Particles and Fields

– Lecture Notes –

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Preface

Lecture notes are only a largely incomplete replacement of face-to-face lectures – as many of us have learned during the past one and a half years of the pandemic.

I hope, the present lecture notes will still be useful for many, as they cover a set of topics that may not have been assembled in a single textbook before. They represent my attempt at creating a course that fits in between a standard theory curriculum as is taught in many places worldwide during a physics Bachelor program and a more advanced graduate program on theoretical physics including, in particular, quantum field theory.

This course assumes a solid knowledge of classical and analytical mechanics, electrodymamics, and a good knowledge of quantum theory, but anticipates no experience with quantum field theory – even though the highlights of this course may unfold their beauty only once some knowledge on quantum field theory may have been acquired. Despite its title with its appeal to elementary particle physics, the present course cannot replace an experimental or phenomenological course on particle and/or nuclear physics. Still, most of the applications and examples concern the realm of elementary particle physics and may serve as a motivation to learn more about our current understanding of the building blocks of nature and their interactions.

These notes are based on a set of handwritten lecture notes prepared for the first version of the course held in the winter term 2016/2017 with extensions and improvements over the years. I am extraordinarily grateful to Johannes Schmechel for his initiative to typeset these notes in LaTeX also including TikZ versions of the figures. His contin-

uous work on this project has been a strong motivation to turn the handwritten version into this – hopefully more useful – polished version. This new version also contains a few additions and is planned to replace the handwritten version from now on[1].

Jena, October 2021 Holger Gies

[1] Comments, suggestions, and hints at typos are more than welcome!

1 Introduction

1.1 Why Particles and Fields?

This course is meant to be a preparatory course for an in depth lecture course on *quantum field theory* (QFT). In fact, QFT has become *the* language of modern physics. Most prominently, QFT describes the physics of elementary particles and their interactions at the most fundamental level that is currently accessible to observations in the laboratory (i.e. at colliders) or in astrophysical or cosmological data. QFT even has the potential to describe systems to arbitrarily short-distance or arbitrarily high-energy scales (in contrast to classical mechanics, electrodynamics or quantum mechanics). Moreover, QFT provides also for useful tools for the description of condensed-matter systems, many-body physics, critical phenomena, statistical systems, phase transitions, etc.

It is therefore not astonishing that QFT exhibits a deep level of structural and technical complexity, challenging both – students and teachers – in a compact lecture course.

The purpose of this course hence is to remove a large part of this complexity by ignoring quantization. The remaining body of classical field theory still offers a comprehensive playground where many physical concepts and moreover observable physical phenomena can be learned and understood.

Though the mathematics of this course deals with classical field theory, the goal (behind the horizon) is QFT and its application to particle physics. Hence, some applications and discussions center around elementary particle physics. As QFT supersedes the point-particle concept, the word *particle* in the title does not allude to classical point particles, but to the modern understanding of particles as quantized excitations of fields. As we stay within the realm of classical physics in this course, a particle should be thought of as a classical excitation of a field, such as a localized propagating wave.

1.2 Examples of classical field theories

In classical field theory, each point in spacetime $x \equiv (t, \vec{x})$ is associated with a function ϕ (field amplitude)

$$x \to \phi. \tag{1.1}$$

Depending on the system, ϕ could be a real or a complex number, $\phi \in \mathbb{R}$ or $\phi \in \mathbb{C}$, or an N-tupel of such numbers ϕ^a, $a = 1, \ldots, N$. Examples are given by the electrostatic potential $\varphi(x) \in \mathbb{R}$ in classical electrostatics, or the vector potential $\vec{A}(x)$ consisting of 3 components, giving rise to a magnetic field $\vec{B}(x) = \vec{\nabla} \times \vec{A}(x)$.
We typically assume $\phi(x)$ to be sufficiently smooth and differentiable (e.g. $\phi \in C^2$) such that its dynamics can be governed by a differential equation, the *field equation* or *equation of motion* (EoM). This abstract notion is already familiar from classical electrodynamics, being a paradigmatic example for a classical field theory.
The field equations for the electric and magnetic field components, $\vec{E}(x)$ and $\vec{B}(x)$, are given by the Maxwell equations, which in vacuum read

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 0 \qquad & \vec{\nabla} \times \vec{B} - \frac{\partial}{\partial t}\vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{B} &= 0 \qquad & \vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}\vec{B} &= 0. \end{aligned} \tag{1.2}$$

Here, we have already used the convention $c \equiv 1$ (i.e. all velocity-like quantities are measured in fractions of the speed of light, or lengths are measured by the time that light takes to propagate some distance).
Mathematically, the field equations are (coupled) partial differential equations (PDEs), the solutions of which requires suitable boundary conditions or/and initial data.
The Maxwell equations form a rather peculiar example, as the information encoded in the 6 functions $E_k(x)$, $B_k(x)$, $k \in \{1, 2, 3\}$ can also be parametrized by the above mentioned 4 auxiliary functions of the

electrostatic potential $\varphi(x)$ and the vector potential $\vec{A}(x)$, where

$$\begin{aligned} \vec{B}(x) &= \vec{\nabla} \times \vec{A}(x) \\ \vec{E}(x) &= -\vec{\nabla}\varphi(x) - \frac{\partial}{\partial t}\vec{A}(x). \end{aligned} \tag{1.3}$$

Inserting (1.3) into (1.2), and using $\vec{\nabla} \times \vec{\nabla}\varphi = 0$ and $\vec{\nabla} \times \vec{\nabla} \times \vec{A} = 0$ (for smooth φ and $\vec{A}$), the 2nd line of (1.2) is evidently satisfied, while the 1st line boils down to

$$\begin{aligned} &\vec{\nabla}^2\varphi + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = 0 \\ &\vec{\nabla}^2\vec{A} - \frac{\partial^2}{\partial t^2}\vec{A} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A} + \frac{\partial}{\partial t}\varphi) = 0, \end{aligned} \tag{1.4}$$

forming 4 PDEs for the 4 unknown components of the fields φ and $\vec{A}$. This parametrization in terms of the potentials φ and $\vec{A}$ is even more peculiar, as the choice of φ and $\vec{A}$ is not unique. For instance, if φ and $\vec{A}$ are shifted according to

$$\begin{aligned} \varphi(x) &\to \varphi'(x) = \varphi(x) - \frac{\partial}{\partial t}\lambda(x) \\ \vec{A}(x) &\to \vec{A}'(x) = \vec{A}(x) + \vec{\nabla}\lambda(x) \end{aligned} \tag{1.5}$$

with an arbitrary function $\lambda(x) \in \mathbb{R}$, the $\vec{E}$ and $\vec{B}$ fields in (1.3) remain the same. While $\vec{E}$ and $\vec{B}$ can be measured in terms of forces acting on (moving) charged particles, the values of φ and $\vec{A}$ at a given point x can be shifted by (1.5) to any value and thus have no locally observable meaning. This invariance under local shifts a la (1.5) is called a *gauge symmetry* and characterizes a very special (and very important) class of field theories.

For our present purpose, it is useful to choose $\lambda(x)$ in such a way that φ' and $\vec{A}'$ satisfy the following auxiliary condition (Lorenz gauge condition):

$$\vec{\nabla} \cdot \vec{A}' + \frac{\partial}{\partial t}\varphi' = 0. \tag{1.6}$$

If so, the field equations (1.4) for φ' and $\vec{A}'$ simplify to

$$\begin{aligned} \vec{\nabla}^2\varphi' - \frac{\partial^2}{\partial t^2}\varphi' &= 0 \\ \vec{\nabla}^2\vec{A}' - \frac{\partial^2}{\partial t^2}\vec{A}' &= 0, \end{aligned} \tag{1.7}$$

or simply $\Box\varphi' = 0$, $\Box\vec{A}' = 0$, where $\Box = -\vec{\nabla}^2 + \frac{\partial^2}{\partial t^2}$ is the d'Alembert operator.

Eqs. (1.7) are wave equations for all 4 field functions which hence admit plane wave solutions:

$$\varphi',\ \vec{A}' \quad \sim \quad \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\vec{k}\cdot\vec{x}}, \qquad \text{with } \omega^2 = \vec{k}^2 \tag{1.8}$$

(for complexified fields, or $\Re/\Im$ of $\mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\vec{k}\cdot\vec{x}}$ for real fields).

In addition to gauge invariance, Maxwell's equations also have an invariance with respect to the choice of coordinate systems. The corresponding invariance is a relativistic invariance, and the corresponding transformations between coordinate systems moving relative to each other at constant speed $\beta = \frac{v}{c} \equiv v$ are the Lorentz transformations. For instance, if two coordinate systems move relative to each other along their common x direction, the Lorentz transformation reads

$$\begin{aligned} t' &= \gamma(t - \beta x) \qquad & y' &= y \\ x' &= \gamma(x - \beta t) \qquad & z' &= z, \qquad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}}. \end{aligned} \tag{1.9}$$

Summarizing the spacetime coordinates in a ('contravariant') 4-vector $x^\mu = (t, x, y, z) \equiv (x^0, x^1, x^2, x^3)$ to be understood as column vector, this transformation can be written in a matrix form

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu \qquad \text{(summation over } \nu \text{ is implicity understood)}, \tag{1.10}$$

where

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.11}$$

Of course, by suitably applying rotation matrices, $\vec{x}' = R\vec{x}$, $R^T R = \mathbb{1}$, $R \in \mathrm{SO}(3)$, the Lorentz transformations generalize to 'boosts' along any other direction $\vec{\beta}$, as well as to coordinate systems spatially rotated relative to each other (as in classical mechanics). Recall that (1.9) follows from Einstein's postulate that the wave front of a flash of light starting at a common origin of the coordinate systems propagate at the same speed as measured in both systems. The position of such a (spherical) wave front after time t (t') is at

$$0 = t^2 - (x^2 + y^2 + z^2), \qquad 0 = t'^2 - (x'^2 + y'^2 + z'^2) \tag{1.12}$$

respectively. This suggests to introduce the Minkowski metric ,

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.13}$$

to write the propagation distance of the wave front in both systems as

$$0 = x^{\mu}\, g_{\mu\nu}\, x^{\nu} = x'^{\mu}\, g_{\mu\nu}\, x'^{\nu}. \tag{1.14}$$

Using (1.10), we get

$$\begin{aligned} x'^{\mu}\, g_{\mu\nu}\, x'^{\nu} &= \Lambda^{\mu}{}_{\kappa}\, x^{\kappa}\, g_{\mu\nu}\, \Lambda^{\nu}{}_{\lambda}\, x^{\lambda} \\ &\overset{\mu\Leftrightarrow\kappa\,,\,\nu\Leftrightarrow\lambda}{=} x^{\mu}\, \Lambda^{\kappa}{}_{\mu}\, g_{\kappa\lambda}\, \Lambda^{\lambda}{}_{\nu}\, x^{\nu}. \end{aligned} \tag{1.15}$$

Note that, in this context, x^{μ} is not just any position in spacetime, but a *vector* specifying the distance of the wave front from the origin. From (1.15) we read off that Lorentz transformations Λ of such vectors satisfy

$$g_{\mu\nu} = g_{\kappa\lambda}\, \Lambda^{\kappa}{}_{\mu}\, \Lambda^{\lambda}{}_{\nu}. \tag{1.16}$$

It is straightforward to verify that (1.11) satisfies this condition.

More generally, we call any 4×4 matrix Λ that satisfies (1.16) for the metric (1.13) a Lorentz transformation. Hence, (1.16) has the same status for Lorentz transformations, as $R^T R = \mathbb{1}$ ($\delta_{ij} = \delta_{kl}\, R^{k}{}_{i}\, R^{l}{}_{j}$) has for rotations. The corresponding matrix group is SO(3, 1). We will discuss this group in more detail in chapter 4.

Any 4-tupel v^{μ}, $\mu = 0, 1, 2, 3$, that transforms under changes of the Lorentz system as

$$v^{\mu} = \Lambda^{\mu}{}_{\nu}\, v^{\nu} \tag{1.17}$$

is called a Lorentz 4-vector. Correspondingly, objects $\mathrm{T}^{\mu_1,\mu_2\cdots\mu_n}$ that transform as

$$\mathrm{T}'^{\mu_1,\mu_2\cdots\mu_n} = \Lambda^{\mu_1}{}_{\nu_1}\, \Lambda^{\mu_2}{}_{\nu_2}\, \ldots\, \Lambda^{\mu_n}{}_{\nu_n}\, \mathrm{T}'^{\nu_1,\nu_2\cdots\nu_n} \tag{1.18}$$

are called Lorentz tensors of rank n. It is useful to introduce 'covariant' vectors by defining

$$x_\mu \quad := \quad g_{\mu\nu}\, x^\nu \quad = \quad (t, -\vec{x}). \tag{1.19}$$

With this notation, the light-front position discussed above can be written as
$0 = x_\mu\, x^\mu = x'_\mu\, x'^\mu$, which makes it obvious that expressions with pair-wise contracted upper and lower indices are Lorentz invariant. For instance, the argument of the plane wave in (1.8) can be written as

$$-\mathrm{i}\omega t + \mathrm{i}\vec{k}\cdot\vec{x} = -\mathrm{i}k^\mu\, x_\mu, \tag{1.20}$$

where $k^\mu = (\omega, \vec{k})$.
[NB: the fact that ω and $\vec{k}$ indeed transform as components of a 4-vector is a manifestation of the relativistic Doppler effect.]
Hence, the plane-wave form of (1.8) is a relativistic invariant. This translates into the invariance of the corresponding wave operator

$$\Box = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2. \tag{1.21}$$

The trivial fact that

$$\frac{\partial}{\partial x^\mu} x^\nu = \begin{cases} 1 \text{ for } \mu = \nu \\ 0 \text{ otherwise} \end{cases} \tag{1.22a}$$

implies that

$$\frac{\partial}{\partial x^\mu} x^\mu = 4 \tag{1.22b}$$

holds in any Lorentz frame. This suggests to interpret $\frac{\partial}{\partial x^\mu}$ as a co-variant vector: ∂_μ

$$\partial_\mu x^\mu = 4, \qquad \partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{1.23}$$

The corresponding contravariant vector operator is

$$\partial^\mu = g^{\mu\nu}\partial_\nu, \qquad \partial^\mu = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right), \tag{1.24}$$

where $g^{\mu\nu}$ denotes the inverse of $g_{\mu\nu}$. Obviously, we have $(g^{-1})^{\mu\nu} = g_{\mu\nu}$ component-wise. We write

$$g^{-1}g = \mathbb{1}, \quad \text{or in components} \quad g^{\mu\nu}g_{\nu\kappa} = \delta^{\mu}_{\kappa}. \tag{1.25}$$

With this notation, we have

$$\Box = \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 = \partial_\mu \partial^\mu \tag{1.26}$$

which makes Lorentz invariance manifest.

To conclude the discussion of classical electrodynamics, the form invariance of Maxwell's equations under Lorentz transformations becomes manifest by noticing that $\varphi(x)$ and $\vec{A}(x)$ also transform as components of a 4-vector

$$A^\mu(x) = \left(\varphi(x), \vec{A}(x)\right). \tag{1.27}$$

The Lorenz gauge condition (1.6) is hence Lorentz invariant,

$$\partial_\mu A^\mu = 0. \tag{1.28}$$

From (1.3) it is clear that $\vec{E}$ and $\vec{B}$ cannot be arranged into 4-vectors. Instead, their components can be arranged into a Lorentz tensor, the field strength tensor

$$\begin{aligned} F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\ (F)^{\mu\nu} &= \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \end{aligned} \tag{1.29}$$

such that the 1st line of Maxwell's equations read

$$\partial_\mu F^{\mu\nu} = 0. \tag{1.30}$$

This is a set of 4 equations, $\nu = 0, 1, 2, 3$, that transform as a 4-vector under Lorentz transformations. In order to write the 2nd line of (1.2) into 4-notation, it is useful to introduce the Minkowskian analogue of the Levi-Civita symbol

$$\epsilon^{\mu\nu\kappa\lambda} = \begin{cases} 1 \text{ for } \mu = 0, \nu = 1, \kappa = 2, \lambda = 3 \text{ and even permutations} \\ -1 \text{ for odd permutations} \\ 0 \text{ if two indices are equal} \end{cases} . \tag{1.31}$$

This allows to introduce the dual field strength tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda} F_{\kappa\lambda}, \tag{1.32}$$

where $F_{\kappa\lambda} = g_{\kappa\mu} F^{\mu\nu} g_{\nu\lambda}$. More explicitly,

$$\left(\tilde{F}\right)^{\mu\nu} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & E_3 & -E_2 \\ B_2 & -E_3 & 0 & E_1 \\ B_3 & E_2 & -E_1 & 0 \end{pmatrix}. \tag{1.33}$$

By construction, we have

$$0 = \partial_\mu \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\partial_\mu \left(\partial_\kappa A_\lambda - \partial_\lambda A_\kappa\right), \tag{1.34}$$

since the partial derivatives commute. This is also called the Bianchi identity, which reproduces the 2nd line of (1.2). We close this section on electrodynamics by noting that the whole formalism can be generalized to non-vanishing charges and currents. Combining the charge density ρ and the current density $\vec{\jmath}$ into a 4-vector $j^\mu = (\rho, \vec{\jmath})$, the Maxwell equation (1.30) reads (in Heaviside-Lorentz units)

$$\partial_\mu F^{\mu\nu} = j^\nu, \tag{1.35}$$

while (1.34) remains as it is. Since $F^{\mu\nu}$ (as well as $\tilde{F}^{\mu\nu}$) is antisymmetric by construction, $F^{\mu\nu} = -F^{\nu\mu}$, current conservation is manifest:

$$0 = \partial_\mu \partial_\nu F^{\mu\nu} = \partial_\nu j^\nu = \frac{\partial}{\partial t}\rho + \vec{\nabla} \cdot \vec{\jmath} \tag{1.36}$$

Classical electrodynamics is an obvious example for a classical field theory with a high degree of structure both due to gauge symmetry as

well as the vector and tensor nature of the field variables.
With this insight, we can 'guess' a much simpler field theory that satisfies relativistic invariance:

$$\Box\phi(x) = 0, \tag{1.37}$$

where $\phi(x)$ is a scalar field that transforms trivially under Lorentz transformations $\phi(x) \to \phi'(x') \equiv \phi(x)$.
In fact, (1.37) is identical to the Klein-Gordon equation

$$\left(\Box + m^2\right)\phi(x) = 0 \tag{1.38}$$

for the special case of vanishing mass m. (Here we use also the convention $\hbar = 1$.)

From our advanced quantum mechanics course, we know that the Klein-Gordon equation also admits plane wave solutions,

$$\phi \sim \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\vec{k}\cdot\vec{x}} = \mathrm{e}^{-\mathrm{i}k^\mu x_\mu}, \tag{1.39}$$

where

$$k_\mu k^\mu = m^2. \tag{1.40}$$

The last equation is equivalent to

$$\omega^2 = \vec{k}^2 + m^2 \tag{1.41a}$$

which according to our conventions is identical to

$$E^2 = \vec{p}^2\mathrm{c}^2 + \left(m\mathrm{c}^2\right)^2, \qquad E = \hbar\omega, \quad \vec{p} = \hbar\vec{k}. \tag{1.41b}$$

This is nothing but the relativistic energy-momentum relation (dispersion relation) of a relativistic point-particle. Of course, in the quantum mechanics course, the Klein-Gordon equation has been motivated by the relativistic dispersion relation (1.41) with the wave equation (1.38) being a consequence of the correspondence principle $E \to \mathrm{i}\partial_t, \quad \vec{p} \to -\mathrm{i}\vec{\partial}_x$. From the viewpoint of field theory, the logic is reversed: we have written down the simplest relativistic field equations in (1.37) and (1.38) which turn out to support wave excitations that obey the dispersion relation of a relativistic point particle.

[NB: in fact, leaving relativity and quantum mechanics aside, the Klein-Gordon equation also appears in continuum mechanics: it describes the propagation of longitudinal waves of (the continuum limit of) a chain or net of oscillators with $\phi(x)$ corresponding to the amplitude of an oscillator at point x; the speed c is related to the spring constants, and m is a measure for a harmonic force pulling each oscillator back to its rest position.]

Comparing the dispersion relation (1.41a/b) to that found for waves in electrodynamics in (1.8), the latter appear to correspond to massless relativistic particles satisfying $\omega^2 = k^2$ or $E = |\vec{p}|\mathrm{c}$, the quantized version of which will be the photons.

Having obtained the (quantum mechanical) Klein-Gordon equation from field theory considerations, it is a perfectly legitimate viewpoint to interpret even the Schrödinger equation (at least mathematically) as a wave equation of a classical field theory,

$$\mathrm{i}\partial_t \psi(x) = -\frac{1}{2m}\vec{\nabla}^2\psi(x) + V(x)\psi(x). \tag{1.42}$$

Obviously, the Schrödinger equation is not invariant under Lorentz transformations, instead it is Galilei invariant (as Newton's classical mechanics). Correspondingly, its excitations give rise to dispersion relations of a non-relativistic point particle $\quad E = \dfrac{p^2}{2m} + \ldots \quad .$

One may justifiably object that there is still a clear distinction between field theories such as electrodynamics on the one hand side, and quantum mechanical field equations on the other hand side, because the quantum mechanical wave functions have a probabilistic interpretation, $P(x) = |\psi(x)|^2$, i.e. first, one needs to square the amplitude,

and second, the result is a probability not a fully deterministic prediction for a single measurement. However, this distinction becomes less meaningful, if we keep in mind that a typical observable for electromagnetic waves is the intensity, $I \sim |\vec{E}|^2, |\vec{B}|^2$, which is also related to the square of the field amplitude.
Moreover, when we approach the regime of very small intensities (and system sizes with actions of the order $S \sim \hbar$), we expect quantum effects to set in. Interestingly, it is not the Maxwell equations which break down in this regime, but it is the interpretation of the amplitudes that break down: the intensity then is related to the probability of measuring radiation (a photon).

An important difference between the quantum mechanical and the field theory viewpoint is the following: in quantum mechanics, we first lift the space coordinates and momenta to operators $\vec{x}, \vec{p} \to \hat{\vec{x}}, \hat{\vec{p}}$ with non-trivial commutation relations. Only once we formulate the Schrödinger equation in position space, the coordinates become 'c-numbers' again. In this manner, there is a fundamental difference between space and time, as the latter t always remains a parameter.
By contrast, both time and space remain parameters on an equal footing in field theory. This holds also true in QFT, where $(t, \vec{x})$ remain parameters; instead the fields themselves are lifted to operators.

All of the examples of field theories mentioned so far are special in the sense that their field equations are linear in the amplitude $\phi(x)$ (or $F^{\mu\nu}, A^\mu, \psi$). As a consequence, the superposition principle holds: if two solutions $\phi_1(x)$ and $\phi_2(x)$ exist, then also

$$\phi(x) = \alpha\phi_1(x) + \beta\phi_2(x) \tag{1.43}$$

is a solution (with $\alpha, \beta = \text{const}$).
This is generally no longer true if we consider non-linear theories. A famous example is Einstein's theory of general relativity, where the field variable is a now dynamical metric $g_{\mu\nu}(x)$ and the field equation

reads (in vacuum without cosmological constant)

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0. \tag{1.44}$$

Here, the Ricci tensor $R_{\mu\nu}$ and Ricci scalar R depend in a nonlinear way on $g_{\mu\nu}$ (and its inverse) and derivatives thereof.

1.3 The action principle for classical field theories

All of the above given examples for field equations can be derived from an action principle in much the same way as Hamilton's principle gives rise to equations of motion in classical mechanics. The corresponding action turns out to be of the form

$$S[\phi] = \int_V \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu\phi) \tag{1.45}$$

Here, the action S is considered to be a functional of the field ϕ. The integration measure d^4x over spacetime is a Lorentz invariant, as the Jacobian of the transformation, $\mathrm{d}^4x \rightarrow \mathrm{d}^4x' = |\det\Lambda|\mathrm{d}^4x$, involves the modulus of the determinant of Λ, which by virtue of (1.16) satisfies $(\det\Lambda)^2 = 1$. If $\mathcal{L}$ transforms as a scalar, S is a Lorentz invariant number for any field ϕ. The integration volume V may be finite or extend over full Minkowski space. Since (1.45) involves a volume integration, $\mathcal{L}$ is called the *Lagrange density*. We assume it to be a function of the field ϕ and its first derivative $\partial_\mu\phi$, since the above given field equations are of second order. As in classical mechanics, we could also allow for higher derivatives at the expense of higher-order field equations.

