Renormalization Group Equations

▷ Idea: study the dependence of correlation functions on the scale k

▷ Various options:

- $k = \Lambda$: Wilson, Wegner-Houghton, "coarse graining"
  constant physics, cutoff/UV insensitivity
- $k = \mu$: Gell-Mann-Low
  constant physics, fixing scale insensitivity
- $k = m_R$: Callan, Symanzik (CS)
  compare theories with different masses

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▷ Advantage of CS idea:

- start with massive theories
  $\Longrightarrow$ suppressed fluctuations
- connect with differential equation to small mass theories
  $\Longrightarrow$ regime with strong correlations

Renormalization Group Equations

- Callan-Symanzik: $m^2 \to m^2 + k^2$
- Flow of master formula:

$$\partial_k Z_k[J] \quad = \quad \int \mathcal{D}\phi \left(-\frac{1}{2} \int d^4x (\partial_k k^2) \phi(x) \phi(x) \right) e^{-S[\phi] - \frac{1}{2} \int \frac{k^2}{2} \phi^2 + \int J\phi}$$

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&= -\frac{1}{2}\int d^4x (\partial_k k^2) G_k^{(2)}(x,x) \\
&= -\frac{1}{2}\mathrm{Tr}\big[(\partial_k k^2) G_k^{(2)}\big]
\end{aligned}$$

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- ▷ Legendre transformation to $\Gamma[\phi]$:

$$\partial_k \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\frac{(\partial_k k^2)}{\Gamma_k^{(2)} + k^2}\right]$$

Problem: still UV divergent in D=4

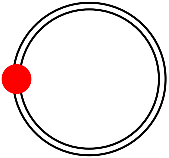
Wetterich Equation

▷ Idea: replace mass deformation by momentum dependent function

$$\frac{1}{2}\int (m^2 + k^2)\phi^2 \to \frac{1}{2}\int d^4p\, \phi(-p)\,(m^2 + R_k(p))\,\phi(p)$$

▷ RG flow equation: (WETTERICH'93)

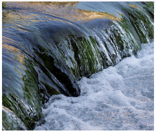
$$\partial_t \Gamma_k \equiv k\partial_k \Gamma_k \;=\; \frac{1}{2}\,\mathrm{Tr}\;\partial_t R_k (\Gamma_k^{(2)} + R_k)^{-1}$$

$$=$$ 

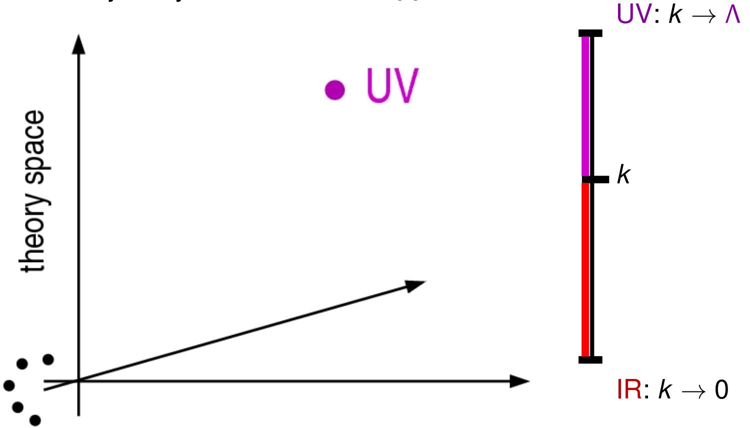
"Exact" Renormalization Group

RG flow in Theory Space

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr} \frac{1}{\Gamma_k^{(2)} + R_k} \partial_t R_k$$



▷ RG trajectory: $\Gamma_{k=\Lambda} = S_{\text{micro}}$



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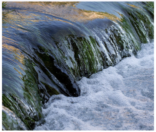

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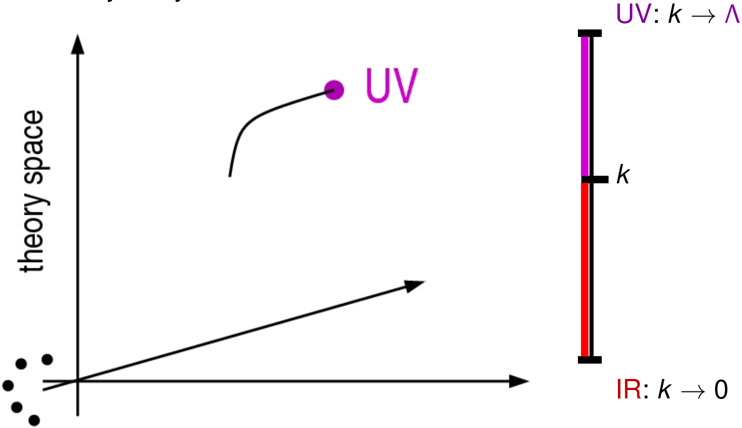

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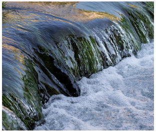

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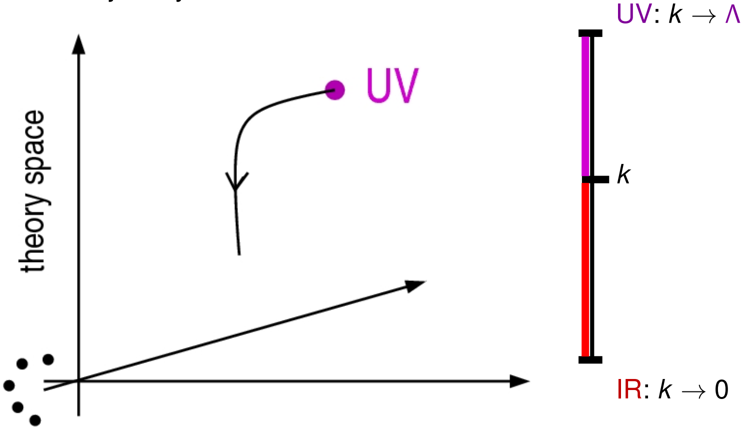

