Physics of Scales

– Lecture Notes –

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Preface

At present, this is a collection of notes that eventually want to become lecture notes. At the moment, the emphasis is on *notes*, sometimes containing item lists rather than full sentences. They are clearly still in its infancy.

These notes are based on a German version of my lecture notes on the physics of scales which I gave for the first time in the summer term 2004 in Heidelberg and which have evolved quite a bit over the years. This new English version also contains a few additions and is planned to replace the handwritten German version in the future.

In comparison to my lecture notes on the functional RG for gauge theories prepared for an ECT* school at Trento in 2006, the present set of notes is meant to cover the more introductory material as a prerequisite for the advanced (and topic-wise more focused) material.

Comments, suggestions, and hints at typos are more than welcome!

Jena, April 2024 Holger Gies

1 Introduction

This lecture course covers several aspects of the "Physics of Scales" as it is formalized within the concept of the renormalization group. In the following, I will try to provide a nontechnical first glance at these aspects, the corresponding guiding questions and resulting pictures.

1.1 Perturbative quantum field theory

Observation: correlation functions in QFT – if computed in (naive) perturbation theory – generically contain divergencies.

- This is technically dealt with by a *regularization* prescription that renders objectsto be computed finite and quantitatively controls the divergencies. An example is given by a UV cutoff Λ, i.e., a high-energy ("ultraviolet") scale beyond which potentially divergent momentum integrals are cut off by hand.

- this is followed by a (still technical) procedure of *renormalization* guided by the idea that physical quantities and correlation functions should be finite and independent or at least insensitive to the precise value of Λ; in the ideal case, these quantities should exist in the limit $\Lambda \to \infty$. This idea is realized by choosing the parameters such as coupling constants, masses, field amplitudes of a given quantum field theory such that correlation functions feature the required properties.

This observation and the resulting procedure goes along with a number of obvious questions:

- Under which conditions is this procedure possible? The answer to this question leads to the classification of perturbatively renormalizable theories.

- Are the predictions of the theory independent of the technical steps such as the regularization procedure?

- Which role does the cutoff play?

- Why are (almost) all theories realized in nature, i.e. in the standard model of elementary particle physics, renormalizable in perturbative QFT?

1.2 Critical Phenomena

Observation: rather different systems with many degrees of freedom and complex interactions exhibit quantitatively identical properties in the vicinity of phase transitions. These properties turn out to be describable with only a few variables and scale relations (*universality*). These scale relations very often follow power laws, e.g., $\phi \sim |t|^\beta$. Here, the physical meaning of ϕ and t depends on the system, but many very different systems can have the same *critical exponent* β, being a number.

As a particularity, β can be non-rational and depend only on the dimensionality and the symmetries of the system (+ very few other details), but does not depend on the details of the interaction or microscopic degrees of freedom.

This universality can be understood using the renormalization group (RG) ideas à la Kadanoff, Wilson, and others.

- start from a microscopic theory (defined in terms of a Hamiltonian or action) and average successively over fluctuations from small to large length scales (*coarse graining*). Based on this procedure, a scale-dependent averaged Hamiltonian or action can be obtained.

- universality results if this averaging procedure (*RG transformation*) exhibits a *fixed point*. If the fixed point has suitable properties, many microscopically different systems may approach the fixed point upon the averaging procedure. Examples are given by ferromagnets, liquid-gas phase transitions, binary mixtures, superfluids, polymeres, thermal SU(2) Yang-Mills theory, etc.

- It turns out that fixed-point properties can often be described by a renormalizable (quantum) field theory!

- In turn, renormalizable quantum field theories can be understood (and defined) as statistical systems at a critical point.