We look for those field configurations that extremize the action S. As in classical mechanics, we assume that the general field can be written as

$$\phi(x,\alpha) = \phi(x) + \alpha\eta(x), \tag{1.46}$$

where $\phi(x)$ is the extremizing solution, α is a parameter and $\eta(x)$ is an *arbitrary* field *variation* that vanishes on the boundary of V:

$$\eta(x)|_{x\in\partial V} = 0. \tag{1.47}$$

(I.e. if the general field has to satisfy specific boundary conditions on ∂V, these boundary conditions are completely carried by $\phi(x)$, i.e. the extremizing field.)
With these assumptions, S has to be stationary at $\alpha = 0$:

$$0 = \left.\frac{\partial S[\phi]}{\partial \alpha}\right|_{\alpha=0} = \int\limits_V \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\eta + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu \eta\right]_{\alpha=0}.$$

Integrating the second term by parts, yields

$$0 = \int\limits_V \mathrm{d}^4x \left\{\left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right]\eta(x)\right\}_{\alpha=0} + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\eta\right]_{\substack{\partial V\\ \alpha=0}}. \tag{1.48}$$

The last term is a surface term (to be evaluated along the normal of the surface) which vanishes because of (1.47). Since the first term has to vanish for any $\eta(x)$, we conclude that

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0.} \tag{1.49}$$

This is the field theory version of the Euler-Lagrange equation, representing a necessary condition for $\phi(x)$ to be a local extremum of the action functional $S[\phi]$. Note that we have not specified the nature of the field ϕ any further. If ϕ represents a multicomponent field $\phi^a, \quad a = 1, \ldots, N$ where a can be any kind of index, we correspondingly obtain N Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} = 0. \tag{1.50}$$

Let us start with the simplest example of a single-component real scalar field $\phi(x) \in \mathbb{R}$. Since $\mathcal{L}$ must be a Lorentz scalar, the simplest term involving $\partial_\mu\phi$ which we can write down is $\sim (\partial_\mu\phi)(\partial^\mu\phi)$. Because of the necessary pairing of the Lorentz indices, this is invariant under the additional symmetry $\phi \to -\phi$ (a $\mathbb{Z}_2$ symmetry, a transformation group consisting of the elements $\mathbb{Z}_2 \hat{=} \{-1, 1\}$). If we wish to maintain this symmetry also for the ϕ-dependent parts, the simplest Lagrange density takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2, \tag{1.51}$$

where the factors of $\frac{1}{2}$ are pure convention and the parameter m has been introduced to let the second term have the same *dimensionality* (units) as the first term. Inserting (1.51) into (1.49), we find

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi. \tag{1.52a}$$

With

$$(\partial_\kappa\phi)(\partial^\kappa\phi) = g^{\kappa\lambda}(\partial_\kappa\phi)(\partial_\lambda\phi) \quad \text{and} \quad \frac{\partial(\partial_\kappa\phi)}{\partial(\partial_\mu\phi)} = \delta_\kappa^\lambda,$$

we get

$$\begin{aligned} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} &= \frac{1}{2}\frac{\partial}{\partial(\partial_\mu\phi)}g^{\kappa\lambda}(\partial_\kappa\phi)(\partial_\lambda\phi) \\ &= \frac{1}{2}g^{\kappa\lambda}\delta_\kappa^\mu\partial_\lambda\phi + \frac{1}{2}g^{\kappa\lambda}\partial_\kappa\phi\delta_\lambda^\mu \\ &= \partial^\mu\phi \end{aligned} \tag{1.52b}$$

$$\Rightarrow \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\partial^\mu\phi = \Box\phi. \tag{1.53}$$

In other words, the Euler-Lagrange equation reads

$$(\Box + m^2)\phi = 0, \tag{1.54}$$

being identical to the Klein-Gordon equation. We conclude, that (1.51) corresponds to the Lagrange density of Klein-Gordon theory. Several comments are in order:

1. We have arrived at (1.51) using *symmetry* arguments (Lorentz, $\mathbb{Z}_2$) and *simplicity*. While symmetry is a clear defined criterion, simplicity (or beauty) is rather vague. While classical field theory has not much to offer as an alternative argument, quantum field theory does have another consistency criterion that can (at least partly) replace simplicity, it goes under the name of *renormalizability* which sounds (and at first sight is) technical, but goes to the very heart of the existence, origin or emergence of quantum field theories (see my lecture notes on 'Physics of Scales'). To *zeroth* approximation, renormalizability is related to dimensionality, see below.

2. Disregarding $\mathbb{Z}_2$ symmetry, an even simpler term would be a linear term $\sim j\phi$ with a parameter j or a function $j(x)$. The resulting field equation would be

$$(\Box + m^2)\phi(x) = j(x). \tag{1.55}$$

   Such a linear term hence would have the meaning of a *source term*. Such a source term would break $\mathbb{Z}_2$ symmetry explicitly.

3. Let us clarify the notion of units or dimensionality in our conventions where $\hbar = \mathrm{c} = 1$. For instance, from the dispersion relation (1.41a), it is clear that energy, momentum and mass all carry the same units which can be expressed in terms of an arbitrary unit scale. In high-energy physics, the typical choice is the energy unit of electron Volts eV with a GeV ($= 10^9$ eV) corresponding approximately to the mass ($\equiv$ rest energy) of the proton. Solely counting mass or energy dimensions, we write

$$[E] = [\omega] = [p_i] = [m] = 1. \tag{1.56a}$$

   Since the action carries the same unit as $\hbar = 1$, the action itself is dimensionless,

$$[S] = 0. \tag{1.56b}$$

   Since position times momentum has the unit of an action (as well as angular momentum), we have

$$[x \cdot p] = 0.$$

With (1.56a) this implies that position carries an inverse mass dimension

$$[x] = -1. \tag{1.56c}$$

Consequently, we have

$$[\mathrm{d}^4 x] = -4, \tag{1.56d}$$

and thus with (1.56b)

$$[\mathcal{L}] = 4, \tag{1.56e}$$

in four spacetime dimensions. From (1.56c) we deduce that

$$[\partial_\mu] = \left[\frac{\partial}{\partial x^\mu}\right] = 1. \tag{1.56f}$$

Combining these findings with the form of $\mathcal{L}$ in (1.51), we see that the field amplitude itself must carry a mass dimension

$$[\phi] = 1. \tag{1.56g}$$

(Exercise: generalize these considerations to a Klein-Gordon field in D dimensional spacetime.)

4. The linearity of the resulting field equation is in one-to-one correspondence with the fact that the action / Lagrangian (1.51) is quadratic in the fields. It is straightforward to construct more general non-linear theories, e.g. by generalizing the mass term to a full function,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi). \tag{1.57}$$

In analogy to classical mechanics, we call $V(\phi)$ a potential. Note, however, that $V(\phi)$ generically does not give preference for a particle / excitation to be at a certain position in space(time), but for the field to have a certain amplitude. Correspondingly, the first term $\sim (\partial_\mu \phi)(\partial^\mu \phi)$ is called a kinetic term. Analogously to mechanics, it is a measure for how much action is stored in variations of the field in time *and* space.

$\mathbb{Z}_2$ symmetry is preserved if the potential satisfies $V(\phi) = V(-\phi)$. Considering its Taylor expansion about the origin in field space

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \dots, \tag{1.58}$$

we encounter a quartic term which, on the level of the equations of motion, turns into a cubic interaction,

$$(\Box + m^2)\phi + \frac{\lambda}{3}\phi^3 + \dots = 0. \tag{1.59}$$

The parameter λ is dimensionless, $[\lambda] = 0$, and serves as a measure for the interaction of the field with itself. For small $\lambda \ll 1$, the dispersion relation of small amplitude fluctuations remains essentially unmodified, and we expect approximate plane wave excitations of mass m. For large couplings and/or large amplitudes, the nonlinearity will lead to sizeable modifications both of the wave form as well as the dispersion relation.

We close this section by listing the actions that give rise to the field equations discussed in the previous section:

1. **Maxwell's electrodynamics**:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu \tag{1.60}$$

in presence of a current J_μ. The signs are chosen such that the above given conventions are met.

2. **Klein-Gordon theory** for a complex field $\phi \in \mathbb{C}$:

$$\mathcal{L} = (\partial_\mu\phi^*)(\partial^\mu\phi) - m^2\phi^*\phi. \tag{1.61}$$

With the decomposition into to real fields

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2), \quad \phi_{1,2} \in \mathbb{R}, \tag{1.62}$$

(1.61) splits into two copies of (1.51).

3. **Schrödinger theory** for $\psi(x) \in \mathbb{C}$:

$$\mathcal{L} = \psi^* \mathrm{i}\partial_t \psi - \frac{1}{2m}(\vec{\nabla}\psi^*)(\vec{\nabla}\psi) - V(x)\psi^*\psi. \tag{1.63}$$

The explicit verification of the corresponding field equations is left as an exercise to the reader.

1.4 Functional differentiation

The variational calculus, introducing a variation parameter *and* an *arbitrary* variation $\eta(x)$, can be most conveniently formulated in terms of functional differentiation. The latter is a directional derivative of a c-number valued functional taken 'into the direction of the function' in function space. Its precise mathematical definition requires a careful discussion of function spaces (see, e.g. *Methods of Mathematical Physics* by Courant, Hilbert '53). For our purposes, it suffices to work with the (mostly) algebraic rules following from its definition (which can equally well be worked out from the variational calculus above): a functional derivative is *linear*

$$\frac{\delta}{\delta\phi(x)}(\alpha F_1[\phi] + \beta F_2[\phi]) = \alpha\frac{\delta F_1[\phi]}{\delta\phi(x)} + \beta\frac{\delta F_2[\phi]}{\delta\phi(x)}, \tag{1.64}$$

and obeys a Leibniz rule

$$\frac{\delta}{\delta\phi(x)}(F_1[\phi]F_2[\phi]) = \frac{\delta F_1[\phi]}{\delta\phi(x)}F_2[\phi] + F_1[\phi]\frac{\delta F_2[\phi]}{\delta\phi(x)}. \tag{1.65}$$

The fundamental nontrivial derivative is

$$\frac{\delta\phi(y)}{\delta\phi(x)} = \delta^{(D)}(y-x), \tag{1.66}$$

where D is the number of spacetime dimensions, and $\delta^{(D)}$ is the δ distribution in the considered function space.
With this tool, let us verify that the extrema of the action $S[\phi]$ satisfy

the Euler-Lagrange equations:

$$
\begin{aligned}
0 = \frac{\delta S}{\delta\phi(x)} &= \int \mathrm{d}^4 y \, \frac{\delta}{\delta\phi(x)} \mathcal{L}(\phi, \partial_\mu\phi; y) \\
&= \int \mathrm{d}^4 y \, \left(\frac{\delta\phi(y)}{\delta\phi(x)} \frac{\partial\mathcal{L}}{\partial\phi}(y) + \frac{\delta\partial^y_\mu\phi(y)}{\delta\phi(x)} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(y) \right) \\
&= \int \mathrm{d}^4 y \, \left(\delta^{(4)}(y-x) \frac{\partial\mathcal{L}}{\partial\phi}(y) + \partial^y_\mu \delta^{(4)}(y-x) \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(y) \right) \\
&\overset{\text{i.b.p.}}{=} \int \mathrm{d}^4 y \, \left[\delta^{(4)}(y-x) \left(\frac{\partial\mathcal{L}}{\partial\phi}(y) - \partial^y_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(y) \right) \right] \\
&= \frac{\partial\mathcal{L}}{\partial\phi(x)} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi(x))} .
\end{aligned} \tag{1.67}
$$

Note that $\mathcal{L}$ is a function of the field and its derivatives and thus only partial derivatives of $\mathcal{L}$ have to be evaluated. The surface term of the partial integration (i.b.p.) does not contribute for obvious reasons as long as x is not on the boundary of the integration volume. If it was, the functional directional derivative would correspond to a change or variation of the boundary conditions imposed on the fields, which we do not want to consider here. This restriction is equivalent to choosing $\eta(x)|_{\partial V} = 0$ in the variational calculus.

Further examples of functional differentiation are discussed in the exercises.

2 Aspects of classical field theory

In the introduction section, we have essentially derived (or rather motivated) the Lagrangian formulation of classical field theory in almost complete analogy to classical mechanics. Let us continue to use this analogy to apply further concepts of classical mechanics to field theory, starting with the Hamiltonian formulation.

2.1 Hamiltonian formulation

Let us use the Klein–Gordon field as a simple example for the following section. As in (1.57), we generalize the mass term to a full potential:

$$\begin{aligned} S[\phi] &= \int \mathrm{d}^4x\, \mathcal{L}(\phi, \partial_\mu \phi), \\ \mathcal{L} &= \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi). \end{aligned} \tag{2.1}$$

Let us first try to find a relativistic (covariant) Hamiltonian, naively generalizing the rules of classical mechanics to field theory. For this, we first define a field momentum conjugate to the field amplitude:

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} \overset{(2.1)}{=} \partial_\mu \phi. \tag{2.2}$$

The corresponding Hamiltonian is then obtained by a Legendre transform:

$$\mathcal{H}_{\text{cov}} = \Pi_\mu \underbrace{\partial^\mu \phi}_{=\Pi^\mu} - \underbrace{\mathcal{L}}_{=\frac{1}{2}\Pi_\mu \Pi^\mu - V(\phi)} = \frac{1}{2}\Pi_\mu \Pi^\mu + V(\phi). \tag{2.3}$$

At first glance, this looks similar to point particle Hamiltonians à la $H = \dfrac{p^2}{2m} + V(x)$. However, there is a problem: with (2.2), the kinetic term corresponds to

$$\frac{1}{2}\Pi_\mu \Pi^\mu = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2.$$

Because of the minus sign, $\mathcal{H}_{\text{cov}}$ is not bounded from below even for bounded potentials $V(\phi)$. Hence, $\mathcal{H}_{\text{cov}}$ cannot be interpreted as an energy quantity related to a given field configuration.
This is not too surprising: By construction, $\mathcal{H}_{\text{cov}}$ is invariant under Lorentz transformations, whereas the field energy is expected to transform as a 0-component of a 4-vector (as for a point particle).

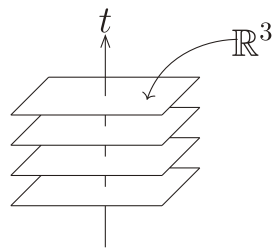
In order to preserve the energy interpretation for the Hamiltonian, we give up manifest covariance for a moment and choose a fixed reference frame with a time t, $x^\mu = (t, \vec{x})$, such that the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi). \tag{2.4}$$

Now we define the canonical momentum as in classical mechanics:

$$\Pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x})} = \dot{\phi}(\vec{x}), \tag{2.5}$$

where the notation should indicate that this definition holds at every space point $\vec{x}$, while the time t is considered as an evolution parameter as in classical mechanics. The Hamiltonian formulation thus induces a foliation of spacetime $\mathbb{M} \to \mathbb{R}^3 \otimes \mathbb{R}$.



Again, we obtain the Hamiltonian by a Legendre transformation,

$$\mathcal{H} = \Pi \underbrace{\dot{\phi}}_{=\Pi} - \underbrace{\mathcal{L}}_{=\frac{1}{2}\Pi^2 - (\vec{\nabla}\phi)^2 - V(\phi)} = \frac{1}{2}\Pi + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi). \tag{2.6}$$

For potentials bounded from below, this is a manifestly bounded function of the field and the momentum. Its units correspond to those of an energy density. Hence, the three terms can be interpreted as the energy densities stored in or required by the time evolution $\sim \Pi^2$, spatial field variations $\sim (\vec{\nabla}\phi)^2$, or the excitation of field amplitudes $\sim V(\phi)$. As will be detailed in the exercises, the equation of motion follow now directly from the corresponding Hamilton equations in complete analogy to classical mechanics. The construction can be briefly summarized as follows:

$\phi(\vec{x})$ and $\Pi(\vec{x})$ span the phase space. Using functional differentiation, we can define Poisson brackets for general phase space functionals $A[\phi, \Pi]\,, B[\phi, \Pi]$:

$$\{A, B\} = \int d^3z \left(\frac{\delta A}{\delta\phi(\vec{z})} \frac{\delta B}{\delta\Pi(\vec{z})} - \frac{\delta B}{\delta\phi(\vec{z})} \frac{\delta A}{\delta\Pi(\vec{z})} \right). \tag{2.7}$$

The fundamental Poisson brackets read

$$\begin{aligned} \{\phi(\vec{x}), \Pi(\vec{y})\} &= \delta^{(3)}(\vec{x} - \vec{y}), \\ \{\phi(\vec{x}), \phi(\vec{y})\} &= 0 = \{\Pi(\vec{x}), \Pi(\vec{y})\}. \end{aligned} \tag{2.8}$$

The canonical equations of motion then yield as usual

$$\dot{\phi}(\vec{x}) = \{\phi(\vec{x}), H\}, \quad \dot{\Pi}(\vec{x}) = \{\Pi(\vec{x}), H\}, \tag{2.9}$$

where we have used the Hamilton functional

$$H = \int d^3y\, \mathcal{H}(\vec{y}). \tag{2.10}$$

Hence, $\mathcal{H}$ is also called the Hamiltonian density.

Inserting (2.6) into (2.9) leads to the field equation

$$0 = \ddot{\phi} - \vec{\nabla}^2\phi + V'(\phi) \equiv \Box\phi + V'(\phi), \tag{2.11}$$

as expected. We emphasize that (2.11) is a covariant field equation, even though the Hamiltonian construction is not manifestly covariant at intermediate stages.

2.2 Symmetries and conservation laws

In classical mechanics, symmetries can be closely related to conserved quantities as is captured by Noether's theorem. In fact, the same relation persists in classical field theory:

Let us consider an infinitesimal deformation of the field

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x), \tag{2.12}$$

where $\delta\phi(x)$ is considered to be an infinitesimal continuous deformation (finite deformations can be generated from successive infinitesimal deformations). Equation (2.12) in considered to be a *symmetry transformation* if the *field equations remain invariant*.

On the level of the Lagrangian, this implies that $\mathcal{L}$ is allowed to change only up to a total derivative:

$$\begin{aligned} \mathcal{L} \quad &\to \quad \mathcal{L}' = \mathcal{L} + \delta\mathcal{L}, \\ \text{where} \quad &\quad \delta\mathcal{L} = \partial_\mu K^\mu. \end{aligned} \tag{2.13}$$

Then, the action changes by a surface term

$$\delta S = \int \mathrm{d}^4x\, \delta\mathcal{L} = \int\limits_{\Omega} \mathrm{d}^4x\, \partial_\mu K^\mu = \int\limits_{\partial\Omega} \mathrm{d}\sigma_\mu K^\mu \tag{2.14}$$

If K^μ is sufficiently localized (which we assume in the following), δS vanishes since $\delta\Omega$ is considered to be the boundary of our spacetime volume Ω at spatial and temporal infinity. This implies that the action is invariant under (2.12) & (2.13) and so are the equations of motion.

Noether's theorem now relates this invariance to a conserved quantity.

Let $\phi \to \phi + \delta\phi$ with $\delta\mathcal{L} = \partial_\mu K^\mu$ be an infinitesimal symmetry transformation. Then, there is a 4-current,

$$\text{Noether current:} \qquad J^\mu = \Pi^\mu \delta\phi - K^\mu \tag{2.15}$$

$$\text{where} \qquad \Pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}, \tag{2.16}$$

$$\text{which is conserved,} \qquad \partial_\mu J^\mu = 0, \tag{2.17}$$

if ϕ satisfies the equations of motion.

Proof:

Varying the Lagrangian yields

$$\begin{aligned} \partial_\mu K^\mu = \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \underbrace{\delta(\partial_\mu\phi)}_{=\partial_\mu\delta\phi} \\ &= \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi + \partial_\mu\left(\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}}_{=\Pi^\mu}\delta\phi\right) \end{aligned} \tag{2.18}$$

Using the equations of motion, the term in [ ]-brackets vanishes, and we find

$$0 = \partial_\mu(\Pi^\mu\delta\phi - K^\mu) =: \partial_\mu J^\mu \tag{2.19}$$

□

If in addition the Noether current vanishes sufficiently fast towards spatial infinity $|\vec{x}| \to \infty$, we find

$$\begin{aligned}0 &= \int \mathrm{d}^3x\, \partial_\mu J^\mu = \partial_t \int \mathrm{d}^3x\, J^0 - \int_{\mathbb{R}^3} \mathrm{d}^3x\, \vec{\nabla} \cdot \vec{J} \\ &= \partial_t \int \mathrm{d}^3x\, J^0 - \underbrace{\int_{\partial\mathbb{R}^3} \mathrm{d}\vec{r} \cdot \vec{J}}_{\to 0} = \partial_t \int \mathrm{d}^3x\, J^0 =: \dot{Q}.\end{aligned} \tag{2.20}$$

The corresponding integral over the zero component of the current is called the Noether charge,

$$Q = \int \mathrm{d}^3x\, J^0, \tag{2.21}$$

which by virtue of (2.20) is conserved. Note that discrete symmetries, such as $\phi \to -\phi$, are not subject to Noether's theorem, as they cannot be formulated infinitesimally.

Let us illustrate the significance of Noethers theorem with the aid of two examples.

Example 1: translations

Translations are part of the space-time symmetries which together with the Lorentz transformations form the *Poincaré group*. Translation invariant systems do not feature a distinguished point in spacetime. A translation

$$x^\mu \to x^{\mu\prime} = x^\mu - a^\mu, \quad a^\mu = \text{const} \tag{2.22}$$

acts on the field as

$$\phi(x) \to \phi'(x) = \phi(x - a). \tag{2.23}$$

For infinitesimal translations, we get

$$\begin{aligned}&\phi(x - a) = \phi(x) - a_\mu \partial^\mu \phi(x) + \mathcal{O}(a^2) \\ &\Rightarrow \delta\phi(x) = -a_\mu \partial^\mu \phi(x).\end{aligned} \tag{2.24}$$

Similarly, we get for the Lagrangian

$$\begin{aligned}\mathcal{L} \to \mathcal{L}(x-a) &= \mathcal{L}(x) - a_\mu \partial^\mu \mathcal{L}(x) + \mathcal{O}(a^2) \\ \Rightarrow \delta\mathcal{L} &= -a_\mu \partial^\mu \mathcal{L}(x) \equiv \partial_\mu(-a^\mu \mathcal{L}) \\ \Rightarrow K^\mu &= -a^\mu \mathcal{L}.\end{aligned} \tag{2.25}$$

From this, we get the Noether current

$$\begin{aligned}J^\mu &= \Pi^\mu \delta\phi - K^\mu = \Pi^\mu(-a_\nu \partial^\nu \phi) + a^\mu \mathcal{L} \\ &= -a_\nu(\Pi^\mu \partial^\nu \phi - g^{\nu\mu}\mathcal{L}) =: -a_\nu T^{\mu\nu},\end{aligned} \tag{2.26}$$

where we have defined the canonical energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu}\mathcal{L} \tag{2.27}$$

which by Noether's theorem satisfies

$$\partial_\mu T^{\mu\nu} = 0. \tag{2.28}$$

The 00-component corresponds to the Hamiltonian density,

$$T^{00} = \Pi^0 \partial^0 \phi - \mathcal{L} \equiv \Pi\dot{\phi} - \mathcal{L} = \mathcal{H}. \tag{2.29}$$

The associated conserved Noether charge

$$\dot{Q} = \partial_t \int \mathrm{d}^3x\, J^0 \qquad \Rightarrow \qquad \partial_t \int \mathrm{d}^3x\, T^{0\nu} =: \frac{\mathrm{d}}{\mathrm{d}t} P^\nu = 0 \tag{2.30}$$

can be interpreted as the physical 4-momentum of the field (not to be confused with the canonical momentum Π^μ),

$$P^\mu := \int \mathrm{d}^3x\, T^{0\mu} = \int \mathrm{d}^3x\, (\Pi \partial^\mu \phi - g^{0\mu}\mathcal{L}), \tag{2.31}$$

the components of which read

$$\begin{aligned}P^0 &= \int \mathrm{d}^3x\, T^{00} = H \qquad \text{(energy)} \\ P^i &= \int \mathrm{d}^3x\, \Pi \partial^i \phi. \qquad \text{(3-momentum)}\end{aligned} \tag{2.32}$$

(e.g. in Maxwell's theory, P^i is related to the Poynting vector.)

Example 2: complex scalar field

In addition to spacetime symmetries also *internal* symmetries can induce conservation laws. Let us consider the case of a complex scalar field

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \tag{2.33}$$

The Lagrangian is invariant under phase rotations, $\delta\mathcal{L} = 0$

$$\phi \to \mathrm{e}^{\mathrm{i}\alpha}\phi, \quad \phi^* \to \mathrm{e}^{-\mathrm{i}\alpha}\phi^* \tag{2.34}$$

for $\alpha = \text{const} \in \mathbb{R}$. Infinitesimally, we have

$$\phi \to \phi + \mathrm{i}\alpha\phi = \phi + \delta\phi, \quad \phi^* \to \phi^* - \mathrm{i}\alpha\phi^* = \phi^* + \delta\phi^*. \tag{2.35}$$

Since $\delta\mathcal{L} = 0$, we have $K^\mu = 0$ as well. Correspondingly, the Noether current is

$$\begin{aligned} J^\mu &= \Pi^\mu \delta\phi + \Pi^{*\mu}\delta\phi^* = \mathrm{i}\alpha(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi) \\ &= 2\alpha\Im(\phi^*\partial^\mu\phi). \end{aligned} \tag{2.36}$$

Apart from the (irrelevant) factor α, we obtain the Klein-Gordon current

$$j^\mu = \frac{J^\mu}{\alpha} = 2\Im(\phi^*\partial^\mu\phi), \tag{2.37}$$

and the corresponding Noether charge

$$Q = \int \mathrm{d}^3x\, j^0 = \mathrm{i}\int \mathrm{d}^3x\,(\phi\partial^0\phi^* - \phi^*\partial^0\phi). \tag{2.38}$$

Both expressions (2.37) & (2.38) are familiar from relativistic quantum mechanics: after reinterpreting the 'negative energy states' as amplitudes, j^μ corresponds to the electromagnetic current generated by a Klein-Gordon wave function, and Q to its electric charge, which upon coupling to a Maxwell field generate $\vec{E}$ and $\vec{B}$ fields.