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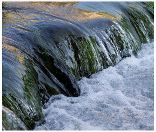
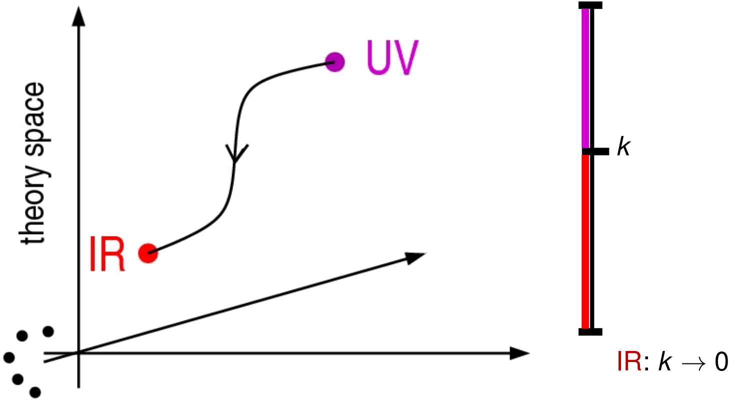


- RG trajectory: $\Gamma_{k \to 0} = \Gamma$



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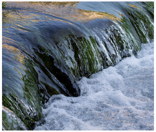
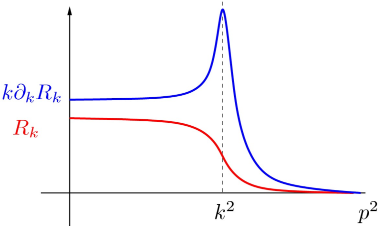
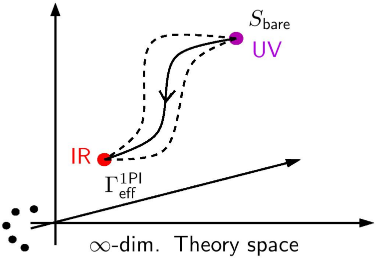


▷ regulator function R_k



▷ RG trajectory:

$$\Gamma_{k=\Lambda} = S_{\text{bare}} \rightarrow \Gamma_{k=0} = \Gamma_{\text{eff}}^{\text{1PI}}$$



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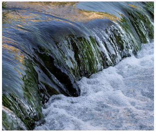
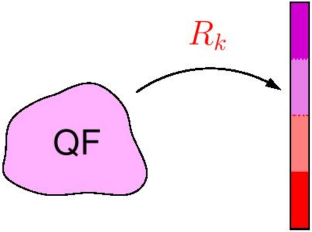


▷ role of the regulator $R_k(\mathcal{O})$



▷ e.g., chiral symmetry OK!

$$R_k^\psi = R_k^\psi(i\partial\!\!\!/)$$

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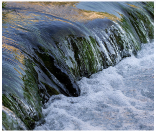

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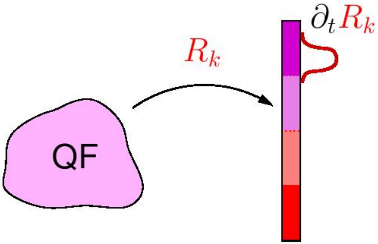

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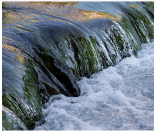

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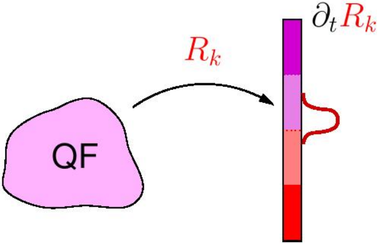

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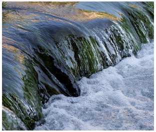

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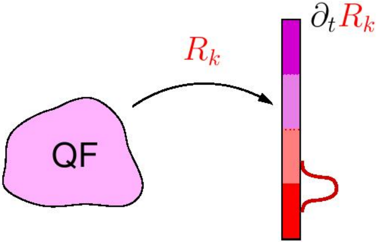

$$R_k^\psi = R_k^\psi(i\partial\!\!\!/)$$

Lesson

- RG flow equation

  ...exact equation

- RG flow equation (+ b.c.)

  ...can serve as definition of QFT

- Wilsonian momentum-shell integration

  ...treats physics scale by scale

- key element: scale-dependent exact propagator:

$$G_k(x,y) = \frac{1}{\Gamma_k^{(2)} + R_k}(x,y)$$

A Simple Example: $0 + 1$ dimensional QFT

▷ classical action of the Euclidean anharmonic oscillator

$$S[x] = \int d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{1}{24}\lambda x^4 \right)$$

▷ truncated effective action

$$\Gamma_k[x] = \int d\tau \left(\frac{1}{2}\dot{x}^2 + V_k(x) \right), \qquad V_k(x) = V_{0,k} + \frac{1}{2}{\omega_k}^2 x^2 + \frac{1}{24}\lambda_k x^4 \dots$$

▷ second functional derivative

$$\Gamma_k^{(2)}[x] = \partial_\tau^2 + V_k''(x)$$

A Simple Example: $0+1$ dimensional QFT

- e.g., linear regulator (Litim'00)

$$R_k(p^2) = (k^2 - p^2)\,\theta(k^2 - p^2)$$

- flow equation

$$\begin{aligned}\partial_t \Gamma_k[\phi] &= \frac{1}{2}\,\mathrm{Tr}\left[\partial_t R_k\,(\Gamma_k^{(2)}[\phi] + R_k)^{-1}\right] \\ &= L_\tau \int \frac{dp_\tau}{2\pi}\,\frac{k^2\,\theta(k^2 - p_\tau^2)}{k^2 + V_k''(x)}\end{aligned}$$

- flow of the potential

$$\partial_t V_k(x) = \frac{1}{\pi}\,\frac{k^3}{k^2 + V_k''(x)}$$

A Simple Example: $0 + 1$ dimensional QFT

▷ polynomial expansion

$$V_k(x) = \frac{1}{2}{\omega_k}^2 x^2 + \frac{1}{24}\,\lambda_k\, x^4 \cdots + E_{0,k} + \text{const.}_k$$

(const.$_k$ fixed such that $\frac{d}{dk} E_{0,k} = 0$ for $\omega_k = 0$)

▷ "coupling" flows ($\sim \beta$ functions)

$$x^0: \qquad \frac{d}{dk} E_{0,k} = \frac{1}{\pi}\left(\frac{k^2}{k^2 + {\omega_k}^2} - 1\right) \tag{1}$$

$$x^2: \qquad \frac{d}{dk}{\omega_k}^2 = -\frac{2}{\pi}\,\frac{k^2}{(k^2 + {\omega_k}^2)^2}\,\frac{\lambda_k}{2} \tag{2}$$

$$x^4: \qquad \frac{d}{dk}\lambda_k = \frac{24}{\pi}\,\frac{k^2}{(k^2 + {\omega_k}^3)^2}\left(\frac{\lambda_k}{2}\right)^2 + \ldots \tag{3}$$

