10. Exercise sheet: Quantum Field Theory

Aufgabe 21:

In the lectures, we found four independent solutions of the free Dirac theory of the form $\psi(x) = u(p)e^{-ipx}$ and $\psi(x) = v(p)e^{ipx}$.

(a) Verify that the algebraic solutions in momentum space,

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\,\xi^s \\ \sqrt{p\cdot\sigma}\,\xi^s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\,\eta^s \\ -\sqrt{p\cdot\sigma}\,\eta^s \end{pmatrix}, \quad s = 1, 2,$$

with ξ^s and η^s denoting a basis of 2-spinors, indeed solve the free Dirac equation; $\bar{\sigma}^\mu = (\mathbb{1}, -\boldsymbol{\sigma})$, and $\sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma})$.

(b) Verify the normalizations

$$u^{r\dagger}(p)u^s(p) = 2E_{\mathbf{p}}\delta^{rs}, \quad \bar{u}^r(p)u^s(p) = 2m\delta^{rs}, \quad v^{r\dagger}(p)v^s(p) = 2E_{\mathbf{p}}\delta^{rs}, \quad \bar{v}^r(p)v^s(p) = -2m\delta^{rs}.$$

(c) Also verify the spin sums

$$\sum_s u^s(p)\bar{u}^s(p) = \gamma\cdot p + m, \quad \sum_s v^s(p)\bar{v}^s(p) = \gamma\cdot p - m,$$

using the fact that

$$\sum_{s=1,2} \xi^s\xi^{s\dagger} = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Aufgabe 22:

Proof the following identities, using only the Clifford algebra of the Dirac matrices (not using a concrete representation),

(a) $\gamma^\mu\gamma_\mu = 4\,\mathbb{1}$,

(b) $\gamma^\mu \not{p} \gamma_\mu = -2\not{p}$,

(c) $\gamma^\mu \not{p}\not{q} \gamma_\mu = 4p\cdot q\,\mathbb{1}$,

where $\not{p} \equiv p^\mu\gamma_\mu$ and $p\cdot q \equiv p^\mu q_\mu$.

Aufgabe 23:

Verify the fundamental anti-commutation relation $\left\{\psi_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{y})\right\} = \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}$, using the representation of the field operator and the ladder operator algebra given in the lectures.