Gauge Theories

– Lecture Notes –

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Preface

These lecture notes have been prepared for a course on gauge theories as part of an advanced theoretical physics curriculum. The course is a small sequel of an introductory course on quantum field theory as it is taught at many German universities during a Master program.

The course assumes a solid knowledge of theoretical physics on the Bachelor level, as well as a good knowledge of quantum field theory including canonical quantization and a bit of the path-integral formalism. The reader should be familiar with the basic concepts of Feynman diagrammar, perturbation theory, and scattering theory. The present notes aim at extending this knowledge towards nonabelian gauge theories, which form the basis of our current understanding of elementary particle physics.

Somewhat different from conventional presentations in textbooks aiming at an exposition of the material for particle phenomology, the present notes are designed to provide a first glance on the nonperturbative aspects of gauge theories focusing on aspects of confinement and dynamical symmetry breaking, and thus on the low-energy sector of the strong nuclear interactions.

These notes are based on various sets of handwritten lecture notes prepared for several summer schools (including a course at the Heidelberg graduate days held together with Jan M. Pawlowski on “Pictures of Confinement” many years back) and have grown and improved over the years. This new version also contains a few additions and is planned to replace the handwritten versions from now on[1].

Jena, August 2025 Holger Gies

[1] Comments, suggestions, and hints at typos are more than welcome!

1 Introduction and basic concepts

1.1 Physical origins of the idea of color

The idea of nonabelian gauge theories has many origins and probably dates back to early attemps by W. Pauli to formulate a gauge theory for the weak interactions. Pauli later appeared to have given up on this idea, since he thought that massless excitations (similar to photons) where an inevitable consequence of such theories, being in contradiction to the short-range nature of the weak interactions.

The concept of nonabelian gauge theories was later independently rediscovered by C.N. Yang and R.L. Mills in 1954 (who were aware of the massless-excitation problem, but didn't care so much about it) and also independently at about the same time by R. Shaw (a student of A. Salam) and R. Utiyama; nowadays, we speak of nonabelian gauge theories and Yang-Mills theories synonymously.

The central idea at the time was to generalize the concept of gauge invariance from electrodynamics with gauge group U(1) to the isospin degree of freedom known from nuclear physics which is described by an SU(2) symmetry. This idea is essentially realized in the electroweak sector of the standard model of particle physics – even though the precise formulation took a while, included many twists and turns, and goes along with many names (G. Glashow, S. Weinberg, A. Salam, P. Higgs, F. Englert, R. Brout, J.C. Taylor, T. Kibble, and many more).

For the present course, we concentrate instead on the motivation for nonabelian gauge theories from the strong interactions. Also in this case, it took a while to realize that the interactions between fundamental constituents of nuclear matter postulated as *quarks* in the 1960s (by M. Gell-Mann and G. Zweig) can be understood as a nonabelian gauge theory. The crucial step was the introduction of a new quantum number, called *color*, by O.W. Greenberg, M.Y. Han and Y. Nambu in 1964.

For the motivation of this quantum number, let us recall the experimental situation in the 1960s where more an more baryon resonances and mesons have been discovered. Attemps at a classification of these states gave a strong hint at the existence of quarks as the fundamental building blocks of hadrons.

The quarks carry spin 1/2 and are fermions (in order to be compatible with the spin 1/2 and fermionic nature of protons and neutrons). Nowadays, we know of 6 types of quarks, called *flavors*: up, down, strange, charm, bottom, and top,

$$\text{flavor} \qquad \begin{array}{ccc} u & c & t \\ d & s & b \end{array} \qquad \begin{array}{l} +\frac{2}{3}\text{electric charge} \\ -\frac{1}{3} \end{array} \tag{1.1}$$

Suitable combinations of these quarks yielded the hadrons known at that time, e.g., the proton (uud) and the neutron (udd). The experimental discovery of spin-3/2 resonances, however, was puzzling. Using a quantum mechanical notation, some of the Δ and Ω resonances can be written as

$$\begin{aligned} |\Delta^{++}\rangle &= |u\uparrow\rangle|u\uparrow\rangle|u\uparrow\rangle \\ |\Omega^{-}\rangle &= |s\uparrow\rangle|s\uparrow\rangle|s\uparrow\rangle \\ |\Delta^{-}\rangle &= |d\uparrow\rangle|d\uparrow\rangle|d\uparrow\rangle \end{aligned} \tag{1.2}$$

With the arrows indicating the alignment of the spins in order to yield the total spin 3/2 of these resonances. However, this configuration seemed to be in contradiction with the Pauli principle which requires an antisymmetric wave function for fermions. The spin-3/2 resonances are fermions (in agreement with the spin-statistics theorem), but the right-hand sides seem to be completely symmetric in both the flavor quantum number as well as the spin quantum number.

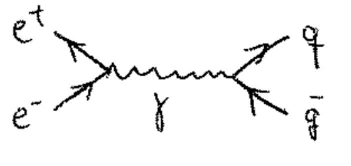
A conclusion drawn at the time was that there must be another quantum number (*color*) with respect to which the wave function is antisymmetric.

An immediate question was how many values this quantum number should take. I.e., how many different colors N_c do quarks come in?

Several experimental hints became available to study this question. One of them is the famous R ratio in electron-positron collisions. It measures the ratio of the cross section for hadron production to the cross section for muon production,

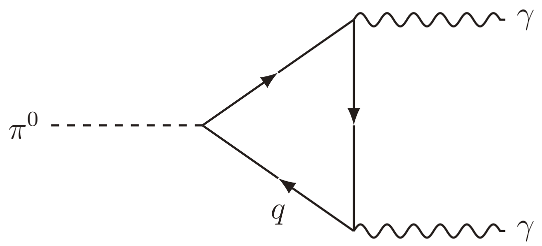
$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \quad \sim N_c^2. \tag{1.3}$$

Since the quarks can come in N_c different colors, the transition amplitudes for the processes is proportional to N_c, such that the cross section as the square of the



amplitude is proportional to N_c^2. The experimental value $R \approx 3$ for energies above the charm threshold thus suggests $N_c = 3$ colors(say red, green, and blue).

Similarly, the decay of the neutral pion into two photons, $\pi^0 \to \gamma\gamma$, is mediated by a quark loop as a virtual intermediate state. The decay rate is proportional to the number of colors N_c since each color contributes equally to the amplitude. The experimental value for the decay rate again suggests $N_\mathrm{c} = 3$.



$$\sim N_c. \tag{1.4}$$

In summary, the experimental evidence strongly suggests that quarks come in three different colors, $N_\mathrm{c} = 3$ and six flavors $N_\mathrm{f} = 6$. Formalising this in terms of a quark wave function $\psi(x)$, this wave function is a vector in a N_c-dimensional color space as well as a N_f-dimensional flavor space,

$$\psi_f^i(x), \quad f = u, d, s, c, b, t, \quad i = 1, 2, 3, \tag{1.5}$$

where f labels the flavor and i labels the color. In the following, we focus on the color degree of freedom and mostly ignore the flavor degree of freedom unless stated otherwise.

1.2 Basic concepts for gauge theories

The above experiments suggest that there is at least a global symmetry in an internal color space by which we can transform the spinors:

$$\psi^i \quad \to \quad \psi'^i = U^{ij}\psi^j. \tag{1.6}$$

The decisive aspect of this symmetry exerting a strong influence on the resulting dynamics, however, is that this symmetry turned out to be a *local* symmetry analogous to the one of QED:

$$\psi'^i(x) = U^{ij}(x)\psi^j(x), \tag{1.7}$$

where $U(x) \in \mathrm{SU}(N_c)$ is a matrix, being an element of the matrix group $\mathrm{SU}(N_c)$. This is the set of complex unitary matrices with $\det(U) = 1$.

This local symmetry property cannot be read off from kinematical observations as the ones given above, but require a close look at the dynamics or bound-state spectra of the system.

1.2.1 Elements of the theory of Lie groups and Lie algebras

Let us first recall a few basic facts about the Lie groups $\mathrm{SU}(N_c)$ and their corresponding Lie algebra $\mathfrak{su}(N_c)$. The complex $N_c \times N_c$ matrices U^{ij} with

$$U^\dagger U = \mathbb{1} = UU^\dagger, \qquad \det(U) = 1 \tag{1.8}$$

form a representation of $\mathrm{SU}(N_c)$. The exponential map

$$U = \mathrm{e}^{\mathrm{i}H}, \qquad \text{where } H = H^\dagger \text{ hermitean } N_c \times N_c \text{ matrix}, \tag{1.9}$$

parametrizes U in terms of

$$N_c^2 - \underset{\uparrow \det U = 1}{1} \tag{1.10}$$

real parameters. This implies that H can be spanned by $N_c^2 - 1$ lineary independent hermitean matrices which serve as generators of $\mathrm{SU}(N_c)$:

$$U = \mathrm{e}^{-\mathrm{i}w_a\tau^a}, \qquad \underset{i,j=1\ldots N_c \quad , \quad a=1\ldots N_c^2-1}{(\tau^a)^{ij} : \text{ generators of } \mathrm{SU}(N_c).} \tag{1.11}$$

Here, w_a are real parameters, and the τ^a can be chosen trace-free since

$$1 = \det(U) = \det(\mathrm{e}^{-\mathrm{i}w_a\tau^a}) = \mathrm{e}^{-\mathrm{i}w_a \operatorname{tr}(\tau^a)}. \tag{1.12}$$

For the commutator $[\tau^a, \tau^b]$, we have

$$\begin{aligned} \operatorname{tr}([\tau^a, \tau^b]) = \operatorname{tr}(\tau^a\tau^b - \tau^b\tau^a) &\underset{\text{(cyclicity)}}{=} 0, && \text{(trace-free)}, \\ [\tau^a, \tau^b]^\dagger = [\tau^{b\dagger}, \tau^{a\dagger}] = [\tau^b, \tau^a] &= -[\tau^a, \tau^b], && \text{(anti-hermitean)}. \end{aligned} \tag{1.13}$$

Hence, we can write $[\tau^a, \tau^b] = \mathrm{i}h$ with h hermitean. Since h can be spanned by τ^a again, we have

$$[\tau^a, \tau^b] = \mathrm{i}f^{abc}\tau^c, \tag{1.14}$$

where the f^{abc}'s are the structure constants of the Lie algebra $\mathfrak{su}(N_c)$ defined by (1.14).

Conventionally, the τ^a's are normalized to

$$\operatorname{tr}(\tau^a\tau^b) = \frac{1}{2}\delta^{ab}. \tag{1.15}$$

A well-known example is given by $N_c = 2$, where $\tau^a = \frac{1}{2}\sigma^a$ (Pauli matrices) such that

$$[\tau^a, \tau^b] = \frac{1}{4}[\sigma^a, \sigma^b] = \frac{1}{4}2\mathrm{i}\epsilon^{abc}\sigma^c = \mathrm{i}\epsilon^{abc}\tau^c. \tag{1.16}$$

In this case the structure constants of $\mathfrak{su}(2)$ are $f^{abc}_{\mathfrak{su}(2)} = \epsilon^{abc}$.