A Simple Example: $0 + 1$ dimensional QFT

▷ truncate to (1) $(\omega_k \to \omega)$

$$E_0 = E_{0,k\to 0} = \frac{1}{2}\,\omega \qquad \text{(HO)}$$

▷ truncate to (1)+(2) and solve perturbatively

$$E_0 = \frac{1}{2}\omega + \frac{3}{4}\,\omega\left(\frac{\lambda}{24\omega^3}\right) - \frac{82}{40}\,\omega\left(\frac{\lambda}{24\omega^3}\right)^2 + \dots$$

▷ compare with perturbation theory (Bender&Wu)

$$E_0 = \frac{1}{2}\omega + \frac{3}{4}\,\omega\left(\frac{\lambda}{24\omega^3}\right) - \frac{105}{40}\,\omega\left(\frac{\lambda}{24\omega^3}\right)^2 + \dots$$

$\Longrightarrow$ "1-loop" exact, "2-loop" $\sim$ 20% error

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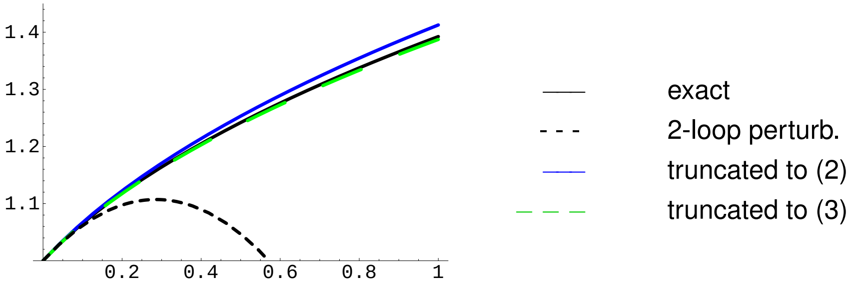
A Simple Example: $0+1$ dimensional QFT

▷ ground state energy E_0/E_{HO} vs. $\frac{\lambda}{24}$ $(\omega = 1,\ m = \frac{1}{2})$



A Simple Example: $0+1$ dimensional QFT

▷ strong-coupling limit

$$E_0 = \left(\frac{\lambda}{24}\right)^{1/3}\left[\alpha_0 + \mathcal{O}(\lambda^{-2/3})\right]$$

$$\alpha_0 = 0.66798\ldots$$ (JANKE& KLEINERT)

$$\alpha_0\big|_{(2)} = 0.6920\ldots$$ error: 4%

$$\alpha_0\big|_{(3)} = 0.6620\ldots$$ error: <1%

Lesson

- RG flow equation

  ... encodes perturbative & nonperturbative physics

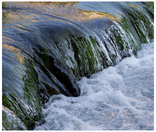
  already in simple approximations

- key element: scale-dependent exact propagator:

$$G_k(x,y) = \frac{1}{\Gamma_k^{(2)} + R_k}(x,y)$$

- many applications ...

RG flow in Theory Space



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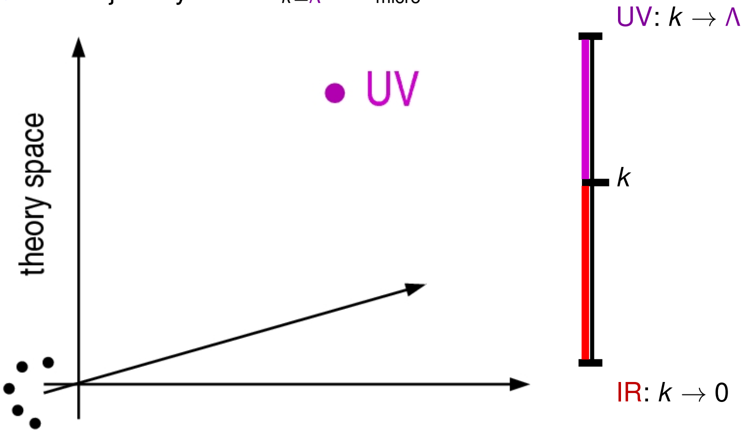

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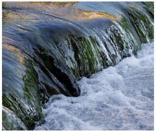

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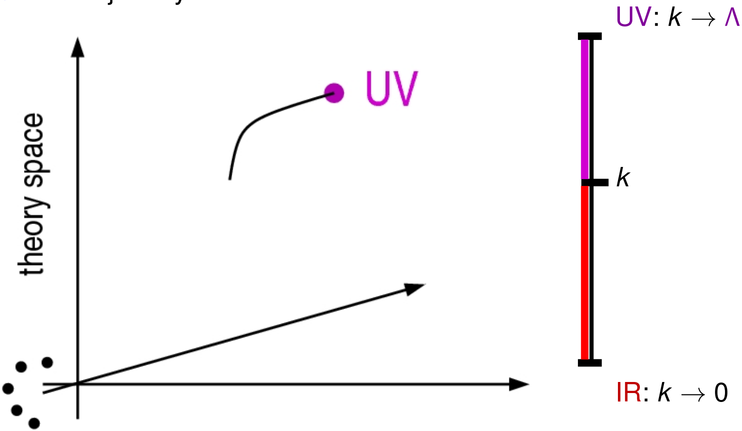

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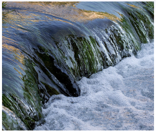

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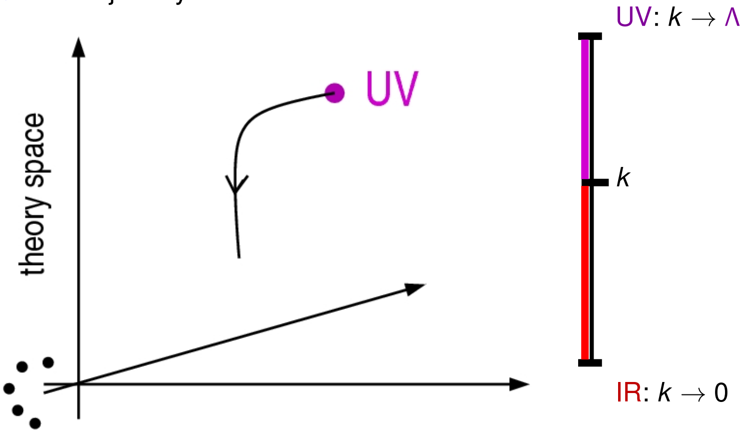

