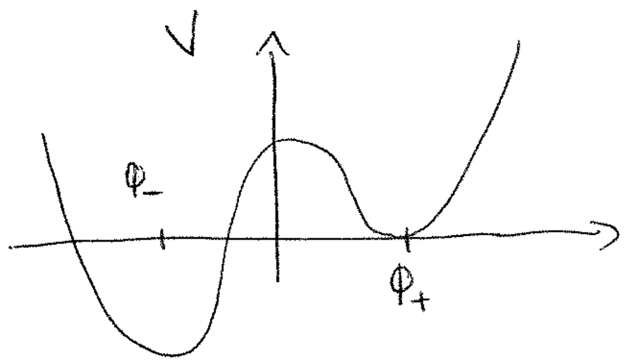
1.5 Decay of the false vacuum

As the universe started from a hot initial state and underwent a rapid evolution with the standard model of particle physics possibly arising as the low-energy effective theory, the effective potential of the standard-model fields (or other so far unknown fields) has certainly also evolved rather rapidly. At a given moment of evolution, such an effective potential may have looked like this



Classically, both local minima correspond to stable states.

Quantum mechanically, φ_+ is only metastable because of the tunnel effect. A quantum mechanical wave function if initially localized around φ_+ can acquire nonzero values near the φ_- minimum, such that a measurement of a quantum particle near φ_- (where φ corresponds to a space coordinate value in quantum mechanics) can give a non-zero value after some time.

Transferring these considerations to field theory where φ is a field amplitude and quantum-mechanical time corresponds to 3+1 dimensional spacetime, we expect φ_--values to arise at some points in spacetime, even if the field VEV initially is located at φ_+. Energetically, we expect these φ_- "bubbles" to grow analogously to water gas bubbles growing inside liquid water at the boiling point. In this classical water analogue, bubbles arise due to thermal fluctuations, whereas here we expect the bubbles to be generated due to quantum fluctuations.

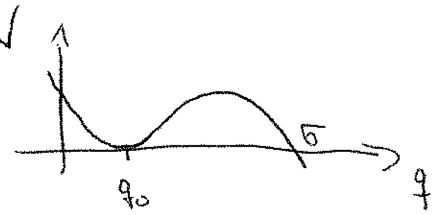
Let us start with the quantum mechanical tunnel effect.

The width of the decay Γ of the metastable state ($\sim$ inverse lifetime) can be determined within the WKB approximation

$$\Gamma = A \, e^{-\frac{B}{\hbar}} \left(1 + \mathcal{O}(\hbar)\right) \qquad (1.111)$$

where $B = 2 \int_{q_0}^{\sigma} dq \, (2V)^{1/2}$

for a 1-dimensional problem where the potential zero point has been chosen such that $E = 0$ for the local minimum q_0 $(\hat{=} \varphi_+)$

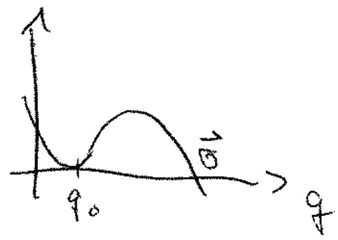


Before we consider the analogous QFT problem, let us schematically discuss the generalization of QM tunneling in more than 1 space dimension. This problem is surprisingly non-trivial and has first been solved by Banks, Bender & Wu in 1973. The main reason why this is non-trivial lies in the fact that while $V(\vec{q}_0) \stackrel{!}{=} 0$ may remain a well defined local minimum, the set of points where $V(\vec{\sigma})=0$ may describe a whole surface of points. The corresponding analysis in a QM-Hamiltonian language is possible and can conveniently be done using an alternative version of the action principle. Here, we will motivate the final expression by a heuristic consideration based on the path integral formulation of QM.

The probability amplitude for a QM particle being localized in a local minimum $\vec{q}_0$ to first tunnel and then propagate to an unspecified asymptotic state is related to the path integral

$$\int\limits_{\vec{q}(t\to-\infty)=\vec{q}_0} \mathcal{D}\vec{q} \quad e^{\frac{i}{\hbar}S[\vec{q}]} \tag{1.112}$$

where $S[\vec{q}]=\int_{-\infty}^{\infty} dt\, L(\vec{q},\dot{\vec{q}})$. At some time which we may choose at $t=0$, the particle has tunneled to the surface $\vec{\sigma}$, where $V(\vec{\sigma})=0$.