3 Nonlinear scalar field theories

In the preceding sections, we have already considered scalar field theories with a general potential $V(\phi)$ as an example for a nonlinear generalization of Klein–Gordon theory, cf. (1.57),

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi). \tag{3.1}$$

This class of models has a wide range of applications (in particle physics, many–body physics, statistical physics, etc.) and features a number of physical mechanisms. In the following, we concentrate on their properties related to symmetry and (spontaneous) symmetry breaking.

3.1 $\mathbb{Z}_2$ model

We have already discussed that (3.1) for a real scalar field entails a $\mathbb{Z}_2$ symmetry under

$$\phi \to -\phi \tag{3.2}$$

if $V(\phi) = V(-\phi)$. E.g., for

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4, \tag{3.3}$$

the equation of motion is

$$\left(\Box + m^2 + \frac{\lambda}{3!}\phi^2\right)\phi = 0 \tag{3.4}$$

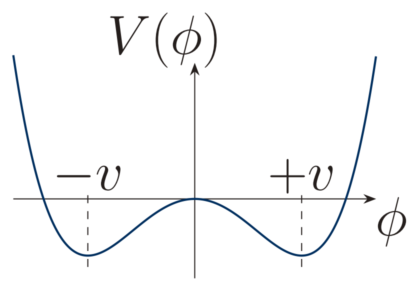
from which it is obvious that for a given solution $\phi_0(x)$ also $-\phi_0(x)$ is a solution of (3.4). (Of course, it may not satisfy the same boundary conditions that have been imposed on $\phi_0(x)$. In general, boundary conditions may break (violate) the $\mathbb{Z}_2$ symmetry explicitly.)

In any case, (3.4) has a trivial solution: $\phi = 0$ which is sometimes called the 'vacuum solution'. Small excitations with amplitude $\phi \lll 1$ propagate to leading order in a λ-expansion according to the 'free' (linear) Klein-Gordon equation
$(\Box + m^2)\phi \approx 0 + \mathcal{O}(\lambda)$, justifying to say that excitations on top of the vacuum have a mass m.

Let us now deform (3.3) a little and consider the potential

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{3.5}$$

At first sight, this looks odd as one may be tempted to say that the theory has a negative mass squared $m^2 \stackrel{?}{=} -\mu^2$. This is, however, not true, as we should study the dispersion relation of excitations on top of the vacuum in order to define a propagating mass.
The form of the potential reveals, that $\phi = 0$ is not a stable solution. Any excitation will drive the system towards one of the minima



$$\phi_0 = \pm\sqrt{\frac{6\mu^2}{\lambda}} =: \pm v. \tag{3.6}$$

Hence, the role of the stable vacuum solution is now played by one of the two cases $\phi_0 = \pm v$. Let us study the excitations on top of the 'right' vacuum:

$$\phi(x) = v + \sigma(x). \tag{3.7}$$

The Lagrangian then reads $\left(v = \sqrt{\dfrac{6\mu^2}{\lambda}}\right)$

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \left[\frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{3!}\lambda v\sigma^3 + \frac{1}{4!}\lambda\sigma^4\right]. \tag{3.8}$$

For small excitations $\sigma \lll 1$, the equations of motion then read

$$(\Box + (2\mu^2))\sigma = 0 + \mathcal{O}(\lambda). \tag{3.9}$$

We conclude that these excitations behave like relativistic point particles with a mass $= \sqrt{2}\mu$. In addition to the quartic $\sim \phi^4$ interaction, σ in (3.8) also exhibits a cubic interaction $\sim \sigma^3$,

$$V_\sigma(\sigma) = \frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{3!}\lambda v \sigma^3 + \frac{1}{4!}\lambda\sigma^4. \tag{3.10}$$

We observe that – while $V(\phi)$ is symmetric under $\phi \to -\phi$ – the potential for σ is not, $V_\sigma(\sigma) \neq V_\sigma(-\sigma)$. This is, of course, not too surprising, because we have made a choice in (3.7) and picked the 'right' vacuum solution $\phi_0 = +v$. If we had picked the 'left' solution, the conclusions about the massive excitation in (3.9) would have been the same, as well as the result that the new potential for σ as the excitation on top of the vacuum $\phi_0 = -v$ would not exhibit a $\mathbb{Z}_2$ symmetry.

The mere fact that the vacuum solution has the property $\phi_0 = \pm v \neq 0$ is already in conflict with the symmetry. In order to be 'in the vacuum' the field has to give preference to either a positive amplitude $\phi_0 = +\phi$ or a negative amplitude $\phi_0 = -\phi$. Once, the vacuum solution has made this choice (we say 'has broken the symmetry') the symmetry is no longer manifest for excitations on top of the vacuum.

It is useful to introduce some more nomenclature: if the vacuum configuration of a field corresponds to a nonzero amplitude, we say that the field *condenses*. The value v of the amplitude in the vacuum is called a *condensate*. As the vacuum configuration no longer respects the symmetry of the Lagrangian, we talk about *spontaneous symmetry breaking*

The attribute 'spontaneous' characterizes the situation that the field, in principle, has two (or, in general, several) options to relax towards a vacuum. This should be contrasted with symmetry breaking induced by boundary conditions or non-symmetric terms in the action, which are imposed *explicitly* in the form of additional conditions or parameters.

3.2 O(N) model

Let us next promote the field ϕ to an N-component vector field

$$\phi^a \in \mathbb{R}, \quad a = 1, \ldots, N$$

with a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)(\partial^\mu \phi^a) - V(\phi), \tag{3.11}$$

where

$$V(\phi) = -\frac{1}{2}\mu^2 \phi^a \phi^a + \frac{\lambda}{4!}(\phi^a \phi^a)^2. \tag{3.12}$$

Equivalently, we could use a vector notation

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})(\partial^\mu \vec{\phi}) - \left[-\frac{1}{2}\mu^2 \vec{\phi} \cdot \vec{\phi} + \frac{\lambda}{4!}(\vec{\phi} \cdot \vec{\phi})^2\right]. \tag{3.13}$$

It is important to note that these vectors $\vec{\phi}$ do not 'point' along certain directions in space or spacetime, but denote directions in an *internal* space $\vec{\phi} \in \mathbb{R}^N$.

In the form of (3.13), it is easy to see that the model is invariant under transformations that leave the Euclidean scalar product in $\mathbb{R}^N$ invariant. These transformations form the group of orthogonal transformations O(N); i.e. the field vector components ϕ^a are transformed by $N \times N$ matrices U^{ab}

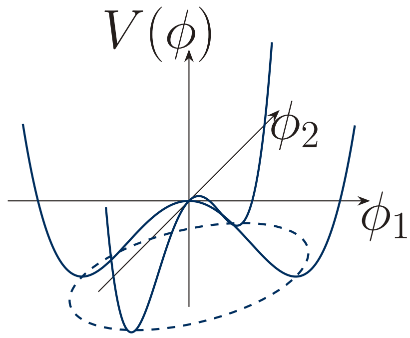
$$\phi^a \to U^{ab} \phi^b, \tag{3.14}$$

which constitute a matrix representation of O(N). The scalar product is invariant, if U satisfies

$$U^{ab} U^{ac} = (U^T)^{ba} U^{ac} = (U^T U)^{bc} = \mathbb{1}^{bc} = \delta^{bc} = (UU^T)bc. \tag{3.15}$$

As the field components ϕ^a are real, U corresponds to an orthogonal $N \times N$ matrix with real components.

For the above case with a 'negative mass-like parameter' $-\mu^2$, the potential has the form as sketched on the right for $N = 2$, where the dashed line marks a circle in field space, where the potential is minimal. For general N, this minimum corresponds to an $(N-1)$-dimensional sphere S^{N-1}, which is defined by



$$\phi_0^a \phi_0^a = v^2 = \frac{6\mu^2}{\lambda}. \tag{3.16}$$

In contrast to the $\mathbb{Z}_2$ model there are not merely two points, but a continuum of possible vacuum solutions. Let us choose a specific one

$$\vec{\phi}_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v \end{pmatrix}, \quad v = \sqrt{\frac{6\mu^2}{\lambda}}. \tag{3.17}$$

Then, the O(N) symmetry is spontaneously broken, since a generic O(N) transformation would rotate $\vec{\phi}_0$ to a different point on S^{N-1}. Still, there is a subset of O(N) transformations that leaves $\vec{\phi}_0$ invariant. This is the set of rotations about the $\vec{\phi}_0$-axis in field space. It is possible to show that this subset forms again a group, namely O($N-1$). We say that the ground state (3.17) breaks the group O(N) spontaneously to O($N-1$).

Now, it is interesting to study the excitations on the top of the vacuum, which we parametrize by

$$\vec{\phi}(x) = \begin{pmatrix} \vec{\pi}(x) \\ v + \sigma(x) \end{pmatrix}, \quad \pi^i, \quad i = 1, \ldots, N-1. \tag{3.18}$$

In terms of the fields $\pi^i(x)$, $\sigma(x)$, the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \pi^i)(\partial^\mu \pi^i) + \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) - V(\sigma, \pi^i), \tag{3.19}$$

where

$$\begin{aligned} V(\sigma, \pi^i) \;=\; & \frac{1}{2}(2\mu^2)\sigma^2 \\ & + \sqrt{\frac{\lambda}{6}}\mu\sigma^3 + \sqrt{\frac{\lambda}{6}}\mu(\pi^i)^2\sigma + \frac{\lambda}{4!}\sigma^4 + \frac{\lambda}{12}(\pi^i)^2\sigma^2 + \frac{\lambda}{4!}\left[(\pi^i)^2\right]^2. \end{aligned} \tag{3.20}$$

Here, we observe:

- a scalar excitation $\sigma(x)$ with mass

$$m_\sigma^2 = 2\mu^2. \tag{3.21}$$

- The π^i and σ fields are interacting as well as self-interacting. This means that the field equations for π^i and σ are mutually coupled and nonlinear.

- The Lagrangian is invariant under transformations of π^i by orthogonal
  $(N-1) \times (N-1)$ matrices

$$\pi^i \to U^{ij}\pi^j \quad \text{where } U \in \mathrm{O}(N-1). \tag{3.22}$$

  This reflects the residual $\mathrm{O}(N-1)$ symmetry.

- The π field *remains massless*, as there is no purely quadratic term in π^i.

The last point is particularly important: the spontaneous breaking of a continuous global symmetry $\mathrm{O}(N) \to \mathrm{O}(N-1)$ yields $N-1$ massless bosons (here: scalars). The latter are called *Nambu-Goldstone bosons* (or only 'Goldstone bosons'), where the nomenclature comes from a QFT / particle-physics context. The phenomenon, however, is equally important in classical field theory, e.g. in applications to statistical models (e.g. spin waves).

The number of Goldstone bosons is related to the symmetry-breaking pattern, more specifically to the 'number of broken generators'. The latter are those generators of $\mathrm{O}(N)$ that generate transformations that would not leave the chosen vacuum invariant. This statement is quantifiable:

$$\begin{aligned}
&\text{\# of O}(N)\text{ generators} && n_{\mathrm{O}(N)} = \frac{1}{2}N(N-1)\\
&\text{\# of O}(N-1)\text{ generators} && n_{\mathrm{O}(N-1)} = \frac{1}{2}(N-1)(N-2)\\
\Rightarrow\quad & n_{\mathrm{O}(N)} - n_{\mathrm{O}(N-1)} = N-1 \;\hat{=}\; \text{\# of } \pi^i \text{ fields.} && (3.23)
\end{aligned}$$

The present example is a special case of the more general Goldstone theorem, see below, relating the appearance of Goldstone bosons and their numbers to the number of spontaneously broken generators; it is not restricted to the present $\mathrm{O}(N)$ case.

The notation in terms of σ and π fields is taken over from low-energy models of Quantum Chromodynamics (QCD): QCD has an approximate chiral symmetry (to be discussed later). In the case, where only 'up' and 'down' quarks are considered, the symmetry corresponds to independent 'flavour' rotations, i.e. unitary transformations, of left- and right-handed components of the Dirac spinor fields. The symmetry group is

$$\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}} \approx \mathrm{O}(4). \tag{3.24}$$

The σ field is also often called a 'radial' excitation, as it characterizes field equations orthogonal to the S^{N-1} sphere (orthogonal to the dashed line in the above figure), while the π^i fields are excitations within the S^{N-1} sphere. The σ excitation has to go 'uphill' in the potential $V(\sigma, \pi^i)$, and thus is massive. In QCD it is supposed to correspond to a heavy scalar mesonic resonance ($\sim \mathcal{O}(1\text{GeV})$). The π^i excitations are excitations within S^{N-1}, i.e. a 'flat' direction in the potential landscape. In QCD, $\pi^1, \pi^2\pi^3$ correspond to the light pions with a mass $\sim$ 135MeV. This small mass arises from the fact that the chiral symmetry is only approximate in QCD. It is also explicitly broken by the quark mass terms. In the literature, O(N) models in the form discussed here are also called 'linear sigma models'.

3.3 Goldstone theorem

The connection between the appearance of massless Goldstone bosons and spontaneously broken symmetries is generally formulated within Goldstone's theorem. It holds both in classical field theory as well as in quantum field theory. In both cases, the proof is essentially identical except for the fact that the classical potential has to be replaced by the effective potential in QFT (NB: the effective potential already includes the effects of all quantum fluctuations.)

We start from the action that we write as

$$S[\phi] = \int \mathrm{d}^4x \, (-V(\phi) + \text{ terms with higher derivatives }). \tag{3.25}$$

We assume that the derivative terms – if nonzero – only result in deviations from the extremum of the action, such that the ground state is homogeneous and thus determined by the minimum of the potential. In other words, we assume that $V(\phi)$ is minimized by $\phi_0^a = \text{const.}$ in space and time. Then

$$\left.\frac{\partial}{\partial \phi^a} V\right|_{\phi^a(x)=\phi_0^a} = 0. \tag{3.26}$$

Expanding about this minimum, we get

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)^a(\phi - \phi_0)^b \frac{\partial^2}{\partial \phi^a \partial \phi^b} V(\phi_0) + \ldots \quad , \tag{3.27}$$

since the linear term vanishes by virtue of (3.26). The coefficient of the quadratic term

$$m_{ab}^2 := \frac{\partial^2}{\partial \phi^a \partial \phi^b} V(\phi_0) \tag{3.28}$$

is a symmetric matrix, the eigenvalues of which specify the masses of the fields. Since ϕ_0 is a minimum, these masses cannot be negative.

Next, we assume that the theory has a continuous symmetry (obeyed by the action as well as the quantization procedure in QFT) with the transformed field of the form

$$\phi^a \to \phi^a + \delta\phi^a, \tag{3.29}$$

where $\delta\phi^a$ can be some function of all fields $\delta\phi^a = \delta\phi^a(\phi)$. Considering only constant fields, the invariance of the action implies invariance of the potential,

$$V(\phi) = V(\phi + \delta\phi) \tag{3.30}$$

$$\Rightarrow \delta\phi^a \frac{\partial}{\partial\phi^a} V(\phi) = 0. \tag{3.31}$$

Differentiating with respect to ϕ^b and setting $\phi = \phi_0$, we get

$$0 = \left.\frac{\partial(\delta\phi^a)}{\partial\phi^b}\right|_{\phi_0} \underbrace{\left(\frac{\partial V(\phi_0)}{\partial\phi^a}\right)}_{=0} + \delta\phi^a(\phi_0) m_{ab} = \delta\phi^a(\phi_0) m_{ab}. \tag{3.32}$$

If the transformation leaves ϕ_0 unchanged, then $\delta\phi^a(\phi_0) = 0$, and (3.32) is trivially satisfied. A spontaneously broken symmetry is precisely one for which $\delta\phi^a(\phi_0) \neq 0$. In this case, $\delta\phi^a(\phi_0)$ is an eigenvector of the mass matrix with *eigenvalue zero.*

This proves *Goldstone's theorem*: every continuous symmetry of the theory that is not a symmetry of the ground state ϕ_0 gives rise to a massless excitation corresponding to a Nambu-Goldstone boson.

3.4 Hidden symmetry & the Higgs mechanism

Though the Goldstone theorem has many applications in field theory in condensed-matter as well as particle physics, it hampered progress in particle physics for quite a while around $\sim$ 1960. While the use of symmetries appeared technically and aesthetically helpful in the construction of models for the weak (and strong) interactions, these symmetries had to be broken in order to match with the data. If the breaking happens spontaneously, Goldstone's theorem seemed to imply the necessary occurrence of massless excitations – which, however, were not observed. On the contrary, the potentially existing bosons seemed to be rather heavy.

The essential breakthrough was stimulated by Anderson's description of superconductivity and the Meissner-Ochsenfeld-Effekt in condensed-matter physics and then was transferred to nonabelian models and particle physics by Brout, Englert, Higgs, Hagen, Kibble and Guralnik, leading to what is now known as the electroweak Higgs sector of the standard model of particle physics.

We will study here the essentials with the aid of a simple model: scalar QED (or abelian Higgs model):

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^* (D^\mu \phi) + \mu^2 \phi^* \phi - \frac{\lambda}{4!} 4 (\phi^* \phi)^2, \tag{3.33}$$

where $\phi = \dfrac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2) \in \mathbb{C}$ is a complex charged scalar field (e.g. the charged pions). The gauge field A_μ occurs in the covariant derivative,

$$D_\mu = \partial_\mu + \mathrm{i} e A_\mu, \tag{3.34}$$

and the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.35}$$

The theory is symmetric under *local* U(1) transformations (gauge transformations)

$$\begin{aligned} \phi(x) &\to \mathrm{e}^{-\mathrm{i}e\Lambda(x)} \phi(x), \quad \mathrm{e}^{-\mathrm{i}e\Lambda(x)} \in \mathrm{U}(1) \\ A_\mu(x) &\to A_\mu + \partial_\mu \Lambda(x), \end{aligned} \tag{3.36}$$

where $\Lambda(x)$ is an arbitrary smooth function of spacetime.

With $\mu^2 > 0$, the potential part of (3.33), $V = -\mu^2 \phi^* \phi + \dfrac{\lambda}{4!} 4 (\phi^* \phi)^2$

has a 'Mexican hat' shape such that the minima of V satisfy

$$\phi_0^*\phi_0 = \frac{1}{2}v^2, \quad v = \sqrt{\frac{6\mu^2}{\lambda}} \tag{3.37}$$

as before (the factor 1/2 takes care of the different normalization of the scalar fields $\in \mathbb{C}$).

The fact that the symmetry is a local symmetry is an essential difference to the purely scalar cases, say the O(2) model, considered before: e.g. choosing ϕ_0 to point into the ϕ_2 direction everywhere is not a meaningful statement, since the local transformation (3.36) can change ϕ_0 independently from one point to another.

The gauge symmetry (3.36) indeed suggests to parametrize $\phi(x)$ differently than before.

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}\frac{\pi(x)}{v}}(v + \sigma(x)) \\ &= \frac{1}{\sqrt{2}}(v + \sigma(x) + \mathrm{i}\pi(x)) + \mathcal{O}(\pi^2).\end{aligned} \tag{3.38}$$

The second line is reminiscent to the linear parametrization used before, however, the complete parametrization in the first line is nonlinear.

For a given field configuration $\phi(x), A_\mu(x)$, we are free to perform a gauge transformation (in the $\mathbb{Z}_2$ model, this corresponds to choosing the 'right' minimum without loss of generality; or in the O(N) model, we chose $\phi_0 = (0, 0, \dots, v)^T$). Here we choose a special gauge transformation with

$$\Lambda(x) = \frac{\pi(x)}{ev}. \tag{3.39}$$

Then:

$$\begin{aligned}\phi(x) \to \phi'(x) &= \mathrm{e}^{-\mathrm{i}e\Lambda(x)}\phi(x) \overset{(3.38)}{\underset{(3.39)}{=}} \frac{1}{\sqrt{2}}(v+\sigma(x)) \\ A_\mu(x) \to A'_\mu(x) &= A_\mu(x) + \frac{1}{ev}\partial_\mu \pi(x).\end{aligned} \tag{3.40}$$

In terms of the new fields $\sigma(x), \pi(x), A'_\mu(x)$, the Lagrangian now reads

$$\begin{aligned}\mathcal{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) + \frac{1}{2}e^2v^2A'_\mu A^{\mu\prime} \\ + \frac{1}{2}e^2(A'_\mu)^2\sigma(2v+\sigma) - \frac{1}{2}(2\mu^2)\sigma^2 + \mathcal{O}(\sigma^3,\sigma^4).\end{aligned} \tag{3.41}$$

We observe:

- σ occurs as a massive scalar as in the purely scalar models
- Additionally, the photon A'_μ has acquired a mass term as in Proca theory
- Most surprisingly, $\pi(x)$ has vanished completely!

This last observation is, in fact, compatible with the counting of propagating degrees of freedom: in the initial formulation, say, with a standard scalar mass parameter $V = +m^2\phi^*\phi\ldots$, we had two real scalar fields (ϕ_1, ϕ_2) and two photon polarization modes (two transverse modes): $2+2=4$.

Now, we find one real scalar field (σ) and three polarization modes of a 'massive' photon (two transverse & one longitudinal). The would-be Nambu-Goldstone boson π has been 'eaten up' by the photon. This highlights the essentials of the Higgs (Anderson, Brout, Englert, Kibble, Hagen, Guralnik) mechanism.

We finally emphasize that the above analysis involved a special choice of gauge which we fixed by hand. The observations made above become particularly transparent in this gauge choice. By choosing a gauge, the gauge symmetry is in some sense explicitly broken by hand. By a somewhat unfortunate nomenclature, the Higgs mechanism is sometimes referred to as the ‘spontaneous breaking of gauge symmetry’. In a strict sense, this is nonsense, as gauge symmetry cannot be broken according to Elitzur’s theorem.

The point here is that particular gauges are convenient to identify the excitations. The gauge symmetry is still intact and we could try to look for the same physics in a different gauge. These circumstances are therefore better referred to by the name ‘hidden symmetry’.

4 Particles and Fields as Representations of the Lorentz group

Even in absence of any internal symmetries, the symmetries of spacetime are an essential property. In relativistic field theories, these are given by the Poincaré group consisting of spacetime translations and Lorentz transformations. Some consequences of both have already been discussed above. In the following, we detail how Lorentz invariance is connected to a classification of fields. Analogous considerations can also be performed for nonrelativistic field theories on the basis of Galilei invariance.

4.1 Lorentz transformations

Let us take a closer look at Lorentz transformations, recalling first some essential properties already listed in chapter 1: a Lorentz transformation is a linear operation on spacetime vectors v^μ,

$$v^\mu \to v^{\mu\prime} = \Lambda^\mu{}_\nu v^\nu, \tag{4.1}$$

that preserves the scalar product in Minkowski space

$$v^2 = g_{\mu\nu} v^\mu v^\nu \equiv v^\mu v_\mu, \quad g = \mathrm{diag}(+,-,-,-). \tag{4.2}$$

The transformation matrix $\Lambda^\mu{}_\nu$ hence satisfies

$$g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}. \tag{4.3}$$

The transformation property of vectors generalizes to transformations of arbitrary contravariant tensors

$$T^{\prime\mu_1\ldots\mu_n} = \Lambda^{\mu_1}{}_{\nu_1}\ldots\Lambda^{\mu_n}{}_{\nu_n} T^{\nu_1\ldots\nu_n} \tag{4.4}$$

of rank n.

There are only two constant invariant tensors. One is given by the metric by virtue of (4.3). The other one is the totally anti-symmetric tensor

$$\epsilon^{\mu\nu\rho\sigma}, \quad \epsilon^{0123} := 1 \tag{4.5}$$

with the usual rules for the Levi-Civita symbol. According to (4.4), it transforms as

$$\begin{aligned} \epsilon'^{\mu\nu\rho\sigma} &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \epsilon^{\alpha\beta\gamma\delta} \\ &= \epsilon^{\mu\nu\rho\sigma} \det \Lambda, \end{aligned} \tag{4.6}$$

where the second step makes use of the construction of the determinant using the ϵ symbol.

From (4.3), we read off

$$(\det \Lambda)^2 = 1 \quad \Rightarrow \quad \det \Lambda = \pm 1. \tag{4.7}$$

So strictly speaking, ϵ is only invariant or those Lorentz transformations that have $\det \Lambda = +1$, but changes sign under those with $\det \Lambda = -1$.

From 3d Euclidean space, we are already familiar with transformations that change the sign of ϵ: these are given by those orthogonal transformations that convert a right-handed basis into a left-handed one. Analogously, this applies to Minkowski space.