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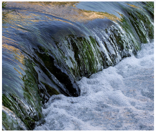

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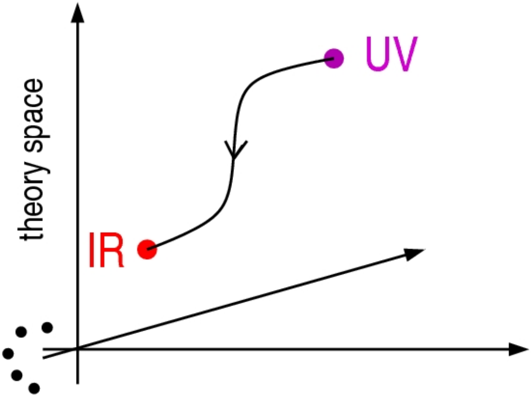

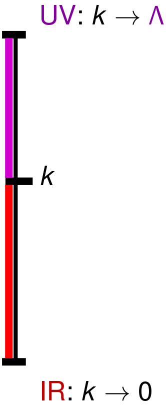

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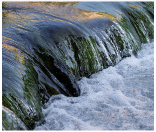
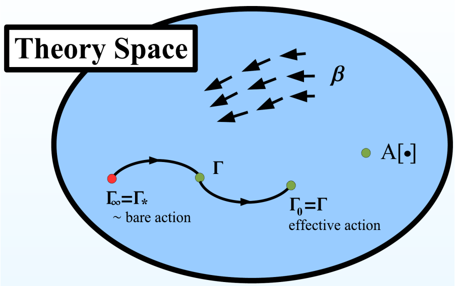


▷ Abstract viewpoint: FRG provides vector field on theory space



[Picture: F. Saueressig]

▷ search for RG trajectories that can be completed to arbitrarily high scales

$\Longrightarrow$ fundamental theory

candidates: fixed points

$\Longrightarrow$ bare action ($\hat{=}$ microscopic action) is a result rather than an initial condition

Quantum field theory $\longleftrightarrow$ gravity

▷ Problem of Physics?

- expected typical scale of QG effects: $M_{\text{Planck}} \sim 10^{19}$GeV
- early/late universe cosmology ?
- astrophysical singularities ?
- hierarchy problems
  (gauge hierarchy, cosmological constant & coincidence problem)

Minimum requirements: compatibility with observed physics

- existence of semiclassical GR regime
- $D = 4 = D_{\text{RG, cr}}$
- compatibility with observed matter content of the universe

QFT $\leftrightarrow$ Gravity

(Goroff,Sagnotti'85'86; van de Ven'92)

▷ perturbative quantization fails

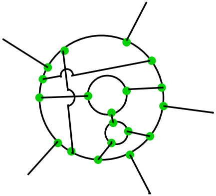
$$\Gamma^{\text{2-loop}}_{\text{div}} = \frac{1}{\epsilon}\frac{209}{2880}\frac{1}{(16\pi^2)^2}\int d^4x\,\sqrt{g}C_{\mu\nu\rho\sigma}C^{\rho\sigma\lambda\tau}C_{\lambda\tau}{}^{\mu\nu}$$

$\Longrightarrow$ Any quantum theory of gravity has to explain the fate of C^3

Spacetime Dimensionality

▷ (perturbative) QFT:

$$\delta(\gamma) = d - \sum_i n_{E_i}[\phi_i] + \sum_\alpha n_{V_\alpha} \delta(V_\alpha)$$



$\Longrightarrow$ RG critical dimension:

$$D_{\text{RG, cr}} = \begin{cases} 4 & \text{(gauge + matter, Yukawa/Higgs)} \\ 2 & \text{(gravity, pure fermionic matter)} \end{cases}$$

▷ (macroscopic) universe:

$$D = 4$$

"It is not known whether the fact that space time has just four dimensions is a mere coincidence or is logically connected with this property." (J. Zinn-Justin, in "QFT and critical phenomena")

Quantizing Gravity

"I know of only one promising approach to this problem . . ."

(S. Weinberg, in "Critical Phenomena for Field Theorists" (1976))

Asymptotic Safety

Necessity of Renormalizability

- IR physics well separated from UV physics

(. . . no/mild cutoff Λ dependence)

- # of physical parameters $\Delta < \infty$ . . . or countably ∞

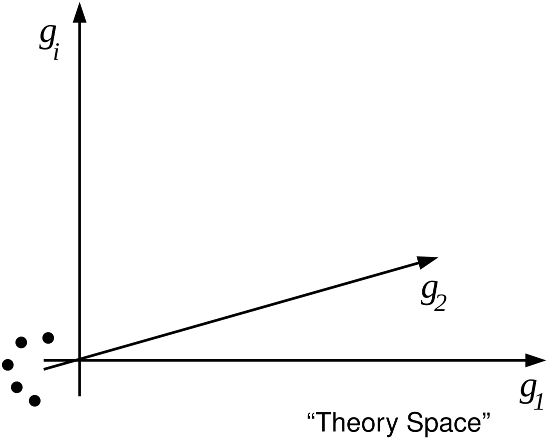
(. . . predictive power)

$\Longrightarrow$ realized by perturbative RG . . .

$\Longrightarrow$ . . . and by "Asymptotic Safety"

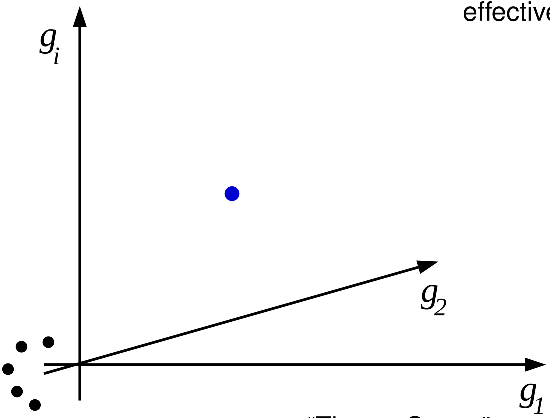
(WEINBERG'76)

