligned} & \left[\frac{\partial F^{\mu\nu}(x(\tau'))}{\partial x_\lambda}\,\delta\left(\tau'-\left[\tau+\frac{\rho}{2}\right]\right)\right]\frac{\delta\exp[K]}{\delta x_\kappa(\tau-\frac{\rho}{2})} \\ = \quad & \left[\left(\partial_\lambda F^{\mu\nu}(0)+\partial_\lambda\partial_\gamma F^{\mu\nu}(0)x^\gamma(\tau')+\mathcal{O}(x^2)\right)\,\delta\left(\tau'-\left[\tau+\frac{\rho}{2}\right]\right)\right] \\ & \times\left[(ie)\,\exp[K]\,\dot{x_\mu}(\tau-\frac{\rho}{2})\,F^{\mu\kappa}(\tau-\frac{\rho}{2})\right], \end{aligned}$$

where we have used that the functional derivative in terms of x_κ of the Wilson loop obeys the relation in the second bracket. We multiply the both brackets and get only terms of the type $x^n\dot{x}$, with $n\in N$. If we consider this fact in connection with the path integral which needs to perfomed, we see that there are no contractions that can give us a nonzero ϵ limit. This is because we never get contractions of the type $\langle\ddot{x}\dot{x}\rangle$ which would lead to a $\dot{\delta}$ function with nonzero ϵ limit. Because of this argument the above expression and the analogous one vanish. Incidentally, the same argument provides for the final justification of (12). We conclude that the only term with a chance to contribute to our worldline path integral is

$$\frac{e}{4}\left(\int d\tau' d\tau''\,\sigma_{\mu\nu}\sigma_{\kappa\lambda}\,F^{\mu\nu}(x(\tau'))\,\left(-\frac{i}{2}\right)\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\,\frac{\delta^2\exp[K]}{\delta x_\lambda(\tau+\frac{\rho}{2})\delta x_\kappa(\tau-\frac{\rho}{2})}\right).$$

The second term is of the same form as the expression that we have calculated earlier. Therefore we get

$$\frac{e^2}{8}\int d\tau' d\tau''\,\sigma_{\mu\nu}\sigma_{\kappa\lambda}\,F^{\mu\nu}(x(\tau'))\,F^{\kappa\lambda}(x(\tau''))\,\exp[K] = \frac{1}{2}\left(\frac{e}{2}\int d\tau\,\sigma F\right)^2\exp[K].$$

This is exactly the second-order term in the expansion of the spin-field term. For all higher orders we can use the identical argument as developed in the above calulation. Only terms of the last type appear in the calculations, all other terms do not contribute to the worldline path integral. Reinstating the path ordering we have therefore shown that

$$\begin{aligned} \langle W_{\text{spin}}[A]\rangle &= \int \mathcal{D}x(\tau)\; e^{ie\oint dxA}\; \text{tr}_\gamma\; \mathcal{P}e^{\frac{e}{2}\int d\tau\sigma F}\; e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} \\ &= \int \mathcal{D}x(\tau) \left[\text{tr}_\gamma\; \mathcal{P}e^{-\frac{i}{2}\int d\tau\sigma \frac{\delta}{\delta s(\tau)}}\; e^{ie\oint dxA}\right]\; e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}}, \end{aligned}$$

is indeed an alternative representation of the familiar worldline representation as was argued in [27],[28]. In principle, it is not surprising that the action of the loop derivative on the holonomy gives an expression in terms of the Faraday tensor, with regard to our comments on the representation theory of the loop algebra. But we had to verify this statement by this straightforward calculation, because we work in coordinate space, which is the basis for expressing the loop derivative. On the other hand we will exploit this result frequently in the progress of this calculation, which is also done in coordinate space. Moreover, we have learnt a great deal about the singularity structure of the loop derivative which is of significant importance for the remainder of this work.

6.4 Spin factor geometry

Let us now integrate the new worldline expression by parts in order to obtain a geometric representation of the spin coupling. We have to do the integration twice, because $\frac{\delta}{\delta s(\tau)}$ is an operator where two derivatives in x appear. Then we get

$$\begin{aligned} \langle W_{\text{spin}}[A]\rangle &= \int \mathcal{D}x(\tau) \left[\text{tr}_\gamma\; \mathcal{P}e^{-\frac{i}{2}\int d\tau\sigma \frac{\delta}{\delta s(\tau)}}\; e^{ie\oint dxA}\right]\; e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} \\ &= \int \mathcal{D}x(\tau) \left[\text{tr}_\gamma\; \mathcal{P}e^{-\frac{i}{2}\int d\tau\sigma \frac{\delta}{\delta s(\tau)}}\; e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}}\right]\; e^{ie\oint dxA}. \end{aligned} \tag{14}$$

We evaluate the action of the loop derivative on the kinetic term. Let us confine ourselves again first to the leading order of the exponential series. Therefore we test the action of $\frac{\delta}{\delta s(\tau)}$ on the kinetic term,

$$\left(-\frac{i}{2}\int d\tau\sigma \frac{\delta}{\delta s(\tau)}\right)\; \exp\left(-\int d\tau \frac{\dot{x}^2(\tau)}{4}\right) = -\frac{i}{2}\int d\tau\sigma \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\; \frac{\delta^2 \exp[I]}{\delta x_\mu(\tau+\frac{\rho}{2})\delta x_\nu(\tau-\frac{\rho}{2})},$$

where we have abbreviated $I = -\int d\tau \frac{\dot{x}}{4}$. We have to perform an anlagous calculation as in the case of the loop derivative of the Wilson loop,

$$\begin{aligned}
\frac{\delta \exp[I]}{\delta x_\nu(\tau - \frac{\rho}{2})} &= \exp[I] \left(-\frac{1}{4}\right) \int d\tau' \frac{\delta \left\{\dot{x}_\mu(\tau')\dot{x}^\mu(\tau')\right\}}{\delta x_\nu(\tau - \frac{\rho}{2})} \\
&= \exp[I] \left(-\frac{1}{2}\right) \int d\tau' \; \dot{x}_\nu(\tau') \; \dot{\delta}\left(\tau' - \left[\tau - \frac{\rho}{2}\right]\right) \\
&= \exp[I] \left(\frac{1}{2}\right) \int d\tau' \; \ddot{x}_\nu(\tau') \; \delta\left(\tau' - \left[\tau - \frac{\rho}{2}\right]\right) \\
&= \frac{1}{2} \exp[I] \; \ddot{x}_\nu(\tau - \frac{\rho}{2}),
\end{aligned}$$

where we have integrated by parts in the third line and used the periodic boundary conditions. The second derivative to this term gives us

$$\begin{aligned}
\frac{\delta \exp[I]}{\delta x_\mu(\tau + \frac{\rho}{2})\delta x_\nu(\tau - \frac{\rho}{2})} &= \frac{\delta}{\delta x_\mu(\tau + \frac{\rho}{2})} \left(\frac{1}{2} \exp[I] \; \ddot{x}_\nu(\tau - \frac{\rho}{2})\right) \\
&= \frac{1}{2}\left(\frac{\delta \exp[I]}{\delta x_\mu(\tau + \frac{\rho}{2})}\right) \ddot{x}_\nu(\tau - \frac{\rho}{2}) + \frac{1}{2} \exp[I] \; \left(\frac{\delta \ddot{x}_\nu(\tau - \frac{\rho}{2})}{\delta x_\mu(\tau + \frac{\rho}{2})}\right) \\
&\Rightarrow \frac{1}{4} \; \ddot{x}_\mu(\tau + \frac{\rho}{2})\ddot{x}_\nu(\tau - \frac{\rho}{2}) \; \exp[I].
\end{aligned}$$

We do not need to take the second term into account, because the contraction over Lorentz indices ensures that this term vanishes $\sigma^{\mu\nu}\delta_{\mu\nu} \; \ddot{\delta}(\rho) = 0$. This follows from the antisymmetry of the σ matrix (this is a trivial symmetry property of σ, beeing unrelated to the full Dirac-algebra structure). At this point, it should be emphasized that the ϵ limit has to be taken at the very end of a calculation as is apparent from its construction, otherwise a term $\sim \ddot{\delta}$ would be ill-defined. We conclude that

$$\left(-\frac{i}{2}\int d\tau \sigma \frac{\delta}{\delta s(\tau)}\right) \; \exp[I] = \left(-\frac{i}{2} \; \int d\tau \; \sigma_{\mu\nu} \; \omega_{\mu\nu}(\tau)\right) \exp[I],$$

where we have defined

$$\omega_{\mu\nu}(\tau) = \frac{1}{4}\lim_{\epsilon\to 0} \int\limits_{-\epsilon}^{\epsilon} d\rho\rho \; \ddot{x}_\mu(\tau + \frac{\rho}{2})\ddot{x}_\nu(\tau - \frac{\rho}{2}).$$

It is this ω term that carries the information that was previously encoded in the field strength tensor. At this point, it is important to stress that our $\omega_{\mu\nu}$ is significantly different from Polyakov's spin factor $\omega_{\mathrm{P}\mu\nu} \sim (\ddot{x}_\mu \dot{x}_\nu - \ddot{x}_\nu \dot{x}_\mu)$, arising in the first-order formalism. For instance, our $\omega_{\mu\nu} = 0$ for any smooth loop, whereas $\omega_{\mathrm{P}\mu\nu}$ is generally nonzero then. The question is now, whether equation (14) has a representation in terms of $\omega_{\mu\nu}$ to all orders in the expansion ? For answering this, let us go to the next order in the Taylor expansion.

Recapitulating, we have to scrutinize if the following relation can be valid

$$\begin{aligned}
&\left(1+\left(-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}\right)+\frac{1}{2}\left(-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}\right)^2+...\right)\exp[I] \\
\stackrel{?}{=}\quad &\left(1+\left(-\frac{i}{2}\int d\tau\;\sigma\;\omega(\tau)\right)+\frac{1}{2}\left(-\frac{i}{2}\int d\tau\;\sigma\;\omega(\tau)\right)^2+...\right)\exp[I].
\end{aligned}$$

As we have shown above this is indeed valid in linear order. Let us now concentrate on the second order term,

$$\begin{aligned}
\left(-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}\right)^2\exp[I] &= \left(-\frac{i}{2}\int d\tau_2\;\sigma\frac{\delta}{\delta s(\tau_2)}\right)\left(-\frac{i}{2}\int d\tau_1\;\sigma\frac{\delta}{\delta s(\tau_1)}\right)\exp[I] \\
&= \left(-\frac{i}{2}\int d\tau_2\;\sigma\frac{\delta}{\delta s(\tau_2)}\right)\left(-\frac{i}{2}\int d\tau_1\;\sigma\;\omega(\tau_1)\right)\exp[I].
\end{aligned}$$

In coordinates this expression transforms into

$$-\frac{1}{4}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}d\eta\;\eta\int d\tau_2 d\tau_1\;\sigma_{\lambda\kappa}\;\sigma_{\mu\nu}\frac{\delta^2(\omega_{\mu\nu}(\tau_1)\exp[I])}{\delta x_\lambda(\tau_2+\frac{\eta}{2})\delta x_\kappa(\tau_2-\frac{\eta}{2})}. \tag{15}$$

Let us analyze the action of the derivatives,

$$\frac{\delta^2(\omega_{\mu\nu}(\tau_1)\exp[I])}{\delta x_\lambda(\tau_2+\frac{\eta}{2})\delta x_\kappa(\tau_2-\frac{\eta}{2})} = \frac{\delta}{\delta x_\lambda(\tau_2+\frac{\eta}{2})}\left[\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\kappa(\tau_2-\frac{\eta}{2})}\exp[I]+\omega_{\mu\nu}(\tau_1)\frac{\delta\exp[I]}{\delta x_\kappa(\tau_2-\frac{\eta}{2})}\right].$$

When we apply the second derivative operator we get

$$\begin{aligned}
&\left[\frac{\delta^2\omega_{\mu\nu}(\tau_1)}{\delta x_\lambda(\tau_2+\frac{\eta}{2})\delta x_\kappa(\tau_2-\frac{\eta}{2})}\exp[I]+\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\kappa(\tau_2-\frac{\eta}{2})}\frac{\delta\exp[I]}{\delta x_\lambda(\tau_2+\frac{\eta}{2})}\right. \\
&\left.+\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\lambda(\tau_2+\frac{\eta}{2})}\frac{\delta\exp[I]}{\delta x_\kappa(\tau_2-\frac{\eta}{2})}+\omega_{\mu\nu}(\tau_1)\frac{\delta^2\exp[I]}{\delta x_\kappa(\tau_2-\frac{\eta}{2})\delta x_\lambda(\tau_2+\frac{\eta}{2})}\right].
\end{aligned}$$

