13. Exercise sheet: Quantum Field Theory

Aufgabe 28:

(a) Show that the following important identity holds for an $N \times N$ matrix A

$$\ln \det A = \operatorname{tr} \ln A.$$

(b) Show that the trace of a function of the Laplace operator in continuous spacetime can be written as a simple integral in Fourier space:

$$\operatorname{Tr}\, f(-\partial^2) = V_D \int \frac{d^D p}{(2\pi)^D}\, f(p^2),$$

where V_D denotes the D dimensional spacetime volume.

Hint: first use the coordinate space representation of the trace, $\operatorname{Tr}[\dots] = \int d^D x \,\langle x| \dots |x\rangle$, and introduce intermediate momentum space states. You may use the following normalization for the coordinate-space–momentum-space transition amplitude: $\langle x|p\rangle = \exp(ixp)/(2\pi)^{D/2}$.

Aufgabe 29:
Consider the functional integral for a theory of a real scalar field with the action

$$S = \int d^D x \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4 \right).$$

(a) Decompose $\phi = \phi_0 + \varphi$, where ϕ_0 is supposed to satisfy the classical equation of motion $\delta S[\phi = \phi_0]/\delta\phi = J$, expand the action to second order in φ, such that the functional integral

$$Z[J] = \int \mathcal{D}\phi\, e^{iS - i\int J\phi}$$

yields a Gauß/Fresnel form. Convince yourself that the integral can be carried out using the continuum version of Exercise 27, implying that this approximation leads to

$$T[J,\phi_0] = \frac{Z[J,\phi_0]}{Z[0,0]} = e^{iS[\phi_0] - i\int J\phi_0}\, \frac{\det^{-\frac{1}{2}}(-\partial^2 - m^2 - \frac{\lambda}{2}\phi_0^2)}{\det^{-\frac{1}{2}}(-\partial^2 - m^2)}.$$

(b) Show that

$$W[\phi_0] = -i \ln T[J = 0, \phi_0]$$

generates all one-particle-irreducible (1PI) diagrams to one-loop order. Hint: use Exercise 28 and expand the determinant (or rather the trace) with respect to ϕ_0.

Aufgabe 30:
Summer project: re-compute the one-loop polarization tensor in QED that has been discussed in the Lectures. For hints, see Peskin-Schroeder, Section 10.3.