For those parts of the path, where the particle is inside the barrier, the action is not minimal, such that $e^{\frac{i}{\hbar}S}$ is wildly fluctuating. Let us assume that a rotation of the time axis $t \to \tau = it$ is possible, then

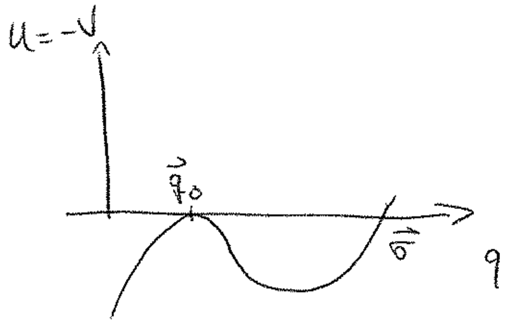
$$S = \int dt \left(\frac{1}{2} \frac{d\vec{q}}{dt} \cdot \frac{d\vec{q}}{dt} - V(q) \right) = -i \int d\tau \left(-\frac{1}{2} \frac{d\vec{q}}{d\tau} \cdot \frac{d\vec{q}}{d\tau} - V \right)$$

$$= i \int d\tau \left(\frac{1}{2} \left(\frac{d\vec{q}}{d\tau} \right)^2 + V \right) =: i \int d\tau \, L_E = i \, S_E \qquad (1.113)$$

such that

$$e^{\frac{i}{\hbar}S} = e^{-\frac{1}{\hbar}S_E} . \qquad (1.114)$$

Now rotating the contour such that $\tau \in \mathbb{R}$, S_E is strictly non-negative and bounded from below.
In the semi-classical limit $\hbar \to 0$, we expect the path integral to be dominated by the "minimal path", i.e., the path that minimizes the Euklidean action.
The Euclidean problem now looks like a standard classical-mechanics problem, however, with inverted classical potential $U = -V$



Hence, the minimum path is that of a classical particle moving in a classical potential well. For the solution, we have to specify the boundary conditions. As $t = 0 = \tau$, we still have $\vec{q}(\tau=0) = \vec{\sigma}$. Since also $E = 0$ has been already specified, the "particle" is at rest at $\tau = 0$

$$\left.\frac{d\vec{q}}{d\tau}\right|_{\vec{\sigma}} = 0 \tag{1.115}$$

From the sketch of the potential $U = -V$ above, it is clear that the metastable point at $\vec{q}_0$ is approached for $\tau \Rightarrow -\infty$: $\vec{q}(\tau \Rightarrow -\infty) = \vec{q}_0$. The Euler-Lagrange equations governing this motion reads

$$\frac{d^2\vec{q}}{d\tau^2} = + \frac{\partial V}{\partial \vec{q}} \tag{1.116}$$

$$E = 0 \Rightarrow \qquad 0 = \frac{1}{2}\,\frac{d\vec{q}}{d\tau}\cdot\frac{d\vec{q}}{d\tau} + U \Rightarrow \frac{1}{2}\left(\frac{d\vec{q}}{d\tau}\right)^2 = V \tag{1.117}$$

The equation of motion (1.116) is obviously symmetric under $\tau \Leftrightarrow -\tau$, hence the particle will move back to $\vec{q}_0$ for asymptotically large Euclidean times $\vec{q}(\tau \Rightarrow +\infty) = \vec{q}_0$. (NB: Here, it becomes obvious that the Euclidean consideration is a technical tool, the regime $\tau \to \infty$ is obviously not similar to $t \to \infty$. The latter will be considered separately, below.)

This Euclidean motion from $\vec{q}_0$ to $\vec{\sigma}$ and back is called the "bounce" (Coleman). Note that even if $V(\vec{\sigma}) = 0$ defines a complicated surface, the Euclidean action principle typically singles out one minimum path (if there is more than one path, the corresponding rate is enhanced by the corresponding multiplicity of this degeneracy.)