If we reinsert this expression into (15) we obtain

$$\begin{aligned}
&-\frac{1}{4}\int d\tau_2 d\tau_1\sigma_{\lambda\kappa}\sigma_{\mu\nu}\left[\left(\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta s_{\lambda\kappa}(\tau_2)}+\omega_{\mu\nu}(\tau_1)\omega_{\lambda\kappa}(\tau_2)\right)\exp[I]\right. \\
&\left.+\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}d\eta\eta\left(\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\kappa(\tau_2-\frac{\eta}{2})}\frac{\delta\exp[I]}{\delta x_\lambda(\tau_2+\frac{\eta}{2})}\right)\left(\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\lambda(\tau_2+\frac{\eta}{2})}\frac{\delta\exp[I]}{\delta x_\kappa(\tau_2-\frac{\eta}{2})}\right)\right].
\end{aligned} \tag{16}$$

We can see that there indeed exists the term we would like to have (the second term in the first line), but there are also further terms. Therefore to all orders we get always the term needed for building the exponential series but we also get many other derivative terms. For the time being we can write the resulting expression in the following way,

$$\begin{aligned}
&\int \mathcal{D}x(\tau) \left[\mathrm{tr}_\gamma\ \mathcal{P}e^{-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}}\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}}\right]\ e^{ie\oint dxA} \\
= \quad &\int \mathcal{D}x(\tau) \left[\mathrm{tr}_\gamma\ \mathcal{P}\left(e^{-\frac{i}{2}\int d\tau\sigma\omega} + \text{derivative terms}\right)\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}}\right]\ e^{ie\oint dxA}.
\end{aligned}$$

It is important to stress that it is absolutely necessary that these derivative terms appear. As we will see these additional terms ensure that the limit $F \to 0$ is actually meaningful. Let us dedicate a short subsection why this special limit (and therefore also the obtained derivative terms) is so important:

6.5 The $F \to 0$ limit of the spin factor representation

Here we want to investigate how the spin factor representation behaves, if we put the field strength to zero. Because we have shown that the "old" worldline path integral and our new expression are equivalent, we expect that there is no distinction in the behaviour of the two expressions if we set F to zero. We will represent this important detail in two parts. The first one is a physically motivated approach that demands that the loop derivative operating on the "free" kinetic term does not contribute to the path integral. The second statement is a pure mathematical one about the behaviour of total derivatives. We have shown above that the following relation is valid

$$\begin{aligned}
&\mathcal{N}\int \mathcal{D}x(\tau)\ e^{ie\oint dxA}\ \mathrm{tr}_\gamma\ \mathcal{P}e^{\frac{e}{2}\int d\tau\sigma F}\ e^{-\int d\tau\frac{\dot{x}^2(\tau)}{4}} \\
= \quad &\mathcal{N}\int \mathcal{D}x(\tau)\left[\mathrm{tr}_\gamma\ \mathcal{P}e^{-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}}\ e^{ie\oint dxA}\right]\ e^{-\int d\tau\frac{\dot{x}^2(\tau)}{4}},
\end{aligned}$$

where we have defined $\mathcal{N}^{-1} = \int \mathcal{D}x(\tau)\ e^{-\int d\tau\frac{\dot{x}^2(\tau)}{4}}$. If we instead send F in the first line to zero we get,

$$\lim_{F\to 0}\ \frac{\int \mathcal{D}x(\tau)\ e^{ie\oint dxA}\ \mathrm{tr}_\gamma\ \mathcal{P}e^{\frac{e}{2}\int d\tau\sigma F}\ e^{-\int d\tau\frac{\dot{x}^2(\tau)}{4}}}{\int \mathcal{D}x(\tau)\ e^{-\int d\tau\frac{\dot{x}^2(\tau)}{4}}} = 1\ .$$

If we now consider the expression with the loop derivative we conclude that all powers of $\frac{\delta}{\delta s(\tau)}$ will vanish in this limit

$$\begin{aligned}
\lim_{F\to 0}\quad &\mathcal{N}\int \mathcal{D}x(\tau)\left[\mathrm{tr}_\gamma\ e^{-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}}\ e^{ie\oint dxA}\right]\ e^{-\int d\tau\frac{\dot{x}^2(\tau)}{4}} \\
= \quad &\mathcal{N}\int \mathcal{D}x(\tau)\ \mathrm{tr}_\gamma\left[1 + \left(-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}\right) + \frac{1}{2}\left(-\frac{i}{2}\int d\tau\sigma\frac{\delta}{\delta s(\tau)}\right)^2 + ...\right] e^{-\int d\tau\frac{\dot{x}^2(\tau)}{4}} \\
\stackrel{!}{=} \quad &1\ ,
\end{aligned}$$

where we have used the association between the Faraday tensor and the connection one form. Additionally we have expanded the exponential function. In this representation it is easy to see that all terms with $\frac{\delta}{\delta s(\tau)}$ have to vanish if both worldline expressions are equivalent. That this is indeed the case can proven by the following argument. Let us consider the total derivative of a Gaußian path integral

$$\int \mathcal{D}x(\tau) \left(\frac{\delta}{\delta x_\mu(\tau)}\right) \exp[-S] = \exp[-S]\Big|_{-\infty}^{+\infty} - \int \mathcal{D}x(\tau)\ 0\ \cdot \exp[-S] = 0\ ,$$

where we have integrated by parts and assumed that the exponential function vanishes at infinity, which is the case for $S = \int d\tau \frac{\dot{x}^2}{4}$. This result can be generalized to any order in the derivative, so we are lead to

$$\int \mathcal{D}x(\tau) \left(\frac{\delta}{\delta s(\tau)}\right)^n \exp[-S] = 0\ , \forall n \in N.$$

This ensures that the limit $F \to 0$ for both cases agrees. The question now is what happens with this limit, if we work with the worldine integral which we obtained after integration by parts of expression (14)

$$\begin{aligned} &\mathcal{N} \int \mathcal{D}x(\tau) \left[\mathrm{tr}_\gamma\ \mathcal{P} e^{-\frac{i}{2}\int d\tau \sigma \frac{\delta}{\delta s(\tau)}}\ e^{ie\oint dxA}\right]\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} \\ = \quad &\mathcal{N} \int \mathcal{D}x(\tau) \left[\mathrm{tr}_\gamma\ \mathcal{P} e^{-\frac{i}{2}\int d\tau \sigma \frac{\delta}{\delta s(\tau)}}\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}}\right]\ e^{ie\oint dxA}. \end{aligned}$$

We have shown in (16) that the following relation is valid in terms of the Taylor expansion. The expanded quantity is in each case the first factor in the square brackets,

$$\begin{aligned} &\mathcal{N} \int \mathcal{D}x(\tau) \left[\mathrm{tr}_\gamma\ \mathcal{P} e^{-\frac{i}{2}\int d\tau \sigma \frac{\delta}{\delta s(\tau)}}\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}}\right]\ e^{ie\oint dxA} \\ = \quad &\mathcal{N} \int \mathcal{D}x(\tau) \left[\mathrm{tr}_\gamma\ \mathcal{P} \left(e^{-\frac{i}{2}\int d\tau \sigma\omega} + \text{derivative terms}\right)\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}}\right]\ e^{ie\oint dxA}\ . \end{aligned}$$

And in the above mentioned $F \to 0$ limit this expression transforms to

$$\mathcal{N} \int \mathcal{D}x(\tau)\ \mathrm{tr}_\gamma \mathcal{P} \left(e^{-\frac{i}{2}\int d\tau \sigma\omega} + \text{derivative terms}\right)\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} \tag{17}$$

At a first glance this expression does not seem to be equal to one, but it has to be, as the $F \to 0$ limit proves. This tells us that the derivative terms must exactly correspond to the self contractions of the spinor factor,

$$\begin{aligned} 1 \quad &= \frac{\int \mathcal{D}x(\tau)\ \mathrm{tr}_\gamma \mathcal{P} \left(e^{-\frac{i}{2}\int d\tau \sigma\omega} + \text{derivative terms}\right)\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}}}{\int \mathcal{D}x(\tau)\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}}} \\ &\equiv \left\langle \mathrm{tr}_\gamma \mathcal{P} \left(e^{-\frac{i}{2}\int d\tau \sigma\omega} - \text{self contractions}\right)\right\rangle . \end{aligned} \tag{18}$$

we can rephrase this fact in terms of an operator language: in order to satisfy (18), the spin factor must be normal ordered:

$$
\begin{aligned}
&\mathcal{N} \int \mathcal{D}x(\tau)\ \mathrm{tr}_\gamma \mathcal{P} \left(e^{-\frac{i}{2}\int d\tau \sigma\omega} + \text{derivative terms} \right)\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} \\
= \quad &\mathcal{N} \int \mathcal{D}x(\tau)\ \mathrm{tr}_\gamma\ \mathcal{P} :e^{-\frac{i}{2}\int d\tau \sigma\omega}:\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} = 1,
\end{aligned}
$$

where $\mathcal{N}^{-1} = \int \mathcal{D}x(\tau)\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}}$. This justifies our claim that the derivative terms ensure the validity of the $F \to 0$ limit. As a next step, we will confirm the identification of the additional derivative terms with the self-contractions explicitly to lowest nontrivial order. Though this identification is proven to all orders by the $F \to 0$ limit, the calculation is instructive and reassuring. To summarize, it is the set of self-contractions that takes care of the fact that all terms which are more singular then $\dot{\delta}$ do indeed cancel, such that all ϵ limits of the loop derivative remain well defined and finite.

6.6 Explicit example: "self-contractions at work"

As we have seen in 6.5 the consistency of our approach can be checked with the limit $F \to 0$ of the worldline integral. With our formal definition for the normalordering procedure we have ensured that this limit is manifestly controlled. Now we want to illustrate explicitly how this is realized by the additional derivative terms that we have recognized as the self-contractions. Let us recall this term to second order in the Taylor expansion (16),

$$-\frac{1}{4}\,\mathcal{P}\int d\tau_2 d\tau_1 \sigma_{\lambda\kappa}\sigma_{\mu\nu}\left(\left(\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta s_{\lambda\kappa}(\tau_2)}+\omega_{\mu\nu}(\tau_1)\omega_{\lambda\kappa}(\tau_2)\right)\exp[I]\right. \tag{19}$$
$$\left.+\lim_{\epsilon_2\to 0}\int\limits_{-\epsilon_2}^{\epsilon_2} d\eta\eta\left[\left(\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\kappa(\tau_2-\frac{\eta}{2})}\frac{\delta\exp[I]}{\delta x_\lambda(\tau_2+\frac{\eta}{2})}\right)+\left(\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\lambda(\tau_2+\frac{\eta}{2})}\frac{\delta\exp[I]}{\delta x_\kappa(\tau_2-\frac{\eta}{2})}\right)\right]\right).$$

We see that this expression can be written as

$$\left[\left(-\frac{i}{2}\int d\tau\sigma\omega(\tau)\right)^2+\text{derivative terms}\right]\exp[I].$$

Next we want to investigate what happens if we perform the path integration of this expression in the limit $F \to 0$. We have observed from this limit that the derivative terms must correspond to the self-contractions of the spin factor. In the following, we intend to confirm this by explicit computation to this order. For this we have to calculate the functional derivatives of the derivative terms

$$\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta s_{\lambda\kappa}(\tau_2)}=\lim_{\epsilon_2\to 0}\int\limits_{-\epsilon_2}^{\epsilon_2} d\eta\eta\;\frac{\delta^2}{\delta x_\kappa(\tau_2-\frac{\eta}{2})\delta x_\lambda(\tau_2+\frac{\eta}{2})}\left(\frac{1}{4}\lim_{\epsilon_1\to 0}\int\limits_{-\epsilon_1}^{\epsilon_1} d\rho\rho\;\ddot{x}_\mu(\tau_1+\frac{\rho}{2})\ddot{x}_\nu(\tau_1-\frac{\rho}{2})\right)$$
$$=\frac{1}{4}\lim_{\epsilon_1,\epsilon_2\to 0}\int\limits_{-\epsilon_1}^{\epsilon_1}\int\limits_{-\epsilon_2}^{\epsilon_2} d\eta d\rho\;\rho\eta\left(\delta_{\mu\lambda}\;\ddot{\delta}\left[\tau_1+\frac{\rho}{2}-(\tau_2+\frac{\eta}{2})\right]\delta_{\nu\kappa}\;\ddot{\delta}\left[\tau_1-\frac{\rho}{2}-(\tau_2-\frac{\eta}{2})\right]\right.$$
$$\left.+\;\delta_{\mu\kappa}\;\ddot{\delta}\left[\tau_1+\frac{\rho}{2}-(\tau_2-\frac{\eta}{2})\right]\delta_{\nu\lambda}\;\ddot{\delta}\left[\tau_1-\frac{\rho}{2}-(\tau_2+\frac{\eta}{2})\right]\right).$$