Asymptotic Safety



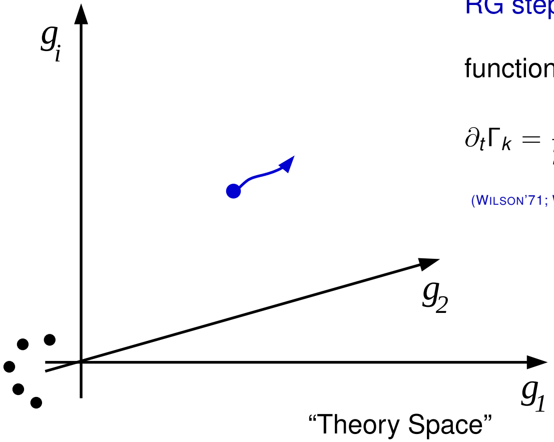
Asymptotic Safety

effective action Γ_k



"Theory Space"

Asymptotic Safety



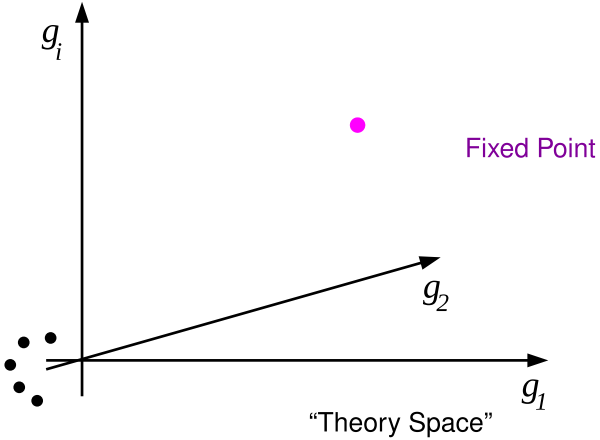
RG step

functional RG: (WETTERICH'93)

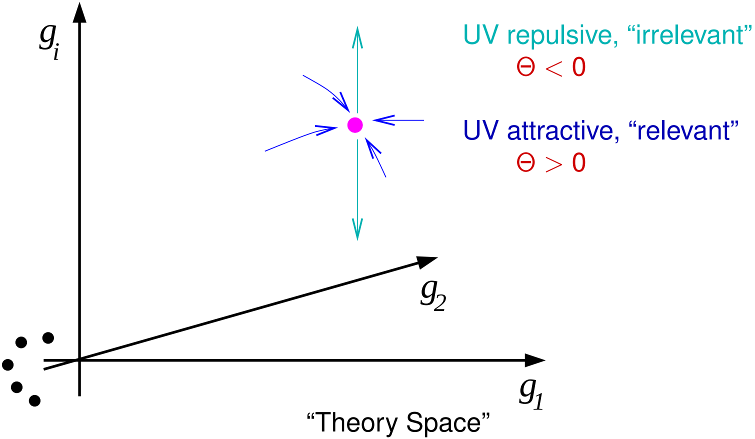
$$\partial_t \Gamma_k = \frac{1}{2} \,\mathrm{Tr}\, \partial_t R_k (\Gamma_k^{(2)} + R_k)^{-1}$$

(WILSON'71; WEGNER,HOUGHTON'73; POLCHINSKI'84)

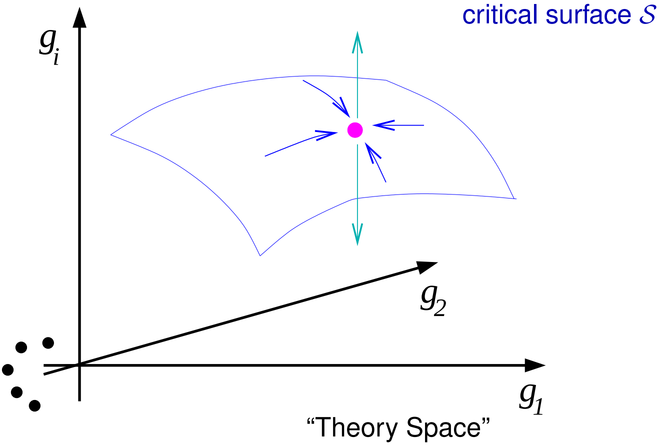
Asymptotic Safety



Asymptotic Safety

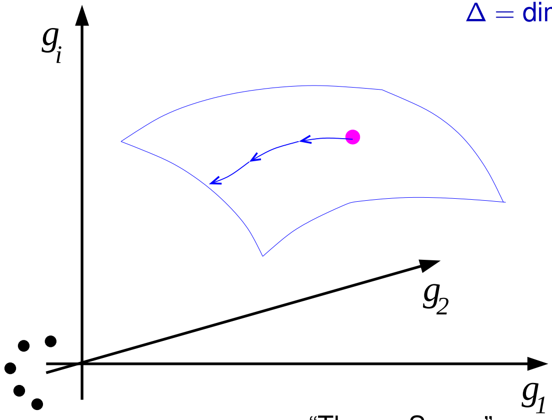


Asymptotic Safety



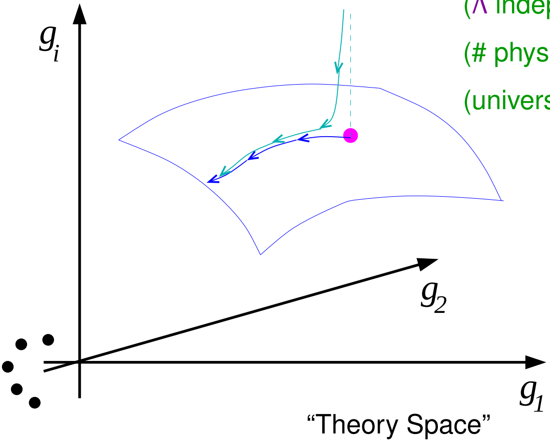
Asymptotic Safety

$\Delta = \dim \mathcal{S}$ = # Θ's>0



"Theory Space"

Asymptotic Safety

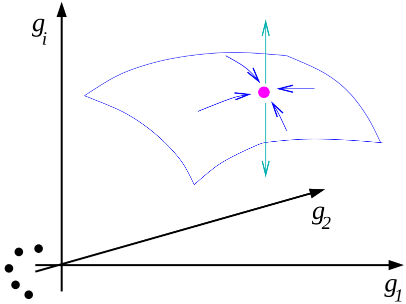


(Λ independence ✓)

(# phys. parameters $< \infty$ ✓)

(universality & predictivity ✓)

Asymptotic Safety



- ▷ FP regime:

$$\partial_t g_i = B_i{}^j \, (g_j - g_{*j}) + \dots$$

- ▷ stability matrix

$$B_i{}^j = \frac{\partial \beta_i(g_*)}{\partial g_j}$$

- ▷ critical exponents:

$$\{\Theta\} = \text{spect}(-B_i{}^j)$$

"Theory Space"