The exponent of the tunnel amplitude in the limit $\hbar \to 0$ corresponds to the Euclidean action of the bounce:

$$\underline{\underline{B = S_E}} = \int_{-\infty}^{\infty} d\tau \, L_E = \int_{-\infty}^{\infty} d\tau \left(\frac{1}{2} \left(\frac{d\vec{q}}{d\tau} \right)^2 + V \right)$$

$ds^2 = d\vec{q} \cdot d\vec{q}$

$$\overset{(1.117)}{=} \int_{-\infty}^{\infty} d\tau \left(\frac{d\vec{q}}{d\tau} \right)^2 = 2 \int_{\vec{q}_0}^{\vec{\sigma}} ds \left(\left(\frac{d\vec{q}}{d\tau} \right)^2 \right)^{1/2}$$

$$\underset{(1.117)}{=} \underline{\underline{2 \int_{\vec{q}_0}^{\vec{\sigma}} ds \, (2V)^{1/2}}} \qquad (1.118)$$

which is also in agreement with the 1-dim. result (1.111).

In a modern language, the bounce corresponds to an "instanton" of the Euclidean theory.

These results can readily be transferred to the corresponding QFT tunneling we are interested in. In the semi-classical limit, we expect the functional integral that describes the transition probability from φ_+ to φ_- to be dominated by field configuration minimizing the Euclidean action. The corresponding tunneling exponent corresponds to the Euclidean action evaluated on this bounce:

$$B = S_E[\varphi] = \int d\tau\, d^3x \left[\frac{1}{2}\left(\frac{\partial \varphi}{\partial \tau}\right)^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) \right], \qquad (1.119)$$

where φ solves the Euclidean equation of motion:

$$\left(\frac{\partial^2}{\partial \tau^2} + \vec{\nabla}^2\right) \varphi = V'(\varphi), \qquad (1.120)$$

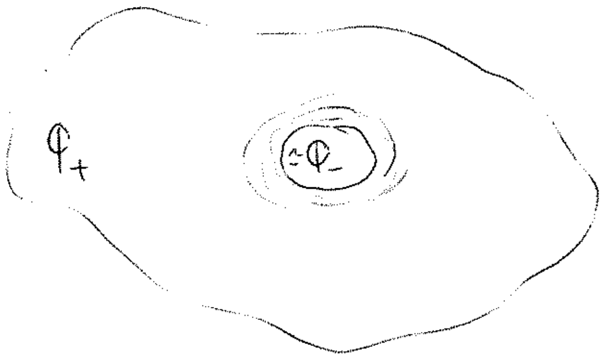
satisfying the boundary condition

$$\lim_{\tau \to \pm\infty} \varphi(\tau, \vec{x}) = \varphi_+ \quad \text{and} \quad \frac{\partial \varphi}{\partial \tau}(0, \vec{x}) = 0 \qquad (1.121)$$

For $B < \infty$, we can extend the first boundary condition to hold at Euclidean infinity

$$\lim_{\sqrt{\tau^2 + \vec{x}^2} \to \infty} \varphi(\tau, \vec{x}) = \varphi_+ . \qquad (1.122)$$

Hence, we expect the bounce to be localized in Euclidean space near $\tau = 0$, at a space region which we can associate with the origin of the spatial coordinates $\vec{x}^2 \simeq 0$.



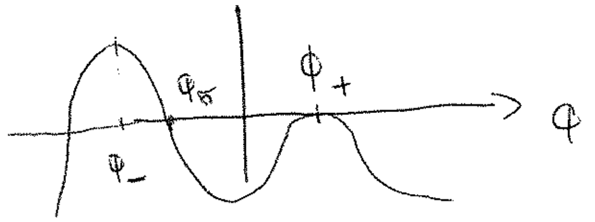
Let us search for solutions under the simplifying assumption that the solution is invariant under Euclidean $O(4)$ rotations. Introducing the spherical coordinate $\varrho = \sqrt{\tau^2 + \vec{x}^2}$, the bounce φ only depends on ϱ, $\varphi = \varphi(\varrho)$. Then the above equations simplify to

$$B = S_E = 2\pi^2 \int_0^\infty d\varrho\, \varrho^3 \left(\frac{1}{2} \left(\frac{d\varphi}{d\varrho}\right)^2 + V \right) \qquad (1.123)$$

with $\varphi(\varrho)$ satisfying

$$\frac{d^2\varphi}{d\varrho^2} + \frac{3}{\varrho}\frac{d\varphi}{d\varrho} = V', \quad \varphi(\varrho \to \infty) = \varphi_+ . \qquad (1.124)$$

This equation is reminiscent to that of a mechanical particle in a potential $-V$ with a Stokes friction coefficient $\sim \frac{1}{\varrho}$



From the mechanical analogue, we infer that there is always a solution that interconnects $\varphi(z \to \infty) = \varphi_+$ with a value $\varphi(0)$ approaching φ_σ near φ_- (without friction, $\varphi(0) = \varphi_\sigma$ would hold identically).