Let us now turn to the other terms

$$\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\kappa(\tau_2-\frac{\eta}{2})}=\frac{1}{4}\lim_{\epsilon\to 0}\int\limits_{-\epsilon}^{\epsilon} d\rho\rho\left(\delta_{\mu\kappa}\;\ddot{\delta}\left[\tau_1+\frac{\rho}{2}-(\tau_2-\frac{\eta}{2})\right]\ddot{x}_\nu(\tau_1-\frac{\rho}{2})\right.$$
$$\left.+\delta_{\nu\kappa}\;\ddot{\delta}\left[\tau_1-\frac{\rho}{2}-(\tau_2-\frac{\eta}{2})\right]\ddot{x}_\mu(\tau_1+\frac{\rho}{2})\right),$$

and

$$\frac{\delta\omega_{\mu\nu}(\tau_1)}{\delta x_\lambda(\tau_2+\frac{\eta}{2})} = \frac{1}{4}\lim_{\epsilon\to 0}\int\limits_{-\epsilon}^{\epsilon} d\rho\rho\,\left(\delta_{\mu\lambda}\;\ddot{\delta}\left[\tau_1+\frac{\rho}{2}-(\tau_2+\frac{\eta}{2})\right]\ddot{x}_\nu(\tau_1-\frac{\rho}{2})\right.$$
$$\left. +\delta_{\nu\lambda}\;\ddot{\delta}\left[\tau_1-\frac{\rho}{2}-(\tau_2+\frac{\eta}{2})\right]\ddot{x}_\mu(\tau_1+\frac{\rho}{2})\right),$$

as well as

$$\frac{\delta\exp[I]}{\delta x_\kappa(\tau_2-\frac{\eta}{2})} = \frac{1}{2}\exp[I]\;\ddot{x}_\kappa(\tau_2-\frac{\eta}{2}).$$

In a totally analogous way we obtain

$$\frac{\delta\exp[I]}{\delta x_\lambda(\tau_2+\frac{\eta}{2})} = \frac{1}{2}\exp[I]\;\ddot{x}_\lambda(\tau_2+\frac{\eta}{2}).$$

We can now insert all these quantities into the expression (19) and perform the path integration where we also use the definition of $\omega_{\mu\nu}(\tau)$ and obtain

$$\lim_{\epsilon_1,\epsilon_2\to 0}\int\limits_{-\epsilon_1}^{\epsilon_1}\int\limits_{-\epsilon_2}^{\epsilon_2} d\rho d\eta\;\rho\eta\int\mathcal{D}x(\tau)$$
$$\times\left(\frac{1}{4}\;\ddot{x}_\mu(\tau_1+\frac{\rho}{2})\;\ddot{x}_\nu(\tau_1-\frac{\rho}{2})\;\ddot{x}_\lambda(\tau_2+\frac{\eta}{2})\;\ddot{x}_\kappa(\tau_2-\frac{\eta}{2})\right.$$
$$+\;\delta_{\mu\lambda}\;\delta_{\nu\kappa}\;\ddot{\delta}\left[\tau_1+\frac{\rho}{2}-(\tau_2+\frac{\eta}{2})\right]\ddot{\delta}\left[\tau_1-\frac{\rho}{2}-(\tau_2-\frac{\eta}{2})\right]$$
$$+\;\delta_{\mu\kappa}\;\delta_{\nu\lambda}\;\ddot{\delta}\left[\tau_1+\frac{\rho}{2}-(\tau_2-\frac{\eta}{2})\right]\ddot{\delta}\left[\tau_1-\frac{\rho}{2}-(\tau_2+\frac{\eta}{2})\right]$$
$$+\;\frac{1}{2}\left(\delta_{\mu\kappa}\;\ddot{\delta}\left[\tau_1+\frac{\rho}{2}-\left(\tau_2-\frac{\eta}{2}\right)\right]\ddot{x}_\nu(\tau_1-\frac{\rho}{2})\ddot{x}_\lambda(\tau_2+\frac{\eta}{2})\right.$$
$$\left.+\;\delta_{\nu\kappa}\;\ddot{\delta}\left[\tau_1-\frac{\rho}{2}-\left(\tau_2-\frac{\eta}{2}\right)\right]\ddot{x}_\mu(\tau_1+\frac{\rho}{2})\ddot{x}_\lambda(\tau_2+\frac{\eta}{2})\right)$$
$$+\;\frac{1}{2}\left(\delta_{\mu\lambda}\;\ddot{\delta}\left[\tau_1+\frac{\rho}{2}-\left(\tau_2+\frac{\eta}{2}\right)\right]\ddot{x}_\nu(\tau_1-\frac{\rho}{2})\ddot{x}_\kappa(\tau_2-\frac{\eta}{2})\right.$$
$$\left.\left.+\;\delta_{\nu\lambda}\;\ddot{\delta}\left[\tau_1-\frac{\rho}{2}-\left(\tau_2+\frac{\eta}{2}\right)\right]\ddot{x}_\mu(\tau_1+\frac{\rho}{2})\ddot{x}_\kappa(\tau_2-\frac{\eta}{2})\right)\right)\exp[I]. \qquad (20)$$

Now let us perform the path integration term by term. After we have calculated all integrals we reconnect the different parts. At first let us have a look at the expression

$$\begin{aligned}
&\int \mathcal{D}x(\tau)\ \frac{1}{4}\ \ddot{x}_\mu(\tau_1+\frac{\rho}{2})\ \ddot{x}_\nu(\tau_1-\frac{\rho}{2})\ \ddot{x}_\lambda(\tau_2+\frac{\eta}{2})\ \ddot{x}_\kappa(\tau_2-\frac{\eta}{2})\ \exp[I] \\
&= \frac{1}{4}\ \big\langle \ddot{x}_\mu(\tau_1+\frac{\rho}{2})\ \ddot{x}_\nu(\tau_1-\frac{\rho}{2})\ \ddot{x}_\lambda(\tau_2+\frac{\eta}{2})\ \ddot{x}_\kappa(\tau_2-\frac{\eta}{2})\big\rangle \\
&= \frac{1}{4}\ \Big[\big\langle \ddot{x}_\mu(\tau_1+\frac{\rho}{2})\ \ddot{x}_\nu(\tau_1-\frac{\rho}{2})\big\rangle \cdot \big\langle \ddot{x}_\lambda(\tau_2+\frac{\eta}{2})\ \ddot{x}_\kappa(\tau_2-\frac{\eta}{2})\big\rangle \\
&\quad + \big\langle \ddot{x}_\nu(\tau_1-\frac{\rho}{2})\ \ddot{x}_\lambda(\tau_2+\frac{\eta}{2})\big\rangle \cdot \big\langle \ddot{x}_\mu(\tau_1+\frac{\rho}{2})\ \ddot{x}_\kappa(\tau_2-\frac{\eta}{2})\big\rangle \\
&\quad + \big\langle \ddot{x}_\mu(\tau_1+\frac{\rho}{2})\ \ddot{x}_\lambda(\tau_2+\frac{\eta}{2})\big\rangle \cdot \big\langle \ddot{x}_\nu(\tau_1-\frac{\rho}{2})\ \ddot{x}_\kappa(\tau_2-\frac{\eta}{2})\big\rangle\Big],
\end{aligned}$$

where we have used Wick's Theorem. Employing $\langle \ddot{x}_\alpha(\tau_1)\ddot{x}_\beta(\tau_2)\rangle = -2\delta_{\alpha\beta}\ \ddot{\delta}(\tau_1-\tau_2)$ from section 6.2 the above expression transforms to

$$\begin{aligned}
&\Big(\delta_{\mu\nu}\ \delta_{\lambda\kappa}\ \ddot{\delta}\Big[\tau_1+\frac{\rho}{2}-\Big(\tau_1-\frac{\rho}{2}\Big)\Big]\ \ddot{\delta}\Big[\tau_2+\frac{\eta}{2}-\Big(\tau_2-\frac{\eta}{2}\Big)\Big] \\
&+\delta_{\nu\lambda}\ \delta_{\mu\kappa}\ \ddot{\delta}\Big[\tau_1-\frac{\rho}{2}-\Big(\tau_2-\frac{\eta}{2}\Big)\Big]\ \ddot{\delta}\Big[\tau_1+\frac{\rho}{2}-\Big(\tau_2-\frac{\eta}{2}\Big)\Big] \\
&+\delta_{\mu\lambda}\ \delta_{\nu\kappa}\ \ddot{\delta}\Big[\tau_1+\frac{\rho}{2}-\Big(\tau_2+\frac{\eta}{2}\Big)\Big]\ \ddot{\delta}\Big[\tau_1-\frac{\rho}{2}-\Big(\tau_2-\frac{\eta}{2}\Big)\Big]\Big).
\end{aligned}$$

First we notice that the first term vanishes upon contraction with $\sigma_{\mu\nu}\sigma_{\lambda\kappa}$ (or its path ordered variant), owing to the antisymmetry of $\sigma_{\mu\nu}$. With the same argument the other part of (20) will transform after path integration to

$$\begin{aligned}
&\Big(\delta_{\mu\lambda}\delta_{\nu\kappa}\ \ddot{\delta}\Big[\tau_1+\frac{\rho}{2}-\Big(\tau_2+\frac{\eta}{2}\Big)\Big]\ \ddot{\delta}\Big[\tau_1-\frac{\rho}{2}-\Big(\tau_2-\frac{\eta}{2}\Big)\Big] \\
&+\ \delta_{\mu\kappa}\delta_{\nu\lambda}\ \ddot{\delta}\Big[\tau_1+\frac{\rho}{2}-\Big(\tau_2-\frac{\eta}{2}\Big)\Big]\ \ddot{\delta}\Big[\tau_1-\frac{\rho}{2}-\Big(\tau_2+\frac{\eta}{2}\Big)\Big] \\
&-\ \delta_{\mu\kappa}\delta_{\nu\lambda}\ \ddot{\delta}\Big[\tau_1+\frac{\rho}{2}-\Big(\tau_2-\frac{\eta}{2}\Big)\Big]\ \ddot{\delta}\Big[\tau_1-\frac{\rho}{2}-\Big(\tau_2+\frac{\eta}{2}\Big)\Big] \\
&-\ \delta_{\nu\kappa}\delta_{\mu\lambda}\ \ddot{\delta}\Big[\tau_1-\frac{\rho}{2}-\Big(\tau_2-\frac{\eta}{2}\Big)\Big]\ \ddot{\delta}\Big[\tau_1+\frac{\rho}{2}-\Big(\tau_2+\frac{\eta}{2}\Big)\Big] \\
&-\ \delta_{\mu\lambda}\delta_{\nu\kappa}\ \ddot{\delta}\Big[\tau_1+\frac{\rho}{2}-\Big(\tau_2+\frac{\eta}{2}\Big)\Big]\ \ddot{\delta}\Big[\tau_1-\frac{\rho}{2}-\Big(\tau_2-\frac{\eta}{2}\Big)\Big] \\
&-\ \delta_{\nu\lambda}\delta_{\mu\kappa}\ \ddot{\delta}\Big[\tau_1-\frac{\rho}{2}-\Big(\tau_2+\frac{\eta}{2}\Big)\Big]\ \ddot{\delta}\Big[\tau_1+\frac{\rho}{2}-\Big(\tau_2-\frac{\eta}{2}\Big)\Big]\Big).
\end{aligned}$$

Now it is easy to see, that all the terms cancel each other and therefore the correct $F \to 0$ limit is manifest in our version of the worldline integral. We have seen explicitly in this example that the additional derivative terms can be identified with self-contractions of the "field" $\omega_{\mu\nu}(\tau)$. We have illustrated that the resulting normal ordering is essential for ensuring the consistency of our spin-factor representation.