Let us add a few remarks to the observation of scaling relations. In fact, scaling relations are also known in classical systems. They are characteristic for systems where only a single scale is relevant. Consider, for instance, Kepler's third law, $T \sim a^{\frac{3}{2}}$ which has a large degree of universality, as it applies to small planets like Mercury as well as to the large and more distant gas giants. This is a scaling relation that follows from a simple consideration:

The Kepler problem is essentially defined in terms of the Newton potential which is inversely proportional to the relative distance of the two gravitationally interacting bodies,

$$V(r) \sim \frac{1}{r}, \quad \text{implying } rV(r) \sim \text{const.} \tag{1.1}$$

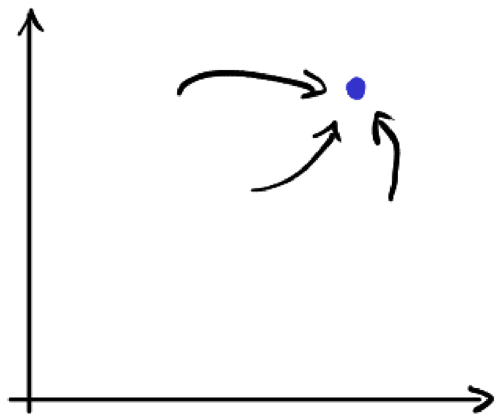

Figure 1.1: Sketch of "Theory Space". Consider an abstract space in which each point corresponds to a theory, e.g. defined in terms of a Hamiltonian or an action. Averaging or coarse graining produces a sequence of effective Hamiltonians. Universality arises if the averaging procedure for very different microscopic theories is attracted by a fixed point in theory space. Near the fixed point, all these different theories are described by similar Hamiltonians.

Now, V is an energy scale, e.g., in SI units we have $[V] = \frac{\text{kg m}^2}{\text{s}^2}$. As the mass units are taken care off by the masses of the bodies together with Newton's constant, we note that $[rV] \sim \frac{\text{m}^3}{\text{s}^2}$. Since $rV(r) \sim$ const., any time scale in the problem must be proportional to $r^{\frac{3}{2}}$, provided r is a proxy for a dominant single length scale in the system. Since characteristic distances r scale with the semi-major axis a, we end up with the scale relation between the period and the semi-major axis $T \sim a^{\frac{3}{2}}$, corresponding to Kepler's third law.

In this example, the other lengths scales, e.g., the radii of the two bodies R_1 and R_2 are much smaller than the characteristic distance scale r and thus "decouple" from the period. The universality expressed through Kepler's third law (applicable to all planets) comes about because of this decoupling of length scales. Note that exponent $\frac{3}{2}$ involved in this scaling relation is a rational number. This is rather generic for classical scaling laws: the corresponding exponents are typically rational or even integer numbers.

For systems dominated by statistical or quantum fluctuations, the exponents found in scaling relations are often non-rational. Consider a ferromagnet near the critical Curie temperature. A microscopic scale is given by the typical distance a of

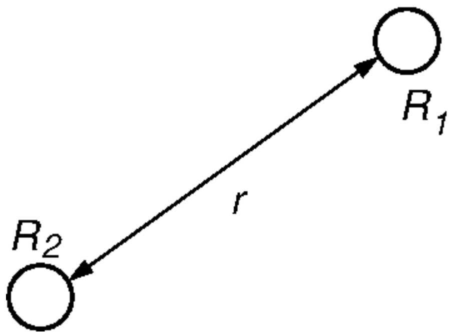

Figure 1.2: In the Kepler problem, there is a clear scale separation between the typical distance r of the two bodies and the radii R_1 and R_2 of the bodies.

the atoms or molecules dominating the microscopic spin. Think of a as a distance of nearest neighbors on a lattice. On the other hand, the correlation length ξ near the critical temperature can grow large, as the spins in large patches of the material tend to be aligned. Let us now study local fluctuations of the spins on some length scale r, where $a \lll r \ll \xi$. It turns out that the correlation function (Green's function) $G(r)$ of the local fluctuations obeys a Laplace equation (as is natural for systems with local interactions). Hence, the Green's function behaves as

$$G(r) \sim \frac{1}{r}, \quad \text{for } a \lll r \ll \xi. \tag{1.2}$$

In analogy to the classical case, we would expect that the microscopic lattice spacing a can be completely neglected compared to the macroscopic scales r and ξ. Hence, a natural ansatz for the Green's function taking ξ into account is

$$G(r) \stackrel{?}{=} \frac{1}{r} f(r/\xi), \tag{1.3}$$

where f is some dimensionless function of a dimensionless argument approaching a constant for small argument. If so, we straightforwardly obtain a prediction for the magnetic susceptibility which is obtained from the correlation function through

$$\chi \sim \int G(r) d^3 r \stackrel{?}{\sim} \xi^2. \tag{1.4}$$

This suggests the simple scaling relation that $\chi \sim \xi^2$ with a "classical" integer exponent. However, this exponent is in contradiction with the experimental result!