For all higher N_c, the generators can be constructed analogously to the Pauli matrices, e.g. $N_c = 3: \quad N_c^2 - 1 = 8, \tau^a = \frac{1}{2}\lambda^a$, where

$$
\begin{aligned}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned}
\tag{1.17}
$$

These are the Gell-Mann matrices. The determination of the structure constants is straightforward:

$$
\begin{array}{cccccccccc}
abc: & 123 & 147 & 156 & 246 & 257 & 345 & 367 & 458 & 678 \\
f^{abc}_{\mathfrak{su}(3)}: & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}
\end{array}
\tag{1.18}
$$

and correspondingly for the permutations of the indices.

The resulting representation of $\mathfrak{su}(N_c)$ in terms of the τ^a is irreducible by construction. It is called the *fundamental representation*. Of course, higher representations of the same algebra (1.14), $[T^a, T^b] = \mathrm{i} f^{abc} T^c$, in terms of higher dimensional matrices T^a also exist. An important one follows directly from the Jacobi identity for the commutator:

$$
\begin{aligned}
& [[\tau^a, \tau^b], \tau^c] + [[\tau^b, \tau^c], \tau^a] + [[\tau^c, \tau^a], \tau^b] = 0 \qquad (1.19) \\
& \Rightarrow f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe} = 0 \\
& \Rightarrow (-\mathrm{i} f^{bad})(-\mathrm{i} f^{edc}) - \underbrace{(-\mathrm{i} f^{bcd})(-\mathrm{i} f^{eda})}_{=(-\mathrm{i} f^{ead})(-\mathrm{i} f^{bdc})} = \mathrm{i} f^{bed}(-\mathrm{i} f^{dac}) \\
& \Rightarrow \left[(-\mathrm{i} f^b), (-\mathrm{i} f^e)\right]^{ac} = \mathrm{i} f^{bed}(-\mathrm{i} f^d)^{ac}. \qquad (1.20)
\end{aligned}
$$

Hence, $(T^a)^{bc} = -\mathrm{i} f^{abc}$ is also a representation of the $\mathfrak{su}(N_c)$ Lie algebra, consequentially generating a corresponding representation of $\mathrm{SU}(N_c)$ in terms of $(N_c^2 - 1) \times (N_c^2 - 1)$ matrices. This is the *adjoint representation*.

1.2.2 Gauge invariant field theory actions

We are aiming at a construction of a classical action for quark fields $\psi(x)$ which is invariant under local gauge transformations (1.7). Since the quarks are fermions appearing as particles and antiparticles, they can be described by Dirac spinors. In absence of any interaction, the kinetic term for Dirac fermions would be given by $S = \int d^4x\, \bar{\psi}(x) \mathrm{i}\partial\!\!\!/ \psi(x)$, where we use the short form $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. However, this kinetic term is not invariant under local gauge transformations (1.7) since

$$\begin{aligned} \bar{\psi}'^i(x) i\partial\!\!\!/ \psi'^i(x) &= \bar{\psi}^j(x) U^{\dagger ji}(x) i\partial\!\!\!/ U^{ik}(x)\psi^k(x) \\ &= \bar{\psi}^j(x) i\partial\!\!\!/ \psi^j(x) + \bar{\psi}^j(x) U^{\dagger ji}(x)(i\partial\!\!\!/ U^{ik}(x))\psi^k. \end{aligned} \tag{1.21}$$

The additional term kann be compensated for by introducing a matrix-valued gauge field

$$A_\mu^{ij}(x) = A_\mu^a(x)(\tau^a)^{ij}, \tag{1.22}$$

where $A_\mu(x)$ needs to transform as

$$A'_\mu = U A_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger. \tag{1.23}$$

Here, we have introduced a coupling constant g for later convenience. With this transformation property, a term of the form $\bar{\psi} ig A\!\!\!/ \psi$ transforms as

$$\begin{aligned} i\bar{\psi}' ig A\!\!\!/' \psi' &= i\bar{\psi} U^\dagger \big(igU A_\mu U^\dagger + (\partial_\mu U)U^\dagger\big)\gamma^\mu U\psi \\ &= i\bar{\psi} ig A\!\!\!/ \psi + \bar{\psi} U^\dagger (i\partial\!\!\!/ U)\psi. \end{aligned} \tag{1.24}$$

Note that the last term corresponds precisely to the last term in Eq. (1.21). The two terms can be combined into a gauge invariant kinetic term involving the so-called *covariant derivative*

$$D_\mu = (\partial_\mu - igA_\mu), \tag{1.25}$$

such that the kinetic term for the quark field can be written as

$$\mathcal{L}_{\mathrm{kin}} = \bar{\psi} i D\!\!\!\!/ \psi. \tag{1.26}$$

In summary, the covariant derivative transforms homogeneously under local gauge transformations,

$$D'_\mu = U D_\mu U^\dagger. \tag{1.27}$$

The local symmetry thus enforces the presence of a gauge field with an adjoint index, i.e., with $N_\mathrm{c}^2 - 1$ components

$$A_\mu^a(x), \quad a = 1, \ldots, N_\mathrm{c}^2 - 1. \tag{1.28}$$

The quantized excitations of this field are called *gluons* in the context of QCD, analogously to photons in electrodynamics. Similarly to electrodynamics, we can construct a gluonic field strength tensor from the commutator of two covariant derivatives,

$$\begin{aligned}[D_\mu, D_\nu] &=: -igF_{\mu\nu} \\ &\Rightarrow \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu. \end{aligned} \tag{1.29}$$

The last step is left as an exercise for the reader. From the definition of the field strength tensor, note that there are several important differences to electrodynamics: first, the field strength tensor is not linear in the gauge field because of the last term in Eq. (1.29). Second, the field strength tensor is not gauge invariant itself, but transforms homogeneously under gauge transformations, as is clear from its definition in terms of covariant derivatives, $F'_{\mu\nu} = UF_{\mu\nu}U^\dagger$. Nevertheless, the corresponding expressions for the case of electrodynamics are reobtained if the gauge group is replaced by an abelian $U(1)$ group, since $f^{abc} = 0$ in this case.

Despite the nontrivial transformation property of the field strength tensor, we can nevertheless use it for the construction of a kinetic term for the gluon field:

$$\begin{aligned}\mathcal{L}_{\text{gluon}} &= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \\ &= -\frac{1}{2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu} \\ &= -\frac{1}{2} \operatorname{tr} U^\dagger U F_{\mu\nu} U^\dagger U F^{\mu\nu} \\ &= -\frac{1}{2} \operatorname{tr} (UF_{\mu\nu}U^\dagger)(UF^{\mu\nu}U^\dagger) \equiv -\frac{1}{2} \operatorname{tr} F'_{\mu\nu} F'^{\mu\nu} \end{aligned} \tag{1.30}$$

In the second to last step, we have used the cyclic property of the trace.

Using the building blocks ψ, $\bar{\psi}$, D_μ, and $F_{\mu\nu}$, we can, in principle, construct many further gauge invariant combinations. However, the ones constructed above represent a minimal set from which we can construct an interacting theory.

Including a mass term m for the quarks (or, more generally, a flavor-dependent mass matrix), we obtain the classical action for chromodynamics, $S_{\text{QCD}} = \int d^d x \, \mathcal{L}_{\text{QCD}}$ with

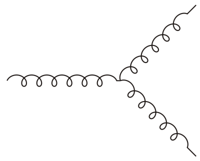
$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \bar{\psi} i \not{D} \psi - m \bar{\psi} \psi. \tag{1.31}$$

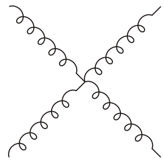
Already the first term alone represents an interacting theory,

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}, \tag{1.32}$$

which is the celebrated Yang-Mills theory, sometimes also called gluodynamics. The quadratic term in the field strength tensor (1.29) leads to self-interactions among the gauge fields, a hallmark of non-abelian gauge theories. Schematically, we find interaction terms of the form

$$\mathcal{L}_{\rm YM}^{\rm int} \quad \sim \quad + \dots \qquad g(\partial_\mu A_\nu)A^\mu A^\nu \qquad + \dots \qquad g^2(A_\mu a_\nu)^2. \tag{1.33}$$





NB: Why do we focus on SU($N_{\rm c}$) and not on U($N_{\rm c}$)?

In principle, we can drop the condition $\det U = 1$ for the symmetry transformations. Then, U can have an additional phase degree of freedom that comes with another generator

$$(\tau^0)_{ij} = \frac{1}{2N_{\rm c}}\delta_{ij}. \tag{1.34}$$

Since this generator comutes with all others, $[\tau^0, \tau^a] = 0$, we have $f^{0ab} = 0$ and thus

$$\begin{aligned} \operatorname{tr} F_{\mu\nu}F^{\mu\nu} &= \frac{1}{2}\sum_{a'=0}^{N_{\rm c}^2-1} F_{\mu\nu}^{a'}F^{a'\mu\nu} \\ &= \frac{1}{2}\sum_{a=1}^{N_{\rm c}^2-1} F_{\mu\nu}^{a}F^{a\mu\nu} + \frac{1}{2}F_{\mu\nu}^{0}F^{0\mu\nu}, \end{aligned} \tag{1.35}$$

where $F^0_{\mu\nu} = \partial_\mu A^0_\nu - \partial_\nu A^0_\mu$ is an abelian field strength tensor. The corresponding gauge field A^0_μ thus decouples from the nonabelian gauge fields A^a_μ with $a = 1, \dots, N_{\rm c}^2 - 1$, and thus is similar to a photon field. The (perturbative) quantization of a U($N_{\rm c}$) gauge theory therefore is nothing but that of a nonabelian Yang-Mills theory with SU($N_{\rm c}$) gauge symmetry and a separate Maxwellian U(1) theory; cf. the group isomorphism $U(1) \times SU(N_{\rm c}) \cong U(N_{\rm c})$. incidentally, in a nonperturbative quantization, there can be additional degrees of freedom associated with the U(1) part of the gauge symmetry depending on the quantization procedure, as we will see in a later section.

1.3 Classical Yang-Mills theory vs. hadron physics

From electrodynamics, we are used to the fact that classical electrodynamics describes the interactions of charged particles with the electromagnetic field very well. Quantum phenomena, where the description in terms of the field in terms of photon quanta becomes important, are typically quantitatively suppressed compared to the classical description.

Let us check in this subsection whether classical Yang-Mills theory can be similarly useful. For this, we take a look at the classical equations of motion for the gluon field as it can be derived from the Euler-Lagrange equations,

$$\frac{\partial \mathcal{L}}{\partial A_\mu^a} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu^a)} = 0. \tag{1.36}$$

In an exercise, we derive the explicit form:

$$D_\mu^{ab} F^{b\mu\nu} \equiv (\partial_\mu \delta^{ab} + g f^{acb} A_\mu^c) F^{b\mu\nu} = j^{a\nu}, \tag{1.37}$$

where $j^{a\nu} = \bar{\psi} g \gamma^\nu \tau^a \psi$. (NB: we obtain the same result also in absence of fermions if we couple the gluon field to a generic source using $\mathcal{L}_{\text{source}} = -j^{a\nu} A_\nu^a$.)