7 Spin factor calculus for the effective action

We have identified in the last section the additional derivative terms arising from the spin-factor with the self-contractions of the "field" $\omega_{\mu\nu}(\tau)$. We are now able to write down the full path integral (the general case with $F \neq 0$),

$$\begin{aligned}
& \int \mathcal{D}x(\tau) \left[\mathrm{tr}_\gamma \; \mathcal{P} e^{-\frac{i}{2}\int d\tau \sigma \frac{\delta}{\delta x(\tau)}} \; e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} \right] e^{ie\oint dx A} \\
= & \int \mathcal{D}x(\tau) \; \mathrm{tr}_\gamma \mathcal{P} \left(e^{-\frac{i}{2}\int d\tau \sigma\omega} - \text{self contractions} \right) e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} e^{ie\oint dx A} \\
= & \int \mathcal{D}x(\tau) \; \mathrm{tr}_\gamma \mathcal{P} : e^{-\frac{i}{2}\int d\tau \sigma\omega} : \; e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} e^{ie\oint dx A}.
\end{aligned}$$

Because this expression is identical to the standart one,

$$\int \mathcal{D}x(\tau) \; \mathrm{tr}_\gamma \; \mathcal{P} e^{\frac{e}{2}\int d\tau \sigma F} e^{ie\oint dx A} \; e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}},$$

we conclude that the following expressions must be the same

$$\left\langle \mathcal{P} e^{\frac{e}{2}\int d\tau \sigma F} e^{ie\int d\tau \dot{x} A} \right\rangle \stackrel{!}{=} \left\langle \mathcal{P} : e^{-\frac{i}{2}\int d\tau \sigma\omega} : e^{ie\int d\tau \dot{x} A} \right\rangle.$$

To summarize, we have found the following representation of one-loop contribution to the effective action for spinor QED involving a purely geometrical spin factor:

$$\Gamma^1_{\text{eff}} = \frac{1}{2} \frac{1}{(4\pi)^{\frac{D}{2}}} \int_0^T \frac{dT}{T^{1+\frac{D}{2}}} \; e^{-m^2 T} \mathcal{N} \int \mathcal{D}x(\tau) \; \mathrm{tr}_\gamma \mathcal{P} : e^{-\frac{i}{2}\int d\tau \sigma\omega} : \; e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} e^{ie\oint dx A}.$$

The obvious advantage of this representation consists in the fact that the dependence on the external gauge field occurs solely in the form of a Wilson loop. As a disadvantage it seems that concrete computations may be plagued by technical difficulties associated with normal ordering. Moreover, even perturbative amplitudes to finite order in A_μ seemingly receive contributions from terms with arbitrarily high products of worldline monomials, $\langle \omega^n \dot{x} ... x \rangle \sim \langle (\dot{x}\ddot{x})^n \dot{x} ... x \rangle$, n arbitrary. However, we will demonstrate in this section that many of these apparent high-order contributions cancel among each other. Practical calculations actually boil down to roughly the same amount of technical work as it occurs for the formalism involving the Pauli term directly. The purpose of this section is the derivation of a few rules for practical calculus based on the spin-factor representation. In view of the variety of possible worldline monomials arising from the expansion of the Wilson loop, the spin factor and the corresponding self contractions (hidden behind the normal ordering), we do not attempt to give a full account of all possible structures and cancelation mechanisms. Instead, we will pick out all those terms that, upon Wick contraction, will lead us back to the full result for the effective action in standart representation. As a result, all possible other terms ultimately have to cancel each other. This will lead us to the general recipe that the spin factor can only contribute if a factor $\sim \int \sigma\omega$ is Wick contracted with a factor $\sim \oint dxA$ from the Wilson loop. For this reason we have to prove that

$$\mathcal{P}\left\{\left(1+\left(-\frac{i}{2}\int d\tau\sigma\omega\right)+\frac{1}{2}\left(-\frac{i}{2}\int d\tau\sigma\omega\right)^2+...\right)\right.$$
$$\left.\left(1+ie\oint dxA+\frac{1}{2}\left(ie\oint dxA\right)^2+...\right)\right\}_{\omega}^{\oint A} \tag{21}$$

gives the same result as

$$\mathcal{P}\left(1+\frac{e}{2}\int d\tau\sigma F+\frac{1}{2}\left(\frac{e}{2}\int d\tau\sigma F\right)^2+...\right)$$
$$\left(1+ie\oint dxA+\frac{1}{2}\left(ie\oint dxA\right)^2+...\right),$$

where $\{\cdots\}_{\omega}^{\oint A}$ indicates that we take only contractions between the complete spin-factor exponent $\sim \int \sigma\omega$ and the Wilson loop exponent $\sim \oint dxA$ into account. To be precise, both $\ddot{x}$'s of one $\omega_{\mu\nu}$ should be Wick contracted with the x dependence of one and the same exponent $\oint dxA$, any other contraction should be dropped. These contractions will turn out to carry all available spin information, whereas all other cancel. Let us now calculate this expression order by order. First we investigate the lowest order, which is trivial,

$$\left\{1\left(1+ie\oint dxA+\frac{1}{2}\left(ie\oint dxA\right)^2+...\right)\right\}_{\omega}^{\oint A}=e^{ie\oint dxA}.$$

Proceeding with the next order we are lead to

$$\left\{\left(-\frac{i}{2}\int d\tau\sigma\omega\right)\left(1+i\oint dxA+\frac{1}{2}\left(i\oint dxA\right)^2+...\right)\right\}_{\omega}^{\oint A}.$$

The first term $\left\{(-\frac{i}{2}\int d\tau\sigma\omega\ )\cdot 1\right\}$ does not contribute because of its Lorentz structure proportional to $\delta_{\mu\nu}\sigma^{\mu\nu}=0$. Therefore let us investigate the contraction

$$\left\{\left(-\frac{i}{2}\int d\tau\sigma\omega\right)\left(ie\int d\tau\dot{x}A\right)\right\}_{\omega}^{\oint A}.$$

For simplicity, we confine ourselves to the constant- field limit for a moment, where $A_\alpha = \frac{1}{2}F_{\beta\alpha}(0)x_\beta$. This will be generalized to arbitrarily inhomogeneous fields later on, where the constant-field limit will simply correspond to the lowest-order Taylor expansion. If we use our definition for $\omega_{\mu\nu}(\tau)$ the above expression transforms into

$$-\frac{e}{16}\int d\tau_1 d\tau_2\ F_{\alpha\beta}(0)\ \sigma_{\mu\nu}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}d\rho\rho\left\langle\ddot{x}_\mu(\tau_1+\frac{\rho}{2})\ddot{x}_\nu(\tau_1-\frac{\rho}{2})\dot{x}_\alpha(\tau_2)x_\beta(\tau_2)\right\rangle. \tag{22}$$

At first we analyze the worldline correlator with aid of Wick's theorem

$$\begin{aligned}\Big\langle \ddot{x}_\mu(\tau_1+\frac{\rho}{2})\ddot{x}_\nu(\tau_1-\frac{\rho}{2})\dot{x}_\alpha(\tau_2)x_\beta(\tau_2)\Big\rangle &= \langle \ddot{x}_\mu(\tau_1+\frac{\rho}{2})\ddot{x}_\nu(\tau_1-\frac{\rho}{2})\rangle\langle \dot{x}_\alpha(\tau_2)x_\beta(\tau_2)\rangle \\ &+ \langle \ddot{x}_\mu(\tau_1+\frac{\rho}{2})\dot{x}_\alpha(\tau_2)\rangle\langle \ddot{x}_\nu(\tau_1-\frac{\rho}{2})x_\beta(\tau_2)\rangle \\ &+ \langle \ddot{x}_\mu(\tau_1+\frac{\rho}{2})x_\beta(\tau_2)\rangle\langle \ddot{x}_\nu(\tau_1-\frac{\rho}{2})\dot{x}_\alpha(\tau_2)\rangle.\end{aligned}$$

Using our expressions for the worldline Green's functions, the right-hand side of the above equation transforms into

$$\begin{aligned}\Big(&+4\ \delta_{\nu\alpha}\delta_{\mu\beta}\left[\dot{\delta}\left(\tau_1-\frac{\rho}{2}-\tau_2\right)\left(-\delta\left(\tau_1+\frac{\rho}{2}-\tau_2\right)+\frac{1}{T}\right)\right] \\ &+4\ \delta_{\mu\alpha}\delta_{\nu\beta}\left[\dot{\delta}\left(\tau_1+\frac{\rho}{2}-\tau_2\right)\left(-\delta\left(\tau_1-\frac{\rho}{2}-\tau_2\right)+\frac{1}{T}\right)\right]\Big),\end{aligned}$$

where we have dropped the first term, beeing proportional to $\sigma_{\mu\nu}\delta_{\mu\nu}=0$. We insert this expression into (22) and get

$$\begin{aligned}-\frac{e}{16}\int d\tau_1 d\tau_2\ \sigma_{\mu\nu}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\ F_{\mu\nu}(0)\Big(&-4\left[\dot{\delta}\left(\tau_1-\frac{\rho}{2}-\tau_2\right)\left(-\delta\left(\tau_1+\frac{\rho}{2}-\tau_2\right)+\frac{1}{T}\right)\right] \\ &+4\left[\dot{\delta}\left(\tau_1+\frac{\rho}{2}-\tau_2\right)\left(-\delta\left(\tau_1-\frac{\rho}{2}-\tau_2\right)+\frac{1}{T}\right)\right]\Big).\end{aligned}$$

If we substitute

$$\rho \mapsto -\rho,$$

in the second term of our expression for the contraction, the whole quantity transforms into

$$\frac{e}{2}\int d\tau_1 d\tau_2 \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\ \sigma_{\mu\nu}F^{\mu\nu}(0)\left[\dot{\delta}\left(\tau_1-\frac{\rho}{2}-\tau_2\right)\left(-\delta\left(\tau_1+\frac{\rho}{2}-\tau_2\right)+\frac{1}{T}\right)\right]. \quad (23)$$

From this expression we will calculate at first the second one $\sim \frac{1}{T}$

$$\begin{aligned}&\int d\tau_1 d\tau_2 \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\ \sigma_{\mu\nu}F^{\mu\nu}(0)\ \dot{\delta}\left(\tau_1-\frac{\rho}{2}-\tau_2\right)\frac{1}{T} \\ \sim &\int d\tau_1 d\tau_2 \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\ \sigma_{\mu\nu}F^{\mu\nu}(0)\ \dot{\delta}\left(\tau_1-\frac{\rho}{2}-\tau_2\right) \\ \sim &\int d\tau_1 d\tau_2 \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\ \sigma_{\mu\nu}F^{\mu\nu}(0)\ \delta\left((\tau_1-\tau_2)-\frac{\rho}{2}\right),\end{aligned}$$

where we have integrated by parts in the last line. We are allowed to write

$$\int_{-\epsilon}^{\epsilon} d\rho\ F_{\mu\nu}(0)\delta\left((\tau_1-\tau_2)+\rho\right) = F_{\mu\nu}(0)\ \theta(\epsilon-|\tau_1-\tau_2|).$$

With this relation the above equation transforms into (modulo prefactors)

$$\lim_{\epsilon\to 0}\int d\tau_1 d\tau_2\ F_{\mu\nu}(0)\ \theta(\epsilon-|\tau_1-\tau_2|) = \int d\tau_2 F_{\mu\nu}(0)\ \lim_{\epsilon\to 0}\int d\tau_1\ \theta(\epsilon-|\tau_1-\tau_2|)$$
$$\sim \int d\tau_2\ F_{\mu\nu}(0)\lim_{\epsilon\to 0}\int d\tau_1'\ \theta(\epsilon-|\tau_1'|) \sim \int d\tau_2\ F_{\mu\nu}(0)\lim_{\epsilon\to 0}\ \epsilon\to 0.$$

This term gives no contribution to the worldline correlator. Let us now turn to the first term of (23)

$$-\frac{e}{2}\int d\tau_1 d\tau_2 \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\ \sigma_{\mu\nu}F^{\mu\nu}(0)\ \dot{\delta}\left(\tau_1-\frac{\rho}{2}-\tau_2\right)\delta\left(\tau_1-(\tau_2-\frac{\rho}{2})\right).$$

Performing the τ_1 integration we arrive at

$$\begin{aligned}-\frac{e}{2}\int d\tau_2 \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\ \sigma_{\mu\nu}F^{\mu\nu}(0)\ \dot{\delta}(-\rho) &= \frac{e}{2}\int d\tau_2 \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\ \sigma_{\mu\nu}F^{\mu\nu}(0)\ \delta(-\rho)\\ &= \frac{e}{2}\int d\tau_2\ \sigma_{\mu\nu}F^{\mu\nu}(0), \qquad (24)\end{aligned}$$

where we have integrated by parts in the first line. We recognize this expression as the leading part of the Taylor expansion of the standart worldline representation in the constant-field limit. But our goal is to show the validity of our conjectured formula in (21) for an arbitrary field configuration. Upon Taylor expansion, we have to show that we arrive at the expression

$$\begin{aligned}\int d\tau\ \sigma F(x(\tau)) &= \int d\tau\ \sigma_{\mu\nu}\left[F_{\mu\nu}(0)+\partial_\beta F_{\mu\nu}(0)x_\beta(\tau)+\frac{1}{2}\ \partial_{\beta\gamma}F_{\mu\nu}(0)x_\beta(\tau)x_\gamma(\tau)+...\right]\\ &= \int d\tau\ \sigma_{\mu\nu}\sum_{n=0}^{\infty}\frac{\partial_{\nu_1}\cdots\partial_{\nu_n}}{n!}\ F_{\mu\nu}(0)\ x_{\nu_1}\cdots x_{\nu_n},\end{aligned}$$

instead of (24) if using an arbitrary A_μ. For this purpose, we generalize the above calculation to the case of an arbitrarly field $A(x(\tau))$ in Schwinger-Fock gauge,

$$\begin{aligned}A_\alpha(x(\tau)) &= \frac{1}{2}x_\lambda(\tau)F_{\lambda\alpha}(0)+\frac{1}{3}x_\lambda(\tau)x_\sigma(\tau)\ \partial_\sigma F_{\lambda\alpha}(0)+...\\ &= \sum_{n=0}^{\infty}\frac{x^\lambda x^{\nu_1}\cdots x^{\nu_n}}{n!(n+2)}\ \partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}\ .\end{aligned}$$

With this general formula we repeat the previous calculation of the expectation value to the next order in the Taylor expansion, in order to gain more intuition for the general case,

$$\left\{\frac{i}{2}\int d\tau\sigma\omega(\tau)\ ie\int d\tau\ \dot{x}(\tau)A(x(\tau))\right\}_\omega^{\oint A} = \left\{-\frac{e}{8}\int d\tau_1\sigma\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\ \ddot{x}_\nu(\tau_1-\frac{\rho}{2})\ddot{x}_\mu\ (\tau_1+\frac{\rho}{2})\right.$$
$$\left.\int d\tau_2\ \dot{x}_\alpha(\tau_2)\left[\frac{1}{2}x_\lambda(\tau)F_{\lambda\alpha}(0)+\frac{1}{3}x_\lambda(\tau)x_\sigma(\tau)\ \partial_\sigma F_{\lambda\alpha}(0)\right]+\mathcal{O}(\partial^2)\right\}_\omega^{\oint A}$$