The reason is that the critical behavior is characterized by fluctuations on *all* length scales. Hence, there are also small-size fluctuations that still know about the lattice spacing a. We therefore need to take a still into account in our ansatz for the correlation function, replacing Eq. (1.3) by

$$G(r) \stackrel{?}{=} \frac{1}{r} f(r/\xi, a/\xi), \tag{1.5}$$

It turns out that the dependence of f on a/ξ follows a power law $\sim (a/\xi)^\eta$ for small a/ξ with η being a small number. Therefore, the magnetic susceptibility becomes

$$\chi \sim \int G(r) d^3r \sim a^\eta \, \xi^{2-\eta}. \tag{1.6}$$

The exponent $2 - \eta$ does not follow from a naive (classical) dimensional analysis of the Green's function equation. Hence, the deviation $\eta > 0$ is an example for an *anomalous dimension*.

1.3 RG-based construction/definition of QFTs

Problem: the analysis of divergencies in correlation functions or matrix elements beyond perturbation theory is difficult; e.g., a full nonperturbative analysis of the Schwinger functional, cf. below, in continuum quantum field theory is not available in general.

Idea: a single RG step is finite. Here, an RG step, for instance, in momentum space corresponds to an integration over a finite momentum interval. Since this integration interval is bounded and the integrand is finite in any momentum interval, a single RG step is well defined. This suggests to aim at a construction of the integration over all fluctuations in terms of an infinite sequence or sum of RG steps. In the limit of infinitesimal momentum intervals this sum becomes an integral. If only the change or a physical quantity from RG step to RG step is monitored we arrive at a formulation in terms of a differential equation.

Technically, this is formulated in the language of *RG flow equations*. It is convenient to formulate such flow equations not for individual physical quantities, but for generating functionals such as the Schwinger functional or the effective action of a QFT, or an effective Hamiltonian of a statistical system. In many parts of these notes, we will focus on the effective action Γ or its fluctuation averaged variant, the effective average action Γ_k that describes the dynamics of field expectation values at an RG momentum scale k. The RG flow equation for Γ_k is given by the *Wetterich equation* introduced in later sections. Its solution represents a continuous *RG trajectory* in theory space interconnecting the microscopic action S_Λ of a system to be quantized, $\Gamma_{k=\Lambda} = S_\Lambda$ with the full quantum effective action $\Gamma = \Gamma_{k=0}$. The latter corresponds to the generating functional of 1PI (one-particle-irreducible) correlation functions.