From (4.3), we can derive another fact:

$$\begin{aligned} \scriptstyle(\rho=0=\sigma) \qquad 1 &= (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 \\ &\Rightarrow (\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_0)^2 \\ &\boxed{\Rightarrow \Lambda^0{}_0 \geq 1 \quad \text{or} \quad \Lambda^0{}_0 \leq -1} \end{aligned} \tag{4.8}$$

Transformations with $\Lambda^0{}_0 \geq 1$ preserve the direction of the time axis, i.e. connect inertial frames where the dynamics evolves from smaller to larger values of the time coordinate. By contrast, transformations with ${\Lambda^0}_0 \leq -1$ flip the direction of the time axis.

The set of all Lorentz transformations forms the group O(3, 1); more precisely: the Λ's discussed here form a matrix representation of this group. This is analogous to the orthogonal transformations O(4) in 4-dimensional Euclidean space, additionally accounting for metric signatures.

Equations (4.7) & (4.8) proof that this set can be decomposed into disconnected components, as there is neither a path (1-parameter family of Λ's) that could possibly continuously interpolate between the $\det \Lambda = +1$ and $\det \Lambda = -1$ transformations nor a path interpolating between the Λ's with ${\Lambda^0}_0 \geq 1$ and ${\Lambda^0}_0 \leq -1$. This makes four disconnected components, out of which those with

$$\det \Lambda = +1, \quad {\Lambda^0}_0 \geq 1 \tag{4.9}$$

are called *orthochronous proper* Lorentz transformations. This is the component that contains the unit element of the group ${\Lambda^\mu}_\nu = \delta^\mu_\nu$.

The other components are related to the orthochronous proper component by a parity transformation (right ↔ left handed basis) and/or a time inversion ($t \to -t$).

Obviously, the infinitesimal Lorentz transformations belong to the orthochronous proper component

$${\Lambda^\mu}_\nu = \delta^\mu_\nu + {\epsilon^\mu}_\nu, \quad {\epsilon^\mu}_\nu \ll 1. \tag{4.10}$$

Expanding (4.3) to first order yields

$$g_{\rho\sigma} + g_{\mu\sigma}{\epsilon^\mu}_\rho + g_{\rho\gamma}{\epsilon^\nu}_\sigma + \mathcal{O}(\epsilon^2) = g_{\rho\sigma} \tag{4.11}$$

$$\Rightarrow \quad \epsilon_{\nu\mu} + \epsilon_{\mu\nu} = 0 \tag{4.12}$$

Thus, $\epsilon_{\mu\nu}$ is an antisymmetric matrix with 6 independent parameters, 3 of which correspond to Lorentz boosts (being parametrized by a spatial velocity vector $\vec{v}$) and further 3 describe spatial rotations (e.g. Euler angles).

It is useful to write an infinitesimal Lorentz transformation as

$$v'^{\mu} = v^{\mu} + \epsilon^{\mu}{}_{\nu} v^{\nu} =: \left(\mathbb{1} - \frac{\mathrm{i}}{2}\epsilon^{\rho\sigma} M_{\rho\sigma}\right)^{\mu}{}_{\nu} V^{\nu} \tag{4.13}$$

where

$$(M_{\rho\sigma})^{\mu}{}_{\nu} = \mathrm{i}(\delta^{\mu}_{\rho} g_{\sigma\nu} - \delta^{\mu}_{\sigma} g_{\rho\nu}) \tag{4.14}$$

This way of writing the transformation separates the parameters $\epsilon^{\rho\sigma}$ from the generators $M_{\rho\sigma}$ of Lorentz symmetry that encode the algebraic structure. For any given set of fixed indices ρ, σ, $M_{\rho\sigma}$ is a 4×4 matrix (with indices $\mu\,\nu$ in (4.14)). These matrices satisfy

$$\boxed{[M_{\mu\nu}, M_{\rho\sigma}] = -\mathrm{i}(g_{\mu\rho} M_{\nu\sigma} - g_{\nu\rho} M_{\mu\sigma} - g_{\mu\sigma} M_{\nu\rho} + g_{\nu\sigma} M_{\mu\rho})}\,. \tag{4.15}$$

Equation (4.15) defines the Lie algebra of the generators of the Lorentz group SO(3, 1) (the 'S' means $\det \Lambda = 1$). From an abstract perspective, Eq. (4.14) defines a particular representation of this algebra in terms of 4×4 matrices. Since $M_{\sigma\rho} = -M_{\rho\sigma}$, there are in total 6 generators of this algebra.

Independently of the representation, we obtain finite Lorentz transformations (within the orthochronous proper component) by the exponential map

$$\Lambda = \mathrm{e}^{-\frac{\mathrm{i}}{2}\epsilon^{\rho\sigma} M_{\rho\sigma}} \approx 1 - \frac{\mathrm{i}}{2}\epsilon^{\rho\sigma} M_{\rho\sigma} + \mathcal{O}(\epsilon^2). \tag{4.16}$$

4.2 Fields as representations of the Lorentz group

Fields being the fundamental degrees of freedom of a field theory can be classified according to their behaviour under Lorentz transformations. So far, we have mainly considered scalar fields which transform trivially,

$$\phi'(x') = \phi(x), \quad x'^\mu = \Lambda^\mu{}_\nu x^\nu. \tag{4.17}$$

We have also already encountered the gauge field $A_\mu(x)$ which transforms as a vector,

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x). \tag{4.18}$$

For a general N-tupel φ_i, $i = 1 \ldots N$, the transformation rule reads

$$\varphi'_i(x') = D(\Lambda)_i{}^j \varphi_j(x), \tag{4.19}$$

where $D(\Lambda)$ should be an $N \times N$ matrix representation of the Lorentz group. Which representations do exist?

Infinitesimally, we have

$$D(\Lambda)_i{}^j = \delta_i^j - \frac{\mathrm{i}}{2}\epsilon^{\mu\nu}(S_{\mu\nu})_i{}^j, \tag{4.20}$$

where $S_{\mu\nu}$ is an $N \times N$ matrix for each fixed set of μ, ν. In order to correspond to a Lorentz transformation, $S_{\mu\nu}$ has to satisfy the Lorentz algebra (4.15), $S_{\mu\nu} \equiv D(M_{\mu\nu})$. Our goal is to classify all possible finite dimensional choices of $S_{\mu\nu}$. For this, we first go back to the representation $M_{\mu\nu}$ and introduce

$$\begin{aligned} J_i &:= \frac{1}{2}\epsilon_{ijk}M^{jk}, \\ K_i &:= M_{i0} = -M_{0i}, \quad i, j, k = 1, 2, 3. \end{aligned} \tag{4.21}$$

Using (4.15), it is straightforward to verify

$$\begin{aligned} [J_i, J_j] &= \mathrm{i}\epsilon_{ijk}J_k, \\ [J_i, K_j] &= \mathrm{i}\epsilon_{ijk}K_k, \\ [K_i, K_j] &= -\mathrm{i}\epsilon_{ijk}J_k. \end{aligned} \tag{4.22}$$

$\vec{J}$ satisfies the angular momentum algebra and hence is evidently related to the generator of spatial rotations. $\vec{K}$ in turn corresponds to the generator of Lorentz boosts.

It is instructive to change the basis of generators once more and introduce

$$\vec{A} = \frac{1}{2}(\vec{J} + \mathrm{i}\vec{K}), \quad \vec{B} = \frac{1}{2}(\vec{J} - \mathrm{i}\vec{K}). \tag{4.23}$$

These satisfy

$$\begin{aligned} [A_i, A_j] &= \mathrm{i}\epsilon_{ijk}A_k, \\ [B_i, B_j] &= \mathrm{i}\epsilon_{ijk}B_k, \\ [A_i, B_j] &= 0. \end{aligned} \tag{4.24}$$

Therefore, the Lorentz algebra is equivalent to two sets of angular momentum algebras which we call $\vec{A}$ and $\vec{B}$ spins. These spin algebras obviously commute. We conclude that we can classify all possible representations of the Lorentz algebra simply in terms of all possible representations of these angular momentum algebras. The latter are countable in terms of the eigenvalue of the squared spins $\vec{A}^2, \vec{B}^2$. For a given total spin, the eigenvectors can further be labelled by the eigenvalues of a spin component, say A_3 and B_3

$$\begin{aligned} \vec{A}^2|Aa\rangle &= A(A+1)|Aa\rangle, & A_3|Aa\rangle &= a|Aa\rangle, \\ \vec{B}^2|Bb\rangle &= B(B+1)|Bb\rangle, & B_3|Bb\rangle &= b|Bb\rangle, \\ a &= -A, \ldots, A, & b &= -B, \ldots, B. \end{aligned} \tag{4.25}$$

For a given set of total spin quantum numbers A and B, the representation space is spanned by $|Aa, Bb\rangle = |Aa\rangle \otimes |Bb\rangle$ and is

$$N = (2A+1)(2B+1) \tag{4.26}$$

dimensional. Hence, the index i of the N-tuple field φ_i simply labels all possible values of a and b

$$i = (a, b). \tag{4.27}$$

In this fashion, we have found all possible irreducible representations of the Lorentz algebra. Of course, by means of tensor products, we can combine different representations to form further reducible representations.

4.3 Spinors

Apart from the trivial scalar representation, the simplest representation is a spin $\frac{1}{2}$ representation, e.g.

$$(A,B)=\left(0,\frac{1}{2}\right) \tag{4.28}$$

$$\Rightarrow \quad D(\vec{A})=0, \quad D(\vec{B})=\frac{\vec{\sigma}}{2} \quad (\sigma_i : \text{Pauli matrices}).$$

The corresponding fields have two components,

$$\varphi_i \to \xi_\alpha, \quad \alpha=1,2. \tag{4.29}$$

The representation of $\vec{J}$ and $\vec{K}$ are

$$D(\vec{J})=\frac{\vec{\sigma}}{2}, \quad D(\vec{K})=\mathrm{i}\frac{\vec{\sigma}}{2}. \tag{4.30}$$

We can summarize the parameters $\epsilon^{\mu\nu}$ of the Lorentz transformation into two 3-vectors:

$$(\epsilon_{23},\epsilon_{31},\epsilon_{12}) =: -\vec{\theta}, \quad (\epsilon_{10},\epsilon_{20},\epsilon_{30}) =: -\vec{\omega}, \tag{4.31}$$

such that the representation of the Lorentz transformation is given by

$$D(\Lambda)=\mathrm{e}^{\mathrm{i}\vec{\theta}\cdot D(\vec{J})+\mathrm{i}\vec{\omega}\cdot D(\vec{K})}, \tag{4.32}$$

or explicitly

$$a_\alpha{}^\beta := D(\Lambda)_\alpha{}^\beta = \left[\mathrm{e}^{\frac{\mathrm{i}}{2}\vec{\theta}\cdot\vec{\sigma}-\frac{1}{2}\vec{\omega}\cdot\vec{\sigma}}\right]_\alpha{}^\beta \tag{4.33}$$

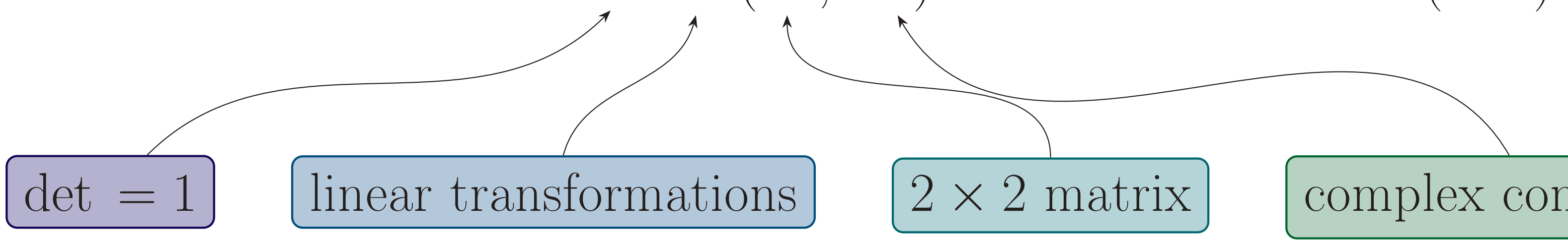
$$\Rightarrow \quad \xi'_\alpha(x')=a_\alpha{}^\beta \xi_\beta(x). \tag{4.34}$$

As can be verified explicitly, the matrix a is a 2×2 matrix with complex entries and satisfies

$$\det a = 1. \tag{4.35}$$

Thus it has 6 real parameters which are exhausted by $\vec{\theta}$ and $\vec{\omega}$. The set of matrices of this type form the matrix group

$$\mathrm{S\ L\ (\ 2\ ,\ \mathbb{C}\ )}. \tag{4.36}$$



We call the field $\xi_\alpha(x)$ also an SL$(2,\mathbb{C})$ spinor. The above equations (4.32) and (4.33) describe a homomorphism between the Lorentz group SO$(3,1)$ and SL$(2,\mathbb{C})$, where SL$(2,\mathbb{C})$ covers each element of SO$(3,1)$ twice (as is already familiar from SU$(2) \leftrightarrow$ SO(3) in quantum mechanics). Let $\vec{\omega} = 0$. If we rotate θ_1 by 2π, we have $\Lambda_\mu{}^\nu = \delta^\nu_\mu$, whereas $a \to -a$ in SL$(2,\mathbb{C})$. The identity is reached again after a 4π rotation.

To close this section, we can also study the complex conjugate spinor $(\xi_\alpha)^* \equiv \xi_{\dot{\alpha}}$ ('dotted' spinor), which transforms as

$$\eta'{}_{\dot{\alpha}}(x') = a^*{}_{\dot{\alpha}}{}^{\dot{\beta}}\eta_{\dot{\beta}}(x) \qquad \left(a^*{}_{\dot{\alpha}}{}^{\dot{\beta}} \equiv \left(a_\alpha{}^\beta\right)^*\right) \tag{4.37}$$

From the complex conjugate form of a in (4.33) we can deduce backwards that this corresponds to a representation

$$D(\vec{A}) = -\frac{\vec{\sigma}^*}{2}, \quad D(\vec{B}) = 0 \tag{4.38}$$

which is an $(A,B) = \left(\frac{1}{2}, 0\right)$ representation.

4.4 Spinors and 4-vectors

Since the dimension of a representation of the Lorentz group is given by
$N = (2A+1)(2B+1)$, 4-vectors (being related to integer spins) have to be related to the mixed representation:

$$2 \times 2^* : \quad \left(0, \frac{1}{2}\right) \times \left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}, \frac{1}{2}\right). \tag{4.39}$$

In practise, this implies that there must be a relation between an object with indices $(\alpha, \dot{\beta})$ and one with index μ. For this, we define the auxiliary objects

$$(\sigma_\mu)_{\alpha\dot{\beta}} = (\mathbb{1}, \vec{\sigma}), \quad (\bar{\sigma}_\mu)^{\dot{\alpha}\beta} = (\mathbb{1}, -\vec{\sigma}). \tag{4.40}$$

It is suggestive to use the 2d ϵ tensor as a metric in spinor space, e.g.

$$(\bar{\sigma}_\mu)_{\dot{\alpha}\beta} := \epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon_{\beta\delta}(\bar{\sigma}_\mu)^{\dot{\gamma}\delta}. \tag{4.41}$$

Then it can straightforward be checked that σ_μ and $\bar{\sigma}_\mu$ are related by

$$(\bar{\sigma}_\mu)_{\dot{\alpha}\beta} = \left[(\sigma_\mu)_{\alpha\dot{\beta}}\right]^*. \tag{4.42}$$

With these definitions, it also follows that

$$\frac{1}{2}\,\mathrm{Tr}\,(\bar{\sigma}^\mu\sigma_\nu) = \delta^\mu_\nu, \quad (\sigma^\mu)_{\alpha\dot{\beta}}(\bar{\sigma}_\mu)^{\dot{\gamma}\delta} = 2\delta^\delta_\alpha\delta^{\dot{\gamma}}_{\dot{\beta}}, \tag{4.43}$$

and

$$\sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu = \bar{\sigma}_\mu\sigma_\nu + \bar{\sigma}_\nu\sigma_\mu = 2g_{\mu\nu}. \tag{4.44}$$

Using the explicit representation (4.33) for a Lorentz transformation ${a_\alpha}^\beta$, we obtain the important formula

$$\sigma_\mu{\Lambda^\mu}_\nu = a\sigma_\nu a^\dagger. \tag{4.45}$$

This equation connects the Lorentz transformation of a 4-vector, ${\Lambda^\mu}_\nu$, with the transformation matrices a and $a^\dagger$ of a spinor and its complex conjugate. This suggests to define the spinor representation of a 4-vector

$$\underline{x} := x^\mu\sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix}. \tag{4.46}$$

Eq. (4.45) now gives us the transformation properties

$$\begin{aligned} \underline{x}' = \underbrace{x'^\mu}_{={\Lambda^\mu}_\nu x^\nu} \sigma_\mu = \sigma_\mu{\Lambda^\mu}_\nu x^\nu \overset{(4.45)}{=} a\sigma_\nu a^\dagger x^\nu \\ = a\underline{x}a^\dagger. \end{aligned} \tag{4.47}$$

In turn, we can construct a 4-vector out of two independent spinors $\xi_\alpha, \eta_{\dot\alpha}$:

$$V_\mu := \xi^\alpha (\sigma_\mu)_{\alpha\dot\beta} \eta^{\dot\beta}. \tag{4.48}$$

By an argument inverse to (4.47), it is possible to show that V_μ transforms as a 4-vector under Lorentz transformations if ξ_α and $\eta_{\dot\alpha}$ transform as spinors.

The general relation between a vector and a mixed spinor object is hence given by

$$V_{\alpha\dot\beta} = V^\mu (\sigma_\mu)_{\alpha\dot\beta}, \quad V^\mu = \frac{1}{2} (\bar\sigma^\mu)^{\dot\beta\alpha} V_{\alpha\dot\beta}. \tag{4.49}$$

So far, we have written the Lorentz transformations a and a^* of the $\mathrm{SL}(2,\mathbb{C})$ spinors explicitly in terms of Pauli matrices. However, there is also a representation of the generators in terms of objects that satisfy the Lorentz algebra directly. These are given by

$$\begin{aligned} (\sigma^{\mu\nu})_\alpha{}^\beta &:= \frac{\mathrm{i}}{2} (\sigma^\mu \bar\sigma^\nu - \sigma^\nu \bar\sigma^\mu)_\alpha{}^\beta, \\ (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} &:= \frac{\mathrm{i}}{2} (\bar\sigma^\mu \sigma^\nu - \bar\sigma^\nu \sigma^\mu)^{\dot\alpha}{}_{\dot\beta} \end{aligned} . \tag{4.50}$$

Each of these two objects satisfy the Lorentz algebra (4.15) with $M^{\mu\nu} \to \sigma^{\mu\nu}$ or $\bar\sigma^{\mu\nu}$. So we have $D_{\frac{1}{2}}(M^{\mu\nu}) \hat{=} \sigma^{\mu\nu}$ or $\bar\sigma^{\mu\nu}$.

Hence, the Lorentz transformation can be written as

$$\xi'_\alpha(x') = a_\alpha{}^\beta \xi_\beta(x) = \left[\mathrm{e}^{-\frac{\mathrm{i}}{4}\epsilon^{\mu\nu}\sigma_{\mu\nu}}\right]_\alpha{}^\beta \xi_\beta(x), \tag{4.51}$$

or for $\eta^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta} \eta_{\dot\beta}$ as

$$\eta'^{\dot\alpha}(x') = (\epsilon a^* \epsilon^T)^{\dot\alpha}{}_{\dot\beta} \eta^{\dot\beta}(x) = \left[\mathrm{e}^{-\frac{\mathrm{i}}{4}\epsilon^{\mu\nu}\bar\sigma_{\mu\nu}}\right]^{\dot\alpha}{}_{\dot\beta} \eta^{\dot\beta}(x). \tag{4.52}$$

4.5 Some aspects of spinor calculus

For a given spinor ξ_α, we wish to identify the dual spinor ξ^α such that the inner product of the two forms a scalar product which is invariant under Lorentz transformations. As already suggested in the preceding section, this metric is given by the anti-symmetric tensor in two dimensions,

$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \epsilon_{\alpha\beta} = \epsilon_{\dot\alpha\dot\beta} = \mathrm{i}\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.53}$$

such that

$$\xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \quad \eta^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\eta_{\dot\beta}. \tag{4.54}$$

The resulting Lorentz invariance of the inner product $\xi^\alpha\xi_\alpha = \epsilon^{\alpha\beta}\xi_\beta\xi_\alpha$ will be discussed in the exercises. Since ϵ is anti-symmetric, some *care* is necessary, as some manipulations seem non-obvious if compared to vector calculus in $\mathbb{R}^3$ or $\mathbb{M}$. For instance,

$$\begin{aligned} \xi_\alpha &= -\epsilon_{\alpha\beta}\xi^\beta, & \eta_{\dot\alpha} &= -\epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta} \\ &= \xi^\beta\epsilon_{\beta\alpha} & &= \eta^{\dot\beta}\epsilon_{\dot\beta\dot\alpha} \end{aligned}, \tag{4.55}$$

because: $-\epsilon_{\alpha\beta}\xi^\beta = -\underbrace{\epsilon_{\alpha\beta}\epsilon^{\beta\gamma}}_{=-\delta_\alpha^\gamma}\xi_\gamma = \xi_\alpha.$

In (4.55), we observe that no explicit sign appears if the indices are arranged such that they are contracted from upper-left to lower-right, or 'NW - SO' (North-West to South-East).

I.e., if we wish to drop the indices in our notation, we have to agree on this convention:

$$\xi\zeta := \xi^\alpha\zeta_\alpha = -\xi_\alpha\zeta^\alpha. \tag{4.56}$$

Another useful notation is inspired by matrix multiplication rules (e.g. also the scalar product of two Euclidean vectors $\vec{x}$ and $\vec{y}$, $\vec{x}\cdot\vec{y}$, can be viewed as a matrix multiplication where the left vector is considered as transposed vector $\vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}$), where we consider the left-hand spinor (not the dual spinor!) as a transposed spinor:

$$\begin{aligned}\xi\zeta = \xi^\alpha \zeta_\alpha = \epsilon^{\alpha\beta} \xi_\beta \zeta_\alpha = \xi_\beta \epsilon^{\alpha\beta} \zeta_\alpha \\ = -\xi_\beta \epsilon^{\beta\alpha} \zeta_\alpha = -\xi^T \epsilon \zeta = \xi^T \epsilon^T \zeta,\end{aligned} \tag{4.57}$$

where

$$\epsilon^T = -\epsilon. \tag{4.58}$$

In this latter notation, we can write Lorentz transformations in the following manner:

$$\xi'^\alpha = a_\alpha{}^\beta \xi_\beta \quad \Rightarrow \quad \xi' = a\xi, \tag{4.59a}$$

$$\text{or} \qquad \xi'^T = \xi^T a^T. \tag{4.59b}$$

5 Simple Spinor field theories

Having identified the spinor fields $\xi_\alpha(x), \eta^{\dot{\alpha}}(x)$ as the simplest nontrivial representations of the Lorentz group, let us try to construct field theories for these spinors by means of Lorentz-invariant Lagrangians.

5.1 Kinetic part

Using (4.49), we can immediately write a derivative in spinor space:

$$\partial_\mu \to \partial_{\alpha\dot{\beta}} = (\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu. \tag{5.1}$$

Whereas scalar fields involved always two derivatives to form a Lorentz scalar $(\partial_\mu\phi)(\partial^\mu\phi)$, it is already possible to write down a single derivative term in the spinor case which is nevertheless bilinear in the fields and thus no total derivative:

$$\eta^{*\alpha}(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu\eta^{\dot{\beta}} = \eta^\dagger\sigma^\mu\partial_\mu\eta, \tag{5.2}$$

$$\text{where} \qquad (\eta^{\dot{\alpha}})^* = (\eta^*)^\alpha.$$

Since the spinor fields are complex, Eq. (5.2) is not guaranteed to be real. Hence, we may try

$$\begin{aligned}\mathcal{L} &\stackrel{?}{=} \eta^\dagger\sigma^\mu\partial_\mu\eta + \text{h.c.} = \eta^\dagger\sigma^\mu\partial_\mu\eta + (\partial_\mu\eta^\dagger)\sigma^\mu\eta \\ &= \partial_\mu(\eta^\dagger\partial^\mu\eta). \end{aligned} \tag{5.3}$$

However, this combination projecting on the real part is a total derivative and hence does not give rise to nontrivial equations of motion.