Dimensional Balancing

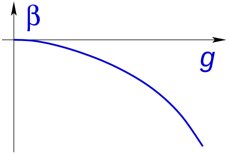
Non-Gaußian Fixed Points

▷ coupling $\bar{g}$ with canonical dimension $\delta_{\bar{g}}(D)$ in spacetime dimension D:

$$[\bar{g}] = \delta_{\bar{g}}(D)$$

▷ critical RG dimension:

$$\delta_{\bar{g}}(D_{\text{RG, cr}}) = 0$$



▷ perturbative β function in $D_{\text{RG, crit}}$, e.g.:

$$\beta_{\bar{g}} = b_0\,\bar{g}^2 + \ldots$$

$\Longrightarrow$ if $b_0 < 0$: theory is asymptotically free (and safe)

Non-Gaußian Fixed Points

▷ away from $D_{\text{RG, cr}}$ (+ analyticity in D):

$$\beta_{\bar{g}} = \frac{b_0(D)}{k^{\delta_{\bar{g}}(D)}}\,\bar{g}^2 + \dots$$

▷ dimensionless coupling in units of a given scale k

$$g = \frac{\bar{g}}{k^{\delta(\bar{g};D)}}$$

▷ RG flow of dimensionless coupling:

$$k\frac{d}{dk}g \equiv \beta_g = -\delta_{\bar{g}}(D)g + b_0(D)\,g^2$$

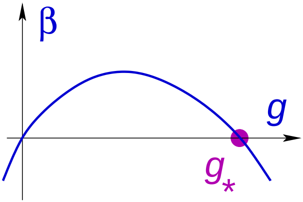
Non-Gaußian Fixed Points

▷ RG flow of dimensionless coupling:

$$k\frac{d}{dk}g \equiv \beta_g = \underbrace{-\delta_{\bar{g}}(D)g}_{\text{dimensional running}} \quad \underbrace{+b_0(D)\,g^2}_{\text{fluctuation-induced running}}$$

$\Longrightarrow$ NGFP g_* for: $\quad \text{sign}(\delta_{\bar{g}}(D)) = \text{sign}(b_0(D))$

$\Longrightarrow$ $g_* > 0$ for: $\quad \delta_{\bar{g}}(D), b_0(D) < 0$



Example: Fermionic Systems

▷ for instance, Nambu–Jona-Lasinio / Gross-Neveu in 3 dimensions:

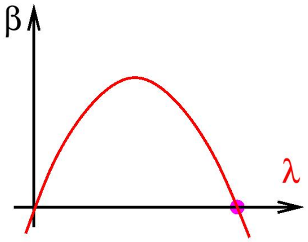
$$\Gamma_k = \int d^3x \bar{\psi} i\partial\!\!\!/ \psi + \frac{1}{2}\bar{\lambda}(\bar{\psi}\psi)^2 + \dots \quad , \quad [\bar{\lambda}] = -1$$

▷ dim'less coupling $\lambda = k\,\bar{\lambda}$

$$\partial_t \lambda = \lambda - c\,\lambda^2$$

▷ UV fixed point $\lambda_* = 1/c$

▷ critical exponent $\Theta = 1$



$\longrightarrow$ "χSB"

$\Longrightarrow$ asymptotically safe

▷ proven to all orders in $1/N_f$ expansion:

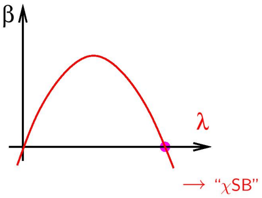
(Gawedzki, Kupiainen'85; Rosenstein, Warr, Park'89; de Calan et al.'91)

Example: 3D Gross-Neveu model

▷ exact mapping to Yukawa model: (STRATONOVICH'58,HUBBARD'59)

$$\Gamma[\psi,\sigma] = \int d^3x \left(\frac{N_f Z_\sigma}{2} (\partial\sigma)^2 + \bar{\psi}(Z_\psi i\partial\!\!\!/ + i\bar{h}\sigma)\psi + N_f U(\sigma) \right)$$

▷ non-Gaußian fixed point in Yukawa model: (BRAUN,HG,SCHERER'10)



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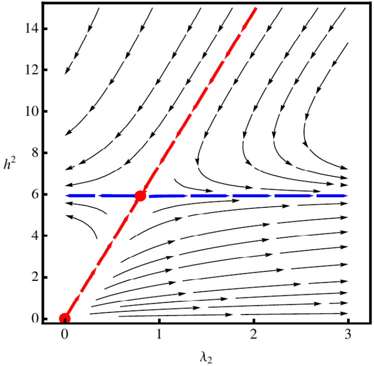


Example: 3D Gross-Neveu model

▷ exact large-N_f fixed point effective potential (BRAUN,HG,SCHERER'10)

$$u_*(\rho) = -\frac{2d-8}{3d-4}\rho\, {}_2F_1\left(1-\frac{d}{2}, 1; 2-\frac{d}{2}; \frac{(d-4)(d-2)}{6d-8}\frac{d}{d_\gamma v_d}\rho\right), \rho = \frac{\sigma^2}{2}$$

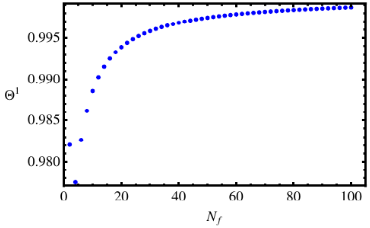
▷ exact critical exponents:

$$\Theta = 1, -1, -1, -3, -5, -7, \ldots$$

$\Longrightarrow$ dim $\mathcal{S} = 1$ physical parameter

Example: 3D Gross-Neveu model

▷ critical exponents beyond large-N_f ($\nu = 1/\Theta^1$) (BRAUN,HG,SCHERER'10)



N_f	Θ^1	Θ^2	Θ^3	Θ^4	Θ^5	Θ^6
2	0.9821	−0.8722	−1.0916	−3.5135	−6.0514	−8.5820
4	0.9775	−0.9240	−1.1010	−3.3910	−5.7739	−8.2429
12	0.9903	−0.9735	−1.0506	−3.1810	−5.3665	−7.6004
50	0.9975	−0.9936	−1.0143	−3.0510	−5.1062	−7.1789
100	0.9987	−0.9968	−1.0073	−3.0263	−5.0550	−7.0934
∞	1	−1	−1	−3	−5	−7