From now on, we concentrate on a specific example that can be analyzed within simplifying assumptions. More complicated examples can, of course, straightforwardly studied by numerical integration.

Let us start with a $\mathbb{Z}_2$-symmetric scalar potential of φ^4-type:

$$V_{sym} = -\frac{1}{2}\mu^2\varphi^2 + \frac{\lambda}{4!}\varphi^4 + \frac{3}{2}\frac{\mu^4}{\lambda} = \frac{\lambda}{4!}\left(\varphi^2 - 6\frac{\mu^2}{\lambda}\right)^2 \tag{1.125}$$

that satisfies

$$V'_{sym}(\pm v) = 0 \quad , \quad v^2 = 6\frac{\mu^2}{\lambda} \tag{1.126}$$

at the two local and degenerate minima $\pm v$. Excitations on top of these minima have the

$$\text{mass} \quad V''_{sym}(\pm v) = 2\mu^2 . \tag{1.127}$$

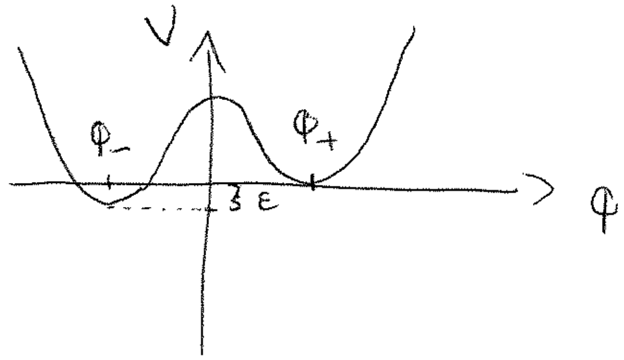
Now, we lift this degeneracy of the vacua by adding a Z_2-symmetry breaking term:

$$V = V_{sym} + \frac{\varepsilon}{2v}(\phi - v) \quad , \quad \varepsilon > 0 \qquad (1.128)$$

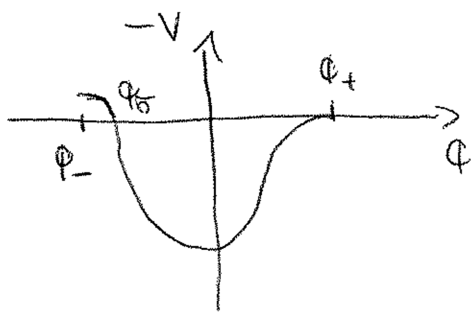
The two minima are still located at

$$\phi_\pm = \pm v + \mathcal{O}(\varepsilon) \qquad (1.129)$$

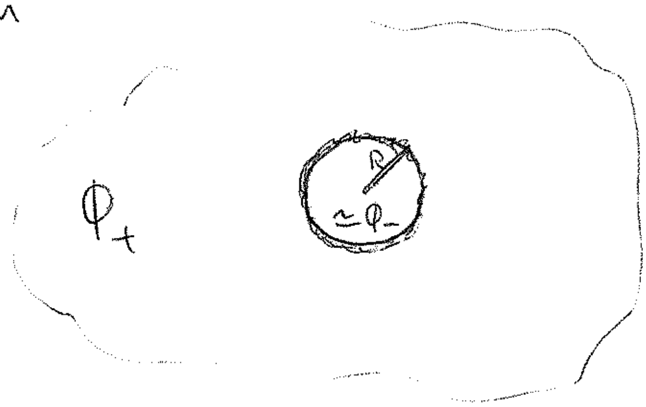
The parameter ε denotes the energy difference between the true vacuum ϕ_- and the false vacuum at ϕ_+ (local minimum)



The related Euclidean problem deals with the inverted potential $-V$. As discussed in the general case, the Euclidean bounce dominating the semi-classical approximation is characterized by $\phi(\varsigma \to \infty) = \phi_+$, whereas $\phi(\varsigma)$



approaches φ_0 for $\rho \to 0$; the precise value of $\varphi(\rho \to 0)$ depends on the influence of the friction term. Let us introduce a characteristic length scale $\rho = R$, being defined by the ρ value, where $\varphi(\rho)$ passes the minimum point of the Euclidean potential. In the present example, we have $\varphi(R) \simeq 0 + \mathcal{O}(\varepsilon)$. The scale R corresponds to the characteristic radius of the bubble of true vacuum inside the false vacuum