Therefore, the only combination left is the imaginary part

$$\mathcal{L}_{\mathrm{L}}^{\mathrm{kin}} = \frac{\mathrm{i}}{2}(\eta^\dagger \sigma^\mu \partial_\mu \eta - (\partial_\mu \eta^\dagger)\sigma^\mu \eta)$$
$$\boxed{=: \frac{\mathrm{i}}{2}\eta^\dagger \sigma^\mu \overset{\leftrightarrow}{\partial}_\mu \eta \ .} \tag{5.4}$$

This is the simplest possible kinetic term. Here, we have introduced the derivative operator $\overset{\leftrightarrow}{\partial}$ acting to the right as well as to the left including a minus sign. Similarly, we obtain for ξ:

$$\mathcal{L}_{\mathrm{R}}^{\mathrm{kin}} = \frac{\mathrm{i}}{2}\xi^*{}_{\dot\beta}(\bar\sigma^\mu)^{\dot\beta\alpha} \overset{\leftrightarrow}{\partial}_\mu \xi_\alpha$$
$$\boxed{= \frac{\mathrm{i}}{2}\xi^\dagger \bar\sigma^\mu \overset{\leftrightarrow}{\partial}_\mu \xi \ .} \tag{5.5}$$

Both Lagrangians exhibit a continuous symmetry of phase transformations,

$$\begin{aligned} \xi'(x) &= \mathrm{e}^{\mathrm{i}\theta}\xi(x) \quad , \quad \xi^{*\prime}(x) = \mathrm{e}^{-\mathrm{i}\theta}\xi^*(x), \\ \eta'(x) &= \mathrm{e}^{\mathrm{i}\theta'}\eta(x) \quad , \quad \eta^{*\prime}(x) = \mathrm{e}^{-\mathrm{i}\theta'}\eta^*(x), \end{aligned} \tag{5.6}$$

that leave the action invariant. These symmetries are also called *chiral* symmetries, each one forming a U(1) group: $\mathrm{U}(1)_{\mathrm{R}}$, $\mathrm{U}(1)_{\mathrm{L}}$.

5.2 Mass term

Analogously to bosonic field theories, we expect that a mass term in the Lagrangian has to be quadratic in the fields such that excitations propagate according to the relativistic dispersion relation of a point particle. As the kinetic term is linear in derivatives ($\sim$ 4-momenta), we expect the quadratic term in the Lagrangian to be linear in the mass.

The simplest quadratic Lorentz scalars are

$$\eta^{\dot\alpha}\epsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta} = \eta^T\epsilon\eta, \quad \xi_\alpha\epsilon^{\alpha\beta}\xi_\beta = \xi^T\epsilon\xi. \tag{5.7}$$

Explicitly, this yields, e.g.

$$\eta^{\dot{\alpha}}\epsilon_{\dot{\alpha}\dot{\beta}}\eta^{\dot{\beta}} = \eta^1\eta^2 - \eta^2\eta^1. \tag{5.8}$$

If the components η^1 and η^2 are ordinary commuting numbers, this expression is identically zero.

However, with a glimpse into quantum theory, we expect the the connection between spin and statistics eventually implies that the excitations of the spinor fields obey Fermi-Dirac statistics (spin-statistics theorem): in a quantum setting, we will associate η^1 and η^2 with operators that create a spinor excitation above the vacuum. Since these excitations have to anti-commute, we are actually forced to impose $\eta^1\eta^2 = -\eta^2\eta^1$.

For operators, this property seems straightforwardly implementable. Nevertheless, here we do not plan to quantize, but stay within classical field theory. Still, we wish to realize the correct statistical properties of the excitations.

Evidently, both requirements cannot be satisfied by pure numbers $\eta^1, \eta^2 \in \mathbb{C}$. Still, there exists a consistent set of numbers, defined in terms of conventional algebraic axioms, that even facilitates the definition of derivatives and integrals, with the special property that these numbers *anti-commute*. These are the *Grassmann numbers*. If we interpret $\eta^1, \eta^2, \xi^1, \xi^2$ to be Grassmann-valued, we have $\eta^1\eta^2 = -\eta^2\eta^1$, and thus (5.8) is nonzero. (Grassmann numbers can be treated abstractly; if we still wish to represent them in terms the body of the real numbers, we are lead to matrix representations, see exercises.)

Hence, a real mass term is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{L}}^{m} &= -\frac{1}{2}(m_{\mathrm{L}}\eta^{T}\epsilon\eta - m_{\mathrm{L}}^{*}\eta^{\dagger}\epsilon\eta^{*}),\\ \mathcal{L}_{\mathrm{R}}^{m} &= -\frac{1}{2}(m_{\mathrm{R}}\xi^{\dagger}\epsilon\xi^{*} - m_{\mathrm{R}}^{*}\xi^{T}\epsilon\xi),\end{aligned} \tag{5.9}$$

where the mass parameters m_{L} and m_{R} may be complex. Here we have used $(\theta\chi)^{*} = \chi^{*}\theta^{*}$ for Grassmann numbers (as is familiar from matrices). Also, $\epsilon^{\dagger} = \epsilon^{T} = -\epsilon$ has been used.

These mass terms are called *Majorana* masses. The Majorana mass breaks the chiral symmetry $\mathrm{U}(1)_{\mathrm{L}}$ or $\mathrm{U}(1)_{\mathrm{R}}$ completely. If a Majorana mass exists, the corresponding Noether charges are not conserved. In particle physics, no Majorana mass term has been verified (yet). Still, the mass of the neutrinos may be associated with a Majorana mass term; if so, the non-conservation of the Noether charge would translate into violations of lepton number conservation. A possible signature in terms of a neutrinoless double β decay is actively searched for.
In condensed-matter systems, Majorana fermions can arise as an effective degree of freedom. This is currently a very active field of research.

Whereas the above kinetic and mass terms can exist for each representation $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ separately, there is another possible mass term, which exists in the simultaneous presence of the two spinors:

$$\mathcal{L}_{\mathrm{D}}^{m} = -(m\xi^{\dagger}\eta + m^{*}\eta^{\dagger}\xi). \tag{5.10}$$

This is the *Dirac* mass term. It does not break the chiral symmetries completely: choosing $\theta = \theta'$ in (5.6), the spinors transform as

$$\begin{pmatrix}\eta' \\ \xi'\end{pmatrix} = \mathrm{e}^{\mathrm{i}\theta}\begin{pmatrix}\eta \\ \xi\end{pmatrix}, \quad \begin{pmatrix}\eta^{*\prime} \\ \xi^{*\prime}\end{pmatrix} = \mathrm{e}^{-\mathrm{i}\theta}\begin{pmatrix}\eta^{*} \\ \xi^{*}\end{pmatrix}. \tag{5.11}$$

These simultaneous $\mathrm{U}(1)_{\mathrm{L}}$ and $\mathrm{U}(1)_{\mathrm{R}}$ transformations form also a $\mathrm{U}(1)$ group which is called a 'vector' $\mathrm{U}(1)_{\mathrm{V}}$. The corresponding Noether charge is positive for η, ξ and negative for η^{*}, ξ^{*}. Hence, this symmetry is similar to the U(1) symmetry for a complex scalar. The Noether charge can be associated with 'particle number' or electric charge upon coupling to a Maxwell field.

5.3 The Dirac spinor

Whereas the kinetic terms as well as the Majorana mass term can be formulated for each $SL(2, \mathbb{C})$ spinor ξ or η (Weyl spinors) separately, the Dirac mass term requires the simultaneous presence of both Weyl spinors and provides for a bilinear coupling. Hence, it is useful to introduce the combined 4-spinor

$$\psi(x) = \begin{pmatrix} \eta^{\dot{\alpha}}(x) \\ \xi_\alpha(x) \end{pmatrix}, \tag{5.12}$$

which is a *Dirac spinor*, obviously belonging to the $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ representation of the Lorentz group. We obtain a compact notation for the Lagrangians by also summarizing the (generalized) Pauli matrices as

$$\gamma^\mu := \begin{pmatrix} 0 & (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \\ (\sigma^\mu)_{\alpha\dot{\beta}} & 0 \end{pmatrix}, \tag{5.13}$$

or more explicitly

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}. \tag{5.14}$$

These are the Dirac matrices. They occur here in the so-called chiral representation (several other representations are also used in the literature). Independently of the representation, the γ matrices satisfy (c.f. (4.44))

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \cdot \mathbb{1}. \tag{5.15}$$

Equation (5.15) can be viewed as the defining property of the Dirac matrices. Mathematically, the Dirac matrices generate a matrix representation of a *Clifford* algebra, i.e. an algebra of elements that close under the anti-commutator.

The generator of Lorentz transformations in the Dirac representation can also be constructed from those of the Weyl spinors, c.f. (4.50):

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} (\bar{\sigma}^{\mu\nu})_\alpha{}^\beta & 0 \\ 0 & (\sigma^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}. \tag{5.16}$$

now 4×4 $\quad$ 2×2

Hence, the Lorentz-transformed spinor reads

$$\psi'(x') = D_{(\frac{1}{2},0)\otimes(0,\frac{1}{2})}(\Lambda)\psi(x) = \left(\mathrm{e}^{-\frac{\mathrm{i}}{4}\epsilon^{\mu\nu}\sigma_{\mu\nu}}\right)\psi(x) =: A\psi(x). \tag{5.17}$$

The 4×4 matrix A is the direct analogue of the transformation matrix a. From (4.51) and (4.52), we can read off

$$(A\psi(x))^{\dot{\alpha}}_{\alpha} = \begin{pmatrix} (\epsilon\alpha^*\epsilon^T)^{\dot{\alpha}}{}_{\dot{\beta}} & 0 \\ 0 & a_\alpha{}^\beta \end{pmatrix} \begin{pmatrix} \eta^{\dot{\beta}}(x) \\ \xi_\beta(x) \end{pmatrix}. \tag{5.18}$$

Here, we have used the SL$(2,\mathbb{C})$ spinor indices $\dot{\alpha}, \alpha$ in order to make the SL$(2,\mathbb{C})$ content explicit. Of course, working directly with the Dirac spinor, it is more natural to summarize the components for $\dot{\alpha} = 1, 2$, $\alpha = 1, 2$ into one index

$$\psi^\gamma(x), \quad \gamma = 1, 2, 3, 4$$

of the 4-component spinor $\psi(x)$.

With the aid of another definition,

$$\bar{A} := \gamma_0 A^\dagger \gamma_0, \tag{5.19}$$

together with (4.45), it is straightforward to verify that

$$\bar{A}\gamma^\mu A = \Lambda^\mu{}_\nu \gamma^\nu. \tag{5.20}$$

This equation emphasises the relation between the Lorentz transformations of the Dirac spinor and that of the '4-vector' of Dirac matrices γ^μ, as the transformations of the spinor indices of the γ's (LHS) can be written as a Lorentz transformation of the vector index (RHS). The bar symbol in (5.19) is used to denote *Dirac conjugation*. In addition to complex conjugation, it involves a multiplication with γ_0 for each index. It is useful to think of γ_0 as a spin metric, i.e., it relates spinor space to its corresponding dual vector space. The elements of this dual space are Dirac conjugated spinors:

$$\bar{\psi} := \psi^\dagger \gamma_0. \tag{5.21}$$

In fact, this spinor occurs naturally if we consider the kinetic Lagrangian for the Dirac spinor

$$\begin{aligned} \mathcal{L}_{\mathrm{D}}^{\mathrm{kin}} &:= \mathcal{L}_{\mathrm{L}}^{\mathrm{kin}} + \mathcal{L}_{\mathrm{R}}^{\mathrm{kin}} = \frac{\mathrm{i}}{2}\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}\psi \\ &= \frac{\mathrm{i}}{2}(\bar{\psi}\gamma^\mu\partial_\mu\psi) - \frac{\mathrm{i}}{2}((\partial_\mu\bar{\psi})\gamma^\mu\psi) \\ &= \mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{\mathrm{i}}{2}\partial_\mu(\bar{\psi}\gamma^\mu\psi). \end{aligned} \tag{5.22}$$

Hence, the action can be written as

$$\boxed{S_{\mathrm{D}}^{\mathrm{kin}} = \int \mathrm{d}^4x \, \mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi \ .} \tag{5.23}$$

Similarly, the Dirac mass term (for a real mass $m = m^*$) can compactly written as

$$\begin{aligned} \mathcal{L}_{\mathrm{D}}^{m} &= -m\bar{\psi}\psi \\ \boxed{S_{\mathrm{D}}^{m} = -\int \mathrm{d}^4x \, m\bar{\psi}\psi \ .} \end{aligned} \tag{5.24}$$

Let us start analyzing the symmetries of Dirac's theory by briefly verifying the manifest Lorentz invariance:

$$S_{\mathrm{D}} = \int \mathrm{d}^4x \, (\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi). \tag{5.25}$$

Since $\psi' = A\psi$, it follows $\bar{\psi}' = \bar{\psi}\bar{A}$ (using $\gamma_0^2 = \mathbb{1}$). Let us explicitly study the kinetic part:

$$\boxed{\bar{\psi}'\gamma^\mu\partial'_\mu\psi'} = \bar{\psi}\bar{A}\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu A\psi$$

$$= \bar{\psi}\bar{A}\gamma^\mu A\Lambda_\mu{}^\nu\partial_\nu\psi \overset{(5.20)}{=} \bar{\psi}\Lambda^\mu{}_\rho\gamma^\rho \underbrace{\Lambda_\mu{}^\nu\partial_\nu\psi}_{g_{\mu\sigma}\Lambda^\sigma{}_\nu\partial^\nu} = \bar{\psi}\underbrace{g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\sigma{}_\nu}_{g_{\rho\nu}}\gamma^\rho\partial^\nu$$

$$\boxed{= \bar{\psi}\gamma^\nu\partial_\nu\psi.} \tag{5.26}$$

Of course, the invariance was already clear from the SL$(2,\mathbb{C})$ construction. But this example shows manifestly that invariant scalars arise if both vector as well as Dirac spinor indices are fully contracted. From the invariance of the Dirac mass term in the SL$(2,\mathbb{C})$ construction, it follows that $\bar{A}A = \mathbb{1}$ (which can be verified straightforwardly), such that

$$\bar{\psi}'\psi' = \bar{\psi}\psi \tag{5.27}$$

transforms as a *scalar*. Similarly, we can justify that $\bar{\psi}\gamma_\mu\psi$ transforms as a vector and $\bar{\psi}\sigma_{\mu\nu}\psi$ as a tensor under Lorentz transformations.

For an analysis of chiral symmetries in the Dirac notation, it is useful to introduce

$$\gamma_5 := \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.28}$$

where the first equality holds in general, and the second is particular for the chiral representation.

In the chiral representation, it is obvious that γ_5 can be used to define chiral projectors

$$P_{\mathrm{R}} = \frac{1+\gamma_5}{2}, \quad P_{\mathrm{L}} = \frac{1-\gamma_5}{2}, \tag{5.29}$$

satisfying

$$P_{\mathrm{R,L}}^2 = P_{\mathrm{R,L}}, \quad P_{\mathrm{R}}P_{\mathrm{L}} = 0 = P_{\mathrm{L}}P_{\mathrm{R}} \quad \text{and} \quad P_{\mathrm{R}} + P_{\mathrm{L}} = \mathbb{1}, \tag{5.30}$$

such that

$$\psi_{\mathrm{R}} := P_{\mathrm{R}}\psi = \begin{pmatrix} 0 \\ \xi_\alpha \end{pmatrix}, \quad \psi_{\mathrm{L}} := P_{\mathrm{L}}\psi = \begin{pmatrix} \eta^{\dot\alpha} \\ 0 \end{pmatrix}. \tag{5.31}$$

Since we have $\gamma_5 \to -\gamma_5$ under parity $x^i \to -x^i$, the combination $\bar\psi\gamma_5\psi$ is a pseudovector under Lorentz transformations. Note that only the open Lorentz indices are relevant for this classification. With respect to spinor space, all these expressions are scalars anyway.

Concerning the chiral transformations $\mathrm{U}(1)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}}$ of ξ and η, these can equivalently be represented by their linear combinations:

$$\begin{aligned} \theta = \theta' : \quad & \mathrm{U}(1)_{\mathrm{V}} : \quad \psi' = \mathrm{e}^{\mathrm{i}\theta}\psi \quad , \ \bar\psi' = \bar\psi \mathrm{e}^{-\mathrm{i}\theta}, \\ \theta = -\theta' : \ & \mathrm{U}(1)_{\mathrm{A}} : \quad \psi' = \mathrm{e}^{\mathrm{i}\gamma_5\theta}\psi \quad , \ \bar\psi' = \bar\psi \mathrm{e}^{\mathrm{i}\gamma_5\theta}. \end{aligned} \tag{5.32}$$

As discussed in the exercises, γ_5 anticommutes with all γ_μ's:

$$\{\gamma^\mu, \gamma_5\} = 0. \tag{5.33}$$

With this property, we can verify the invariance of the kinetic term under $\mathrm{U}(1)_{\mathrm{A}}$, the so-called *axial* transformations:

$$\begin{aligned} \mathrm{U}(1)_{\mathrm{A}} : \quad \bar\psi'\gamma^\mu\partial_\mu\psi &= \bar\psi \mathrm{e}^{\mathrm{i}\gamma_5\theta}\gamma^\mu\partial_\mu \mathrm{e}^{\mathrm{i}\gamma_5\theta}\psi \\ &= \bar\psi\gamma^\mu \mathrm{e}^{-\mathrm{i}\gamma_5\theta}\partial_\mu \mathrm{e}^{\mathrm{i}\gamma_5\theta}\psi = \bar\psi\gamma^\mu\partial_\mu\psi. \end{aligned} \tag{5.34}$$

The mass term $\sim -m\bar\psi\psi$, however, is not invariant under axial transformations.
By contrast, both kinetic and mass term are invariant under the *vector* transformations $\mathrm{U}(1)_{\mathrm{V}}$ in agreement with the observations in the $\mathrm{SL}(2,\mathbb{C})$ formalism.

5.4 Dirac Equation

Since the Dirac spinor is a complex object (complex-Grassmann-valued), we can use the same trick as for complex scalar fields and treat ψ and $\bar\psi$ as formally independent for the variational principle. Hence, we obtain the equation of motion by varying the action (5.25) e.g. with respect to $\bar{\psi}$:

$$0 = \frac{\delta}{\delta\bar{\psi}} S_{\mathrm{D}} = (\mathrm{i}\gamma^\mu \partial_\mu - m)\psi(x) = 0. \tag{5.35}$$

This is the Dirac equation. In the following, let us just recall a few basic properties of this relativistic spinor theory. In order to verify that m indeed has the meaning of mass in the sense of a relativistic point particle, let us multiply (5.35) with $(-\mathrm{i}\gamma^\nu\partial_\nu - m)$:

$$\begin{aligned} 0 &= (-\mathrm{i}\gamma^\nu\partial_\nu - m)(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi \\ &= (\underbrace{\gamma^\mu\gamma^\nu}_{=\underbrace{\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}}_{=g^{\mu\nu}} + \underbrace{\frac{1}{2}[\gamma^\mu,\gamma^\nu]}_{\text{antisym.}}} \overbrace{\partial_\mu\partial_\nu}^{\text{sym.}} + m^2)\psi \\ &= (\partial^2 + m^2)\psi(x). \end{aligned} \tag{5.36}$$

Here, we have used that a product of a symmetric and an antisymmetric tensor vanishes. Hence, the solutions of the Dirac equation also satisfy the Klein-Gordon equation and thus the solutions obey the relativistic energy-momentum relation with mass m.

This suggests as an ansatz

$$\psi(x) = u(p)\mathrm{e}^{-ipx}, \quad \text{where } p^2 = m^2. \tag{5.37}$$

In the chiral basis, the spinor $u(p)$ has to satisfy the algebraic equation

$$\left[\begin{pmatrix} 0 & \bar{\sigma}\cdot p \\ \sigma\cdot p & 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \right] u(p) = 0. \tag{5.38}$$

We observe that

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p_\mu p_\nu \underbrace{\frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)}_{=g^{\mu\nu}} = p^2 = m^2,$$

and hence the Dirac equation is solved by

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\xi \\ \sqrt{p\cdot\sigma}\xi \end{pmatrix}, \tag{5.39}$$

where ξ is an arbitrary SL$(2,\mathbb{C})$ spinor.

Let us verify this result explicitly:

$$\left[\begin{pmatrix} 0 & \bar\sigma\cdot p \\ \sigma\cdot p & 0\end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m\end{pmatrix}\right]\begin{pmatrix}\sqrt{p\cdot\bar\sigma}\xi \\ \sqrt{p\cdot\sigma}\xi\end{pmatrix} = \begin{pmatrix}\bar\sigma p\sqrt{p\cdot\sigma}\xi - m\sqrt{p\cdot\bar\sigma}\xi \\ \sigma p\sqrt{p\cdot\bar\sigma}\xi - m\sqrt{p\cdot\sigma}\xi\end{pmatrix}$$
$$= \begin{pmatrix}\sqrt{p\cdot\bar\sigma}(\sqrt{(\bar\sigma p)(\sigma p)} - m)\xi \\ \sqrt{p\cdot\sigma}(\sqrt{(\sigma p)(\bar\sigma p)} - m)\xi\end{pmatrix} \overset{\sqrt{(\bar\sigma p)(\sigma p)}=m}{=} 0.$$

Possible base spinors are $\xi^s = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ (times a Grassmann-valued number) such that (5.39) represents two solutions corresponding to spin-up ξ^1 or spin-down ξ^2 along the 3-direction, i.e., eigenvalues to $p_3\sigma_3 = \begin{pmatrix}p_3 & 0\\ 0 & -p_3\end{pmatrix}$. The solutions are normalized to

$$\begin{aligned} \bar u^r(p)u^s(p) &= 2\,m\,\delta^{rs}, \\ \text{or}\quad u^{\dagger r}(p)u^s(p) &= 2\,E_{\vec p}\,\delta^{rs}, \quad E_{\vec p} = \sqrt{\vec p^2 + m^2}. \end{aligned} \tag{5.40}$$

which is straightforwardly verifiable. In addition, there are also 'negative frequency' solutions

$$\psi(x) = v(p)\mathrm{e}^{ipx}, \quad p^2 = m^2, \quad p^0 > 0, \tag{5.41}$$

where $\quad v(p) = \begin{pmatrix}\sqrt{p\cdot\bar\sigma}\eta^s \\ -\sqrt{p\cdot\bar\sigma}\eta^s\end{pmatrix}\quad$ with spin base vectors η^s, $s = 1, 2$

The latter are normalized to

$$\bar v^r(p)v^s(p) = -2m\delta^{rs}, \quad v^{\dagger r}(p)v^s(p) = -2\,E_{\vec p}\,\delta^{rs}. \tag{5.42}$$

The u and v spinors are also mutually orthogonal,

$$\bar u^r(p)v^s(p) = \bar v^r(p)u^s(p) = 0. \tag{5.43}$$

In particle-physics processes, one is often interested in spin-summed results (e.g. if the spin of a single particle is not measured by the detector). For these, let us finally mention the following spin sums

$$\sum_s u^s(p)\bar{u}^s(p) = \gamma \cdot p + m$$
$$\sum_s v^s(p)\bar{v}^s(p) = \gamma \cdot p - m \tag{5.44}$$

The frequently occurring combination $\gamma_\mu p^\mu = \gamma \cdot p$ is often abbreviated by the Feynman slash

$$\gamma_\mu p^\mu = \not{p}.$$

5.5 Rarita-Schwinger spinors

So far, we have encountered the trivial spin-0 (scalar fields), and the nontrivial spin-$\frac{1}{2}$ (Weyl spinors, Dirac spinors) representations of the Lorentz group. In classical field theory, it is straightforward to construct higher-spin representations and their corresponding free theories; interacting theories which satisfy all consistency criteria can be more difficult.

As an example, let us study the spin-$\frac{3}{2}$ case. More concretely, we wish to compose a field ψ_μ such that it unifies properties of a Dirac spinor (with 4 spinor components with suppressed indices) as well as a vector field with index $\mu = 0, 1, 2, 3$. So, in total ψ_μ has 16 complex components. Since vectors belong to the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation, and Dirac spinors to the $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ representation, the general object ψ_μ is an element of the tensor product space

$$\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right) = \left[\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\frac{1}{2}, 0\right)\right] \oplus \left[\left(\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\right.\right. \tag{5.45}$$

Now, recall from the summation of the angular momenta that the tensor product of two spin-$\frac{1}{2}$ gives a spin-1 as well as a scalar spin-0 component:

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0,$$

or, using the notation that counts the dimension of the Hilbert spaces,

$$\underline{2} * \underline{2} = \underline{3} + \underline{1}. \tag{5.46}$$

Hence, Eq. (5.45) yields

$$\begin{aligned}\left(\frac{1}{2},\frac{1}{2}\right) \otimes \left(\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)\right) &= \left(1,\frac{1}{2}\right) \oplus \left(0,\frac{1}{2}\right) \oplus \left(\frac{1}{2},0\right) \oplus \left(\frac{1}{2},\right. \\ &= \underbrace{\left[\left(1,\frac{1}{2}\right) \oplus \left(\frac{1}{2},1\right)\right]}_{\text{Rarita-Schwinger}} \oplus \underbrace{\left[\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right.\right.}_{\text{Dirac spinor}}\end{aligned} \tag{5.47}$$

We observe that this tensor product contains Dirac spinors as well as the new $\left[\left(1,\frac{1}{2}\right) \oplus \left(\frac{1}{2},1\right)\right]$ terms, and thus is reducible into a Dirac part that we already know, and a new part which we will call a Rarita-Schwinger spinor (incidentally, Rarita and Schwinger's original 3/4-page paper deals with the full reducible object).