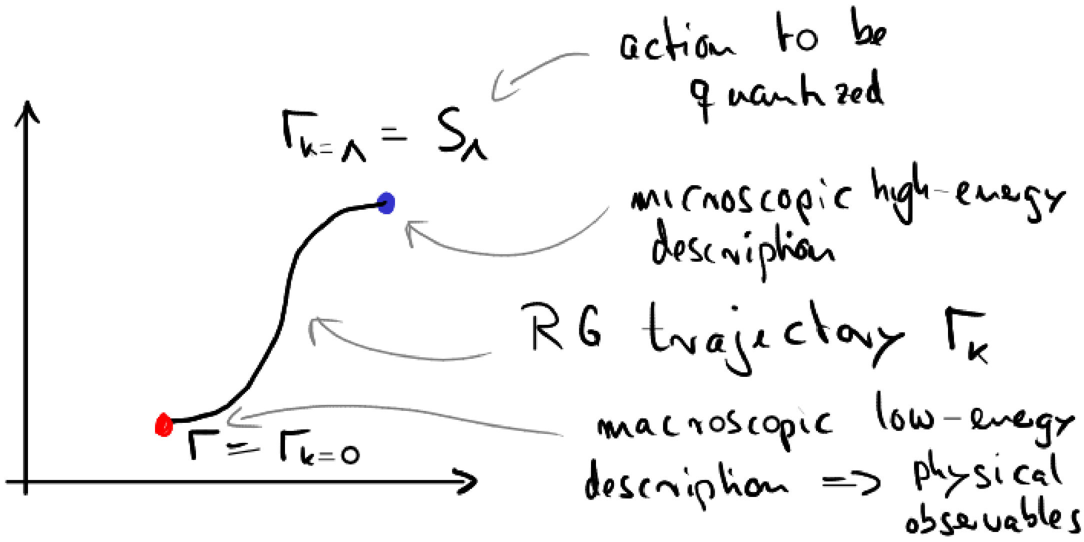

Figure 1.3: Illustration of the RG flow of a generating functional such as the effective action Γ in an abstract theory space. Given a starting point in terms of a microscopic action to be quantized, the flow equation describes an RG trajectory in the space of all possible actions parametrized by a momentum scale parameter k. Once, all fluctuations are averaged over, i.e., at $k = 0$ corresponding to macroscopic length scales or low momentum scales, we arrive at the effective action that describes the physics observable in low-energy experiments.

1.4 QFTs in the high-energy limit

The possibility to formulate general quantum field theories with a UV cutoff Λ or some similar regularization scale allows us to consider rather general – not necessarily perturbatively renormalizable – quantum field theories. In such a general case, the UV cutoff can become a physical parameter, i.e., it characterizes a scale below which the QFT description is valid, being replaced by a different theory beyond Λ. Such cases where Λ remains as a physical parameter are called *effective (quantum) field theories (EFT)*. Historically, Fermi's theory of β decay has been such an effective field theory. It is a valid and useful description of the weak nuclear force at low energies, but needs to be replaced by the full electroweak Higgs sector of the standard model at higher energies.

Perturbatively renormalizable theories have the additional property that physical observables do not depend on Λ, and therefore a large part of the dependence on the detailed form of the microscopic theory at the high scale Λ drops out and is irrelevant for experimental measurements. Still, the question persists as to whether the limit $\Lambda \to \infty$ can be taken within such a QFT. There are two possible answers, both are interesting:

- Yes, the limit $\Lambda \to \infty$ can be taken. In this case, the corresponding QFT gives meaningful predictions at any energy scale. Therefore, it can, in principle, be valid to arbitrarily high-energy scales. It is therefore a candidate for a truly fundamental description of nature.
- No, the limit $\Lambda \to \infty$ may lead to inconsistencies or contradictions with experiment. In this case, the theory intrinsically predicts the existence of a maximum possible value of the UV scale $\Lambda_{\rm max}$, beyond which the theory cannot be extended. In this case, the corresponding QFT predicts its own breakdown, i.e., the true description of nature must be replaced by a different theory at or already below this scale $\Lambda_{\rm max}$. More precisely, the corresponding QFT cannot make any predictions that probe energy or momentum scales that exceed $\Lambda_{\rm max}$.

In the first case, we may have arrived at a microscopically complete description of nature (beware: hubris!), even though this does not exclude the possibility that the theory is ultimately still replaced by some other description at higher scales not yet tested by experiment.

In the second case, we encounter a rather particular property of quantum field theories: namely, a quantum field theory with a $\Lambda_{\rm max}$ predicts its own failure. This is a clear indication for the necessary existence of *new physics*, i.e., for a yet to be discovered better theory. In the standard model of particle physics, the hypercharge U(1) sector of the standard-model gauge theories appears to have such a $\Lambda_{\rm max}$. So far, this is the only intrinsic hint that the standard model cannot be a complete and consistent description of nature.

While perturbative renormalizability is a useful and systematic tool to analyze the RG behavior of QFTs, there is no robust reason for nature to be perturbatively describable at high energies. Another obvious question therefore is as to whether the classification of pertubatively renormalizable is incomplete and may miss theories which are nonperturbatively renormalizable, i.e., consistent and potentially higher-energy complete, but may look nonrenormalizable from a perturbative perspective.

In fact, such scenarios have by now been established – for some theories even with a certain amount of mathematical rigor. A prominent example is given by Weinberg's *asymptotic safety* scenario.

Whether or not Einstein's gravity theory or certain variants of gravity theories belong to the class of nonperturbatively renormalizable theories is a subject of contemporary research. So far, gravity is the only known (and observed) fundamental interaction that is not perturbatively renormalizable. While it works very well as an effective quantum field theory below a high-energy scale (presumably the Planck scale), perturbative quantization and reasoning leads to inconsistencies. Therefore, gravity is a playground where nonperturbative quantization methods can make a real difference in our comprehension of nature.