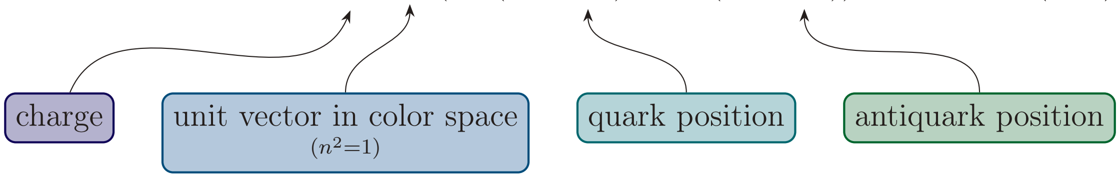
In (1.37), we encounter the covariant derivative in the adjoint representation:

$$\begin{aligned} D_\mu^{ab} &= \partial_\mu \delta^{ab} + g f^{acb} A_\mu^c = (\partial_\mu - \mathrm{i} g T^c A_\mu^c)^{ab} \\ \text{with } (T^c)^{ab} &= -\mathrm{i} f^{cab}, \end{aligned} \tag{1.38}$$

cf. Eq. (1.20).

Let us, for example, consider a static quark-anti-quark pair as a simple model for a meson,

$$j^{a0} = Q \; n^a \, (\delta^{(3)}(\vec{x} - \vec{x}_1) - \delta^{(3)}(\vec{x} - \vec{x}_2)). \tag{1.39}$$



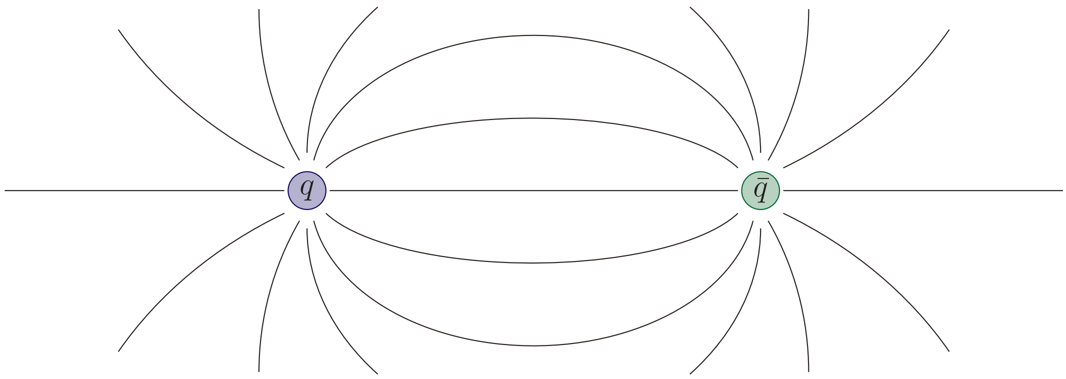
The equation of motion can fully be mapped onto classical electrodynamics, by noting that a pseudo-abelian ansatz

$$A_\mu^a = n^a a_\mu, \qquad F_{\mu\nu}^a = n^a f_{\mu\nu}, \tag{1.40}$$

with $f^{abc} n^b n^c = 0$ leads to

$$\partial_\mu f^{\mu\nu} = j^{a\nu} n^a. \tag{1.41}$$

Hence, the solution is fully equivalent to that of a classical dipole field for the n^a components of the chromoelectric field:



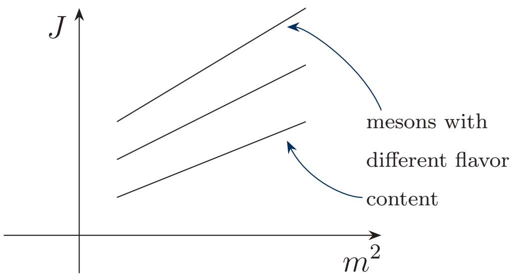
Correspondingly, the static potential corresponds to the Coulomb potential

$$V(r) \sim \frac{1}{r}. \tag{1.42}$$

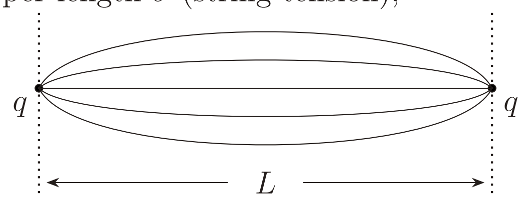
However, this is in contradiction with the experimental observation. For instance, if higher mesonic excitations with higher angular momentum J are studied, one observes that their total (squared) mass is proportional to J:

$$J \sim m^2. \tag{1.43}$$

These lines of proportionality are called *Regge trajectories*.
In contrast to the classical analysis given above, this observation can be described by a string model for the field distribution of a meson. Let us define this model based on two simple assumptions:



- the gluon field of a meson is stringlike with a constant energy per length σ (string tension),



- for higher excitations, the quarks on both ends rotate at almost the speed of light.

Then the energy/mass of the system is ($R = L/2$)

$$m \equiv E = 2\int_0^R \frac{\sigma}{\sqrt{1-v(r)^2}}\mathrm{d}r = 2\int_0^R \frac{\sigma\mathrm{d}r}{\sqrt{1-(r/R)^2}} = \pi\sigma R, \tag{1.44}$$

whereas the angular momentum is

$$J = 2\int_0^R \frac{\sigma r v(r)}{\sqrt{1 - v(r)^2}}\mathrm{d}r = \frac{2}{R}\sigma \int_0^R \frac{r^2 \mathrm{d}r}{\sqrt{1 - (r/R)^2}} = \frac{1}{2}\pi\sigma R^2, \tag{1.45}$$

from which we read off that

$$J = \frac{1}{2\pi\sigma}m^2. \tag{1.46}$$

This is in agreement with the experimental observation. The slope of the Regge trajectories gives

$$\begin{aligned} \alpha' = \frac{1}{2\pi\sigma} \simeq 0.9(\mathrm{GeV})^{-2} \\ \text{or } \sigma \simeq (420\mathrm{MeV})^2. \end{aligned} \tag{1.47}$$

A stringlike color electric field distribution can be associated with a linear potential,

$$V(r) \sim r. \tag{1.48}$$

This line-like field distribution between two quarks and the corresponding impossibility to isolate a single quark is called *confinement*. The comparison with our conclusion from the classical equation of motion shows that classical Chromodynamics is insufficient to describe this basic experimentally verified property of the strong interactions. Therefore: Quantum effects modify the dynamics of QCD qualitatively (not only quantitatively).

2 Quantization of gauge theories

Let us first recall some basic elements of field quantization. In this course, we focus on functional integral quantization techniques as they are most suitable for gauge theories. For a more detailed introduction to quantum field theory (QFT), we refer the reader to standard textbooks or lecture notes.

2.1 Elements of quantum field theory

In quantum field theory (QFT), all physical information is stored in correlation functions. For instance, consider a collider experiment with two incident beams and $(n-2)$ scattering products. All information about this process can be obtained from the *n-point function*, a correlator of n quantum fields. In QFT, we obtain this correlator by definition from the product of n field operators at different spacetime points $\varphi(x_n)$ averaged over all possible field configurations (quantum fluctuations).

In Euclidean QFT, the field configurations are weighted with an exponential of the action $S[\varphi]$,

$$\langle \varphi(x_1) \dots \varphi(x_n) \rangle := \mathcal{N} \int \mathcal{D}\varphi \, \varphi(x_1) \dots \varphi(x_n) \, e^{-S[\varphi]}, \tag{2.1}$$

where we fix the normalization $\mathcal{N}$ by demanding that $\langle 1 \rangle = 1$. (NB: In canonical quantization, Eq. (2.1) is related to the expectation value of the time-ordered product of Heisenberg-picture field operators in the vacuum state.) We assume that Minkowski-valued correlators can be defined from the Euclidean ones by analytic continuation. We also assume that a proper regularized definition of the measure can be given (for instance, using a spacetime lattice discretization), which we formally write as $\int \mathcal{D}\varphi \to \int_\Lambda \mathcal{D}\varphi$; here, Λ denotes an ultraviolet (UV) cutoff. This regularized measure should also preserve the symmetries of the theory: for a symmetry transformation U which acts on the fields, $\varphi \to \varphi^U$, and leaves the action invariant, $S[\varphi] \to S[\varphi^U] \equiv S[\varphi]$, the invariance of the measure implies

$$\int_\Lambda \mathcal{D}\varphi \to \int_\Lambda \mathcal{D}\varphi^U \equiv \int_\Lambda \mathcal{D}\varphi. \tag{2.2}$$

For simplicity, let φ denote a real scalar field. The following discussion also holds for other fields such as fermions with minor modifications; the more elaborate modifications required for the case of a gauge field will be subject to subsequent sections. All n-point correlators are summarized by the generating functional $Z[J]$,

$$Z[J] \equiv e^{W[J]} = \int \mathcal{D}\varphi\, e^{-S[\varphi]+\int J\varphi}, \tag{2.3}$$

with source term $\displaystyle\int J\varphi = \int d^D x\, J(x)\varphi(x)$. All n-point functions are obtained by functional differentiation:

$$\langle \varphi(x_1)\ldots\varphi(x_n)\rangle = \frac{1}{Z[0]}\left(\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right)_{J=0}. \tag{2.4}$$

Once the generating functional is computed, the theory is solved.

In Eq. (2.3), we have also introduced the generating functional of *connected correlators*[1], $W[J] = \ln Z[J]$, which, loosely speaking, is a more efficient way to store the physical information. An even more efficient information storage is obtained by a Legendre transform of $W[J]$: the *effective action* Γ:

$$\Gamma[\phi] = \sup_J \left(\int J\phi - W[J]\right). \tag{2.5}$$

For any given ϕ, a special $J \equiv J_{\text{sup}} = J[\phi]$ is singled out for which $\displaystyle\int J\phi - W[J]$ approaches its supremum. Note that this definition of Γ automatically guarantees that Γ is convex. At $J = J_{\text{sup}}$, we get

$$\begin{aligned} 0 \ &\overset{!}{=}\ \frac{\delta}{\delta J(x)}\left(\int J\phi - W[J]\right) \\ \Rightarrow\ \ \ &\phi = \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J} = \langle\varphi\rangle_J. \end{aligned} \tag{2.6}$$

This implies that ϕ corresponds to the expectation value of φ in the presence of the source J. The meaning of Γ becomes clear by studying its derivative at $J = J_{\text{sup}}$

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -\int_y \frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\phi(x)} + \int_y \frac{\delta J(y)}{\delta\phi(x)}\phi(y) + J(x) \overset{(2.6)}{=} J(x). \tag{2.7}$$

This is the *quantum equation of motion* by which the effective action $\Gamma[\phi]$ governs the dynamics of the field expectation value, taking the effects of all quantum fluctuations into account.

[1] In this short introduction, we use but make no attempt at fully explaining the standard QFT nomenclature; for the latter, we refer the reader to any standard QFT textbook, such as [?, ?].

From the definition of the generating functional, we can straightforwardly derive an equation for the effective action:

$$e^{-\Gamma[\phi]} = \int_\Lambda \mathcal{D}\varphi \, \exp\left(-S[\phi+\varphi] + \int \frac{\delta\Gamma[\phi]}{\delta\phi}\,\varphi\right). \tag{2.8}$$

Here, we have performed a shift of the integration variable, $\varphi \to \varphi + \phi$. We observe that the effective action is determined by a nonlinear first-order functional differential equation, the structure of which is itself a result of a functional integral. An exact determination of $\Gamma[\phi]$ and thus an exact solution has so far been found only for rare, special cases.