▷ matches even with $N_f \to 0$ limit (Ising model),

▷ excellent agreement with lattice simulations (available for $N_f = 2$)

(KARKKAINEN ET AL.'93)

Quantum Einstein Gravity

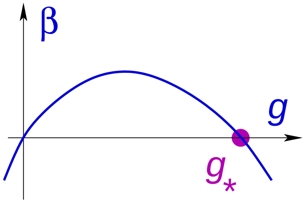
(Reuter'96)

▷ effective action in Einstein-Hilbert truncation

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^D x \sqrt{g}(-R + 2\Lambda_k)$$

▷ running dim'less Newton's constant
in $D = 4$: $g = k^2 G_k$, $\Lambda_k = 0$

$$\partial_t g = 2\,g - c\,g^2 + \mathcal{O}(g^3), \quad c = c[R_k] > 0$$



(Dou,Percacci'97)

(Souma'99)

(Lauscher,Reuter'01'02)

(Reuter,Saueressig'01)

(Niedermaier'02)

(Litim'03)

(Codello,Percacci'06)

(Codello,Percacci,Rahmede'07'08)

(Machado,Saueressig'07)

(Benedetti,Machado,Saueressig'09)

(Eichhorn,HG,Scherer'09)

(Donkin,Pawlowski'12)

(Christiansen,Litim,Pawlowski,Rodigast'12)

(Christiansen,Meibohm,Pawlowski,Reichert'15)

(Falls'15;Christiansen'16)

(Denz,Pawlowski,Reichert'16)

Quantum Einstein Gravity

(Reuter'96)
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(Falls'15;Christiansen'16)
(Denz,Pawlowski,Reichert'16)

▷ effective action in Einstein-Hilbert truncation

$$\Gamma_k = \frac{1}{16\pi G_k} \int d^D x \sqrt{g}(-R + 2\Lambda_k)$$

▷ running G_k and Λ_k
in $D = 4$: $g = k^2 G_k$, $\lambda = \Lambda_k / k^2$
e.g., sharp cutoff

$$\partial_t g = (2+\eta) g$$

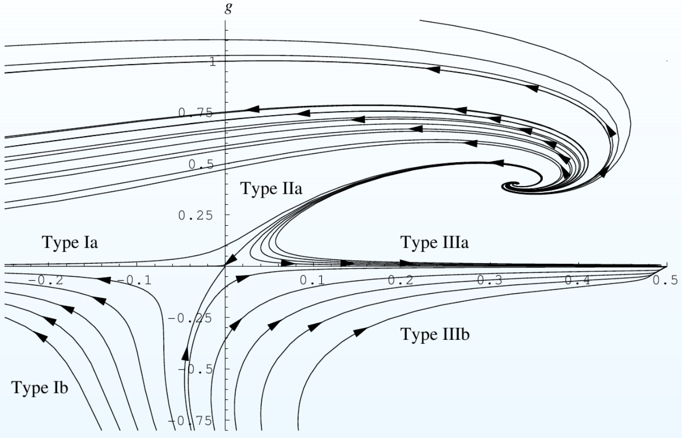
$$\partial_t \lambda = -2(2-\eta)\lambda - \frac{g}{\pi}\left[5\ln[1-2\lambda] - 2\zeta(3) + \frac{5}{2}\eta\right]$$

anomalous graviton dimension:

$$\eta = -\frac{2g}{6\pi + 5g}\left[\frac{18}{1-2\lambda} + 5\ln(1-2\lambda) - \zeta(2) + 6\right]$$

RG flow of Quantum Einstein Gravity

(Reuter,Saueressig'01)

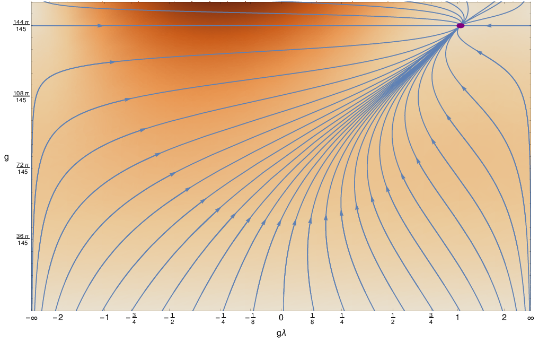


▷ critical exponents:

$$\mathrm{Re}\Theta_{1,2} \simeq 2$$

From Quantum to Classical Gravity

(HG, Knorr, Lippoldt'15)



▷ RG trajectories
interconnecting the transplanckian and classical regimes exist

▷ : physical trajectory:

$$g\lambda|_{k\to\text{"today"}} \simeq +3 \times 10^{-122}$$

Fate of two-loop counterterm

▷ operator expansion with background field method

$\vdots$			
R^8	$\dots$		
R^7	$\dots$		
R^6	$\dots$		
R^5	$\dots$		
R^4	$\dots$		
R^3	$C_{\mu\nu}{}^{\rho\sigma} C_{\rho\sigma}{}^{\kappa\lambda} C_{\kappa\lambda}{}^{\mu\nu}$	$R \Box R$	+ 7 more
R^2	$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$	$R_{\mu\nu} R^{\mu\nu}$	
R			
$\mathbb{1}$			

[PICTURE: F. SAUERESSIG]

Flow of Einstein-Hilbert + Goroff-Sagnotti

▷ effective action (HG,Knorr,Lippoldt,Saueressig'16)

$$\Gamma_k = \Gamma_k^{EH} + \Gamma_k^{GS}$$

▷ Einstein-Hilbert

$$\Gamma_k^{EH} = \frac{1}{16\pi G_k} \int d^D x \sqrt{g}(-R + 2\Lambda_k)$$

▷ Goroff-Sagnotti:

$$\Gamma_k^{GS} = \bar{\sigma}_k \int d^D x \sqrt{g}\, C_{\mu\nu\rho\sigma} C^{\rho\sigma\lambda\tau} C_{\lambda\tau}{}^{\mu\nu}$$

▷ dimensionless coupling constants

$$g = k^2 G_k, \quad \lambda = \Lambda_k / k^2, \quad \sigma = \bar{\sigma}_k k^2$$