It is, in fact, easy to get rid of the Dirac part by noting that the object $(\gamma^\mu \psi_\mu)$ is a 'scalar' with respect to the Lorentz index structure but still features a Dirac index. Hence for a general ψ_μ, the object $\chi = \gamma^\mu \psi_\mu$ transforms as a Dirac spinor and thus corresponds to the $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ part of ψ_μ.
In turn, those fields ψ_μ that satisfy the *irreducibility condition*

$$\gamma^\mu \psi_\mu = 0 \tag{5.48}$$

do not contain Dirac spinor elements and hence transform as $\left(1,\frac{1}{2}\right) \oplus \left(\frac{1}{2},1\right)$ representation of the Lorentz group.
The irreducibility condition (5.48) has important consequences for the construction of a Lagrangian. For instance, one might naively try to

write down a symmetrically looking mass term:

$$\begin{aligned}\bar{\psi}^{\mu}\psi_{\mu} &= \bar{\psi}_{\mu}g^{\mu\nu}\psi_{\nu} = \frac{1}{2}\bar{\psi}_{\mu}\{\gamma^{\mu},\gamma^{\nu}\}\psi_{\nu} \\ &= \frac{1}{2}\bar{\psi}_{\mu}\gamma^{\mu}\underbrace{\gamma^{\nu}\psi_{\nu}}_{=0}+\frac{1}{2}\bar{\psi}_{\mu}\underbrace{\gamma^{\nu}\gamma^{\mu}}_{=\gamma^{\mu}\gamma^{\nu}-[\gamma^{\mu},\gamma^{\nu}]}\psi_{\nu} \\ &= \frac{1}{2}\bar{\psi}_{\mu}\gamma^{\mu}\gamma^{\nu}\psi_{\nu}-\frac{1}{2}\bar{\psi}_{\mu}[\gamma^{\mu},\gamma^{\nu}]\psi_{\nu} \\ &= \mathrm{i}\bar{\psi}_{\mu}\sigma^{\mu\nu}\psi_{\nu}, \qquad \sigma^{\mu\nu}=\frac{\mathrm{i}}{2}[\gamma^{\mu},\gamma^{\nu}]. \end{aligned} \tag{5.49}$$

We observe the mass term, in fact, has to be antisymmetric in the Lorentz indices of the Rarita-Schwinger field. A similar argument applies to the building block of the kinetic term:

$$\bar{\psi}_{\mu}\gamma_{\nu}\partial_{\kappa}\psi_{\lambda},$$

implying that all indices must be contracted in an antisymmetric fashion. This is possible with the aid of the ϵ tensor. In order to preserve parity invariance, we amend this building block with another γ_5 factor. The resulting Lagrangian for the Rarita-Schwinger field reads

$$\boxed{\mathcal{L} = -\frac{1}{2}\bar{\psi}_{\mu}(\epsilon^{\mu\nu\kappa\lambda}\gamma_5\gamma_{\nu}\partial_{\kappa} - \mathrm{i}m\sigma^{\mu\nu})\psi_{\lambda}.} \tag{5.50}$$

Correspondingly, the field equation yields

$$\boxed{(\epsilon^{\mu\nu\kappa\lambda}\gamma_5\gamma_{\nu}\partial_{\kappa} - \mathrm{i}m\sigma^{\mu\nu})\psi_{\lambda} = 0.} \tag{5.51}$$

Spin-$\frac{3}{2}$ fields are indeed known and used in physics for the description of spin-$\frac{3}{2}$ bound states in the theory of strong interactions. An example is given by the Δ resonances of the nucleon which are bound states of 3 quarks with all spins $\frac{1}{2}$ aligned to yield a spin-$\frac{3}{2}$ state (Δ^-: $|\mathrm{ddd}\rangle$, Δ^0: $|\mathrm{udd}\rangle$, Δ^+: $|\mathrm{uud}\rangle$, Δ^{++}: $|\mathrm{uuu}\rangle$), each having a lifetime $\sim 5\cdot 10^{-24}$s and commonly decaying to (p^+, n^0) and (π^+, π^-, π^0) depending on the charge state.

Elementary particles of spin-$\frac{3}{2}$ which are not boundstates have not been observed so far. In fact, a straightforward pertubative quantization of spin-$\frac{3}{2}$ fields leads to inconsistencies (such theories turn out to be pertubatively nonrenormalizable). These inconsistencies can be (partly) resolved in supersymmetric theories, where the Rarita-Schwinger spinor becomes the superpartner of the graviton and is called gravitino.

6 Interacting field theories with spinors

6.1 Yukawa theories

For the construction of scalar theories, we have used a criterion of simplicity. For the interactions this has been partly related to the dimensionality of the interaction terms, e.g. the $\lambda\phi^4$-term in $d = 4$ dimensions has a dimensionless coupling constant $[\lambda] = 0$.
For a similar argument for spinor theories, we first need the dimensionality of the spinor field. With regard to the kinetic term

$$S_{\mathrm{D}}^{\mathrm{kin}} = \int \underbrace{\mathrm{d}^4 x}_{-4}\, \mathrm{i}\bar{\psi}\gamma^\mu \underbrace{\partial_\mu}_{1} \psi, \tag{6.1}$$

we read off that $[\bar{\psi}\psi] = 3$ and hence

$$[\psi] = \frac{3}{2}. \tag{6.2}$$

The same result follows from the mass term

$$-\int \underbrace{\mathrm{d}^4 x}_{-4} \underbrace{m}_{1} \underbrace{\bar{\psi}\psi}_{\Rightarrow 3}.$$

Recalling that scalar fields have mass dimension $[\phi] = 1$, the simplest interaction term which yields a Lorentz scalar is

$$S_{\mathrm{Yuk}} = -\int \underbrace{\mathrm{d}^4 x}_{-4}\, h \underbrace{\phi}_{1} \underbrace{\bar{\psi}\psi}_{3}. \tag{6.3}$$

This is the so-called Yukawa interaction describing the interaction of two Dirac spinors with a scalar field. Here, h denotes a coupling constant, which we call Yukawa coupling in the following, that is dimensionless, $[h] = 0$. The Yukawa interaction therefore satisfies also our quantifiable criterion of simplicity. Historically, this has first been used for the description of the pion (scalars) - nucleon (spinors) interaction. The full action of a typical (simple) Yukawa theory is

$$S = \int \mathrm{d}^4x \left[\bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - h\phi\bar{\psi}\psi - V(\phi) \right]. \tag{6.4}$$

Here we have ignored a possible Dirac mass term which would break the axial symmetry. Actually, also the Yukawa interaction (6.3) breaks the chiral symmetry because

$$\bar{\psi}\psi \xrightarrow{\text{axial U(1)}} \bar{\psi}\mathrm{e}^{\mathrm{i}\gamma_5\theta}\mathrm{e}^{\mathrm{i}\gamma_5\theta}\psi = \bar{\psi}\mathrm{e}^{2\mathrm{i}\gamma_5\theta}\psi. \tag{6.5}$$

For generic $\theta \in [0, 2\pi]$, $\mathrm{e}^{2\mathrm{i}\gamma_5\theta}$ is a nontrivial 4×4 matrix which cannot be compensated by a transformation of a real scalar field $\phi \in \mathbb{R}$. However, if we choose $\theta = \frac{\pi}{2}$, we have

$$\begin{aligned} & & \mathrm{e}^{2\mathrm{i}\gamma_5\theta} &= \cos(2\theta) + \mathrm{i}\gamma_5\sin(2\theta) \qquad \text{(in general)} \\ \theta = \frac{\pi}{2}: & & \mathrm{e}^{\mathrm{i}\pi\gamma_5} &= \cos(\pi) = -1, \end{aligned} \tag{6.6}$$

and hence: $\bar{\psi}\psi \to -\bar{\psi}\psi$.

If we now combine this specific axial transformation with the $\mathbb{Z}_2$-symmetry of the scalar field $\phi \to -\phi$ (provided that $V(\phi)$ is $\mathbb{Z}_2$ symmetric), the Yukawa theory of (6.4) is invariant under the discrete symmetry:

$$\begin{aligned} \phi &\to -\phi, \\ \psi &\to \mathrm{e}^{\mathrm{i}\frac{\pi}{2}\gamma_5}\psi, \\ \bar{\psi} &\to \bar{\psi}\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\gamma_5}. \end{aligned} \tag{6.7}$$

Note that the Dirac mass term would not be compatible with (6.7).

In turn, if we impose the symmetry (6.7), the spinor field is massless. The mass of the scalar field depends on the parameters in the potential, e.g. if we have

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4, \tag{6.8}$$

the scalar field is massive.

Now, we know that the $\mathbb{Z}_2$ symmetry in the scalar sector can be broken spontaneously if $V(\phi)$ has minima different from $\phi = 0$, e.g. for

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{6.9}$$

$$\Rightarrow \phi_{\text{min}} = \pm v = \pm\sqrt{\frac{6\mu^2}{\lambda}}. \tag{6.10}$$

Let us assume that ϕ picks the value $\phi_{\text{min}} = v$ as its ground state. Expanding ϕ about this ground state $\phi(x) = v + \sigma(x)$, we find the action (c.f. Eq. (3.8))

$$S = \int \mathrm{d}^4x \left[\bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - hv\bar{\psi}\psi \right. \\ \left. -h\sigma\bar{\psi}\psi - \left[\frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{3!}\lambda v\sigma^3 + \frac{1}{4!}\lambda\sigma^4\right]\right]. \tag{6.11}$$

Here, we can read off the mass $m_\sigma^2 = 2\mu^2$ of the scalar excitation σ. In addition, we observe the occurrence of a Dirac mass term $-(hv)\bar{\psi}\psi$, such that the Dirac spinors have also acquired a mass

$$m_\psi = hv = h\sqrt{\frac{6\mu^2}{\lambda}}. \tag{6.12}$$

The remaining terms are interactions of Yukawa type $\sim \sigma\bar{\psi}\psi$ or scalar self-interactions.

We conclude that the breaking of the $\mathbb{Z}_2$ symmetry in the scalar sector also extends to the Yukawa sector, spontaneously generating a mass for the Dirac spinor. The spinor mass is otherwise kept zero if the symmetry is preserved. This is a first simple but non-trivial example for the fact that Dirac spinor masses can be zero on the level of the action but then be generated by spontaneous symmetry breaking in a scalar sector.

The present model is often used as a toy-model for the sector of the Standard model of particle physics involving only the Higgs boson and

the top quark (as the heaviest quark). As the model only features a discrete symmetry, no Goldstone bosons occur in the broken phase (as is also true for the standard model, however, by virtue of the Higgs mechanism involving a gauge symmetry).
It is instructive to also study this (toy-) standard model application of the present model on the level of parameters and numbers. On the level of the Lagrangian, we have 3 parameters: h, μ^2, λ. This corresponds to the number of measurable quantities in the top-Higgs sector of the standard model:

$$\begin{aligned} \text{Higgs boson mass}: \quad & m_{\mathrm{H}} && \simeq 125\mathrm{GeV} \quad && (\text{date: 2} \\ \text{top quark mass}: \quad & m_{\mathrm{t}} && \simeq 173\mathrm{GeV} \quad && (\text{date: 2} \\ \underset{\sim\text{Higgs vacuum expectation value}}{\text{Fermi-constant}}: \quad & v = \left(\sqrt{2}G_{\mathrm{F}}\right)^{-\frac{1}{2}} && \simeq 246\mathrm{GeV} \quad && (\text{date: 2} \end{aligned}$$

Using the identification with our model parameters

$$\begin{aligned} m_{\mathrm{H}} \quad &\leftrightarrow \quad m_\sigma = \sqrt{2\mu^2} = \sqrt{2}\mu, \\ m_{\mathrm{t}} \quad &\leftrightarrow \quad m_\psi = hv = h\sqrt{\frac{6\mu^2}{\lambda}}, \\ v \quad &\leftrightarrow \quad v = \sqrt{\frac{6\mu^2}{\lambda}} \qquad \Leftrightarrow \lambda = \frac{6\mu^2}{v^2}, \end{aligned} \tag{6.13}$$

we find

$$\begin{aligned} \mu &\simeq 88\mathrm{GeV}, \\ h &\simeq 0.70, \\ \lambda &\simeq 0.77. \end{aligned} \tag{6.14}$$

We observe that both coupling constants are of the order $\mathcal{O}(1)$. However, λ comes with a factor of $(4!)^{-1}$ in the action. This is not the case for the top-Yukawa coupling h. Even though the top-quark is very short-lived with a lifetime of $\sim 5 \cdot 10^{-25}$s and was difficult to discover due to its high mass (discovery 1995 by CDF and DØ)[1], it plays the most important role for the dynamics of the theory at high energies

[1] Collider Detector at Fermilab (CDF) and the DØ experiment were two major experiments at the Tevatron Collider at the Fermi National Accelerator Laboratory (Fermilab).

among all the other quarks and leptons. Of course, for a proper discussion in the context of particle physics, a full quantization of the theory is necessary.

6.2 Yukawa vs. fermionic theories

In the purely scalar case, we have been able to construct a whole class of models by promoting the real scalar $\phi \in \mathbb{R}$ to a vector ϕ^a in an internal symmetry space $\mathrm{O}(N)$. Naively, one may try to do the same for Yukawa systems by promoting $\phi \to \phi^a$ and similarly promoting the Dirac spinor to multiple copies $\psi \to \psi^a$, which are often called *flavors* in the fermionic context, $a = 1 \dots N_{\mathrm{f}}$.
However, it is not fully trivial to construct a Yukawa interaction from such rather arbitrary building blocks (e.g. you may try to contract the indices to get a scalar). Moreover, since $\phi^a \in \mathbb{R}^N$ for $a = 1 \dots N$, ϕ^a transforms under $\mathrm{O}(N)$ whereas $\psi^a, \bar{\psi}^a$ are complex fields and hence $\bar{\psi}^a \psi^a$ is invariant under the unitary group $\mathrm{U}(N_{\mathrm{f}})$. So, the symmetries would not fit for arbitrary contractions of fermionic and scalar indices.
In the above example, we have considered the action

$$S = \int \mathrm{d}^4 x \left[\bar{\psi} \mathrm{i} \partial\!\!\!/ \psi + \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - h \phi \bar{\psi} \psi - V(\phi) \right], \tag{6.15}$$

being invariant under $\mathbb{Z}_2$ symmetry. However, the symmetry acted rather differently on ϕ and ψ, c.f. Eq. (6.7). On the other hand, the symmetry transformation looks equivalent on the level of ϕ and the fermion bilinear

$$\begin{aligned} \phi &\to -\phi, \\ \bar{\psi}\psi &\to -\bar{\psi}\psi. \end{aligned} \tag{6.16}$$

In fact, this can become a general construction principle for theories with spinors and further fields for featuring invariance under bigger continuous symmetries.

This construction principle becomes even more visible in a certain limit of the above theory. Let us take a look at the equations of motion:

$$\begin{aligned}(\mathrm{i}\not{\partial} - h\phi)\psi &= 0 \\ \partial^2\phi + V'(\phi) + h\bar{\psi}\psi &= 0.\end{aligned} \tag{6.17}$$

Obviously, we have two coupled partial differential equations featuring a high degree of nonlinearity.

Let us study a particularly simple limit : let us assume that ϕ is slowly varying or almost constant in spacetime $\phi \simeq$ const. Then, with $\partial^2\phi \simeq 0$, we get

$$h\bar{\psi}\psi + V'(\phi) = 0. \tag{6.18}$$

For the simple case $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$, we have

$$h\bar{\psi}\psi + m^2\phi + \frac{\lambda}{3!}\phi^3 = 0. \tag{6.19}$$

Let us further assume that $\frac{\lambda}{3!} \ll 1$, then

$$\phi = -\frac{h}{m^2}\bar{\psi}\psi, \tag{6.20}$$

which is naturally compatible with the symmetry. Even if we do not assume $\frac{\lambda}{3!} \ll 1$ but include a full potential $V(\phi)$, (6.18) can in principle be expressed as $\phi = f(\bar{\psi}\psi)$ at least locally connecting the scalar to a fermion bilinear. It is instructive to study the action (6.4) in the simple limit $\lambda \ll 1$, $\partial\phi \simeq 0$:

$$S = \int \mathrm{d}^4x \left[\bar{\psi}\mathrm{i}\not{\partial}\psi - h\phi\bar{\psi}\psi - \frac{1}{2}m^2\phi^2\right]. \tag{6.21}$$

Using the equation of motion (6.20) for ϕ, we get an action depending solely on the spinor field:

$$\begin{aligned} S &= \int \mathrm{d}^4x \left[\bar{\psi}\mathrm{i}\partial\!\!\!/\psi - h\left(-\frac{h}{m^2}\bar{\psi}\psi\right)\bar{\psi}\psi - \frac{1}{2}m^2\left(-\frac{h}{m^2}\bar{\psi}\psi\right)^2\right] \\ &= \int \mathrm{d}^4x \left[\bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \frac{h^2}{2m^2}(\bar{\psi}\psi)^2\right] \\ &= \int \mathrm{d}^4x \left[\bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \frac{g}{2}(\bar{\psi}\psi)^2\right], \qquad g = \frac{h^2}{m^2}. \end{aligned} \tag{6.22}$$

This is the famous Gross-Neveu model, introduced by Gross and Neveu in 1974 in two dimensions as a model with analogies to the strong interactions. The precise statement is that the theory defined by (6.22) purely in terms of spinors and that of (6.21) defined in terms of spinors and scalars are completely identical by virtue of the equations of motion (6.20) of the scalar field.
Of course, beyond the limit $\lambda \to 0$ and for non vanishing scalar kinetic terms, the equivalence is only approximate.
Incidentally in the quantized version, the exact equivalence between (6.22) and (6.21) persists to hold. Moreover, the equivalence can even hold upon inclusion of interactions and derivative terms for properties of the long-range physics. This is an example of *universality*.

In turn, if we had started with the Gross-Neveu model (6.22), we could have used the inverse construction, defining a scalar field

$$\phi = -g\bar{\psi}\psi \tag{6.23}$$

in order to write the action as

$$S = \int \mathrm{d}^4x \left[\bar{\psi}\mathrm{i}\partial\!\!\!/\psi - \phi\bar{\psi}\psi - \frac{1}{2}\frac{1}{g}\phi^2\right]. \tag{6.24}$$

Writing $g = \dfrac{h^2}{m^2}$ and rescaling $\phi \to h\phi$ would have lead to (6.21) again. This construction that converts a non-linear fermionic theory into a bilinear (Gaußian) action is known as Hubbard-Stratonovich transformation. Again, this transformation can also be performed on the quantum level.

Let us use this construction to introduce Yukawa models with higher symmetries. E.g. it is straightforward to upgrade the spinor content to $N_{\rm f}$ flavors ψ^a, $a = 1 \dots N_{\rm f}$:

$$S = \int d^4x \left[\bar{\psi}^a i\partial\!\!\!/ \psi^a + \frac{g}{2}(\bar{\psi}^a\psi^a)^2 \right]. \tag{6.25}$$

This theory is invariant under flavor rotations,

$$\begin{aligned} \psi^a &\to U^{ab}\psi^b \\ \bar{\psi}^a &\to \bar{\psi}^b U^{\dagger ba}, \end{aligned} \tag{6.26}$$

such that

$$U^\dagger U = \mathbb{1}, \qquad \text{i.e. } U \in \mathrm{U}(N_{\rm f}).$$

In absence of a mass term $\sim \bar{\psi}^a\psi^a$ (which would also be $\mathrm{U}(N_{\rm f})$ invariant), the model also has the discrete $\mathbb{Z}_2$ axial symmetry (6.7), transforming $\bar{\psi}^a\psi^a \to -\bar{\psi}^a\psi^a$.

The structure of the interaction suggests to introduce a scalar field

$$\phi = -g\bar{\psi}^a\psi^a, \tag{6.27}$$

leading, as before, to the equivalent action

$$S = \int d^4x \left[\bar{\psi}^a i\partial\!\!\!/ \psi^a - \phi\bar{\psi}^a\psi^a - \frac{1}{2}\frac{1}{g}\phi^2 \right]. \tag{6.28}$$

Now, we can add kinetic terms and interaction terms for the scalar field to arrive at a new Yukawa theory for $N_{\rm f}$ spinor flavors:

$$S_{\rm Yuk} = \int d^4x \left[\bar{\psi}^a i\partial\!\!\!/ \psi^a - h\phi\bar{\psi}^a\psi^a + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi) \right]. \tag{6.29}$$

The model still has the full $\mathrm{U}(N_{\rm f})$ flavor symmetry. However, the scalar sector is the same as before. In order to preserve the $\mathbb{Z}_2$ symmetry of the fermionic system, we only need a real scalar $\phi \in \mathbb{R}$ and a $\mathbb{Z}_2$ symmetric potential $V(-\phi) = V(\phi)$.

Upon spontaneous symmetry breaking by a suitable potential with a minimum at $\phi_{\text{min}} = v \neq 0$, all flavors of fermions acquire the same mass term:

$$-m_\psi \bar{\psi}^a \psi^a, \qquad m_\psi = hv. \tag{6.30}$$

Most importantly, the breakdown of the $\mathbb{Z}_2$ symmetry does not imply the breakdown of flavor symmetry. The mass term preserves the $\mathrm{U}(N_\mathrm{f})$ symmetry.
In order to arrive at a more complex scalar sector, the axial/chiral symmetry on the fermionic side has to be more complex as well.

6.3 Models with continuous chiral symmetry

In the exercises, we had already studied a fermionic model with continuous chiral symmetry:

$$S_{\text{NJL}} = \int \mathrm{d}^4x \left(\bar{\psi}\mathrm{i}\partial\!\!\!/\psi - \frac{g}{2}\left((\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2\right)\right). \tag{6.31}$$

This is the famous Nambu-Jona-Lasinio model for the case of one fermion flavor $N_\mathrm{f} = 1$. The model has been inverted by Nambu and Jona-Lasimio (and independently by Vaks and Larkin) in 1961 by transferring ideas from the BCS theory of superconductivity to the description of nucleons and mesons in elementary particle physics. Up to the present day it is frequently used as an effective low-energy model of the strong interactions (low-energy QCD). The model is invariant under

$$\begin{aligned} \mathrm{U_V}(1): &\qquad \psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi, &\qquad \bar{\psi} \to \bar{\psi}\mathrm{e}^{-\mathrm{i}\alpha}, \\ \mathrm{U_A}(1): &\qquad \psi \to \mathrm{e}^{\mathrm{i}\alpha\gamma_5}\psi, &\qquad \bar{\psi} \to \bar{\psi}\mathrm{e}^{\mathrm{i}\alpha\gamma_5}, \end{aligned} \tag{6.32}$$

as discussed in detail in the exercises. Hence it is also invariant under both chiral symmetries $\mathrm{U_L}(1), \mathrm{U_R}(1)$, which are a linear combination of (6.32).

In the spirit of the Hubbard-Stratonovich transformation, it is natural to introduce two scalar fields,

$$\phi_1 = -g(\bar{\psi}\psi), \qquad \phi_2 = -\mathrm{i}g(\bar{\psi}\gamma_5\psi), \tag{6.33}$$

in order to rewrite (6.31) as

$$S_{\mathrm{NJL}} = \int \mathrm{d}^4x \left[\bar{\psi}\mathrm{i}\partial\!\!\!/\psi - \phi_1\bar{\psi}\psi - \mathrm{i}\phi_2\bar{\psi}\gamma_5\psi - \frac{1}{2}\frac{1}{g}(\phi_1^2 + \phi_2^2)\right]. \tag{6.34}$$

Since $\bar{\psi}\psi$ as well as $\bar{\psi}\gamma_5\psi$ are separately invariant under $\mathrm{U_V}(1)$, the fields ϕ_1 and ϕ_2 transform trivially under this symmetry: $\phi_{1,2} \to \phi_{1,2}$. The Noether charge of this $\mathrm{U_V}(1)$ corresponds to particle number. This implies that ϕ_1 and ϕ_2 do not carry particle number ($\hat{=}$ electric charge) and hence can be considered as neutral. In order to identify their transformation under $\mathrm{U_A}(1)$, we note that

$$\mathrm{e}^{\mathrm{i}\alpha\gamma_5} = \mathbb{1}\cos(\alpha) + \mathrm{i}\gamma_5\sin(\alpha). \tag{6.35}$$

This implies that

$$\begin{aligned} \bar{\psi}\psi &\to \bar{\psi}\mathrm{e}^{\mathrm{i}\alpha\gamma_5}\mathrm{e}^{\mathrm{i}\alpha\gamma_5}\psi = \bar{\psi}\mathrm{e}^{2\mathrm{i}\alpha\gamma_5}\psi = \bar{\psi}\psi\cos(2\alpha) + \mathrm{i}\sin(2\alpha)\bar{\psi}\gamma_5\psi, \\ \bar{\psi}\gamma_5\psi &\to \bar{\psi}\mathrm{e}^{\mathrm{i}\alpha\gamma_5}\gamma_5\mathrm{e}^{\mathrm{i}\alpha\gamma_5}\psi = \bar{\psi}\gamma_5\psi\cos(2\alpha) + \mathrm{i}\sin(2\alpha)\bar{\psi}\psi. \end{aligned} \tag{6.36}$$

We observe that the combination

$$\phi_1\bar{\psi}\psi + \mathrm{i}\phi_2\bar{\psi}\gamma_5\psi$$

is invariant under $\mathrm{U_A}(1)$, iff ϕ_1 and ϕ_2 transform as

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to \begin{pmatrix}\cos(2\alpha) & \sin(2\alpha)\\ -\sin(2\alpha) & \cos(2\alpha)\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{6.37}$$

Interpreting ϕ_a, $a = 1, 2$ as an element of $\mathbb{R}^2$, Eq. (6.37) corresponds to an SO(2) rotation in the ϕ_1, ϕ_2 plane. This rotation also leaves the scalar mass term $\sim (\phi_1^2 + \phi_2^2)$ invariant as it corresponds to the scalar product in $\mathbb{R}^2$. Since the symmetry groups $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ are isomorphic to one another the complex transformations of ψ and the real transformations of ϕ_1, ϕ_2 fit perfectly. Note, however that a full axial rotation in $U_A(1)$ from $\alpha = 0$ to $\alpha = 2\pi$ covers the SO(2) rotations twice: $2\alpha = 0$ to $2\alpha = 4\pi$. In the language of Noether charges this implies that the scalar carries twice the axial charge of the spinor.