As a first example of a functional technique, a solution of Eq. (2.8) can be attempted by a *vertex expansion* of $\Gamma[\phi]$,

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^D x_1 \dots d^D x_n \, \Gamma^{(n)}(x_1,\dots,x_n)\, \phi(x_1)\dots\phi(x_n), \tag{2.9}$$

where the expansion coefficients $\Gamma^{(n)}$ correspond to the *one-particle irreducible (1PI) proper vertices*. Inserting Eq. (2.9) into Eq. (2.8) and comparing the coefficients of the field monomials results in an infinite tower of coupled integro-differential equations for the $\Gamma^{(n)}$: the Dyson-Schwinger equations. This functional method of constructing approximate solutions to the theory via truncated Dyson-Schwinger equations, i.e., via a finite truncation of the series Eq. (2.9) has its own merits and advantages; their application to gauge theories is well developed; see, e.g., [**?**, **?**, **?**, **?**]. We will come back to this and other functional methods in later sections.

2.2 Quantization of gauge fields

The naive attempt to define the quantum field theory of gluodynamics,

$$Z[J] = \int \mathcal{D}A \, e^{-S_{\mathrm{YM}}[A] + \int J^a_\mu A^a_\mu}, \tag{2.10}$$

fails and generically leads to ill-defined quantities plagued by infinities. The reason is that the measure $\mathcal{D}A$ contains a huge redundancy, since many gauge-field configurations A^a_μ are physically equivalent.

In practice, the problems arise from the fact that already the free propagator following from the quadratic part of the action in Eq. (2.10) is ill-defined:

$$\begin{aligned} S_{\mathrm{YM}}[A] &= \frac{1}{4}\int d^D x\, F^a_{\mu\nu} F^a_{\mu\nu} \\ &= \frac{1}{2}\int d^D x\, A^a_\mu \left[-\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu\right] A^a_\nu + O(A^3). \end{aligned} \tag{2.11}$$

The operator in square brackets corresponds to the inverse of the free gluon propagator which we call D_A, schematically,

$$D_A \stackrel{?}{=} [-\partial^2 \mathbb{1} + \partial \otimes \partial]^{-1} \quad \xrightarrow{\text{Fourier space}} \quad [p^2 \mathbb{1} - p \otimes p]^{-1}. \tag{2.12}$$

The right-hand side of Eq. (2.12) does not exist, since the operator $p^2\mathbb{1} - p \otimes p$ has a zero eigenvalue with eigenvector $\sim p_\mu$:

$$(p^2 \delta_{\mu\nu} - p_\mu p_\nu) p_\nu = p^2 p_\mu - p_\mu p^2 = 0. \tag{2.13}$$

Defining the projector

$$(P_\mathrm{L})_{\mu\nu} = \partial_\mu \frac{1}{\partial^2} \partial_\nu, \tag{2.14}$$

the eigenvector with zero eigenvalue corresponds to the *longitudinal* component of the gauge field:

$$A^a_{\mathrm{L},\mu} = (P_\mathrm{L})_{\mu\nu} A^a_\nu \quad \Rightarrow \quad (-\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu) A^a_{\mathrm{L},\mu} = 0. \tag{2.15}$$

Already from the abelian case, we know that the longitudinal component of the gauge field is a pure gauge degree of freedom which can be removed by a gauge transformation, as with $A_\mu \to A_\mu + \partial_\mu \omega$, the gauge function ω can be chosen such that $A^a_{\mathrm{L},\mu} = 0$. Therefore, it is not astonishing that this causes a zero-mode of the inverse propagator also in the nonabelian case.

For a well-defined generating functional, the redundant degrees of freedom need to be removed. Ideally, we would like to remove this redundancy completely by picking one representative gauge field out of each set of gauge-equivalent fields. The latter set is called a gauge orbit:

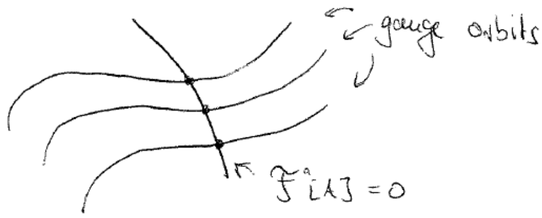
$$[A^\mathrm{orbit}_\mu] = \Big\{ A^\omega_\mu \,|\, A_\mu = A^\mathrm{ref}_\mu, U(\omega) \in SU(N) \Big\}, \tag{2.16}$$

where A^ref_μ is a reference gauge field which is representative for the orbit, and A^ω_μ is the gauge-transformed field,

$$\begin{aligned} A^\omega_\mu &= U(\omega) A^\mathrm{ref}_\mu U(\omega)^\dagger - \frac{i}{g} \left(\partial_\mu U(\omega) \right). \\ &\quad \text{with } U(\omega) = e^{-ig\omega^a \tau^a}, \ \omega^a = \omega^a(x). \end{aligned} \tag{2.17}$$

For the quantum theory, we would like to have a measure $\mathcal{D}A$ which picks exactly one representative gauge-field configuration out of each gauge orbit. This is intended by choosing a *gauge condition* (or gauge-fixing condition),

$$\mathcal{F}[A] = \tau^a \mathcal{F}^a[A] = 0, \tag{2.18}$$



for instance,

$$\mathcal{F}^a[A] = \partial_\mu A^a_\mu, \tag{2.19}$$

which is called the *Lorenz gauge*. Ideally, Eq. (2.18) should be satisfied by exactly one representative gauge field A^a_μ of each orbit. Unfortunately, this is actually impossible for standard smooth gauge-fixing conditions, owing to topological obstructions.

To keep things simple, some essence of this is captured by the following simplified example. Consider the "action"

$$S[r] = -\frac{1}{2}\frac{r^2}{L^2}, \tag{2.20}$$

where $r = \sqrt{x_1^2 + x_2^2}$, and a "gauge invariance" corresponding to rotations about the origin

$$\mathbf{x}' = U(\omega)\mathbf{x} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \omega \in [0, 2\pi) =: I_{2\pi}. \tag{2.21}$$

"Gauge-invariant" observables $\mathcal{O}(r)$ have an expectation value defined by

$$\langle \mathcal{O} \rangle = \int dx_1 dx_2 \, \mathcal{O}(r) e^{-S[r]}. \tag{2.22}$$

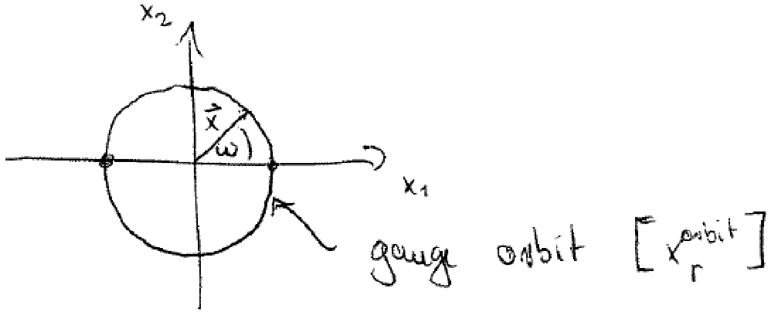
Of course, no problem arises here from the angular redundancy, and we could even decompose the measure into gauge-invariant and gauge-variant degrees of freedom by going to polar coordinates,

$$dx_1 dx_2 = r dr d\omega. \tag{2.23}$$

with $\omega = \arctan(x_2/x_1)$.

Since this is difficult in real gauge theories, let us try to solve this problem by "gauge fixing" fully formulated in terms of x_1 and x_2; e.g.,

$$0 = \mathcal{F}(\mathbf{x}) = x_2(\omega). \tag{2.24}$$



Owing to the topology of the gauge orbit $\sim S^1$, the gauge condition is satisfied by two points on the orbit.

Now consider the Faddeev-Popov determinant:

$$\Delta_{\rm FP}^{-1}[\mathbf{x}] = \int_{I_{2\pi}} d\omega\, \delta\big[\mathcal{F}[\mathbf{x}^\omega]\big], \tag{2.25}$$

which is gauge invariant:

$$\begin{aligned} \Delta_{\rm FP}^{-1}[\mathbf{x}^{\bar\omega}] &= \int_{I_{2\pi}} d\omega\, \delta\big[\mathcal{F}[\mathbf{x}^{\bar\omega+\omega}]\big] \\ &= \int_{I_{2\pi}} d(\bar\omega+\omega)\, \delta\big[\mathcal{F}[\mathbf{x}^{\bar\omega+\omega}]\big] \\ &= \int_{I_{2\pi}} d\omega\, \delta\big[\mathcal{F}[\mathbf{x}^{\omega}]\big] = \Delta_{\rm FP}^{-1}[\mathbf{x}]. \end{aligned} \tag{2.26}$$

The Faddeev-Popov trick consists in inserting the following identity into the integral:

$$\begin{aligned} \langle\mathcal{O}\rangle &= \int dx_1 dx_2\, \mathcal{O}(r) e^{-S[r]}\cdot 1 \\ &= \int dx_1 dx_2\, \mathcal{O}(r) e^{-S[r]} \Delta_{\rm FP}[\mathbf{x}] \int_{I_{2\pi}} d\omega\, \delta\big[\mathcal{F}[\mathbf{x}^\omega]\big]. \\ &= \int_{I_{2\pi}} d\omega \int dx_1 dx_2\, \mathcal{O}(r) e^{-S[r]} \Delta_{\rm FP}[\mathbf{x}] \delta\big[\mathcal{F}[\mathbf{x}^\omega]\big]. \end{aligned} \tag{2.27}$$

Now we use the gauge invariance of the measure in the plane, of the action, and of the Faddeev-Popov determinant to write

$$\begin{aligned} \langle\mathcal{O}\rangle &= \int_{I_{2\pi}} d\omega \int dx_1^\omega dx_2^\omega\, \mathcal{O}(r) e^{-S[r^\omega]} \Delta_{\rm FP}[\mathbf{x}^\omega] \delta\big[\mathcal{F}[\mathbf{x}^\omega]\big] \\ &= \left(\int_{I_{2\pi}} d\omega\right) \int dx_1 dx_2\, \mathcal{O}(r) e^{-S[r]} \Delta_{\rm FP}[\mathbf{x}] \delta\big[\mathcal{F}[\mathbf{x}]\big]. \end{aligned} \tag{2.28}$$

In the last line, the integral over the gauge orbit $\sim S^1$ factorizes and can be absorbed into the normalization; the remaining integral is a gauge-fixed integral.