Flow of Einstein-Hilbert + Goroff-Sagnotti

▷ Fluctuations with GS vertex: (HG,Knorr,Lippoldt,Saueressig'16)

$$\Gamma^{GS(2)} \sim \sigma C^{\mu\nu}_{\alpha\beta} + \mathcal{O}(R^2)$$

▷ BUT:

$$\mathrm{tr} C^{\mu\nu}_{\alpha\beta} = 0$$

$\Longrightarrow$ Two-loop counterterm does not directly feed back into EH

$\Longrightarrow$ Fixed point in Einstein-Hilbert sector is maintained

asymptotic safety

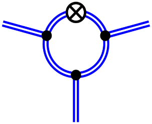
Flow of Einstein-Hilbert + Goroff-Sagnotti

▷ Einstein-Hilbert: (HG,Knorr,Lippoldt,Saueressig'16)

$$\begin{aligned}\partial_t g &= (2+\eta_N)\, g \\ \partial_t \lambda &= (\eta_N - 2)\, \lambda + \frac{g}{2\pi}\left(\frac{5}{1-2\lambda} - 4 - \frac{5}{6}\eta_N \frac{1}{1-2\lambda}\right)\end{aligned}$$

▷ Goroff-Sagnotti

$$\partial_t \sigma = c_0 + (2 + c_1)\,\sigma + c_2\,\sigma^2 + c_3\,\sigma^3$$



$\sim (10^3)^3$ terms

$$c_0 = \frac{1}{64\pi^2(1-2\lambda)}\left(\frac{2-\eta_N}{2(1-2\lambda)} + \frac{6-\eta_N}{(1-2\lambda)^3} - \frac{5\eta_N}{378}\right),$$

$$c_1 = \frac{3g}{16\pi(1-2\lambda)^2}\left(5(6-\eta_N) + \frac{23(8-\eta_N)}{8(1-2\lambda)} - \frac{7(10-\eta_N)}{10(1-2\lambda)^2}\right),$$

$$c_2 = \frac{g^2}{2(1-2\lambda)^3}\left(\frac{233(12-\eta_N)}{10} - \frac{9(14-\eta_N)}{7(1-2\lambda)}\right),$$

$$c_3 = \frac{6\pi g^3(18-\eta_N)}{(1-2\lambda)^4} \neq 0\,.$$

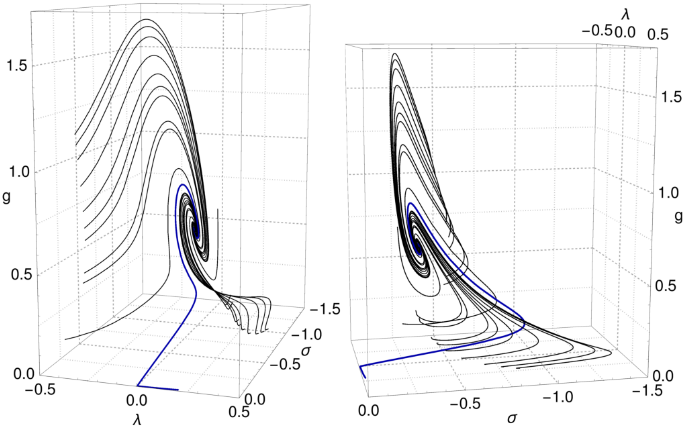
$$g_* > 0 \implies c_3 > 0$$

$\implies$ asymtotic safety

extends to (irrelevant) GS term

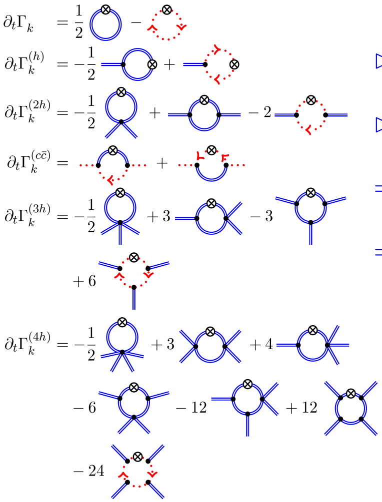
Phase portrait

(HG, KNORR, LIPPOLDT, SAUERESSIG'16)



▷ crossover to classical regime persists

Towards apparent convergence in Quantum Gravity



(Denz, Pawlowski, Reichert'16)

▷ systematic vertex expansion $n \leq 4$

▷ fully dynamical propagators

$\Longrightarrow$ asymptotically safe fixed point

$\Longrightarrow$ 3 relevant directions

Λ

R

R^2

$R_{\mu\nu}R^{\mu\nu}$ irrel.

$\Longrightarrow$ promising path towards systematic scheme

establishing asymptotic safety

Lesson

- FRG facilitates a search for quantizable theories

  candidates: RG fixed points

- Wilsonian renormalization extends beyond perturbative realm

  UV complete and predictive

- Asymptotic safety in fermion systems and in gravity

  ...perturbatively nonrenormalizable, but nonperturbatively renormalizable

- Functional RG interconnects all physical scales

  ...from Planck to Hubble

Lesson

- FRG facilitates a search for quantizable theories

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  ...perturbatively nonrenormalizable, but nonperturbatively renormalizable

- Functional RG interconnects all physical scales

  ...from Planck to Hubble

- Why D=4? ...no news yet

The flow of the Renormalization Group . . .

. . . holds, I think, the supreme position among the laws of Nature.

Sir Athur Eddington (1927) paraphrased by V. Rivasseau (2011)

The flow of the Renormalization Group . . .

If someone points out to you that your pet theory
of the universe
is in disagreement with Maxwell's equations . . .

Sir Athur Eddington (1927) paraphrased by V. Rivasseau (2011)

The flow of the Renormalization Group . . .

. . . then so much the worse
for Maxwell's equations.

Sir Athur Eddington (1927) paraphrased by V. Rivasseau (2011)

The flow of the Renormalization Group . . .

If it is found to be contradicted by observation

. . .

Sir Athur Eddington (1927) paraphrased by V. Rivasseau (2011)

The flow of the Renormalization Group . . .

. . . well, these experimentalists
do bungle things sometimes.

Sir Athur Eddington (1927) paraphrased by V. Rivasseau (2011)

The flow of the Renormalization Group . . .

But if your theory
is found to be against
the flow of the renormalization group . . .

Sir Athur Eddington (1927) paraphrased by V. Rivasseau (2011)

The flow of the Renormalization Group . . .

I can give you no hope;
there is nothing for it
but to collapse in deepest humiliation.

Sir Athur Eddington (1927) paraphrased by V. Rivasseau (2011)