These symmetry considerations allow us to finally construct a Yukawa theory that exhibits the chiral symmetry of the NJL model,

$$S_{\text{Yuk-NJL}} = \int \mathrm{d}^4x \left[\bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \frac{1}{2}(\partial_\mu\phi_a)(\partial^\mu\phi_a) - h(\phi_1\bar{\psi}\psi + \mathrm{i}\phi_2\bar{\psi}\gamma_5\psi) - V \tag{6.38}$$

where $V(\phi)$ depends on ϕ_a only through the scalar product $\phi_a\phi_a$. Note that the symmetry fixes both Yukawa interactions to have the same coupling h.

Let us now study the predictions of this model for the particle and mass spectrum in the phase with spontaneous symmetry breaking if $V(\phi)$ develops a vacuum expectation value at

$$\phi_{0,a}\phi_{0,a} = v^2. \tag{6.39}$$

Parametrizing the field as

$$\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} v + \sigma(x) \\ \pi(x) \end{pmatrix}, \tag{6.40}$$

the action (6.38) becomes for $V(\phi) = -\frac{1}{2}\mu^2\phi_a\phi_a + \frac{\lambda}{4!}(\phi_a\phi_a)^2$:

$$\begin{aligned} S_{\text{Yuk-NJL}} = \int \mathrm{d}^4x \bigg[&\bar{\psi}\mathrm{i}\partial\!\!\!/\psi + \frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) + \frac{1}{2}(\partial_\mu\pi)(\partial^\mu\pi) \\ &- hv\bar{\psi}\psi - h(\sigma\bar{\psi}\psi + \mathrm{i}\pi\bar{\psi}\gamma_5\psi) - V(\sigma,\pi)\bigg], \end{aligned} \tag{6.41}$$

where $V(\sigma, \pi)$ is the same potential that we have studied in the context of $O(N)$ models in Eq. (3.20) for the case of only one π field. Hence, we obtain the mass spectrum

$$\begin{aligned} m_\psi &= hv, \\ m_\sigma &= \sqrt{2\mu^2}, \\ m_\pi &= 0. \end{aligned} \tag{6.42}$$

The masslessness of the π field is in agreement with Goldstones theorem. The fermions become massive. As the π field couples to $\bar{\psi}\gamma_5\psi$ which is a pseudoscalar fermion bilinear, also π must transform as a pseudoscalar, i.e., with a minus sign under parity transformation.

In their original publications Nambu and Jona-Lasinio associated the ψ's with the nucleon (proton/neutron), the π-field with a light pion and thus predicted the sigma meson as a heavy nucleon/anti-nucleon bound state. Of course, quarks had not yet been invented in 1961. In the modern use of the NJL model, ψ denotes the quarks and hence m_ψ is interpreted as the constituent quark mass $m_\psi \simeq 300\text{MeV}$.

With regard to the Hubbard-Stratonovich transformation $\phi_1 = v + \sigma \sim -g(\bar{\psi}\psi)$, the nonvanishing expectation value of ϕ_1 is also interpreted as a nonvanishing *chiral condensate* $\langle\bar{\psi}\psi\rangle$ (in quantum notation). Since the mesons $(\sigma, \pi, \dots)$ are bound states and not fundamental in contrast to the quarks, the formation of a bilinear condensate is sometimes also called *dynamical symmetry breaking*. Quantitatively, the vacuum expectation value is related to the pion decay constant $f_\pi = v \simeq 93\text{MeV}$.

7 Field theories of matter and gauge interactions

The most characteristic feature of particle physics is that the interactions among fermionic matter building blocks is mediated by gauge bosons such as the photon.[1] The underlying *local* gauge symmetry that we have already encountered in Maxwell's theory is largely reponsible for the resulting structures. In G. 't Hooft's words, we are *under the spell of the gauge principle.*

7.1 (Quantum) Electrodynamics (QED)

Starting from the Maxwell Lagrangian (1.60) known from classical electrodynamics,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu, \tag{7.1}$$

let us try to add fermionic electron/positron degrees of freedom in the form of a Dirac spinor field $\psi(x)$, while preserving the local gauge symmetry under gauge transformations:

$$A_\mu(x) \quad \to \quad A_\mu(x) + \partial\Lambda(x), \qquad \Lambda(x) : \text{arbitrary}. \tag{7.2}$$

[1]NB: the Higgs boson is somewhat Janus-faced, it carries matter properties as well as mediates a force via Yukawa interactions.

Assuming that the interaction can be written in terms of a suitable choice for the source $J_\mu = J_\mu[\bar{\psi}, \psi]$, the action remains invariant,

$$\begin{aligned} S = \int \mathrm{d}^4x \left[-\frac{1}{4} \underbrace{F_{\mu\nu}F^{\mu\nu}}_{\text{gauge invariant}} - J_\mu A^\mu \right] \\ \rightarrow \quad \int \mathrm{d}^4x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu - J_\mu \partial^\mu \Lambda \right] \\ \stackrel{\text{i.b.p.}}{=} \int \mathrm{d}^4x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu + \Lambda \partial^\mu J_\mu \right] \end{aligned} \tag{7.3}$$

if the source is conserved,

$$\partial^\mu J_\mu = 0. \tag{7.4}$$

Indeed, the free Dirac theory

$$S_{\mathrm{D}} = \int \mathrm{d}^4x \; [\bar{\psi} \mathrm{i} \partial\!\!\!/ \psi - m \bar{\psi} \psi] \tag{7.5}$$

offers a conserved source: the Noether current j^μ associated with $\mathrm{U_V}(1)$ vector symmetry

$$\psi \rightarrow \mathrm{e}^{\mathrm{i}\alpha} \psi, \qquad \bar{\psi} \rightarrow \bar{\psi} \mathrm{e}^{-\mathrm{i}\alpha}. \tag{7.6}$$

We have determined the resulting Noether current in the exercises:

$$j^\mu = \bar{\psi} \gamma^\mu \psi, \qquad \partial_\mu j^\mu = 0. \tag{7.7}$$

This suggest to identify J^μ with the Noether current,

$$J^\mu = e j^\mu, \tag{7.8}$$

where we have allowed for a coupling constant e that parametrizes the strenght of the interactions between the Maxwell and the Dirac field. Upon insertion of (7.8) into (7.1) and adding the Dirac action (7.5), we arrive at

$$S_{\mathrm{QED}} = \int \mathrm{d}^4x \left[-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi} \mathrm{i} D\!\!\!\!/\,[A] \psi - m \bar{\psi} \psi \right], \tag{7.9}$$

where we have used the covariant derivative (c.f. (3.34))

$$D_\mu[A] = \partial_\mu + \mathrm{i}eA_\mu, \quad \text{and} \quad \not{D} = \gamma^\mu D_\mu. \tag{7.10}$$

Equation (7.9) denotes the classical action of Quantum Electrodynamics (which becomes *Quantum*, of course, only upon quantization of the fields).

Our construction guarantees, that S_{QED} is invariant under the local gauge symmetry (7.2) as well as the global vector symmetry (7.6) separately. However, the interesting observation now is that S_{QED} is fully invariant under a simultaneous local transformation of both fields:

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) + \partial_\mu \Lambda(x), \\ \psi(x) &\to \mathrm{e}^{-\mathrm{i}e\Lambda(x)}\psi(x), \\ \bar{\psi}(x) &\to \bar{\psi}(x)\mathrm{e}^{\mathrm{i}e\Lambda(x)}\psi(x). \end{aligned} \tag{7.11}$$

This is the same type of *local* U(1) symmetry that we have already encountered for the abelian Higgs model in (3.33) and ff.

The essential building block is the covariant derivative $D_\mu[A]$, which guarantees that

$$(D_\mu[A]\psi) \quad \to \quad \mathrm{e}^{-\mathrm{i}e\Lambda(x)}(D_\mu[A]\psi) \tag{7.12}$$

– despite the partial derivative – transforms with a simple U(1) phase factor. It is an instructive computation to verify (7.12) explicitly.

Already on this classical level, the theory (7.9) is useful, as (together with a proton field) it offers the relativistic version of the quantum mechanical hydrogen-problem, describing relativistic effects in atomic physics rather accurately (c.f. your course on advanced quantum mechanics).

QED, however, celebrates its greatest successes in the quantized version, e.g., for the quantitative description of the anomalous magnetic moment of the electron or Lamb shift effects in atoms.

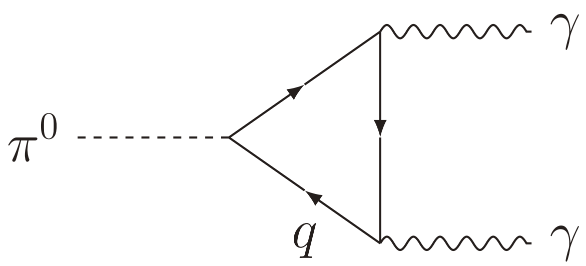
Here, we plan to go beyond and wish to view this theory as a first simple example of a gauge theory.

7.2 (Quantum) Chromodynamics (QCD)

The necessity of a further quantum number, i.e., another type of charge for elementary constituents became clear from the observation of Baryon resonances with three quarks in the same flavor and spin state,

$$\begin{aligned} |\Delta^{++}\rangle &= |u\uparrow\rangle\,|u\uparrow\rangle\,|u\uparrow\rangle, \\ |\Omega^{-}\rangle &= |s\uparrow\rangle\,|s\uparrow\rangle\,|s\uparrow\rangle, \\ |\Delta^{-}\rangle &= |d\uparrow\rangle\,|d\uparrow\rangle\,|d\uparrow\rangle, \end{aligned} \tag{7.13}$$

seemingly contradicting Pauli's exclusion principle. Upon adding a further quantum number, the required antisymmetrization for the fermionic constituents can be realized with respect to this new quantum number, called *color*. As a consequence, processes which can proceed via different internal values of this quantum number become proportional to it. An example is given by pion decay into two photons,



$$\sim N_c. \tag{7.14}$$

According to QFT, the decay proceeds via an internal quark fluctuation. As the quarks now can occur in differently colored versions, the process is proportional to the *number of colors* N_c. The experimental result is $N_c = 3$. I.e. in addition to the different quark *flavors* $f = u, d, s, c, b, t$ quarks also carry a *color* index $i = 1, 2, 3$:

$$\psi(x) \triangleq \psi^i_f(x). \tag{7.15}$$

In the following, we ignore the flavor and concentrate on the color index $i = 1, 2, 3$.

The above experiments suggest that there is at least a global symmetry in an internal color space by which we can transform the spinors:

$$\psi^i \quad \to \quad \psi'^i = U^{ij}\psi^j. \tag{7.16}$$

The decisive aspect of this symmetry exerting a strong influence on the resulting dynamics, however, is that this symmetry turned out to be a *local* symmetry analogous to the one of QED:

$$\psi'^i(x) = U^{ij}(x)\psi^j(x), \tag{7.17}$$

where $U(x) \in \mathrm{SU}(N_c)$ is a matrix, being an element of the matrix group $\mathrm{SU}(N_c)$. This is the set of complex unitary matrices with $\det(U) = 1$. This local symmetry property cannot be read off from kinematical observations as the ones given above, but require a close look at the dynamics or bound-state spectra of the system.

Let us first recall a few basic facts about the Lie groups $\mathrm{SU}(N_c)$ and their corresponding Lie algebra. The complex $N_c \times N_c$ matrices U^{ij} with

$$U^\dagger U = \mathbb{1} = UU^\dagger, \qquad \det(U) = 1 \tag{7.18}$$

form a representation of $\mathrm{SU}(N_c)$. The exponential map

$$U = \mathrm{e}^{\mathrm{i}H}, \qquad \text{where } H = H^\dagger \text{ hermitean } N_c \times N_c \text{ matrix}, \tag{7.19}$$

parametrizes U in terms of

$$N_c^2 - \underset{\uparrow \det U = 1}{1} \tag{7.20}$$

real parameters. This implies that H can be spanned by $N_c^2 - 1$ lineary independent hermitean matrices which serve as generators of $\mathrm{SU}(N_c)$:

$$U = \mathrm{e}^{-\mathrm{i}w_a\tau^a}, \qquad \underset{i,j=1\ldots N_c\ ,\ \ a=1\ldots N_c^2-1}{(\tau^a)^{ij} : \text{ generators of } \mathrm{SU}(N_c).} \tag{7.21}$$

Here, w_a are real parameters, and the τ^a can be chosen trace-free since

$$1 = \det(U) = \det\left(\mathrm{e}^{-\mathrm{i}w_a\tau^a}\right) = \mathrm{e}^{-\mathrm{i}w_a \operatorname{tr}(\tau^a)}. \tag{7.22}$$

For the commutator $[\tau^a, \tau^b]$, we have

$$\operatorname{tr}([\tau^a,\tau^b]) = \operatorname{tr}(\tau^a\tau^b - \tau^b\tau^a) \underset{\text{(cyclicity)}}{=} 0, \qquad \text{(trace-free)},$$

$$[\tau^a,\tau^b]^\dagger = [\tau^{b\dagger},\tau^{a\dagger}] = [\tau^b,\tau^a] = -[\tau^a,\tau^b], \qquad \text{(anti-hermitean)}.$$

Hence, we can write $[\tau^a, \tau^b] = \mathrm{i}h$ with h hermitean. Since h can be spanned by τ^a again, we have

$$[\tau^a,\tau^b] = \mathrm{i}f^{abc}\tau^c, \tag{7.23}$$

where the f^{abc}'s are the structure constants of the Lie algebra su(N_c) defined by (7.23). Conventionally, the τ^a's are normalized to

$$\operatorname{tr}(\tau^a\tau^b) = \frac{1}{2}\delta^{ab}. \tag{7.24}$$

A well-known example is given by $N_c = 2$, where $\tau^a = \frac{1}{2}\sigma^a$ (Pauli matrices) such that

$$[\tau^a,\tau^b] = \frac{1}{4}[\sigma^a,\sigma^b] = \frac{1}{4}2\mathrm{i}\epsilon^{abc}\sigma^c = \mathrm{i}\epsilon^{abc}\tau^c. \tag{7.25}$$

In this case the structure constants of su(2) are $f^{abc}_{\mathrm{su}(2)} = \epsilon^{abc}$. For all higher N_c, the generators can be constructed analogously to the Pauli matrices, e.g. $N_c = 3: \quad N_c^2 - 1 = 8, \tau^a = \frac{1}{2}\lambda^a$, where

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{7.26}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

These are the Gell-Mann matrices. The determination of the structure constants is straightforward:

$abc:$	123	147	156	246	257	345	367	458	678
$f^{abc}_{\mathrm{su}(3)}:$	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

(7.27)

and correspondingly for the permutations of the indices.

The resulting representation of $\mathrm{su}(N_c)$ in terms of the τ^a is irreducible by construction. It is called the *fundamental representation*. Of course, higher representations of the same algebra (7.23), $[T^a, T^b] = \mathrm{i} f^{abc} T^c$, in terms of higher dimensional matrices T^a also exist. An important one follows directly from the Jacobi identity for the commutator:

$$\begin{aligned}
&[[\tau^a,\tau^b],\tau^c] + [[\tau^b,\tau^c],\tau^a] + [[\tau^c,\tau^a],\tau^b] = 0\\
&\Rightarrow f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0\\
&\Rightarrow (-\mathrm{i} f^{bad})(-\mathrm{i} f^{edc}) - \underbrace{(-\mathrm{i} f^{bcd})(-\mathrm{i} f^{eda})}_{=(-\mathrm{i} f^{ead})(-\mathrm{i} f^{bdc})} = \mathrm{i} f^{bed}(-\mathrm{i} f^{dac})\\
&\Rightarrow \left[(-\mathrm{i} f^b), (-\mathrm{i} f^e)\right]^{ac} = \mathrm{i} f^{bed} (-\mathrm{i} f^d)^{ac}. \qquad (7.28)
\end{aligned}$$

Hence, $(T^a)^{bc} = -\mathrm{i} f^{abc}$ is also a representation of the $\mathrm{su}(N_c)$ Lie algebra, consequentially generating a corresponding representation of $\mathrm{SU}(N_c)$ in terms of $(N_c^2 - 1) \times (N_c^2 - 1)$ matrices. This is the *adjoint representation*.

Now, let us start with a free Dirac theory for a massive quark field occurring in N_c colors:

$$\mathcal{L}_{\mathrm{D}} = \bar{\psi}^i \mathrm{i} \partial\!\!\!/ \psi^i - m \bar{\psi}^i \psi^i, \qquad i = 1 \ldots N_c. \qquad (7.29)$$

As noted before, this theory is invariant under unitary *global* rotations in color space,

$$\psi^i \to U^{ij} \psi^j, \qquad \bar{\psi}^i \to \bar{\psi}^j (U^\dagger)^{ji}. \qquad (7.30)$$

Using the representation (7.21), it is straightforward to show that the corresponding Noether current is given by

$$j^{\mu a} = \bar{\psi}^i \gamma^\mu \tau^a_{ij} \psi^j, \qquad \partial_\mu j^{\mu a} = 0. \tag{7.31}$$

Identifying $J^{\mu a} = -g j^{\mu a}$ with a coupling constant[2] $g > 0$ as the vector-color current that we wish to couple to a photon-like color gauge field, we recognize that this color gauge field also has to carry an *adjoint* index:

$$\mathcal{L}_J = -JA = -J^{\mu a} A^a_\mu, \qquad a = 1 \dots N_c^2 - 1. \tag{7.32}$$

Adding the current term to the free Dirac theory, we obtain the Lagrangian

$$\mathcal{L} = \bar{\psi}^i \mathrm{i} \not{D}_{ij}[A] \bar{\psi}^j - m \bar{\psi}^i \psi^i, \tag{7.33}$$

where the covariant derivative now takes the form

$$\not{D}_{ij} = \gamma^\mu D_{\mu\, ij} = \gamma^\mu (\partial_\mu \delta_{ij} - \mathrm{i} g \tau^a_{ij} A^a_\mu). \tag{7.34}$$

Incidentally, note that – in order to preserve the invariance of (7.33) under global color rotations – A^a_μ is not allowed to remain unmodified under a global rotation. Writing

$$A_{\mu\, ij} := \tau^a_{ij} A^a_\mu \tag{7.35}$$

or A_μ in short, the color gauge field has to transform as

$$A_\mu \quad \to \quad U A_\mu U^\dagger \tag{7.36}$$

under global color rotations. Note that this is still in line with QED, as for a U(1) symmetry the generator is a number, say $\tau|_{\mathrm{U}(1)} = 1$, such that $U A_\mu U^\dagger = A_\mu$ for QED.

However, inspired by QED we now wish to promote the invariance to a *local* invariance. This is possible if the covariant derivative of the spinor transforms as

$$\not{D}\psi \quad \to \quad U(x) \not{D}\psi \tag{7.37}$$

[2] In QED, we have $e < 0$.

analogously to QED, cf. (7.12) such that

$$\bar{\psi}\mathrm{i}\not{D}\psi \quad \rightarrow \quad \mathrm{i}\bar{\psi}\underbrace{U^\dagger U}_{=\mathbb{1}}\not{D}\psi = \bar{\psi}\mathrm{i}\not{D}\psi. \tag{7.38}$$

This condition for the covariant derivative is met if we generalize (7.36) to the *local* transformation rule

$$A_\mu \quad \rightarrow \quad A'_\mu = UA_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger. \tag{7.39}$$

Check:

$$\begin{aligned} D_\mu\psi \rightarrow (\partial_\mu - \mathrm{i}gA'_\mu)\psi' &= (\partial_\mu - \mathrm{i}gUA_\mu U^\dagger - (\partial_\mu U)U^\dagger)U\psi \\ &= U(\partial_\mu - \mathrm{i}gA_\mu)\psi + \cancel{(\partial_\mu U)\psi} - \cancel{(\partial_\mu U)U^\dagger U\psi} \\ &= UD_\mu\psi. \end{aligned} \tag{7.40}$$

Having introduced a field A_μ that couples to the color charge of the quarks similar to the photon-electron coupling, we finally need to specify its dynamics by constructing a kinetic term for A_μ on the level of the action.

For this, we first note that the field strength in electrodynamics follows from the commutator of covariant derivatives,

$$\mathrm{U}(1): \qquad [D_\mu, D_\nu] = \mathrm{i}eF_{\mu\nu}. \tag{7.41}$$

Taking the different sign conventions for the coupling into account, we similarly define the field strength for $\mathrm{SU}(N_c)$ gauge theory using the covariant derivatives:

$$F_{\mu\nu} := \frac{1}{\mathrm{i}g}[D_\mu, D_\nu], \qquad F_{\mu\nu} \equiv F^a_{\mu\nu}\tau^a. \tag{7.42}$$

This field strength $F_{\mu\nu}$ is matrix-valued in the $\mathrm{su}(N_c)$ algebra. As discussed in the exercises, this leads us to

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu. \tag{7.43}$$

Since the covariant derivative transforms homogeneously,

$$D_\mu \quad \rightarrow \quad UD_\mu U^\dagger \qquad (\text{cf. } (7.37)), \tag{7.44}$$

also the field strength transforms homogeneously,

$$F_{\mu\nu} \quad \to \quad UF_{\mu\nu}U^\dagger = F'_{\mu\nu}, \tag{7.45}$$

and is thus not invariant componentwise in contrast to electrodynamics.

Still, we can straightforwardly construct a gauge-invariant action

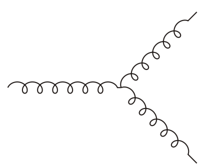
$$\begin{aligned}\mathcal{L}_{\mathrm{YM}} &= -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} \overset{(7.24)}{\equiv} -\frac{1}{2}F^a_{\mu\nu}F^{b\mu\nu}\,\mathrm{Tr}\,(\tau^a\tau^b) \\ &= -\frac{1}{2}\,\mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) \\ &= -\frac{1}{2}\,\mathrm{Tr}\,(U^\dagger UF_{\mu\nu}U^\dagger UF^{\mu\nu}) \\ &\overset{\text{cyclicity}}{=} -\frac{1}{2}\,\mathrm{Tr}\,((UF_{\mu\nu}U^\dagger)(UF^{\mu\nu}U^\dagger)) \\ &= -\frac{1}{2}\,\mathrm{Tr}\,(F'_{\mu\nu}F'^{\mu\nu}).\end{aligned} \tag{7.46}$$

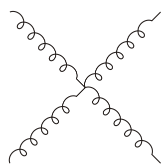
This is the celebrated Lagrangian of Yang-Mills theory, an $\mathrm{SU}(N_c)$ bosonic theory of a vector field (spin-1) with a local symmetry. It is important to realize that this action not only defines the kinetic terms for A^a_μ,

$$\mathcal{L}^{\mathrm{kin}}_{\mathrm{YM}} \simeq -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) \tag{7.47}$$

but also contains self-interaction terms which are enforced by gauge invariance (schematically)

$$\mathcal{L}^{\mathrm{int}}_{\mathrm{YM}} \quad \sim \quad + \ldots \qquad g(\partial_\mu A_\nu)A^\mu A^\nu \qquad + \ldots \qquad g^2(A_\mu a_\nu)^2. \tag{7.48}$$





Therefore, already the pure Yang-Mills part is a highly non-trivial interacting theory unlike the pure Maxwell part. The gauge field excitations are also called gluons, hence (7.46) describes *gluodynamics*.

Read together with the Dirac part of the quarks (7.33), we arrive at the classical action defining Quantum Chromodynamics (QCD)

$$S_{\mathrm{QCD}} = \int \mathrm{d}^4 \left[-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi} \mathrm{i} \not{D} \psi - m \bar{\psi} \psi \right]. \tag{7.49}$$

Upon the inclusion of different quark flavors, each flavor may have a different mass parameter m.