Let us now compute the Faddeev-Popov determinant explicitly for our particular gauge fixing (2.24):

$$\begin{aligned}\Delta_{\mathrm{FP}}^{-1}[\mathbf{x}] &= \int_{I_{2\pi}} d\omega\, \delta\big[\mathcal{F}[\mathbf{x}^\omega]\big] \\ &= \int_{I_{2\pi}} d\omega\, \frac{1}{\left|\frac{\delta \mathcal{F}[\mathbf{x}^\omega]}{\delta\omega}\right|} \sum_i \delta(\omega-\omega_i). \end{aligned} \tag{2.29}$$

Here, we have used a functional notation in order to make contact with the field-theoretic setting later; but, of course, in the present problem, all derivatives are ordinary ones and the determinant notation, representing the Jacobian in the multi-/infinite-dimensional case, is redundant here. The ω_i are the solutions of the gauge condition

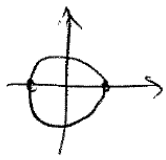
$$0 = \mathcal{F}[\mathbf{x}^\omega] \tag{2.30}$$

We get explicitly

$$\begin{aligned}\frac{\delta \mathcal{F}[\mathbf{x}^\omega]}{\delta\omega} &= \frac{d}{d\omega}\,(x_2^\omega) = \frac{d}{d\omega}\,(x_1 \sin\omega + x_2\cos\omega) \\ &= x_1\cos\omega - x_2\sin\omega. \\ &= \cos\omega(x_1 - x_2\tan\omega) \\ &= \cos\omega\left(x_1 - \frac{x_2^2}{x_1}\right) \end{aligned} \tag{2.31}$$

The corresponding solutions of the gauge condition are

$$\omega_1 = 0, \quad \omega_2 = \pi, \quad (x_2 = 0). \tag{2.32}$$



For the Faddeev-Popov determinant, we find

$$\begin{aligned}\Delta_{\mathrm{FP}}^{-1}[\mathbf{x}] &= \int_{I_{2\pi}} d\omega\, \frac{1}{|x_1\cos\omega|}\Big(\delta(\omega) + \delta(\omega-\pi)\Big) \\ &= \frac{2}{|x_1|}, \\ \Rightarrow \quad \Delta_{\mathrm{FP}}[\mathbf{x}] &= \frac{|x_1|}{2}. \end{aligned} \tag{2.33}$$

In the full system, we get contributions from both gauge copies at $\omega = 0$ and $\omega = \pi$. Imagine, we would only be interested in "perturbation theory" near $\omega = 0$; then $\Delta_{\mathrm{FP}}[\mathbf{x}]$ would reduce to

$$\Delta_{\mathrm{FP}}^{-1}[\mathbf{x}]|_{\mathrm{pert}} = \int_{-\epsilon}^{\epsilon} d\omega\, \delta\Big(x_2(\omega)\Big) = \frac{1}{x_1}. \tag{2.34}$$

Since $x_1 > 0$ near $\omega = 0$, we can drop the absolute-value prescription. In this case, we can represent the Faddeev-Popov determinant by a Grassmann integral,

$$\Delta_{\mathrm{FP}}[\mathbf{x}]|_{\mathrm{pert}} = x_1 \hat{=} \det \frac{\delta \mathcal{F}}{\delta \omega}\bigg|_{\omega=0} = \int d\bar{c} dc\, e^{-\bar{c} \frac{\delta \mathcal{F}}{\delta \omega}|_{\omega=0} c}, \tag{2.35}$$

such that $\Delta_{\mathrm{FP}}[\mathbf{x}]$ can be written in terms of a QFT contribution with a *local* action

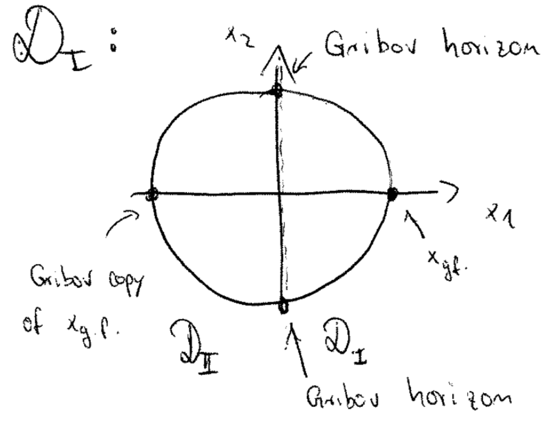
$$S_{\mathrm{gh}} = \bar{c} \frac{\delta \mathcal{F}}{\delta \omega} c, \tag{2.36}$$

the so-called *ghost action*. We emphasize that this construction does not hold in the same way beyond perturbation theory:

$$\Delta_{\mathrm{FP}}^{-1}[\mathbf{x}] \overset{?}{=} \int_{I_{2\pi}} d\omega \frac{1}{\det \frac{\delta \mathcal{F}}{\delta \omega}|_{\omega_i}} \sum_i \delta(\omega - \omega_i) = \frac{1}{x_1} + \frac{1}{-x_1} = 0. \tag{2.37}$$

We observe that dropping the absolute-value prescription in Eq. (2.33) has dramatic consequences. This would correspond to an insertion of ∞ into the integral instead of an identity in Eq. (2.27).

In order to maintain the local ghost-action form, but still arrive at a nonperturbatively valid definition of the "theory", we can confine the integral to the "first Gribov region" $\mathcal{D}_I$, The first Gribov region includes the perturbative origin



$x_1 > 0, x_2 = 0$ and is bounded by the Gribov horizon which is defined by those configuration for which $\det \frac{\delta \mathcal{F}}{\delta \omega}$ vanishes. Restricting the integral to this region allows us to drop the absolute-value prescription for the definition of gauge invariant expectation values:

$$\langle \mathcal{O} \rangle = \int_{\mathcal{D}_I} dx_1 dx_2 \, \mathcal{O}(r) e^{-S[r]} \det \frac{\delta \mathcal{F}}{\delta \omega} \delta\big[\mathcal{F}[\mathbf{x}]\big]. \tag{2.38}$$

This is a nonperturbative definition which – except for a normalization – corresponds to the full original integral.

Now, let's go back to the field-theoretic case. In fact, the same type of reasoning applies to the Faddeev-Popov quantization of gauge theories. As we have already used a somewhat redundant notation for the toy model, most of the formulas given above can literally be taken over to the gauge-theory case. For the gauge-fixed generating functional, we get

$$Z[J] = \int_{\mathcal{D}_I} \mathcal{D}A \, \frac{1}{n[A]} \, \det \frac{\delta \mathcal{F}^a}{\delta \omega^b} \delta\big[\mathcal{F}[A]\big] e^{-S_{\mathrm{YM}}[A] + \int J^a_\mu A^a_\mu}, \tag{2.39}$$

where $n[A]$ takes care of the fact that there might be $n[A]$ gauge-equivalent configurations to a reference field A_μ even within the first Gribov region. Therefore, $n[A]$ counts the number of Gribov copies within the first Gribov region $\mathcal{D}_I$ for a given gauge field A_μ. Equation (2.39) represents a generating functional for gauge theories with a well-defined gauge-fixing procedure.

For a quantitative treatment of Eq. (2.39), it is useful to write as many terms as possible in the form of local contributions to the action. For instance, we can use a Gaussian representation of the delta functional,

$$\delta\big[\mathcal{F}[A]\big] \sim e^{-\frac{1}{2\alpha} \int (\mathcal{F}[A])^2} \Big|_{\alpha \to 0}, \tag{2.40}$$

which yields another contribution to the action,

$$\Rightarrow \quad e^{-S_{\mathrm{gf}}}, \quad S_{\mathrm{gf}} = \frac{1}{2\alpha} \int d^D x \, (\mathcal{F}^a[A])^2. \tag{2.41}$$

As discussed above, the Faddeev-Popov determinant can be exponentiated by means of Grassmann-valued (anti-commuting) fields $c^a, \bar{c}^a$ called *ghost fields*:

$$\det \frac{\delta \mathcal{F}^a}{\delta \omega^b} = \int \mathcal{D}\bar{c}\mathcal{D}c \, e^{-S_{\mathrm{gh}}}, \quad S_{\mathrm{gh}} = \int d^D x \, \bar{c}^a \frac{\delta \mathcal{F}^a}{\delta \omega^b} c^b. \tag{2.42}$$

These ghost fields live in the adjoint representation of the gauge group and transform homogeneously under gauge transformations, e.g., $c' = UcU^\dagger$, where $c = c^a \tau^a$.

Unfortunately, no local description for $n[A]$ is known. In practice, one usually ignores this factor and hopes that it does not affect the quantities of interest. This is indeed the case in perturbation theory, where $n[A] = 1$ for all gauge fields near the perturbative vacuum $A_\mu = 0$. Beyond perturbation theory, however, this is an approximation.

Let us study a concrete example. We start with the Lorenz gauge,

$$\mathcal{F}^a[A] = \partial_\mu A^a_\mu. \tag{2.43}$$

The corresponding gauge-fixing action reads

$$\begin{aligned}
\Rightarrow \qquad & \delta\big[\mathcal{F}[A]\big] \to e^{-S_{\mathrm{gf}}}\Big|_{\alpha\to 0}, \\
S_{\mathrm{gf}} &= \frac{1}{2\alpha}\int d^Dx\,(\partial_\mu A^a_\mu)^2 = -\frac{1}{2\alpha}\int d^Dx\, A^a_\mu \partial_\mu\partial_\nu A^a_\nu \\
&= \frac{1}{2\alpha}\int A^a_\mu (P_{\mathrm{L}})_{\mu\nu} A^a_\nu \\
&= \frac{1}{2\alpha}\int A^a_{\mathrm{L}\mu}(-\partial^2) A^a_{\mathrm{L}\mu},
\end{aligned} \tag{2.44}$$

where the limit $\alpha \to 0$ is implicitly understood. This limit corresponds to the Lorenz-Landau gauge or simply Landau gauge. (NB: In fact, it turns out that perturbation theory is independent of α; thus, one can also work at finite α. Frequently used choices are $\alpha = 1$ (Feynman gauge) or $\alpha = -3$ (Yennie gauge).)

The gauge-fixing action involves the longitudinal projector P_{L} defined in Eq. (2.14) and gives a contribution only to the longitudinal part of the gauge field. In the Landau-gauge limit, $\alpha \to 0$, all contributions from the $A^a_{\mathrm{L},\mu}$ components are suppressed in the functional integral and decouple from physical amplitudes.

For the Faddeev-Popov operator, it is useful to study *infinitesimal* gauge transformations first:

$$U(\omega) = e^{-ig\omega^a\tau^a} \approx 1 - ig\omega^a\tau^a + O(\omega^2). \tag{2.45}$$

From this, we can work out the infinitesimal gauge transformation of the gauge field:

$$\begin{aligned}
A^\omega_\mu &= U A_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger \\
&= (1 - ig\omega)A_\mu(1 + ig\omega) - (\partial_\mu\omega) + O(\omega^2) \\
&= A_\mu - ig\underbrace{[A_\mu,\omega]}_{if^{abc}\omega^a A^b_\mu \tau^c} - \partial_\mu\omega + O(\omega^2) \\
&= A^a_\mu\tau^a + g f^{abc}\omega^a A^b_\mu \tau^c - \partial_\mu\omega^a\tau^a + O(\omega^2).
\end{aligned} \tag{2.46}$$

Writing $A^\omega_\mu = A_\mu + \delta A_\mu + O(\omega^2)$, we can read off the infinitesimal variation of the gauge field:

$$\delta A^a_\mu = -\partial_\mu \omega^a + g f^{abc} \omega^b A^c_\mu. \tag{2.47}$$

In terms of the adjoint covariant derivative (1.38), this can be written as

$$\delta A^a_\mu = -D^{ab}_\mu [A] \omega^b. \tag{2.48}$$

From here, we can immediately compute the Faddeev-Popov operator

$$\frac{\delta \mathcal{F}^a}{\delta \omega^b} = \frac{\delta(\partial_\mu A^{\omega,a}_\mu)}{\delta \omega^b} = \partial_\mu \frac{\delta A^{\omega,a}_\mu}{\delta \omega^b} = -\partial_\mu D^{ab}_\mu [A]. \tag{2.49}$$

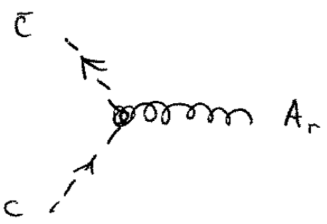
In the Landau gauge (and actually in any Lorenz gauge independently of the value of α), the ghost action thus reads

$$S_{\text{gh}} = \int d^D x\, \bar{c}^a (-\partial_\mu D^{ab}_\mu [A]) c^b = \int d^D x\, \partial_\mu \bar{c}^a D^{ab}_\mu [A] c^b. \tag{2.50}$$

Separating this into a free and an interaction part, we get

$$S_{\text{gh}} = \int d^D x\, \left(\partial_\mu \bar{c}^a \partial_\mu c^a + g f^{abc} \partial_\mu \bar{c}^a A^b_\mu c^c \right). \tag{2.51}$$

The last term obviously corresponds to a ghost-gluon interaction. In abelian gauge



theories, the structure constants f^{abc} vanish, and there is no ghost-gluon interaction. In this case, the ghosts completely decouple and can be ignored. In nonabelian gauge theories, however, the ghosts are an essential ingredient of the theory.