We recognize the familiar first term in this expression. Therefore we confine our attention to the second term:

$$\int_{-\epsilon}^{\epsilon} d\rho\rho \, \frac{1}{3} \, \partial_\sigma F_{\lambda\alpha}(0) \; x_\sigma(\tau_2) \left\{ \ddot{x}_\nu(\tau_1 - \frac{\rho}{2}) \ddot{x}_\mu(\tau_1 + \frac{\rho}{2}) \dot{x}_\alpha(\tau_2) x_\lambda(\tau_2) \right\}_\omega^{\oint A} .$$

According to our rule for the $\{...\}_\omega^{\oint A}$ bracket, the $\ddot{x}$ from $\omega_{\mu\nu}$ are Wick contracted with the x dependence of $\oint dxA$, the remaining $x_\sigma(\tau_2)$ can finally be contracted with an x from a different term. This quantity has three different possibilities of permutation

1. $$\partial_\sigma F_{\lambda\alpha} \; x_\sigma(\tau_2) \left\{ \ddot{x}_\nu(\tau_1 - \frac{\rho}{2}) \ddot{x}_\mu(\tau_1 + \frac{\rho}{2}) \dot{x}_\alpha(\tau_2) x_\lambda(\tau_2) \right\}_\omega^{\oint A}$$

2. $$\partial_\sigma F_{\lambda\alpha} \; x_\lambda(\tau_2) \left\{ \ddot{x}_\nu(\tau_1 - \frac{\rho}{2}) \ddot{x}_\mu(\tau_1 + \frac{\rho}{2}) \dot{x}_\alpha(\tau_2) x_\sigma(\tau_2) \right\}_\omega^{\oint A}$$

3. $$\partial_\sigma F_{\lambda\alpha} \; \dot{x}_\alpha(\tau_2) \left\{ \ddot{x}_\nu(\tau_1 - \frac{\rho}{2}) \ddot{x}_\mu(\tau_1 + \frac{\rho}{2}) x_\lambda(\tau_2) x_\sigma(\tau_2) \right\}_\omega^{\oint A} ,$$

where the $\{...\}_\omega^{\oint A}$ brackets demand that the $\ddot{x}$'s are only contracted with x or $\dot{x}$ but not with each other (the latter would anyway vanish upon contraction with $\sigma_{\mu\nu}$). Using Wick's theorem, it is easy to obtain the Green's functions, but for our purposes we are mainly interested in the Lorentz structure of the above expressions. So if we arrange the contraction we get beside the propagators the following Lorentz structures. For the first of the above quantities we get

$$\begin{aligned} &\frac{1}{24} \int d\tau_1 d\tau_2 \; \sigma_{\mu\nu} \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} d\rho\rho \; \partial_\sigma F_{\lambda\alpha} x_\sigma(\tau) \\ &\times \Big(\delta_{\nu\alpha}\delta_{\mu\lambda} \; 2\dot{\delta}\left(\tau_1 - \frac{\rho}{2} - \tau_2\right) \left[-2\delta\left(\tau_1 + \frac{\rho}{2} - \tau_2\right)\right] \\ &\quad + \delta_{\nu\lambda}\delta_{\mu\alpha} \; 2\dot{\delta}\left(\tau_1 + \frac{\rho}{2} - \tau_2\right) \left[-2\delta\left(\tau_1 - \frac{\rho}{2} - \tau_2\right)\right] \Big) . \end{aligned}$$

Let us investigate the Lorentz structure. We get

$$\partial_\sigma F_{\lambda\alpha} x_\sigma \; \delta_{\nu\alpha}\delta_{\mu\lambda} = \partial_\sigma F_{\mu\nu} x_\sigma,$$

as well as

$$\partial_\sigma F_{\lambda\alpha} x_\sigma \; \delta_{\nu\lambda}\delta_{\mu\alpha} = -\partial_\sigma F_{\mu\nu} x_\sigma.$$

Therefore we are lead to

$$\begin{aligned} \frac{1}{24} \int d\tau_1 d\tau_2 \; \sigma_{\mu\nu} \Bigg[&\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} d\rho\rho \; 4 \; \partial_\sigma F_{\lambda\alpha} x_\sigma(\tau) \; \dot{\delta}\left(\tau_1 - \frac{\rho}{2} - \tau_2\right) \left(-\delta\left(\tau_1 + \frac{\rho}{2} - \tau_2\right)\right) \\ &+ \lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} d\rho\rho \; -4 \; \partial_\sigma F_{\lambda\alpha} x_\sigma(\tau) \; \dot{\delta}\left(\tau_1 + \frac{\rho}{2} - \tau_2\right) \left(-\delta\left(\tau_1 - \frac{\rho}{2} - \tau_2\right)\right) \Bigg]. \end{aligned}$$

If we substitute $\rho \to -\rho$ in the second term we get in total

$$\frac{1}{6}\int d\tau_1 d\tau_2\ \sigma_{\mu\nu} \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} d\rho\rho\ 2\ \partial_\sigma F_{\lambda\alpha} x_\sigma(\tau)\ \dot{\delta}\left(\tau_1 - \frac{\rho}{2} - \tau_2\right)\left(-\delta\left(\tau_1 + \frac{\rho}{2} - \tau_2\right)\right).$$

We have calculated an analogous expression earlier. Using the results of the calulation which lead to (24) we get for the above expression

$$\frac{2}{6}\int d\tau\ \sigma_{\mu\nu}\ \partial_\sigma F_{\mu\nu} x_\sigma(\tau).$$

In correspondence to this calculation the other possible permutations contribute a factor $\frac{1}{6}\int d\tau\sigma_{\mu\nu}\ \partial_\beta F_{\mu\nu} x_\beta(\tau)$ to this solution (as we will show in the discussion of the Lorentz structure below). Therefore we get the next order of the Taylor expansion

$$\frac{1}{2}\int d\tau\sigma_{\mu\nu}F_{\mu\nu} = \int d\tau\sigma\left(\frac{1}{2}F_{\mu\nu} + \frac{1}{2}\partial_\sigma F_{\mu\nu}x_\sigma + \dots\right).$$

The essence of this calculation lies in the analysis of the Lorentz structure. For the other permutations the discussion is analogous. Let us condense this statement into a compact notation.

1.

$$\begin{aligned}
\partial_\sigma F_{\lambda\alpha}\ x_\sigma\left(\delta^{\nu\alpha}\delta^{\mu\lambda}\Big|_\rho \oplus \delta^{\nu\lambda}\delta^{\mu\alpha}\Big|_{-\rho}\right) &= \partial_\sigma F_{\lambda\nu}\ x_\sigma\ \delta^{\mu\lambda}\Big|_\rho \oplus \partial_\sigma F_{\lambda\mu}\ x_\sigma\ \delta^{\nu\lambda}\Big|_{-\rho} \\
&= -\partial_\sigma F_{\nu\mu}\ x_\sigma\Big|_\rho \oplus -\partial_\sigma F_{\mu\nu}\ x_\sigma\ \Big|_{-\rho} \\
&= \partial_\sigma F_{\mu\nu}\ x_\sigma\Big|_\rho \oplus \partial_\sigma F_{\mu\nu}\ x_\sigma\ \Big|_\rho \\
&= 2\ \partial_\sigma F_{\mu\nu}\ x_\sigma\Big|_\rho,
\end{aligned}$$

where the notation $\Big|_\rho$ and $\Big|_{-\rho}$ takes care of the signs of ρ in the arguments of the δ functions, the transformation $\Big|_{-\rho}$ to $\Big|_\rho$ can be performed by a partial integration. This is also the reason why we denote the addition in the Lorentz structure as "$\oplus$". The Lorentz structure of the second term is indicated as

2.

$$\begin{aligned}
\partial_\sigma F_{\lambda\alpha}\ x_\lambda\left(\delta^{\nu\alpha}\delta^{\mu\sigma}\Big|_\rho \oplus \delta^{\nu\sigma}\delta^{\mu\alpha}\Big|_{-\rho}\right) &= \partial_\sigma F_{\lambda\nu}\ x_\lambda\ \delta^{\mu\sigma}\Big|_\rho \oplus \partial_\sigma F_{\lambda\mu}\ x_\lambda\ \delta^{\nu\sigma}\Big|_{-\rho} \\
&= \partial_\mu F_{\lambda\nu}\ x_\lambda\Big|_\rho \oplus \partial_\nu F_{\lambda\mu}\ x_\lambda\Big|_{-\rho} \\
&= \partial_\mu F_{\lambda\nu}\ x_\lambda\Big|_\rho \oplus \partial_\nu F_{\mu\lambda}\ x_\lambda\Big|_\rho \\
&= \partial_\lambda F_{\mu\nu}\ x_\lambda\Big|_\rho,
\end{aligned}$$

where we have used the Bianchi identity

$$\begin{aligned}
0 &= \partial_\mu F_{\lambda\nu} + \partial_\lambda F_{\nu\mu} + \partial_\nu F_{\mu\lambda} \\
\Rightarrow \partial_\mu F_{\lambda\nu} + \partial_\nu F_{\mu\lambda} &= -\partial_\lambda F_{\nu\mu} = \partial_\lambda F_{\mu\nu}.
\end{aligned}$$

Let us now investigate the lorentz structure of the third permutation.

3.

$$\begin{aligned}\partial_\sigma F_{\lambda\alpha}\ \dot{x}^\alpha \left(\delta^{\nu\lambda}\delta^{\mu\sigma} \oplus \delta^{\nu\sigma}\delta^{\mu\lambda}\right) &= -\partial_\sigma F_{\alpha\nu}\ \dot{x}^\alpha\delta^{\mu\sigma} \oplus \partial_\nu F_{\lambda\alpha}\ \dot{x}^\alpha\delta^{\mu\lambda} \\ &= -\dot{x}^\alpha \left(\partial_\mu F_{\alpha\nu} \oplus \partial_\nu F_{\alpha\mu}\right) \\ &= -\dot{x}^\alpha \left(\partial_\mu F_{\alpha\nu} \ominus \partial_\mu F_{\alpha\nu}\right) \\ &= 0.\end{aligned}$$

Where we have used that after the exchange of indices $\mu \leftrightarrow \nu$ the quantity $\sigma_{\mu\nu}\ \partial_\nu F_{\alpha\mu} \rightarrow \sigma_{\nu\mu}\ \partial_\mu F_{\alpha\nu}$ and we get

$$\sigma_{\nu\mu}\ \partial_\mu F_{\alpha\nu} = -\sigma_{\mu\nu}\ \partial_\mu F_{\alpha\nu}\ .$$

Furthermore we have neglected the sign-of-ρ dependence since the δ function combination occuring here is symmetric under $\rho \rightarrow -\rho$.

So far we have only analyzed the contraction of the second term beyond leading order. Next we want show that we indeed are able to derive the right formular for the full field strength tensor $F(x(\tau))$. For this reason we have to analyze the expression

$$\begin{aligned}&\left\{\frac{e}{2}\int d\tau\sigma\omega(\tau)\int d\tau\ \dot{x}(\tau)A(x(\tau))\right\}_\omega^{\oint A} \\ &= \left\{\frac{e}{2}\int d\tau_1 d\tau_2 \sigma_{\mu\nu}\sum_{n=0}^{\infty}\frac{\ddot{x}_\nu\ddot{x}_\mu\dot{x}_\alpha x_\lambda x_{\nu_1}\cdots x_{\nu_n}}{n!(n+2)}\ \partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}(0)\right\}_\omega^{\oint A}.\end{aligned}$$

We have the following number of permutations:

1. terms with

$$\partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}(0)\ x_{\nu_1}\cdots x_{\nu_n}\langle\ddot{x}_\nu\ddot{x}_\mu\dot{x}_\alpha x_\lambda\rangle$$

appear twice and therefore give a factor $2\ \partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}(0)\ x_{\nu_1}\cdots x_{\nu_n}$.

2. terms with

$$\begin{aligned}&\partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}(0)\ x_\lambda x_{\nu_2}\cdots x_{\nu_n}\langle\ddot{x}_\nu\ddot{x}_\mu\dot{x}_\alpha x_{\nu_1}\rangle \\ &. \\ &. \\ &. \\ &\partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}(0)\ x_\lambda x_{\nu_1}\cdots x_{\nu_{n-1}}\langle\ddot{x}_\nu\ddot{x}_\mu\dot{x}_\alpha x_{\nu_n}\rangle\end{aligned}$$

exist n times and contribute with a factor $n\ \partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}(0)\ x_{\nu_1}\cdots x_{\nu_n}$ to the sum.