As $\varphi(R)$ is at the minimum of the Euclidean potential, the bounce has acquired its maximum speed at R:

$$\frac{d\varphi}{d\rho}(R) = \max_{\rho} \frac{d\varphi(\rho)}{d\rho} . \qquad (1.130)$$

Correspondingly, the friction term is expected to play an important role here (if at all). If the bounce passes rapidly through this potential minimum, the transition region between true and false vacuum is small. In other words, the bubble wall is thin in this case. Let us assume that the parameters ε, μ and λ can be chosen such that (for solving the eq. of motion)

- the friction term can be neglected
- the ε-term in the potential can be neglected

This is called the "thin-wall" approximation. Then, the Euclidean equation of motion

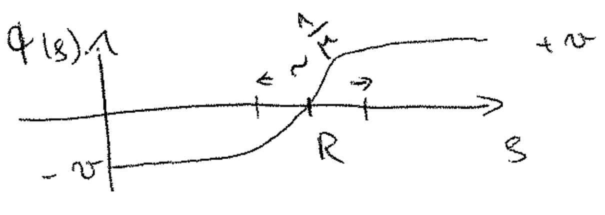
$$\frac{d^2\phi}{d\varsigma^2} = V'_{sym}(\phi)$$

can be solved by using energy conservation:

$$(\varsigma - R) = \int_0^{\phi} d\phi' \frac{1}{(2V_{sym}(\phi'))^{1/2}}, \qquad (1.131)$$

yielding

$$\phi(\varsigma) = v \tanh \frac{1}{\sqrt{2}} \mu (\varsigma - R). \qquad (1.132)$$



Here, we can read off that the width of the bubble wall has a characteristic length scale of $\sim \frac{1}{\mu}$.
The thin-wall approximation hence becomes self-consistent

in the limit $\frac{1}{\mu} \ll R$ and $\varepsilon \ll v$ (if R can be adjusted appropriately)

On the basis of this bounce solution, we can now determine the tunnel exponent,

$$B = S_E = 2\pi^2 \int_0^\infty d\varrho\, \varrho^3 \left[\frac{1}{2}\left(\frac{d\varphi}{d\varrho}\right)^2 + V(\varphi) \right]$$

$$\underset{\varepsilon \text{ kept finite}}{\simeq} \underbrace{-\frac{1}{2}\pi^2 R^4 \varepsilon}_{\text{bulk}} + \underbrace{\pi^2 R^3 \frac{2\sqrt{2}\,\mu^3}{\lambda}}_{\text{wall}} \qquad (1.133)$$

where the first term is proportional to the volume of the bubble (denoting a bulk term) and the second is proportional to the surface of the bubble (denoting the action "to be paid" for the non-constant field region).

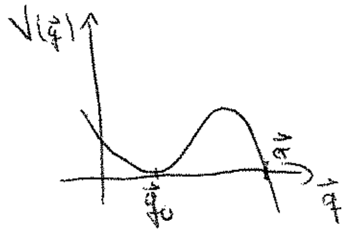
So far, R has been a yet-to-be-determined parameter which finally follows from the fact that the bounce is the minimum of the Euclidean action, $\frac{dS_E}{dR} = 0$

$$\Rightarrow \quad R = \frac{6\sqrt{2}\,\mu^3}{\lambda\varepsilon} \qquad (1.134)$$

Hence, ε has to be chosen small enough to justify the thin-wall approximation. In this limit, the tunnel exponent yields

$$B = S_E \simeq \frac{2^5 \cdot 3^3 \cdot \pi^2 \mu^{12}}{\varepsilon^3 \lambda^4} \qquad (1.135)$$

So far, we have worked in Euclidean space. Let us now try to interpret these results in Minkowski space. Again, we start with quantum mechanics.



The QM particle is supposed to reside in the metastable state $\vec{q}_0$ for $t \to -\infty$. We have defined the time $t=0$ as the time of the tunneling event (e.g. when the wave function at $\vec{\sigma}$ reaches its maximum). For $t>0$, the QM particle propagates in the classically allowed region according to Schrödinger's wave equation.