Expanding the action about $A = 0$ in perturbation theory, the Faddeev-Popov operator reduces to the negative Laplacian $-\partial^2$ which is a positive operator (e.g., on L_2). (NB: it has only trivial zero modes corresponding to constant functions which play no role.) Hence, the Gribov ambiguity is irrelevant in perturbation theory.

2.3 Background gauge and perturbation theory

In practice, it is difficult to make use out of the defining generating functional (2.39) in particular because of the lack of an explicit representation of $n[A]$ and the restriction to the Gribov region. In order to get acquainted with the gauge-fixed quantization, let us first confine ourselves to perturbation theory around the trivial vacuum $A_\mu = 0$ where the problems related to the Gribov ambiguity are absent; we will come back to these in the later parts of this course.

Let us first analyze perturbation theory to lowest (nontrivial) order, starting from Eq. (2.8), i.e., ignoring the complications from gauge fixing for a second,

$$e^{-\Gamma[\phi]} = \int_\Lambda \mathcal{D}\varphi\, e^{-S[\phi+\varphi]+\int \frac{\delta\Gamma[\phi]}{\delta\phi}\varphi}. \tag{2.52}$$

Perturbation theory corresponds to a steepest-descent/saddle-point approximation of the integral, for which we need

$$\begin{aligned} S[\phi+\varphi] - \int \frac{\delta\Gamma[\phi]}{\delta\phi}\varphi &= S[\phi] + \int \left(\frac{\delta S[\phi]}{\delta\phi} - \frac{\delta\Gamma[\phi]}{\delta\phi}\right)\varphi \\ &\quad + \frac{1}{2}\int\int \varphi \underbrace{\frac{\delta^2 S[\phi]}{\delta\phi\delta\phi}}_{=:S^{(2)}[\phi]} \varphi + O(\varphi^3) \end{aligned} \tag{2.53}$$

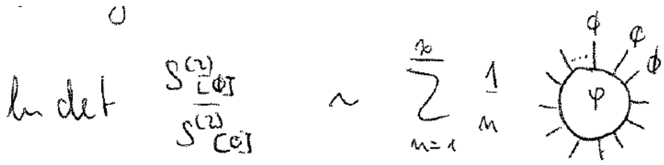
At the saddle point $\phi = \phi_{\rm sp}$, the linear term in φ vanishes. Truncating the series at quadratic order leaves us with a Gaussian integral:

$$\begin{aligned} e^{-\Gamma[\phi]} &\simeq e^{-S[\phi]} \int_\Lambda \mathcal{D}\varphi\, e^{-\frac{1}{2}\int\int \varphi S^{(2)}[\phi]\varphi} \\ &= e^{-S[\phi]}\, \mathcal{N}\, \det{}_\Lambda^{-\frac{1}{2}} S^{(2)}[\phi]. \end{aligned} \tag{2.54}$$

The normalization of the correlator $\langle 1\rangle = 1$ implies $\Gamma[0] = 0$ and thus fixes the constant $\mathcal{N} = \left(\det{}_\Lambda^{\frac{1}{2}} S^{(2)}[0]\right)^{-1}$. In conlusion, we have

$$\Gamma[\phi] = S[\phi] + \frac{1}{2}\ln\det\frac{S^{(2)}[\phi]}{S^{(2)}[0]} + \dots \tag{2.55}$$

As can be seen upon expansion of the ln det in powers of ϕ, the ln det term corresponds to a sum of all possible one-loop diagrams with arbitrary number of external legs, The ellipsis in Eq. (**??**) denotes higher-loop contributions which we will neglect in the following. Since the loop expansion corresponds to a coupling expansion, we expect this expansion to hold at weak coupling.



Now we could try to do a saddle-point approximation of the full gauge-fixed generating functional (2.39) along the same lines. However, here we encounter a conceptual problem: On the one hand, we expect that a properly quantized gauge theory results in a *gauge-invariant* effective action $\Gamma[A] \equiv \Gamma[A^\omega]$. On the other hand, gauge fixing is necessary for integrating over the fluctuations.

This seeming paradox can be resolved with the aid of the background-field gauge (or background gauge for short). In this gauge, we decompose the gauge field A into a background field $\bar{A}$ and a fluctuation field Q,

$$A_\mu = \bar{A}_\mu + Q_\mu, \tag{2.56}$$

From a "quantum-field viewpoint", the background field $\bar{A}$ is just an external parameter; gauge symmetry on the quantum level is carried by the fluctuation field Q. This is expressed by the quantum transformation (QT):

$$\begin{aligned} \bar{A}'_\mu &= \bar{A}_\mu, \\ Q'_\mu &= U(\bar{A}_\mu + Q_\mu)U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger - \bar{A}_\mu. \end{aligned} \tag{2.57}$$

with $(\bar{A}_\mu + Q_\mu)$ transforming in total as usual. This is the symmetry which we have to gauge fix for being able to do the functional integral. For this, we choose the gauge condition

$$\bar{\mathcal{F}}^a[\bar{A}, Q] = D_\mu^{ab}[\bar{A}]Q_\mu^b \equiv \bar{D}_\mu^{ab} Q_\mu^b = 0. \tag{2.58}$$

The resulting gauge-fixed action reads

$$S_{\rm gf}[\bar{A}, Q] = \frac{1}{2\alpha} \int d^4x \, (\bar{D}_\mu^{ab} Q_\mu^b)^2, \tag{2.59}$$

where we have again introduced a gauge parameter α. The important observation now is that $S_{\rm gf}$ – even though not being gauge invariant under QT – is invariant under an additional symmetry, the background gauge transformation (BT):

$$\begin{aligned} \bar{A}'_\mu &= U\bar{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger, \\ Q'_\mu &= UQ_\mu U^\dagger. \end{aligned} \tag{2.60}$$

Again, $(\bar{A}_\mu + Q_\mu)$ transforms as usual. The invariance of Eq. (**??**) is obvious, since $\bar{D}_\mu$ as well as Q_μ transform homogeneously under BT.

The integration measure is also invariant under BT, $\mathcal{D}A \to \mathcal{D}Q$, since it correspond to a shift in field space. For the Faddeev-Popov term, we need the infinitesimal version of QT, using $U \simeq 1 + ig\omega^a T^a$,

$$\begin{aligned} Q'^{\,a}_\mu &= Q^a_\mu + gf^{abc}\omega^b(\bar{A}^c_\mu + Q^c_\mu) - \partial_\mu\omega^a \\ &= Q^a_\mu - \bar{D}^{ab}_\mu\omega^b. \end{aligned}$$

Thus, the Faddeev-Popov operator reads

$$\frac{\delta\bar{\mathcal{F}}^a[\bar{A},Q]}{\delta\omega^b} = -D^{ac}_\mu[\bar{A}]\bar{D}^{cb}_\mu[\bar{A}+Q] \quad \Rightarrow \quad \Delta_{\mathrm{FP}}[\bar{A},Q] = \det\left(-D^{ac}_\mu[\bar{A}]D^{cb}_\mu[\bar{A}+Q]\right). \tag{2.61}$$

Aiming at the quantum correlations, the source term in the generating functional is coupled to Q only, $e^{\int jA} \to e^{\int jQ}$. We end up with the generating functional in background gauge,

$$\bar{Z}[j,\bar{A}] = \int \mathcal{D}\,Q\,\Delta_{\mathrm{FP}}[\bar{A},Q]\,e^{-S_{\mathrm{YM}}[\bar{A}+Q]-S_{\mathrm{gf}}[\bar{A},Q]+\int jQ}, \tag{2.62}$$

where we have ignored issues related to the Gribov problem for the time being. Note that $\bar{Z}[j,\bar{A}]$ is invariant under BT provided that the source j transforms homogeneously, $j' = UjU^\dagger$. Even a ghost-field representation of the Faddeev-Popov determinant Δ_{FP} is invariant under BT since it is reasonable to demand that the ghost fields transform homogeneously as well.

Manifest BT invariance holds also for the effective action

$$\bar{\Gamma}[\bar{A},Q] = \sup_j\left[-\ln\bar{Z}[j,\bar{A}] + \int jQ\right]. \tag{2.63}$$

The crucial question now is how $\bar{\Gamma}[\bar{A},\bar{Q}]$ is related to the original effective action $\Gamma[A]$ which we are actually interested in.

To answer this question, let us shift the integration variable $Q \to Q - \bar{A}$. Then Eq. (**??**) becomes

$$\bar{Z}[j,\bar{A}] = Z[j]e^{-\int j\bar{A}}, \tag{2.64}$$

where $Z[j]$ is the standard generating functional with an unusual gauge condition,

$$\mathcal{F}[Q] \equiv \bar{\mathcal{F}}[\bar{A}, Q-\bar{A}]. \tag{2.65}$$

Correspondingly, the Legendre transform yields:

$$\bar{\Gamma}[\bar{A},Q] = \sup_j\left[-\ln Z[j] + \int j(Q+\bar{A})\right] = \Gamma[Q+\bar{A}]. \tag{2.66}$$

Here, we rediscover the standard effective action with an unusual argument. From the BT invariance of $\bar{\Gamma}[\bar{A}, Q]$, we thus conclude that for $A = \bar{A}$, i.e., $Q = 0$,

$$\Gamma[\bar{A}] = \bar{\Gamma}[\bar{A} = A, Q = 0] \tag{2.67}$$

is manifestly invariant under

$$A_\mu \to A'_\mu = U A_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger. \tag{2.68}$$

In conclusion, $\Gamma[A]$ in the background gauge inherits the manifest gauge invariance from the background invariance. Furthermore, diagrams with $Q =$, i.e., vacuum diagrams with Q fluctuations only as internal lines and $\bar{A}$ only as external legs are sufficient to analyze the structure of the theory.