3. terms with

$$\partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}(0)\;\dot{x}_\alpha x_{\nu_2}\cdots x_{\nu_n}\langle\ddot{x}_\nu\ddot{x}_\mu x_\lambda x_{\nu_1}\rangle$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}(0)\;\dot{x}_\alpha x_{\nu_1}\cdots x_{\nu_{n-1}}\langle\ddot{x}_\nu\ddot{x}_\mu x_\lambda x_{\nu_n}\rangle,$$

vanish and therefore give no contribution (note that we have used the results of the previous calculation for this discussion).

From this investigation we can conclude that our expectation value takes the following representation

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{\partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}(0)}{(n+2)n!}\right)\;\mathrm{perm}(x_{\nu_1}\cdots x_{\nu_n}\langle\ddot{x}_\nu\ddot{x}_\mu\dot{x}_\alpha x_\lambda\rangle,\nu_1...\nu_m\alpha\lambda)\\
=\quad&\sum_{n=0}^{\infty}\left(\frac{\partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}(0)}{(n+2)n!}\;x_{\nu_1}\cdots x_{\nu_n}\right)(n+2)\\
=\quad&\sum_{n=0}^{\infty}\frac{\partial_{\nu_1}\cdots\partial_{\nu_n}F_{\lambda\nu}(0)}{n!}\;x_{\nu_1}\cdots x_{\nu_n}=F_{\mu\nu}(x(\tau)).
\end{aligned}$$

Because the last line represents the Taylor expansion of the function $F(x(\tau))$ we can finally write:

$$\left\{-\frac{i}{2}\int d\tau\sigma\omega\;ie\int d\tau\;\dot{x}_\nu(\tau)A_\nu(x(\tau))\right\}_\omega^{\oint A}=\frac{e}{2}\int d\tau\sigma_{\mu\nu}F^{\mu\nu}(x(\tau))\;. \tag{25}$$

At this point we only dealt with the first term of the Taylor series of

$$\mathcal{P}\left\{\left(1+\left(-\frac{i}{2}\int d\tau\sigma\omega\right)+\frac{1}{2}\left(-\frac{i}{2}\int d\tau\sigma\omega\right)^2+...\right)\right.$$
$$\left.\left(1+ie\oint dxA+\frac{1}{2}\left(ie\oint dxA\right)^2+...\right)\right\}_\omega^{\oint A}.$$

We have shown that the contraction

$$\left\{\left(-\frac{i}{2}\int d\tau\sigma\omega\right)\left(ie\int d\tau\dot{x}A\right)\right\}_\omega^{\oint A},$$

gives indeed the linear term of the expansion

$$\left(1+\frac{1}{2}\int d\tau\sigma F+\frac{1}{2}\left(\frac{1}{2}\int d\tau\sigma F\right)^2+...\right)\left(1+ie\oint dxA+\frac{1}{2}\left(ie\oint dxA\right)^2+...\right).$$

Let us now investigate what happens when we go to the next order (the quadratic order of the Wilson loop) and beyond. Considering the contraction of the spin-factor term with the next quadratic term of the Wilson loop, we recognize that this calculation is similar to the calculation for the linear term. The only difference is that we have to consider the possible permutations of the contractions. Then we have the following structure

$$-\frac{i}{2}\int d\tau_1 \sigma_{\mu\nu}\,\omega_{\mu\nu}\frac{(ie)^2}{2}\left(\int d\tau_2 d\tau_3\ \oint dxA \oint dxA\right). \tag{26}$$

Because their exist two possible permutations of non vanishing contractions we get

$$\left\{\left(\frac{1}{2}\right)2\int d\tau_1\ \sigma_{\mu\nu}F_{\mu\nu}\ \frac{ie^2}{2}\oint dxA\right\}_{\omega}^{\oint A} = \frac{e}{2}\int d\tau\ \sigma_{\mu\nu}F_{\mu\nu}\ ie\oint dxA\ .$$

When we go to the next order we get an expression of the form

$$\left\{-\frac{i}{2}\int d\tau_1\ \sigma_{\mu\nu}\ \omega_{\mu\nu}\ \frac{(ie)^3}{3!}\oint dxA\oint dxA\oint dxA\right\}_{\omega}^{\oint A}.$$

After contracting, this transforms into

$$\left(\frac{e}{2}\right)3\int d\tau_1\ \sigma_{\mu\nu}F_{\mu\nu}\ \frac{1}{3!}\ (ie)^2\oint dxA\oint dxA.$$

When we continue this stringently we see that we construct the series of the Wilson loop with the spin field coupling as pre-factor:

$$\left(\frac{e}{2}\int d\tau\sigma_{\mu\nu}F_{\mu\nu}\right)\left(1+ie\int d\tau\dot{x}A+\frac{1}{2}\left(ie\int d\tau\dot{x}A\right)^2+...\right).$$

The above permutation argumentation will hold to all orders of the expansion of $\mathcal{P}e^{-\frac{i}{2}\int d\tau\sigma\omega}$, therefore we find that order by order the exponential series

$$\begin{aligned}&\mathcal{P}\left(1+\frac{e}{2}\int d\tau\sigma F+\frac{1}{2}\left(\frac{e}{2}\int d\tau\sigma F\right)^2+...\right)\left(1+ie\int d\tau\dot{x}A+\frac{1}{2}\left(ie\int d\tau\dot{x}A\right)^2+...\right)\\ =\ &\mathcal{P}\exp\left(\frac{e}{2}\int d\tau\sigma F\right)\exp\left(ie\oint dxA\right)\end{aligned}$$

will be built up. We conclude that the seemingly extensive spin-factor calculus can be summarized in a simple recipe: among the manifold contractions arising from Wick's theorem involving Wilson loop, spin factor and its self contractions, only those terms have to be accounted for which arise from contractions of one $\sigma\omega$ factor with one and the same $\oint dxA$ factor. This excludes already many Wick contractions, in particular, these where the two $\ddot{x}$'s out of one $\omega_{\mu\nu}$ are contracted with two different objects. All these terms of the latter type cancel each other or vanish by the ϵ limit. Our basic rule for spin-factor calculus can be summarized by the formular

$$\begin{aligned} \mathcal{N}\int \mathcal{D}x\ e^{-\int d\tau \frac{\dot{x}^2}{4}}\ \mathrm{tr}_\gamma \mathcal{P} : e^{-\frac{i}{2}\int d\tau\sigma\omega} : e^{ie\oint dxA} &= \left\langle \mathrm{tr}_\gamma \mathcal{P} : e^{-\frac{i}{2}\int d\tau\sigma\omega} : e^{ie\oint dxA} \right\rangle \\ &= \left\langle \left\{ \mathrm{tr}_\gamma \mathcal{P} e^{-\frac{i}{2}\int d\tau\sigma\omega} e^{ie\oint dxA} \right\}_\omega^{\oint A} \right\rangle, \quad (27) \end{aligned}$$

with the $\{...\}_\omega^{\oint A}$ bracket denoting the restriction to pure $\int \sigma\omega \leftrightarrow \oint dxA$ contractions as detailed above. Note that this recipe also dispenses us from considering normal ordering or a detailed analysis of the self contraction terms, since these cannot contribute to the $\{...\}_\omega^{\oint A}$ bracket by construction. The spin-factor calculus developed in this section has a physical interpretation: the spin factor is only operating at those space-time points where the fluctuating particle interacts with the external field. The spin of the fluctuation does not generate self-interactions of the fluctuation with its own worldline, nor does spin interact nonlocally with the external field at two different spacetime points simultaneously. In the following section we demonstrate the applicability of the spin-factor calculus developed here by rederiving the classic Heisenberg-Euler effective action with this new formalism.

8 Heisenberg-Euler action in the loop approach

After we have developed some aspects of spin-factor calculus for our new representation of the spinorial worldline path integral, we now calculate the Heisenberg-Euler action in the loop approach. To obtain this action we perform the calculation in Fourier space. Therefore we substitute all x's and functions depending on x in the integral

$$\mathcal{N} \int \mathcal{D}x(\tau)\ \mathrm{tr}_\gamma \mathcal{P} : e^{-\frac{i}{2}\int d\tau\sigma\omega} :\ e^{i\oint dxA}\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}},$$

by their Fourier representation. It is possible to write the Fourier series of $x_\mu(\tau)$ as follows

$$x_\mu(\tau) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{T}}\ a_{n\mu}\ e^{\frac{2\pi i n\tau}{T}}.$$

It is easy to compute the derivatives of this expression so we are lead to

$$\dot{x}^2(\tau) = \sum_{n,m=-\infty}^{\infty} \frac{1}{T} \left(\frac{2\pi i}{T}\right)^2 (nm)\ a_{n\mu}\ a_{q\mu}\ e^{\frac{2\pi i\tau(n+m)}{T}},$$

as well as

$$\ddot{x}_\mu(\tau + \frac{\rho}{2}) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{T}} \left(\frac{2\pi i n}{T}\right)^2\ a_{n\mu}\ e^{\frac{2\pi i n(\tau+\frac{\rho}{2})}{T}}.$$

Therefore the exponential functions in the worldline integral transforms as follows

$$e^{-\frac{i}{2}\int d\tau\sigma\omega} \mapsto \exp\left[\frac{1}{4}\lim_{\epsilon\to 0} \sum_{n=-\infty}^{\infty} \left(\frac{2\pi n}{T}\right)^2\ a^*_{n\mu}\ a_{n\nu}\ \sigma_{\mu\nu}\ g_n(\epsilon)\right],$$

with

$$g_n(\epsilon) = \left(\frac{2\pi n}{T}\epsilon\right) \cos\left(\frac{2\pi n}{T}\epsilon\right) - \sin\left(\frac{2\pi n}{T}\epsilon\right),$$

arising from the ρ integration. We have also used the reality conditions of the Fourier coefficients $a_{-n\mu} = a^*_{n\mu}$ as well as the orthonormality of the plane-wave basis

$$\int_0^T d\tau\ e^{\frac{2\pi i\tau(n+m)}{T}} = T\delta_{n,-m}.$$

In almost the same manner we treat the other expressions. Therefore the other quantities can be written as follows

$$e^{-\int d\tau\frac{\dot{x}^2(\tau)}{4}} \mapsto \exp\left[-\frac{1}{4}\sum_{n=-\infty}^{\infty}\left(\frac{2\pi i}{T}\right)^2 (-1)\ n^2\ a^*_{n\mu}\ a_{n\mu}\right],$$

as well as

$$e^{i\oint dxA(x(\tau))} \mapsto \exp\left[i\sum_{n=-\infty}^{\infty}\left(\frac{2\pi i(-n)}{T}\right)a^*_{n\mu}\ \tilde{A}_{\mu,n}[x]\right].$$

Considering the constant-field case only we can write the gauge field as follows

$$\begin{aligned} A_\mu(x) &= -\frac{1}{2}F_{\mu\nu}x_\nu(\tau) \\ &= -\frac{1}{2}F_{\mu\nu}\ \frac{1}{\sqrt{T}}\sum_{n=-\infty}^{\infty} a_{\nu n}\ e^{\frac{2\pi i n\tau}{T}} \\ &= \frac{1}{\sqrt{T}}\sum_{n=-\infty}^{\infty}\tilde{A}_{\mu n}\ e^{\frac{2\pi i n\tau}{T}} \\ \Rightarrow\ \tilde{A}_{\mu n} &= -\frac{1}{2}F_{\mu\nu}\ a_{\nu n}. \end{aligned}$$

Therefore we conclude that in the constant-field case the Wilson loop transforms as

$$e^{i\oint dxA(x(\tau))} \mapsto \exp\left[-\frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\frac{2\pi n}{T}\right)a^*_{\mu n}\ F_{\mu\nu}\ a_{\nu n}\right].$$

If we insert these expressions into the path integral we find ($\mathcal{D}x = \mathcal{D}a$)

$$\mathcal{N}\int\mathcal{D}a\ \mathrm{tr}_\gamma\left\{\exp\left[-\frac{1}{2}\sum_n a^*_{\mu n}\left(\frac{1}{2}\left(\frac{2\pi}{T}\right)^2 n^2\delta_{\mu\nu} + \frac{2\pi n}{T}F_{\mu\nu} - \frac{1}{2}\left(\frac{2\pi n}{T}\right)^2\sigma_{\mu\nu}\ g_n(\epsilon)\right)a_{\nu n}\right]\right\}_\omega^{\oint A},$$

where we have replaced the normal ordering by the rule for spin-factor calculus derived in the last section. Note that we have also dropped the path ordering which is irrelevant in the case of a constant electromagnetic field. If we split the Fourier coefficients as follows

$$\begin{aligned} a_{\mu n} &= b_{\mu n} + ic_{\mu n}, \\ a^*_{\mu n} &= b_{\mu n} - ic_{\mu n}, \end{aligned}$$

with real b, c, as well as using the reality condition of the Fourier coefficients $a^*_n = a_{-n} \Rightarrow b_n = b_{-n}, c_n = -c_{-n}$, we can write $M = \left(\frac{1}{2}\left(\frac{2\pi}{T}\right)^2 n^2\delta_{\mu\nu} + \frac{2\pi n}{T}F_{\mu\nu} - \frac{1}{2}\left(\frac{2\pi n}{T}\right)^2\sigma_{\mu\nu}\ g_n(\epsilon)\right)$ of the above path integral in (b, c) components as

$$M = \begin{pmatrix} \frac{1}{2}\left(\frac{2\pi}{T}\right)^2 n\ \delta_{\mu\nu} & i\frac{2\pi}{T}F_{\mu\nu} - \frac{i}{2}\left(\frac{2\pi n}{T}\right)^2\sigma_{\mu\nu}g_n(\epsilon) \\ -i\frac{2\pi}{T}F_{\mu\nu} + \frac{i}{2}\left(\frac{2\pi n}{T}\right)^2\sigma_{\mu\nu}g_n(\epsilon) & \frac{1}{2}\left(\frac{2\pi}{T}\right)^2 n\ \delta_{\mu\nu} \end{pmatrix}.$$