Let us finally take a quick look at the classical field equations. As discussed in the exercises, the Euler-Lagrange equations for the gauge/gluon field lead to

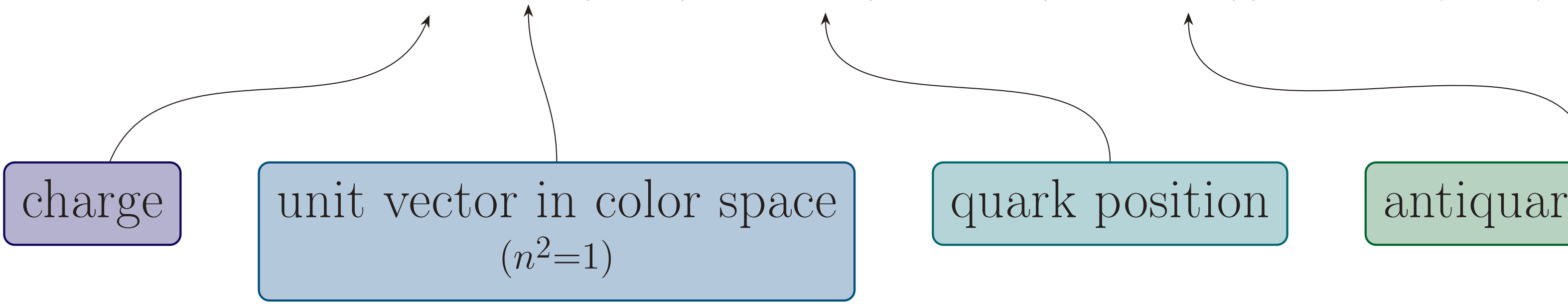
$$D^{ab}_{\mu} F^{b\mu\nu} \equiv (\partial_{\mu} \delta^{ab} + g f^{acb} A^c_{\mu}) F^{b\mu\nu} = j^{a\nu}, \tag{7.50}$$

where $j^{a\nu} = \bar{\psi} g \gamma^{\nu} \tau^a \psi$. Here, we encounter the covariant derivative in the adjoint representation:

$$\begin{aligned} D^{ab}_{\mu} &= \partial_{\mu} \delta^{ab} + g f^{acb} A^c_{\mu} = (\partial_{\mu} - \mathrm{i} g T^c A^c_{\mu})^{ab} \\ \text{with } (T^c)^{ab} &= -\mathrm{i} f^{cab}. \end{aligned} \tag{7.51}$$

Let us, for example, consider a static quark-anti-quark pair as a simple model for a meson,

$$j^{a0} = Q \; n^a \, (\delta^{(3)}(\vec{x} - \vec{x}_1) - \delta^{(3)}(\vec{x} - \vec{x}_2)). \tag{7.52}$$



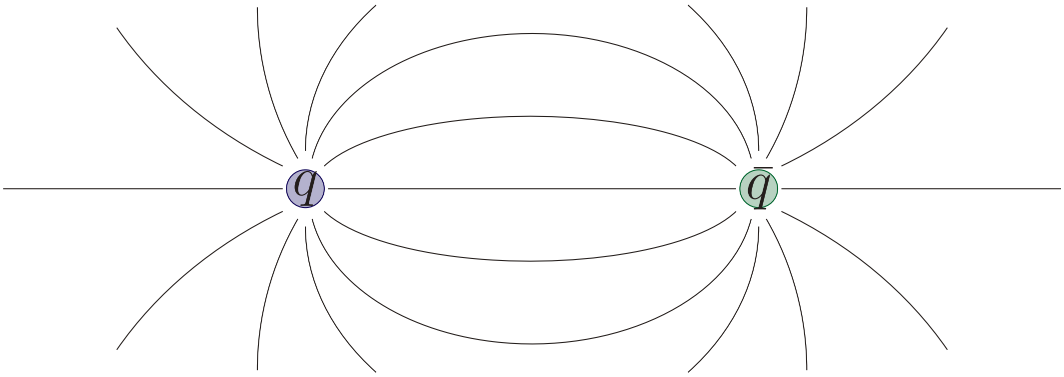
The equation of motion can fully be mapped onto classical electrodynamics, by noting that a pseudo-abelian ansatz

$$A^a_{\mu} = n^a a_{\mu}, \qquad F^a_{\mu\nu} = n^a f_{\mu\nu}, \tag{7.53}$$

with $f^{abc} n^b n^c = 0$ leads to

$$\partial_{\mu} f^{\mu\nu} = j^{a\nu} n^a. \tag{7.54}$$

Hence, the solution is fully equivalent to that of a classical dipole field for the n^a components of the chromoelectric field:



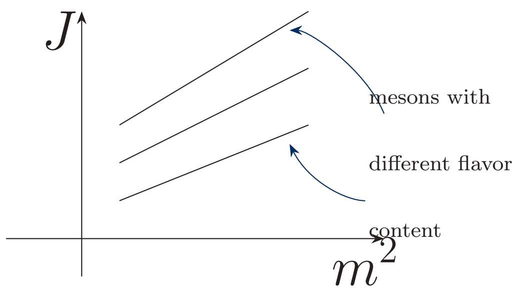
Correspondingly, the static potential corresponds to the Coulomb potential

$$V(r) \sim \frac{1}{r}. \tag{7.55}$$

However, this is in contradiction with the experimental observation. For instance, if higher mesonic excitations with higher angular momentum J are studied, one observes that their total (squared) mass is proportional to J:

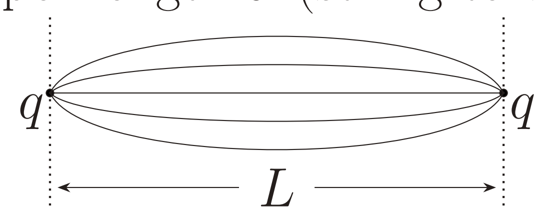
$$J \sim m^2. \tag{7.56}$$

These lines of proportionality are called *Regge trajectories*.



In contrast to the classical analysis given above, this observation can be described by a string model for the field distribution of a meson. Let us define this model based on two simple assumptions:

- the gluon field of a meson is stringlike with a constant energy per length σ (string tension),



- for higher excitations, the quarks on both ends rotate at almost the speed of light.

Then the energy/mass of the system is ($R = L/2$)

$$m \equiv E = 2\int\limits_0^R \frac{\sigma}{\sqrt{1-v(r)^2}}\mathrm{d}r = 2\int\limits_0^R \frac{\sigma\mathrm{d}r}{\sqrt{1-(r/R)^2}} = \pi\sigma R, \quad (7.57)$$

whereas the angular momentum is

$$J = 2\int\limits_0^R \frac{\sigma r v(r)}{\sqrt{1-v(r)^2}}\mathrm{d}r = \frac{2}{R}\sigma\int\limits_0^R \frac{r^2\mathrm{d}r}{\sqrt{1-(r/R)^2}} = \frac{1}{2}\pi\sigma R^2, \quad (7.58)$$

from which we read off that

$$J = \frac{1}{2\pi\sigma}m^2. \quad (7.59)$$

This is in agreement with the experimental observation. The slope of the Regge trajectories gives

$$\begin{aligned} \alpha' = \frac{1}{2\pi\sigma} \simeq 0.9(\mathrm{GeV})^{-2} \\ \text{or } \sigma \simeq (430\mathrm{MeV})^2. \end{aligned} \quad (7.60)$$

A stringlike color electric field distribution can be associated with a linear potential,

$$V(r) \sim r. \quad (7.61)$$

This line-like field distribution between two quarks and the corresponding impossibility to isolate a single quark is called *confinement*. The comparison with our conclusion from the classical equation of motion shows that classical Chromodynamics is insufficient to describe this basic experimentally verified property of the strong interactions. Therefore: Quantum effects modify the dynamics of QCD qualitatively (not only quantitatively).

8 Classical Field Theory for Particle Physics - an example -

In this course, we have mainly discussed classical field theory aspects which are relevant for particle physics. This included mainly the aspects of possible degrees of freedom (scalars, spinors, vectors, ...), their symmetries (external spacetime and internal symmetries), and the construction of interactions on the level of the classical action. However, a thorough discussion of particle physics applications typically involves quantization, as it is the quantized excitations of these fields which are relevant for computing observables. Also, some aspects which could, in principle, be discussed on the classical level ('tree-level processes'), follow much more elegantly within the quantized formalism making it less worthwhile to deal with the classical equations of motion.

Still, the language of classical field theory does become even more useful than the quantum notion of Fock spaces etc., as soon as the corresponding experimental situation involves coherent classical fields. In the following, we want to illustrate this with an example from experimental searches for new particles.

8.1 Axion Electrodynamics

The standard model of particle physics has various shortcomings, a prominent one being the rather large number of parameters such as fermion masses which do not seem to follow a natural pattern. Even more serious is the fact that some parameters which, in principle, are allowed to be sizeable seem to be zero or at least unnaturally small.

Most prominently, there is an angle type of parameter θ (a combination of a QCD parameter and the phase of the determinant of the quark mass matrix) which would physically induce CP violation in the strong interactions. If so, QCD bound states would be expected to show CP-violating properties. An example would be given by an electric dipole moment of the neutron $\vec{d}_n$. Measurements so far have only found an upper bound on a possibly non-zero value: $|d_n| < 3 \cdot 10^{-26}$ ecm (data from 2015). The precise relation between $|d_n|$ and θ is difficult to compute as is any bound-state property of QCD from first principles. However, simple estimates translate the bound for $|d_n|$ as follows into a bound for θ: given the diameter of the neutron $\sim 10^{-15}$ m and assuming a linear dependence on θ, we may estimate

$$|d_n| \simeq \mathrm{c}\theta \cdot \mathrm{e}\, 10^{-15}\,\mathrm{m} = \mathrm{c}\theta \cdot 10^{-13}\,\mathrm{ecm}, \tag{8.1}$$

where c is a constant to be determined from a full calculation. Generic field theory computations often yield factors inversely proportional to the phase space and thus to the volume of the 4-sphere. So a small number one typically gets is $\mathrm{c} \simeq 1/(32\pi^2) \simeq 10^{-3}$. Hence we conclude that $\theta \lesssim 10^{-10}$. As θ is an angle $\in [0, 2\pi]$ we would naturally expect it to be of $\mathcal{O}(1)$, rendering $\theta = 10^{-10}$ or smaller rather unnatural. This is the '*strong-CP*' problem.

Note that the strong-CP problem is not at all a problem of mathematical consistency, but rather a problem of unlikeliness: if nature can choose any value for θ in the interval 0 to 2π, why should it choose some value so close or equal to zero. Of course, $\theta = 0$ is not a mathematically nor logically excluded choice, but from a physicist's perspective, it seems to be on *unnatural* choice. Therefore, the strong-CP problem is an example of a *naturalness problem.*

One possibility to 'explain' $\theta \simeq 0$ is to impose a suitable symmetry. This is not completely trivial as θ receives contributions from two different origins (QCD + quark mass matrix). All requirements are ultimately satisfied by models that lift θ to be the expectation value of a dynamical field that acquires a suitable potential in a dynamical fashion. Ultimately, these models do not only predict (or post-dict) $\theta = 0$ but also feature the possibility of having excitations on top of the vacuum, corresponding to a pseudo-scalar field: the '*axion*'.

To cut a long story short: the so-far only valid solution to the strong-CP problem predicts another pseudo-scalar particle ϕ which in many respects behave like the neutral pion π^0. In particular, it has a non-zero mass m and can couple to two photons $\pi^0 \leftrightarrow 2\gamma$. The corresponding effective classical field theory is:

$$\begin{aligned}\mathcal{L}_{axED} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \\ &-\frac{1}{4}g\phi F_{\mu\nu}\tilde{F}^{\mu\nu}, \qquad \text{`Axion electrodynamics'},\end{aligned} \tag{8.2}$$

which involves a coupling between the axion and the pseudo-scalar invariant

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \vec{E}\cdot\vec{B}. \tag{8.3}$$

This effective field theory involves two parameters m and g. Dimensional analysis reveals that g must have an inverse mass dimension, so g^{-1} corresponds to a mass scale.

In order to solve the strong-CP problem, g and m are related:

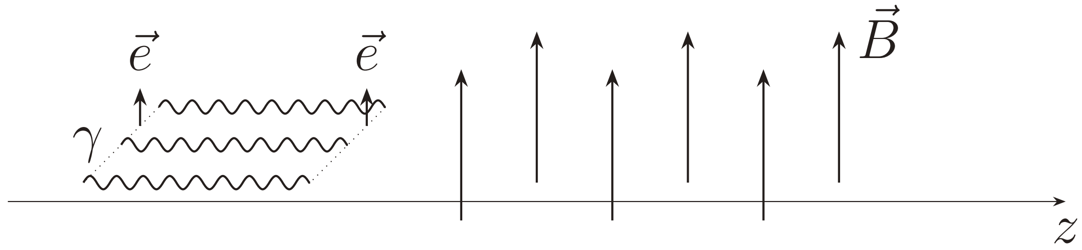
$$\frac{m}{[1\,\mathrm{meV}]} \sim \frac{g}{[10^{13}\,\mathrm{geV}]^{-1}}. \tag{8.4}$$

The precise relation between g and m depends on the details of the underlying model that embeds the additional symmetry into the context of the standard model of particle physics. The fact that we haven't observed any direct signature of the axion puts severe constraints on the coupling. Hence, the axion can be expected to be rather light (if it exists).

8.2 Photon-Axion conversion

Now, the coupling $\sim \phi\vec{E}\vec{B}$ inspires to look at the following process: consider a plain wave with electric field component $\vec{e}$ propagating across a magnetic field $\vec{B}$ with $e \parallel \vec{B}$. Then, this interaction allows for a mixing of the plane wave $\vec{e}$ with the axion field ϕ. So, even if we start initially with a pure plane wave, the axion field will acquire a non-zero amplitude after some distance of propagation inside the magnetic field. A quantitative analysis follows from the field equations. Using a Weyl-Coulomb gauge ($A_0 = 0, \nabla \cdot \vec{A} = 0$), the plane wave field can be parametrized by a pure vector potential $\vec{a}$, $\vec{e} = -\dot{\vec{a}}$.

Considering only the relevant case, where $\vec{e} \parallel \vec{B}$, with $\vec{B}$ being a constant field pointing perpendicular to the direction of the plane wave propagation,



we can write the interaction term as

$$\begin{aligned} -\int \mathrm{d}^4x \, \frac{1}{4} g\phi \, F_{\mu\nu}\tilde{F}^{\mu\nu} &= \int \mathrm{d}^4x \, g\phi \, \vec{E} \cdot \vec{B} \\ &= \int \mathrm{d}^4x \, g\phi \, eB = \int \mathrm{d}^4x \, ga \, \dot{\phi} B, \end{aligned} \tag{8.5}$$

where $e = |\vec{e}|$, $B = |\vec{B}|$, $a = |\vec{a}|$. In the last step, we have performed a partial integration.

The interaction term hence contributes to both, the Maxwell as well as the Klein-Gordon equation for $\vec{a}$ and ϕ, respectively:

$$\begin{aligned} \Box\phi + m^2\phi - ge\vec{B} &= 0, \\ \Box a \qquad\quad - g\dot{\phi}B &= 0. \end{aligned} \tag{8.6}$$

We are interested in solutions that propagate along the z direction, hence $a = a(z,t)$, $\phi = \phi(z,t)$

$$\Rightarrow \qquad \Box \to \partial_t^2 - \partial_z^2. \tag{8.7}$$

Though both fields a and ϕ are real, it is useful to formally complexify the fields and perform a Fourier transformation to frequency space:

$$\begin{aligned} a(z,t) &= \int \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega t} a(\omega, z), \\ \phi(z,t) &= -\mathrm{i} \int \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega t} \chi(\omega, z). \end{aligned} \tag{8.8}$$

Then (8.6) turns into equations for the frequency modes $a(\omega, z)$ and $\chi(\omega, z)$:

$$\begin{aligned} (-\omega^2 - \partial_z^2 + m^2)(-\mathrm{i}\chi(\omega, z)) - \mathrm{i} g\omega\, a(\omega, z)\, B = 0, \\ (-\omega^2 - \partial_z^2) a(\omega, z) + g\omega\, \chi(\omega, z)\, B = 0, \end{aligned} \tag{8.9}$$

or in matrix notation

$$\left[\mathbb{1}\left(\omega^2 + \partial_z^2\right) - M\right] \begin{pmatrix} \chi \\ a \end{pmatrix} = 0, \tag{8.10}$$

$$\text{where} \qquad M = \begin{pmatrix} +m^2 & g\omega B \\ g\omega B & 0 \end{pmatrix}. \tag{8.11}$$

Assuming a plane wave form in wave number space

$$\{a, \chi\}(\omega, z) = \{a, \chi\}(\omega) \mathrm{e}^{\mathrm{i}kz} \tag{8.12}$$

leads us to the algebraic equation

$$\left[\mathbb{1}\left(\omega^2 - k^2\right) - M\right] \begin{pmatrix} \chi \\ a \end{pmatrix} = 0. \tag{8.13}$$

Solutions exist if $\det\left(\mathbb{1}\left(\omega^2 - k^2\right) - M\right) = 0$

$$\Rightarrow \qquad (\omega^2 - k^2 - m^2)(\omega^2 - k^2) = (g\omega B)^2, \tag{8.14}$$

the roots of which define the dispersion relations

$$k_\pm^2 = \omega^2 - (m^2 - (gB)^2)\left(\frac{\cos 2\theta \pm 1}{2\cos 2\theta}\right), \tag{8.15a}$$

where

$$\tan 2\theta = \frac{2\omega g B}{m^2 - (gB)^2}. \tag{8.15b}$$

Here, we can see that θ can be interpreted as a mixing angle between axion and photon.

In the limit of vanishing coupling or vanishing magnetic field $gB \to 0$, we have $\theta \to 0$ and hence $k_-^2 = \omega^2$, $k_+^2 = \omega^2 - m^2$. In this limit, k_- corresponds to the wave number of a free photon, and k_+ to that of the massive axion.

In the real experiment, a fixed scale is set by the frequency ω of the propagating laser, and the wave numbers follow from the dispersion relation. The general solution of the equations of motion for a propagating mode along the positive z direction reads

$$\begin{aligned} a(\omega, z) &= a^-(\omega)\mathrm{e}^{\mathrm{i}k_- z} + \tan^2\theta a^+(\omega)\mathrm{e}^{\mathrm{i}k_+ z}, \\ \chi(\omega, z) &= \frac{\omega}{k_-}\tan\theta a^-(\omega)\mathrm{e}^{\mathrm{i}k_- z} - \frac{\omega}{k_+}\tan\theta a^+(\omega)\mathrm{e}^{\mathrm{i}k_+ z}. \end{aligned} \tag{8.16}$$

Let us consider a monochromatic wave, $a^-(\omega) = a^+(\omega) = \mathrm{const.}$ for one fixed ω, and an axion mass much smaller than the optical laser frequency $m^2 \ll \omega^2$. We also confine ourselves to a small mixing angle $\theta \ll 1$. Then, the induced axion amplitude reads ($a^+ = a^- = a_{\mathrm{IN}}$)

$$\chi(\omega, z) = a_{\mathrm{IN}}\theta\left(\mathrm{e}^{\mathrm{i}k_- z} - \mathrm{e}^{\mathrm{i}k_+ z},\right) \tag{8.17}$$

where we keep $k_\pm$ in the phases as the wave numbers can be multiplied by large values of z, but approximate $k_\pm \simeq \omega$ in the prefactor.

Now, we use the fact that the classical field equations lead to amplitudes that can be interpreted as quantum mechanical probability amplitudes. Hence, we arrive at the probability that an initial photon amplitude is converted into an axion as a function of the length L of propagation inside B:

$$\begin{aligned} P(\gamma \to \phi; L) = \frac{|\chi|^2}{(a_{\mathrm{IN}})^2} &= |\phi|^2 \left|\mathrm{e}^{\mathrm{i}k_- z} - \mathrm{e}^{\mathrm{i}k_+ z}\right|^2\Big|_{z=L} \\ &= |\phi|^2(2 - 2\cos((k_+ - k_-)L)). \end{aligned} \tag{8.18}$$

In the above mentioned limits, the occurring quantities yield

$$|\theta|^2 = \left(\frac{\omega g B}{m^2}\right)^2,$$

$$k_+ - k_- = \sqrt{\omega^2 - m^2} - \omega = \omega\left(\sqrt{1 - \frac{m^2}{\omega^2}} - 1\right) \qquad (8.19)$$

$$\simeq \omega\left(1 - \frac{m^2}{2\omega^2} - 1\right) = -\frac{m^2}{2\omega}.$$

Using $2 - 2\cos x = 2\cdot(1-\cos x) = 2\cdot 2\sin^2(x/2)$, we get

$$P(\gamma \to \phi; L) = 4\left(\frac{\omega g B}{m^2}\right)^2 \sin^2\left(\frac{m^2 L}{4\omega}\right). \qquad (8.20)$$

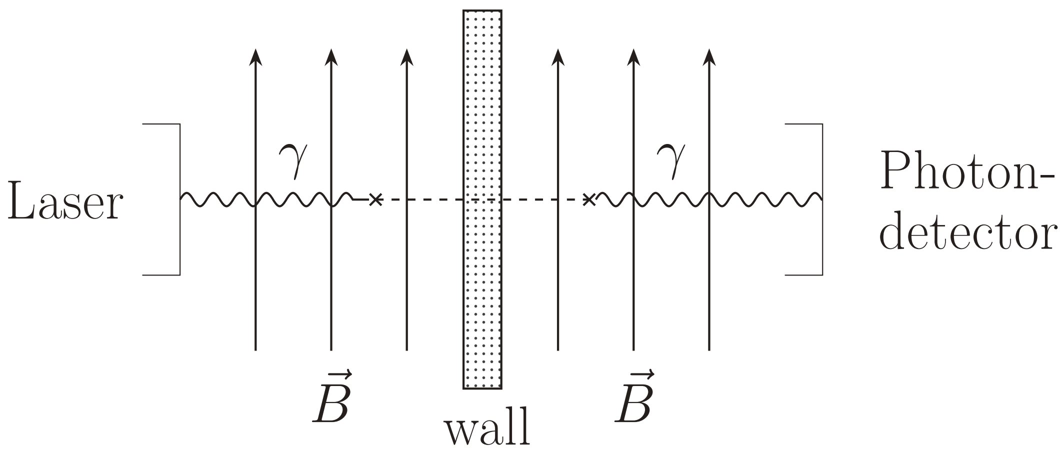
For a given length of the magnetic field, the probability in the small-mass limit becomes

$$P(\gamma \to \phi; L)|_{m\to 0} \simeq \frac{1}{4}(gBL)^2, \qquad (8.21)$$

and thus independent of the mass.

When it comes to discovery experiments, it is not sufficient to convert photons to axions, because we have no 'axiometer' that could measure the axion amplitude. Instead one uses the following idea (Sikivie '83, van Bibber '87):

a 'light-shining-through-wall' experiment



Shine a laser onto a wall and try to observe photons behind the wall. Use a strong magnetic field to convert part of the photon wave (function) into an axion in front of the wall and back into a photon behind the wall. Since the axion is weakly interacting, it can transverse the wall in contrast to photons. This type of experiments has a couple of attractive features: the interaction regions (size of the $\vec{B}$ field) can be macroscopic (in contrast to small collision points in colliders), and can even be enhanced by the use of cavities.

The number of incoming photons can be very large $\gtrsim 10^{20}$, whereas the detection of a single photon can already constitute a signal of 'new physics'. Apart from exceedingly small processes from photon-neutrino-pair processes or photon-graviton conversion, the experiment is essentially background free.

A number of experiments (BFRT, BMV, GammeV, LISPS and ALPS) have been performed. The non-observation of a signal constitute the currently best laboratory bounds on axions, complementing astrophysical bounds. Currently, a major upgrade of ALPS at DESY is in preparation.

To get a rough estimate on the sensitivity, we first note that the back-conversion $\phi \to \gamma$ features the same probability as in (8.21). Assuming that the magnetic field behind and in front of the wall have the same length L, we have

$$P(\gamma \to \phi \to \gamma; L)|_{m\to 0} \simeq \frac{1}{16} C (gBL)^4, \tag{8.22}$$

where C is an enhancement factor if cavities are used in order to enhance the photonic input power. For one cavity in front of the wall $C \sim (N/2)^4$. The finesse N of the cavity can be of order $N \sim 10^3$. The current upgrade of ALPS even plans to put a locked cavity behind the wall, which would give a $C \sim (N/2)^8$ improvement. Converting the units into GeV, we have

$$P \simeq \frac{1}{16} C \left(\frac{g}{[\mathrm{GeV}]^{-1}} \frac{B}{[1\mathrm{Tesla}]} \frac{L}{[1\mathrm{m}]} \right)^4 \tag{8.23}$$

with N_γ being the number of incoming photons per second, the number of reconverted photons per second behind the wall is $N_{\mathrm{obs}} = N_\gamma \cdot P$. Having N_γ in excess of 10^{20}, experiments with $C = 1$ already become sensitive to values of

$$g^{-1} \sim 10^5\,\text{GeV} = 10^2\,\text{Tesla}$$

for meter-size fields and Tesla-strong fields. In fact, ALPS has reached a sensitivity of $g^{-1} \gtrsim 10^7\,\text{GeV}$ which is a factor of 1000 larger than current collider energy scales. This demonstrates that a suitable design of novel non-collider-type experiments can compete with or at least complement collider searches for new hypothetical particles.

The search for axions has been and still is an active research area, also because the axion could have significant relevance in astro physics (stellar cooling) as well as cosmology (dark matter).