2.4 Perturbative effective action

As a first explicit step towards quantum Yang-Mills theory, let us compute the perturbative effective action for gauge theories, using the background field method. For a general field theory, we found in Eq. (2.55)

$$\Gamma[\phi] = S[\phi] + \frac{1}{2} \ln \det \frac{S^{(2)}[\phi]}{S^{(2)}[0]} + \dots \tag{2.69}$$

for the one-loop approximation of $\Gamma[\phi]$. Repeating the argument for gauge theories, we obtain in the background gauge (for the expectation value of the fluctuation field $Q = 0$):

$$\begin{aligned} e^{-\Gamma[\bar{A}]} &= \int \mathcal{D}Q\, e^{-S[\bar{A}+Q]-S_{\mathrm{gf}}[\bar{A}+Q]} \Delta_{\mathrm{FP}}[\bar{A}, Q] \\ &= e^{-S[\bar{A}]} \int \mathcal{D}Q\, e^{-\frac{1}{2}\int Q^a_\mu \left(S+S_{\mathrm{gf}}\right)^{(2)\,ab}_{\mu\nu}[\bar{A}] Q^b_\nu + O(Q^3)} \\ &\quad \times \underbrace{\Delta_{\mathrm{FP}}[\bar{A}, 0]}_{\text{already one ghost loop}} + (\text{higher loops}) \\ &= e^{-S[\bar{A}]} \mathcal{N} \det^{-\frac{1}{2}} M[\bar{A}]\, \Delta_{\mathrm{FP}}[\bar{A}] + \dots \end{aligned} \tag{2.70}$$

where

$$M^{ab}_{\mu\nu}[\bar{A}] = \frac{\delta^2 (S + S_{\mathrm{gf}})[\bar{A}]}{\delta Q^a_\mu \delta Q^b_\nu} = -\bar{D}^{ac}_\alpha \bar{D}^{cb}_\alpha \delta_{\mu\nu} + 2g f^{abc} F^c_{\mu\nu}[\bar{A}] + \left(1 - \frac{1}{\alpha}\right) \bar{D}^{ac}_\mu \bar{D}^{cb}_\nu. \tag{2.71}$$

Hence, the effective action at one-loop order reads

$$\Gamma[\bar{A}] = S[\bar{A}] + \frac{1}{2} \ln \det M[\bar{A}] - \ln \det(-\bar{D}^2) + \text{normalization} + \text{higher loops}, \quad (2.72)$$

where we used Eq. (2.61) for the Faddeev-Popov determinant Δ_{FP} in the background gauge. (NB: If we include quarks, we will get another determinant from the fermionic path integral:

$$-\ln \det(-i\bar{\not{D}} + m). \quad (2.73)$$

where the global minus sign arises from the fermionic statistics of the quarks.)

The evaluation of these functional determinants for arbitrary $\bar{A}$ is difficult. The resulting action will contain nonlinearities and nonlocalities. In the following, we will be satisfied with the exploration of the nonlinearities; the nonlocalities will, for instance, be small for slowly varying fields with

$$\frac{|\partial_\mu F^a_{\nu\rho}|}{|F^a_{\nu\rho}|^{3/2}} \ll 1 (\text{in a smooth gauge}). \quad (2.74)$$

The following further assumptions simplify the calculation; we consider

- a pseudo-abelian field:

$$F_{\mu\nu} = \hat{n}^a F^a_{\mu\nu}, \quad \hat{n}^a \text{ constant in color space}, \quad \hat{n}^a \hat{n}^a = 1, \quad (2.75)$$

- a constant pure magnetic field

$$F_{12} = -F_{21} = B = \text{const.}, \quad F_{\mu\nu} = 0 \quad \text{otherwise}. \quad (2.76)$$

- Feynman gauge $\alpha = 1$.

  (NB: for covariantly constant fields, $D_\alpha F_{\mu\nu} = 0$, the effective action $\Gamma[A]$ is actually independent of α; beyond the Landau gauge $\alpha = 0$, the gauge-fixing condition $\delta[\mathcal{F}^a]$ is not implemented exactly, but in a smeared-out fashion.)

  Here, the gluonic fluctuation operator simplifies to

$$M^{ab}_{\mu\nu}[\bar{A}]\big|_{\alpha=1} = -\bar{D}^{2ab} \delta_{\mu\nu} \delta^{ab} + 2g f^{abc} \bar{F}^c_{\mu\nu}, \quad (2.77)$$

  the first term being the covariant Laplacian, and the second term $2g f^{abc} \bar{F}^c_{\mu\nu} = 2ig (T^c)^{ab} \bar{F}^c_{\mu\nu}$ describes the interaction of the gluon spin with the background field (cf. $\sim \boldsymbol{\mu} \cdot \mathbf{B}$).

Let us perform the calculation within a few steps:

(1) Diagonalization in color space
All color-space dependence comes in the form of

$$(T^c)^{ab}\hat{n}^c = -if^{abc}\hat{n}^c \tag{2.78}$$

which is hermitian and can be diagonalized with eigenvalues

$$\nu_\ell, \quad \ell = 1, \ldots, N_c^2 - 1. \tag{2.79}$$

being real and corresponding to "charges" of the gluons with respect to the color axis $\hat{n}^a$. For SU(2) and SU(3), we have, e.g.,

$$\begin{aligned} \text{SU(2)}: &\quad \nu_\ell = -1, 0, 1 \\ \text{SU(3)}: &\quad \nu_\ell = -1, -\frac{1}{2}, -\frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 1 \quad \text{for } \hat{n}^a = \delta^{a3}. \end{aligned} \tag{2.80}$$

In the following, we will only need the identity

$$\sum_{\ell=1}^{N_c^2-1} \nu_\ell^2 = \operatorname{tr} \hat{n}^c T^c \hat{n}^d T^d = N_c. \tag{2.81}$$

We find ($\bar{A} = A$)

$$\begin{aligned} \Gamma^1[A] &= \frac{1}{2} \ln \det M[A] - \ln \det(-D^2) \\ &= \frac{1}{2} \operatorname{Tr} \ln M[A] - \operatorname{Tr} \ln(-D^2) \\ &= \sum_{\ell=1}^{N_c^2-1} \left\{ \operatorname{Tr} \ln \left[-\hat{D}^2 \delta_{\mu\nu} + 2ig\nu_\ell F_{\mu\nu} \right] - \operatorname{Tr} \ln \left[-\hat{D}^2 \right] \right\} \end{aligned} \tag{2.82}$$

where we have defined

$$\hat{D}_\mu = \partial_\mu - ig\nu_\ell A_\mu. \tag{2.83}$$

(2) Spectrum of the covariant Laplacian
Finding the spectrum of $-\hat{D}^2$ is indeed identical to the quantum mechanical particle with a unit mass in a constant magnetic field B with Hamiltonian $-\hat{D}^2 = 2H$. The solution comes in the form of Landau levels,

$$\text{Spect.}\big\{ -\hat{D}^2 \big\} = p_0^2 + p_z^2 + g|\nu_\ell| B\,(2n+1)\,, \quad n = 0, 1, 2, \ldots \tag{2.84}$$

(3) Spectrum of the gluon spin-field coupling
Since

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.85}$$

we find

$$\text{Spect.}\big\{2igF_{\mu\nu}T^c\hat{n}^c\big\} = \begin{cases} 0 & \text{multiplicity 2} \\ +2g|\nu_\ell|B & \\ -2g|\nu_\ell|B & \end{cases} \tag{2.86}$$

where the first line corresponds to the longitudinal and time-like gluon polarizations, while the second and third line correspond to the two transverse polarizations with spin aligned or anti-aligned with respect to the magnetic field.

From Eq. (2.82), we thus obtain

$$\begin{aligned} \Gamma^1[A] &= \sum_{\ell=1}^{N_c^2-1} \Bigg\{ 2\frac{1}{2}\,\mathrm{Tr}\ln\Big[-\hat{D}^2\Big] + \frac{1}{2}\,\mathrm{Tr}\ln\Big[-\hat{D}^2 + 2g|\nu_\ell|B\Big] \\ &\qquad + \frac{1}{2}\,\mathrm{Tr}\ln\Big[-\hat{D}^2 - 2g|\nu_\ell|B\Big] - \mathrm{Tr}\ln\Big[-\hat{D}^2\Big]\Bigg\} \end{aligned} \tag{2.87}$$

The ghosts thus cancel the contributions from the longitudinal and time-like gluons! This removes the overabundant gauge degrees of freedom.

(4) Transversal gluon modes

$$\text{Spect.}\big\{-\hat{D}^2 \pm 2g|\nu_\ell|B\big\} = \begin{cases} p_0^2 + p_z^2 + g|\nu_\ell|B\,(2n+3) \\ p_0^2 + p_z^2 + g|\nu_\ell|B\,(2n-1) \end{cases}, \quad n = 0,1,2,\ldots \tag{2.88}$$

We observe that the spectrum contains negative modes for $(p_0^2 + p_z^2) < g|\nu_\ell|B$ and $n = 0$ in the second line. These "tachyonic" fluctuations are also called the Nielsen-Olesen unstable mode. Its existence implies that fluctuations with long wavelengths, i.e., $p_0^2+p_z^2$ small, do not cost any action if the gluon spin has a suitable orientation with respect to the magnetic field. The covariant constant magnetic field is thus not a minimum, but a saddle point of the action: We conclude that this covariant constant magnetic field vacuum, also called the Savvidy vacuum, is unstable.

Still, we can view F =const. as a technical assumption to do the calculation. F =const. may still be a reasonable approximation for slowly varying fields.

(5) Trace computation
Using that the trace involves an integral over momentum space as well as a sum over Landau levels,

$$\mathrm{Tr} \to \frac{g|\nu_\ell|BL^2}{2\pi}\sum_{n=0}^{\infty}\int \frac{d^2p}{(2\pi/L)^2}, \tag{2.89}$$

where the prefactor corresponds to the density of states per Landau level, and L is the spatiotemporal box length with $L^4 = \Omega$ being the spacetime volume, the one-loop effective action becomes

$$\Gamma^1[A] = \sum_{\ell=1}^{N_c^2-1} \frac{g|\nu_\ell|BL^2}{2\pi} \sum_{n=0}^{\infty} \int \frac{d^2p}{(2\pi/L)^2} \sum_{\lambda=3,-1} \ln(p_0^2 + p_z^2 + g|\nu_\ell|B(2n+\lambda)). \quad (2.90)$$

Equation (2.90) contains divergences of different types:

- For small momenta, the integral runs into the unstable mode. This can be dealt with by analytic continuation of the log, giving rise to an imaginary part of Γ^1, which can be interpreted as the decay rate of the unstable vacuum. For the present purpose, this part is less relevant.
- For large momenta, the integral diverges logarithmically, since it receives contributions from fluctuations on all scales; these divergences will turn out to be indicative for the running of the coupling.

Let us deal with both types of divergences simultaneously with the aid of a ζ function/propertime regularization. For this, we write the log as

$$\ln x = \lim_{\epsilon\to 0} \left(\frac{1}{\epsilon} - \frac{i^\epsilon \mu^{2\epsilon}}{\epsilon\Gamma(\epsilon)} \lim_{\delta\to 0} \int_0^\infty dT\, T^{\epsilon-1} e^{-ixT} e^{-\delta T} \right), \quad (2.91)$$

for $x \in \mathbb{R}$. Here, the scale μ has been introduced to keep the right-hand side dimensionless; we have $[x] = 2$, $[T] = -2$, and thus $[\mu^{2\epsilon} T^\epsilon] = 0$.