Therefore we can use the well-known formular for calculating Gaussian integrals

$$\mathcal{N}\int\mathcal{D}a\ \mathrm{tr}_\gamma\left\{\exp\left[-\frac{1}{2}\sum_n a^*_{\mu n}\ M_{\mu\nu}\ a_{\nu n}\right]\right\}_\omega^{\oint A} = \mathrm{tr}_\gamma\left\{\mathrm{Det}^{-\frac{1}{2}}\left[\frac{M}{M_0}\right]\right\}_\omega^{\oint A},$$

where M_0 denotes the operator M in the limit $F \to 0$ and the formal limit $g_n(\epsilon) \to 0$. Therefore together with the above arguments we can write these funtional determinant as

$$\begin{aligned}
\mathrm{Det}^{-\frac{1}{2}}\left[\frac{M}{M_0}\right] &= \mathrm{Det}^{-\frac{1}{2}}\begin{pmatrix} \delta_{\mu\nu} & 2i\left(\frac{T}{2\pi n}\right)F_{\mu\nu} - i\sigma_{\mu\nu}g_n(\epsilon) \\ -2i\left(\frac{T}{2\pi n}\right)F_{\mu\nu} + i\sigma_{\mu\nu}g_n(\epsilon) & \delta_{\mu\nu} \end{pmatrix} \\
&= \prod_{\lambda=+/-} \mathrm{Det}^{-\frac{1}{2}}\left[\delta_{\mu\nu} + 2\lambda\left(\frac{T}{2\pi n}\right)F_{\mu\nu} - \lambda\,\sigma_{\mu\nu}g_n(\epsilon)\right],
\end{aligned}$$

where we have diagonalized the 2×2 matrix M in the last step. Using the (ln Det = Tr ln) identity this transforms to

$$\begin{aligned}
& \exp\left[-\frac{1}{2}\sum_{\lambda=+/-}\left\{\mathrm{Tr}\,\ln\left(\delta_{\mu\nu} + 2\lambda\left(\frac{T}{2\pi n}\right)F_{\mu\nu} - \lambda\,\sigma_{\mu\nu}g_n(\epsilon)\right)\right\}_{\omega}^{\oint A}\right] \\
= \quad & \exp\left[-\frac{1}{2}\sum_{\lambda=+/-}\mathrm{tr}_n\mathrm{tr}_L\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left\{\left(2\lambda\left(\frac{T}{2\pi n}\right)F - \lambda\,\sigma g_n(\epsilon)\right)^m_{\mu\nu}\right\}_{\omega}^{\oint A}\right] \\
= \quad & \exp\left[\mathrm{tr}_n\mathrm{tr}_L\sum_{m=1}^{\infty}\frac{1}{2m}\left\{\left(2\left(\frac{T}{2\pi n}\right)F - \sigma g_n(\epsilon)\right)^{2m}_{\mu\nu}\right\}_{\omega}^{\oint A}\right].
\end{aligned}$$

If we write

$$\begin{aligned}
&\left\{\left(2\left(\frac{T}{2\pi n}\right)F - \sigma g_n(\epsilon)\right)^{2m}\right\}_{\omega}^{\oint A} = \sum_{k=0}^{2m}\binom{2m}{k}\left\{\left(2\left(\frac{T}{2\pi n}\right)F\right)^{2m-k}(\sigma g_n(\epsilon))^k\right\}_{\omega}^{\oint A} \\
= \quad & \sum_{k=0}^{m}\binom{2m}{k}\left(2\left(\frac{T}{2\pi n}\right)F\right)^{2m-k}(\sigma g_n(\epsilon))^k,
\end{aligned}$$

where the $\{...\}_{\omega}^{\oint A}$ bracket by definition removes all those terms for which at least one $\sigma g_n(\epsilon)$ term cannot be paired with an F term. This reduces the upper limit of the sum from $2m$ to m. Furthermore, we have used that in the constant field case $[F,\sigma] = 0$ and therefore F and σ can be arranged in an arbitrary order. We compose this sum further by writing the $k = 0$ term separately

$$\left(\frac{TF}{\pi n}\right)^{2m} + \binom{2m}{m}\left(\frac{T}{\pi n}F\sigma g_n(\epsilon)\right)^m + \sum_{k=1}^{m-1}\binom{2m}{k}\left(\frac{TF}{\pi n}\right)^{2m-k}(\sigma g_n(\epsilon))^k. \qquad (28)$$

The first term carries no spin information, it obviously corresponds to the contribution that we would equally encounter in scalar QED. Hence we call it the scalar part. The second term represents a perfect pairing of spin factor and field strength contribution, it will turn out to contain the full spinorial information. The remaining sum has always at least one unpaired F term, even for $k = m - 1$. We will demonstrate below that the accompanying factor of $\frac{1}{n}$ inhabits a sufficiently singular behaviour of the Fourier sum over n, such that this sum vanishes completely in the ϵ limit. Let us now compute the various pieces of (28) separately.

8.1 Scalar part

Let us first consider the scalar part which we calculate analogously to [12]

$$\exp\left[\mathrm{tr}_n \mathrm{tr}_L \sum_{m=1}^{\infty} \frac{1}{2m}\left(\frac{TF}{\pi n}\right)^{2m}\right] = \exp\left[\sum_{m=1}^{\infty} \frac{1}{2m}\left(\frac{T}{\pi}\right)^{2m} \mathrm{tr}_L\left(F^{2m}\right) \sum_{n=1}^{\infty} \frac{1}{n^{2m}}\right],$$

where the n sum runs only from 1 to ∞ corresponding to the independent components of the (b, c) Fourier coefficients that we have integrated over. We recognize the definition of the ζ function, $\sum_{n=1}^{\infty} \frac{1}{n^{2m}} = \zeta(2m) = \frac{(2\pi)^{2m}}{2!(2m)!}|B_{2m}|$. Therefore we can write

$$\begin{aligned}
\exp\left[\sum_{m=1}^{\infty} \frac{1}{2m}\left(\frac{T}{\pi}\right)^{2m} \mathrm{tr}_L\left(F^{2m}\right) \sum_{n=1}^{\infty} \frac{1}{n^{2m}}\right] &= \exp\left[\frac{1}{2}\sum_{m=1}^{\infty} \mathrm{tr}_L \frac{(2FT)^{2m}}{2m(2m)!}\,|B_{2m}|\right] \\
&= \exp\left[-\frac{1}{2}\,\mathrm{tr}_L \sum_{m=1}^{\infty} \frac{(2iFT)^{2m}}{2m(2m)!}\,B_{2m}\right] \\
&= \exp\left[-\frac{1}{2}\,\mathrm{tr}_L \ln\left(\frac{\sin(FT)}{FT}\right)\right] \\
&= \det{}^{-\frac{1}{2}}\left(\frac{\sin(FT)}{FT}\right).
\end{aligned}$$

For the constant B field case we have already calculated this quantity in (6), the result was

$$\det{}^{-\frac{1}{2}}\left[\frac{\sin(FT)}{FT}\right] = \left(\frac{BT}{\sinh(BT)}\right).$$

8.2 Spinor part

We now turn to the second term of (28). The remaining sum (third term) will be discussed after this calculation. The spinor part can be written as

$$\begin{aligned}
&\exp\left[\mathrm{tr}_n \mathrm{tr}_L \sum_{m=1}^{\infty} \frac{(2m-1)!}{(m!)^2}\left(\left(\frac{T}{\pi n}\right) F\sigma g_n(\epsilon)\right)^{m}\right] \\
= \quad &\exp\left[\sum_{m=1}^{\infty} \frac{(2m-1)!}{(m!)^2}\left(\frac{T}{\pi}\right)^{m} \mathrm{tr}_L\left(F\sigma\right)^{m} \sum_{n=1}^{\infty} \frac{g_n(\epsilon)}{n^m}\right].
\end{aligned}$$

Let us discuss the different possibilities for the values of m. For this discussion we use the explicit representation of $g_n(\epsilon)$

$$\sum_{n=1}^{\infty} \frac{g_n(\epsilon)}{n^m} = \sum_{n=1}^{\infty} \frac{\left(\left(\frac{2\pi n\epsilon}{T}\right)\cos\frac{2\pi n}{T}\epsilon - \sin\frac{2\pi n}{T}\epsilon\right)}{n^m}. \tag{29}$$

Let us start with $m = 1$, then we have the following expression

$$\frac{2\pi}{T}\sum_{n=1}^{\infty}\epsilon\,\cos\frac{2\pi n}{T}\epsilon - \sum_{n=1}^{\infty}\frac{\sin\frac{2\pi n}{T}\epsilon}{n}. \tag{30}$$

It is the second term that represents the basic contribution, since it exhibits the proper singularity structure,

$$-\sum_{n=1}^{\infty}\frac{\sin\frac{2\pi n}{T}\epsilon}{n} = -\frac{\pi - \frac{2\pi}{T}\epsilon}{2}.$$

For $\epsilon \to 0$ this goes to $-\frac{\pi}{2}$. It is possible to write the first term of (30)

$$\sum_{n=1}^{\infty}\epsilon\frac{d}{d\epsilon}\frac{\sin\frac{2\pi n}{T}\epsilon}{n} = \epsilon\frac{d}{d\epsilon}\frac{\pi - \frac{2\pi}{T}\epsilon}{2}.$$

If we send ϵ to zero this quantity vanishes. Therefore the spinor part (for $m = 1$) becomes

$$\mathrm{tr}_\gamma\,\exp\left(-\frac{T}{2}\,\mathrm{tr}_L[F\sigma]\right).$$

If we specialize to the constant B field case this leads to

$$\mathrm{tr}_\gamma\,\exp\left(-BT\sigma_{12}\right) = 4\,\cosh BT.$$

If we combine this with the result of the scalar part we get

$$\mathrm{tr}_\gamma\,\mathrm{Det}^{-\frac{1}{2}}\left[\frac{M}{M_0}\right]\Bigg|_{m=1} = (BT)\,\coth(BT). \tag{31}$$

We now turn to the other cases of (29) if m takes values > 1. For this it is more useful to go back to the integral representation of $g_n(\epsilon)$,

$$\sum_{n=1}^{\infty}\frac{g_n(\epsilon)}{n^m} = -i\frac{2\pi^2}{T^2}\int_{-\epsilon}^{\epsilon}d\rho\rho\sum_{n=1}^{\infty}\frac{e^{-\left(\frac{2i\pi\rho}{T}\right)n}}{n^{m-2}} = -\frac{\pi}{T}\int_{-\epsilon}^{\epsilon}d\rho\sum_{n=1}^{\infty}\frac{e^{-\left(\frac{2i\pi\rho}{T}\right)n}}{n^{m-1}}$$

where we integrated by parts. From this we know that any non-zero contribution requires the n sum to result in a $\delta(\rho)$ singularity. As shown above, this is exactly the case for the $m = 1$ term. However, for $m \geq 3$,

$$\sum_{n=1}^{\infty}\frac{e^{-\left(\frac{2i\pi\rho}{T}\right)n}}{n^{m-1}}, \quad \forall m \geq 3,$$

this corresponds to a poly logarithm of degree $m-1 \geq 2$ which is an analytic function for $\rho \to 0$. Hence all $m \geq 3$ terms vanish. The $m=2$ term is more subtle. Here we encounter

$$\sum_{n=1}^{\infty} \frac{e^{-\left(\frac{2i\pi\rho}{T}\right)n}}{n^{m-1}} = \sum_{n=1}^{\infty} \frac{\cos\left(\frac{2\pi\rho}{T}\right)}{n} + i \sum_{n=1}^{\infty} \frac{\sin\left(\frac{2\pi\rho}{T}\right)}{n}.$$

The second term is $\sim \frac{\pi-\rho}{2}$ and vanishes under the ϵ limit. Let us discuss the first term. It is possible to represent this term as follows

$$\sum_{n=1}^{\infty} \frac{\cos(n\frac{2\pi\rho}{T})}{n} = \frac{1}{2} \ln\left(\frac{1}{2(1-\cos\frac{2\pi\rho}{T})}\right).$$

Therefore the ρ integral becomes

$$-\frac{\pi}{2T} \int_{-\epsilon}^{\epsilon} d\rho \, \ln\frac{1}{2(1-\cos\rho)} \approx \frac{\pi}{T} \int_{-\epsilon}^{\epsilon} d\rho \, \ln\rho \to 0.$$

So there is no sufficient singular structure of the integrand and we conclude that the integral vanishes in the $\epsilon \to 0$ limit. Eventually we have to discuss the remaining sum of (28). Concentrating on the n dependence, these can be written as

$$\sum_{k=1}^{m-1} \mathcal{J}_n,$$

where

$$\mathcal{J}_n \sim \frac{1}{n^{2m-k}} \, g_n^k(\epsilon) \sim \int_{-\epsilon}^{\epsilon} d\rho\rho \frac{e^{in\rho}}{n^{2m-k}}. \quad , k = 1, ..., m-1$$

For all $k < m$ this is the same case as discussed before, hence all these terms also vanish in the $\epsilon \to 0$ limit. So only the expression (31) survives and together with the dT integration and pre factors we arrive at the unrenormalized four dimensional Heisenberg-Euler action [29],[30],[31].

$$\Gamma^1_{\text{eff}}[A] \;=\; \frac{1}{8\pi^2} \int_0^{\infty} \frac{dT}{T^3} \, e^{-m^2 T} \, (BT) \coth(BT).$$

We would like to stress that the present derivation of this well-known result is formally independent of other standart calculational techniques, as far as the spinor part is concerned. The spinor contribution arises from the subtle interplay between the purely geometric spin factor and the Wilson loop. Non-zero contributions arise only from terms with a particular singularity structure. Since these singularities cannot arise from smooth worldlines, we conclude that the random zigzag course of the worldlines is an essential ingredient for the coupling between spin and fields.