In the QFT false vacuum decay problem. The tunneling event corresponds to spontaneous bubble creation at $t=0$. Since $t=0$ corresponds to $\tau=0$, the Euclidean configuration serves as the initial condition for the Minkowski evolution for later times:

$$\phi(t=0,\vec{x}) = \phi(\tau=0,\vec{x}) \qquad (1.136)$$

Because of momentum conservation, we can also impose

$$\frac{\partial}{\partial t}\phi(t=0,\vec{x}) = 0 \qquad (1.137)$$

(which holds anyway, see (1.121))

The propagation at later times is then determined by the Klein-Gordon equation

$$-\frac{\partial^2}{\partial t^2}\phi + \vec{\nabla}^2 \phi = V'(\phi) \qquad \text{for } t>0 \qquad (1.138)$$

The solution follows immediately from analytic continuation

$$\phi(t,\vec{x}) = \phi\left(\rho = \sqrt{\vec{x}^2 - t^2}\right) \qquad (1.139)$$

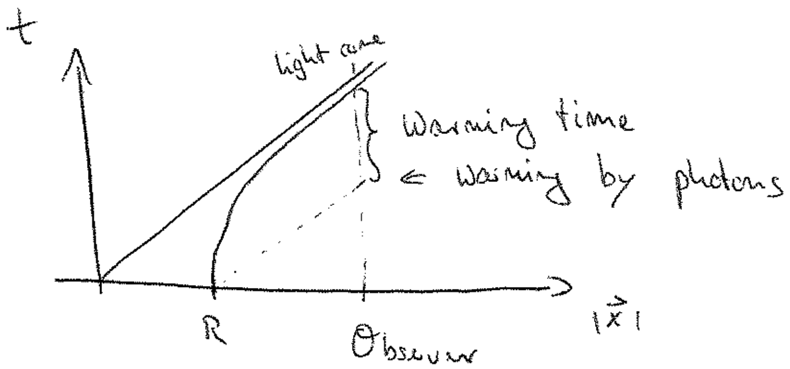
(NB: Since $\frac{\partial}{\partial \tau}\phi(\rho) = 0$, $\phi(\rho)$ has to be continued to negative ρ values (if needed) such that $\phi(\rho)$ is even. Hence, the sign to be chosen in (1.139) for the square root is irrelevant.)

A number of conclusions can now be drawn:

(1) The Euclidean $O(4)$ invariance of the bounce translates into the Lorentzian $O(3,1)$ invariance. Physically, this implies that the bubble growth is the same for all observers.

(2) The location of the bubble wall is at $\varrho = R$ which corresponds to a hyperboloid in Minkowski space

$$R^2 = \vec{x}^2 - t^2 \qquad (1.140)$$



If false vacuum decay happens because of a false vacuum in a particle physics effective potential, we expect R to acquire a typical particle physics scale, e.g.

$$R \simeq \mathcal{O}(1\,\mathrm{fm}) \qquad (1.141)$$

for the strong interactions (or a factor of 1000 shorter for the electroweak interactions).

(3) The warning time in this case for a bubble of true vacuum approaching us then would be $T \simeq R \sim 10^{-21}\,\mathrm{s}$. (1.142)

(4) In the thin-wall approximation, it is straightforward to show that the energy of the propagating bubble wall is

$$E_{wall} = \frac{4\pi}{3} \varepsilon |\vec{x}|^3 \tag{1.143}$$

which equals the total energy released by the false vacuum decay. Hence, all the energy goes into the bubble propagation. As a consequence, hardly any energy goes into particle production (as could be expected naively) inside the wall. Only true vacuum is left behind.

This computation of false vacuum decay is due to Coleman (PRD 15, 2929 (1977)).

In a sequel (together with C. Callan in PRD 16, 1762 ('77)), Coleman showed that the preexponential factor is dominated (semi-classically) by fluctuations about the Euclidean bounce, yielding a functional determinant,

$$\frac{\Gamma}{V} = \left(\frac{B}{2\pi\hbar}\right)^{D/2} e^{-\frac{B}{\hbar}} \left| \frac{\det'\left(-\partial_t^2 + V''(\varphi_{bounce})\right)}{\det\left(-\partial_t^2 + V''(\varphi_+)\right)} \right|^{-1/2} \tag{1.144}$$

where det' denotes the prescription to compute the determinant without the zero modes arising from time translation invariance.

If the universe had undergone several stages of evolution of the effective potential (of some relevant degree of freedom), could we now safely exclude that we are in a true vacuum state?

All we can actually say is that we are at least in a metastable state with a lifetime of the order of the age of the universe. If we happen to live in a false vacuum, a bubble of true vacuum approaching us at almost the speed of light could appear any time ...