9 Conclusions and outlook

In this thesis, we have studied the worldline approach to Quantum Field Theory with gauge symmetry. Guided by the idea that gauge-field information can be solely covered by holonomies, we have investigated the representation of spin-field couplings in spinorial QED. In this example, we have shown that the familiar Pauli term can be reexpressed in terms of a spin factor which is a purely geometric quantity. Our final representation of the fermionic fluctuation determinant, i.e., the one-loop effective action for QED has the following form

$$\Gamma_{\mathrm{Ferm}}[A] = \frac{1}{2}\frac{1}{(4\pi)^{\frac{D}{2}}}\int_0^T \frac{dT}{T^{1+\frac{D}{2}}}\; e^{-m^2 T}\mathcal{N}\int \mathcal{D}x(\tau)\ \mathrm{tr}_\gamma \mathcal{P} : e^{-\frac{i}{2}\int d\tau\sigma\omega} :\ e^{-\int d\tau \frac{\dot{x}^2(\tau)}{4}} e^{ie\oint dx A},$$

where $\mathcal{N}$ indicates the factor of normalization, $\mathcal{P}$ denotes the path ordering and ": ... :" represents the introduced normal ordering procedure. The quantity $\sigma_{\mu\nu}\omega_{\mu\nu}$ encodes the information on the spin field coupling which is contained in $\sigma_{\mu\nu}F_{\mu\nu}$ in the ordinary representation of the fermionic effective action. The advantage of this alternative representation lies in the fact that the dependence on the external gauge field occurs solely in the form of the Wilson loop, i.e., the holonomy. Furhermore we have obtained a geometrical explanation of the spin field coupling in terms of the loop derivative which is closely connected to an alternative formulation of gauge theory refering to equivalence classes of loops. This loop approach to gauge theory is especially demonstrative for the worldline formulation of Quantum Field Theory, because worldlines provide us with an illuminating view on the quantum aspects in terms of "loop clouds". Additionally loop calculus yields an impressivly simple picture of the spin field coupling in terms of the analogon of the curvature tensor in the loop language, the loop derivative. In the affiliation of these geometrical pictures it is possible to get a deeper and more intuitive understanding of quantum behaviour.

It should be interesting to study the question as to whether $\omega_{\mu\nu}$ incorporates topological aspects of the worldlines or their dynamical elements. Maybe there is a hidden relation to the number of windings as for the Polyakov spin factor which arises in the first order representation of the effective action [24]. At present, we observe a number of fundamental differences between our spin factor and Polyakov's spin factor. The most important difference is that our spin factor is zero for smooth loops ($\omega = 0$) and therefore it is not evident how for instance a non-vanishing winding number can be encoded in our representation. Furthermore our worldlines have a Gaussian velocity distribution instead of the unit velocity distribution of Polyakov's approach that relies on the first-order formalism. The possibility of encoded topological properties of the worldline in $\omega_{\mu\nu}$ as well as potential similarities of the two representations of the spin factor have to be analyzed in the future.

The developed formalism can immediately be generalized to quantum fluctuations with higher spin as well as to non abelian gauge theories. This should be readily implementable, because the definition of the loop derivative is independent of the chosen gauge group. As we have shown in this thesis the analyzed gauge theory only depends on the gauge group into which the group of loops is mapped, all geometrical and dynamical properties of the gauge theory like connection or curvature are encoded in the abstract loop calculus and therefore are transfused to the resulting gauge theory.

As is standard in the worldline literature [8],[7],[12] it is also possible to represent the Dirac algebra and the resulting effective action in terms of a Grassmanian path integral in the following representation

$$\Gamma_{\rm Ferm}[A] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2T} \int_{\rm sym} \mathcal{D}x \int_{\rm asym} \mathcal{D}\psi\; e^{-\int_0^T d\tau L_{\rm spin}},$$

with

$$L_{\rm spin} = \frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi_\mu\dot{\psi}^\mu + ie\;\dot{x}^\mu A_\mu - ie\;\dot{\psi}^\mu F_{\mu\nu}\psi^\mu.$$

Starting from this representation, our line of reasoning can immediately be applied, resulting in the following new expression for the QED action

$$\Gamma_{\rm Ferm}[A] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2T} \int_{\rm sym} \mathcal{D}x \int_{\rm asym} \mathcal{D}\psi\; e^{-\int d\tau \frac{\dot{x}^2}{4}} e^{ie\oint dx A} e^{-\int d\tau \frac{\psi\dot{\psi}}{2}} : e^{\int d\tau\; \dot{\psi}\omega\psi} : \;.$$

Normal ordering takes care of the removal of self contractions of the spin factor, whereas the path ordering is automatically guaranteed by the Grassmann integral. An interesting open question of this representation is related to the fate of Supersymmetry. Whereas the standard representation has a worldline Supersymmety, the Supersymmety status of our new representation remains to be investigated.

Another interesting point for future research is the question of the numerical implementation of our formalism. An immediate numerical realization is inhibited by the normal ordering prescription. This requires the study of possible alternatives. If the nature of our spin factor is topological, it should then be possible to classify the worldlines in terms of their topological properties. This would facilitate the implementation of an algorithm that performs a Monte Carlo sampling for each individual topological sector seperately.

To summarize, we have performed a first detailed analysis of spin-factor calculus in the second-order formalism of QED. We believe that this opens the door for many further studies of the interrelation between spin and external fields in a geometric language.

References

[1] R. P. Feynman, Phys. Rev. **80** (1950) 440.

[2] E. S. Fradkin and A. A. Tseytlin, Phys. Lett. B **163** (1985) 123.

[3] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B **298** (1988) 109.

[4] Z. Bern and D. A. Kosower, Nucl. Phys. B **379** (1992) 451.

[5] Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. Lett. **70** (1993) 2677 [arXiv:hep-ph/9302280].

[6] Z. Bern, D. C. Dunbar and T. Shimada, Phys. Lett. B **312** (1993) 277 [arXiv:hep-th/9307001].

[7] M. G. Schmidt and C. Schubert, Phys. Lett. B **318** (1993) 438 [arXiv:hep-th/9309055].

[8] M. G. Schmidt and C. Schubert, Phys. Rev. D **53** (1996) 2150 [arXiv:hep-th/9410100].

[9] M. Reuter, M. G. Schmidt and C. Schubert, Annals Phys. **259** (1997) 313 [arXiv:hep-th/9610191].

[10] M. J. Strassler, Nucl. Phys. B **385** (1992) 145 [arXiv:hep-ph/9205205].

[11] M. J. Strassler, UMI-93-26566

[12] C. Schubert, Phys. Rept. **355** (2001) 73 [arXiv:hep-th/0101036].

[13] H. Gies, K. Langfeld and L. Moyaerts, JHEP **0306** (2003) 018 [arXiv:hep-th/0303264].

[14] K. Langfeld, L. Moyaerts and H. Gies, Nucl. Phys. B **646** (2002) 158 [arXiv:hep-th/0205304].

[15] H. Gies and K. Langfeld, Int. J. Mod. Phys. A **17** (2002) 966 [arXiv:hep-ph/0112198].

[16] M. G. Schmidt and I. O. Stamatescu, Mod. Phys. Lett. A **18** (2003) 1499.

[17] M. G. Schmidt and I. O. Stamatescu, arXiv:hep-lat/0201002.

[18] A. M. Polyakov, Nucl. Phys. B **164** (1980) 171.

[19] A. A. Migdal, Phys. Rept. **102** (1983) 199.

[20] C. Rovelli, Class. Quant. Grav. **8** (1991) 1613.

[21] M. Gaul and C. Rovelli, Lect. Notes Phys. **541** (2000) 277 [arXiv:gr-qc/9910079].

[22] A. Ashtekar, Phys. Rev. D **36** (1987) 1587.

[23] A. Ashtekar, Phys. Rev. Lett. **57** (1986) 2244.

[24] A.M. Polyakov , Mod. Phys. Lett. A **3**, (1988) 335

[25] G. P. Korchemsky, Phys. Lett. B **275** (1992) 375.

[26] I. A. Korchemskaya and G. P. Korchemsky, J. Phys. A **24** (1991) 4511 [Sov. J. Nucl. Phys. **54** (1991 YAFIA,54,1718-1731.1991) 1053].

[27] S. D. Avramis, A. I. Karanikas and C. N. Ktorides, Phys. Rev. D **66** (2002) 045017 [arXiv:hep-th/0205272].

[28] A. I. Karanikas and C. N. Ktorides, JHEP **9911** (1999) 033 [arXiv:hep-th/9905027].

[29] W. Heisenberg and H. Euler, Z. Phys. **98** (1936) 714.

[30] V. Weisskopf, K. Dan. Vidensk. Selsk. Mat. Fy. Medd. **14**, 1 (1936).

[31] J. S. Schwinger, Phys. Rev. **82**, 664 (1951).

[32] W. Dittrich and M. Reuter, Lect. Notes Phys. **220** (1985) 1.

[33] W. Dittrich and H. Gies, Springer Tracts Mod. Phys. **166** (2000) 1.

[34] V. P. Gusynin and I. A. Shovkovy, J. Math. Phys. **40** (1999) 5406 [arXiv:hep-th/9804143].

[35] B. Kors and M. G. Schmidt, arXiv:hep-th/0003171.

[36] B. Kors and M. G. Schmidt, Eur. Phys. J. C **6** (1999) 175 [arXiv:hep-th/9803144].

[37] M. G. Schmidt and C. Schubert, arXiv:hep-th/9810161.

[38] M. G. Schmidt and C. Schubert, arXiv:hep-ph/9412358.

[39] H. Reinhardt, Lecture notes on Yang-Mills theories, Tübingen U. (1998)

[40] R. Gambini and J. Pullin, Loops, Knots, Gauge Theories and Quantum Gravity, Cambridge University Press (1996)

Danke!

Für die unterstützende, motivierende und freundliche Betreuung sowie für das spannende und interessante Thema möchte ich mich herzlichst bei Holger Gies bedanken, der mir jederzeit gestattete seine Arbeit zu unterbrechen und niemals müde wurde mir meine oftmals dummen Fragen zu beantworten. Als weiteren Betreuer danke ich ebenso Prof. Michael G. Schmidt, der sich auch bereit erklärt hat, die Erstkorrektur dieser Arbeit zu übernehmen, ebenso wie dem Zweitkorrektor, welcher zu diesem Zeitpunkt noch nicht fest steht. Weiterhin möchte ich dem anderen Diplomanden der ”Weltlinienarbeitsgruppe” Klaus Klingmüller für die gute Zusammenarbeit und nicht zuletzt für den unermüdlichen Beistand bei Problemen mit der ”eqnarray”-Funktion danken. Auch danke ich allen anderen ”Insassen” des Westzimmers für die oftmals anregenden und erheiternden Diskussionen, die über so manch eisigen Winterabend hinweghalfen, wenn die Heizung im Westzimmer sich mal wieder den Außentemperaturen geschlagen geben mußte.

Ebenfalls bedanke ich mich bei meiner Familie für die motivierende Unterstützung während meines Studiums.

Für eine aufregende Zeit neben der fachlichen Beschäftigung danke ich insbesondere Matthias, Tobias, Sophie, Dennis, Fritz, Lars und Ela. Den Jungs von der U.S. Army ein großes Dankeschön für einige der besten Parties die ich je erlebt habe.

Erklärung:

Ich versichere, daß ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den ....

Unterschrift