The p_0 and p_z integrations and the n summation can now be performed straightforwardly, yielding

$$\begin{aligned} \Gamma^1[A] &= -\lim_{\epsilon\to 0} \frac{\Omega}{16\pi^2} \sum_{\ell=1}^{N_c^2-1} (g|\nu_\ell|B)^2 \left(\frac{\mu^2}{2g|\nu_\ell|B}\right)^\epsilon \frac{1}{\epsilon\Gamma(\epsilon)} \\ &\quad \times \left\{ \int_0^\infty \frac{dT\, T^{\epsilon-2}}{\sinh(T)} + i^\epsilon \int_0^\infty dT\, T^{\epsilon-2} \sin T \right\} \\ &= -\Omega \sum_{\ell=1}^{N_c^2-1} \frac{(g|\nu_\ell|B)^2}{16\pi^2} \left[\frac{11}{6\epsilon} - \frac{11}{6} \ln \frac{gB}{\mu^2} + \text{const.} \right] \end{aligned} \quad (2.92)$$

where the constant also contains an imaginary part. Using $\sum_\ell \nu_\ell^2 = N_c$, the total one-loop effective action becomes

$$\begin{aligned}\Gamma[A] &= S[A] + \Gamma^1[A] \\ &= \int d^4x \left[\frac{1}{2}B^2 + \left(\frac{11}{6\epsilon}\frac{g^2 N_c}{16\pi^2} + \text{const.}\right) B^2 \right. \\ &\qquad \left. + \frac{11}{6} N_{\rm c} \frac{(gB)^2}{16\pi^2} \ln \frac{gB}{\mu^2}\right], \end{aligned} \tag{2.93}$$

which is expressed in terms of unrenormalized fields and coupling (i.e., the field amplitudes and couplings which we introduced at a microscopic scale). The physical parameters are fixed in terms of a renormalization condition for the renormalized quantities. Here, we use the Coleman-Weinberg renormalization condition:

$$\left.\frac{\partial \mathcal{L}}{\partial \left(\frac{1}{2}B_{\rm R}^2\right)}\right|_{g_{\rm R}B_{\rm R}=\mu^2} = 1, \quad \Gamma = \int d^4x\, \mathcal{L}, \tag{2.94}$$

which fixes the residue of the gluon propagator at the renormalization scale μ to be $= 1$.

From the gauge invariance of the background-field effective action, we can derive an important relation. Upon the transition from unrenormalized to renormalized quantities, gauge-covariant objects have to stay gauge covariant, such as

$$D_\mu^{ab} = \partial_\mu \delta^{ab} + g f^{acb} A_\mu^c \equiv \partial_\mu \delta^{ab} + g_{\rm R} f^{acb} A_{{\rm R},\mu}^c. \tag{2.95}$$

Hence, the product $gA_\mu = g_{\rm R} A_{{\rm R},\mu}$ must be RG invariant (in the background gauge!). The RG rescalings of the coupling and the gauge field in Eq. (2.93) thus must be of the form

$$B_{\rm R}^2 = B^2 Z_F^{-1}, \quad g_{\rm R}^2 = g^2 Z_F, \tag{2.96}$$

with a common *wave function renormalization* Z_F. From Eq. (2.93), we observe that Eq. (2.94) is satisfied if

$$Z_F^{-1} = 1 - 2\frac{11}{6\epsilon}\frac{g^2 N_c}{16\pi^2} + \text{const.} \tag{2.97}$$

The action then reads

$$\Gamma[A] = \int d^4x\, \frac{1}{2}B_{\rm R}^2 + \frac{1}{4}b_0 \frac{1}{2}(g_{\rm R}B_{\rm R})^2 \ln \frac{(g_{\rm R}B_{\rm R})^2}{e\mu^4}, \tag{2.98}$$

where

$$b_0 = \frac{11}{3}\frac{N_c}{8\pi^2}. \tag{2.99}$$

Now, the RG scale μ is arbitrary. Changing μ together with an adjustment of $g_{\rm R}$ and $B_{\rm R}$ should leave the physics invariant:

$$\begin{aligned}\Gamma[A] &= \int d^4x \left[\frac{B_{\rm R}^2}{2}\left(1+b_0 g_{\rm R}^2 \ln\frac{\mu'}{\mu}\right)+\frac{1}{4}b_0\frac{(g_{\rm R}B_{\rm R})^2}{2}\ln\frac{(g_{\rm R}B_{\rm R})^2}{e\mu'^4}\right]\\ &= \int d^4x\left[\frac{{B'_{\rm R}}^2}{2}+\frac{1}{4}b_0\frac{(g'_{\rm R}B'_{\rm R})^2}{2}\ln\frac{(g'_{\rm R}B'_{\rm R})^2}{e\mu'^4}\right], \end{aligned} \tag{2.100}$$

where we have introduced the renormalized quantities at the new scale μ',

$$\begin{aligned}{B'_{\rm R}}^2 = {B_{\rm R}}^2(\mu') &= B_{\rm R}^2(\mu)\left(1+b_0 g_{\rm R}^2(\mu)\ln\frac{\mu'}{\mu}\right),\\ {g'_{\rm R}}^2 = {g_{\rm R}}^2(\mu') &= \frac{g_{\rm R}^2(\mu)}{1+b_0 g_{\rm R}^2(\mu)\ln\frac{\mu'}{\mu}}. \end{aligned}\tag{2.101}$$

Equation (2.100) is indeed formally identical to Eq. (2.98). However, since μ and μ' are considered to be physically different scales, Eq. (2.101) tells us how the coupling changes upon a change of scales; we call it the *running* of the coupling. For fixed $g_{\rm R}^2(\mu)$ for a given μ, we observe that ${g_{\rm R}}^2(\mu')$ decreases for increasing $\mu' > \mu$, $g_{\rm R}^2(\mu') \to 0$ for $\mu'/\mu \to \infty$. The coupling becomes asymptotically free in the high-momentum region (in the UV). (Renaming μ' to μ in (2.101) and μ to some initial scale, say μ_0), this is characterized by the negative sign of the β function,

$$\beta_{g^2} := \mu\frac{dg_{\rm R}^2}{d\mu} = -b_0 g_{\rm R}^4 < 0, \tag{2.102}$$

at one-loop order.

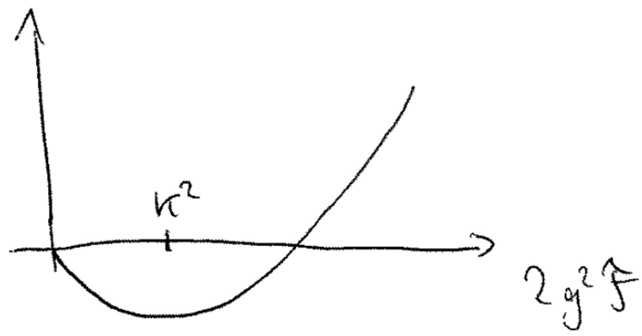
The effective action can be brought into an RG invariant form (from now on, we drop the subscript R):

$$\Gamma[A] = \frac{1}{4}\int d^4x\, b_0 g^2 \mathcal{F}\ln\frac{2g^2|\mathcal{F}|}{e\kappa^2}, \tag{2.103}$$

where $\mathcal{F} = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} = \frac{1}{2}B^2$ (or $= \frac{1}{2}(B^2+E^2)$ for general Euclidean fields) and where we have introduced the RG invariant scale

$$\kappa^2 = \mu^4 e^{-4/(b_0 g^2(\mu))}. \tag{2.104}$$

The invariance of κ under the RG flow can be verified straightforwardly, $\mu\partial_\mu\kappa^2 = 0$. The total result is rather surprising: We started with Yang-Mills theory which is free of any scale. Upon quantization, the coupling turns into a scale-dependent object. Choosing a certain value of g^2 really means fixing g^2 at a certain scale.



So g^2 is traded for a dimensionful scale κ. This phenomenon is called *dimensional transmutation.*

Let us take a look at the plot of the effective action: We observe a minimum of Γ at

$$2g^2\mathcal{F}_{\rm min} = \kappa^2. \tag{2.105}$$

Therefore, the perturbative one-loop vacuum prefers a non-vanishing gluon field strength, a *gluon condensate.*

If we had included $N_{\rm f}$ massless quark flavors, we would have obtained a similar result with

$$b_0 = \frac{1}{8\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_{\rm f}\right). \tag{2.106}$$

The conclusion about asymptotic freedom as well as about the indications for a gluon condensate remain valid as long as $N_{\rm f} < \frac{11}{2}N_c$; for QCD with $N_c = 3$ and $N_{\rm f} = 6$, this is indeed the case.

Whereas the result for the β function is exact at one-loop order, the prediction of asymptotic freedom is solid beyond perturbation theory, because higher loop corrections become exceedingly small in the UV.

However, the indication for a gluon condensate from the minimum of the effective action at one-loop order is less solid for a number of reasons. Let us start with two of these reasons which are already clear at this point:

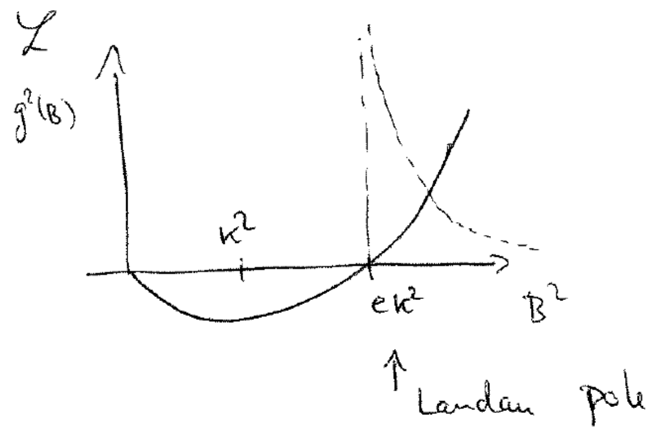
- The Savvidy vacuum with covariant constant field is unstable.
- Covariant constant fields distinguish a direction in spacetime $\sim \mathbf{B}$; hence, Lorentz invariance is broken (contrary to observations in nature).

These two problems may be circumvented by a $\mathbf{B}$ field which is constant within domains of length $L \lesssim \sqrt{gB}^{-1}$, such that the unstable mode's momenta are cut off. If the domains are randomly oriented (cf. Weiss domains in a ferromagnet), Lorentz invariance will be restored on scales $> L$. Together with the conservation of magnetic flux, this leads to the Kopenhagen vacuum or *spaghetti vacuum* picture of the Yang-Mills vacuum. However, this is only a model which has been difficult to handle computationally. But there are more points of criticism:

- The one-loop calculation is only reliable for small couplings $g^2/(4\pi) \ll 1$. Let, for instance, $g_0^2/(4\pi) \ll 1$ for $g_0^2 = g^2(\mu_0^2 = B_0)$ at some large reference scale $\mu_0^2 = B_0 \gg \kappa$. Then, re-expressing μ_0^2 in terms of κ, we find

$$g^2(B) = \frac{g_0^2}{1 + \frac{1}{2} b_0 g_0^2 \ln(B/\mu_0^2)} = \frac{g_0^2}{\frac{1}{4} b_0 g_0^2 \left(\ln \frac{B^2}{e\kappa^2} - 1\right)}. \tag{2.107}$$

  We observe that the coupling diverges already at $B^2 = e\kappa^2$ which is before



  the minimum of the effective action has been reached. Therefore, Γ is no longer valid at the minimum.

- The assumption that the vacuum is dominated by slowly varying fields is unjustified.

- Including quarks, we could estimate the quark condensate (chiral condensate) from

$$\langle \bar{\psi}\psi \rangle = \mathcal{N} \int \mathcal{D}\Phi \, \bar{\psi}\psi \, e^{-\int \cdots - m\bar{\psi}\psi} = -\frac{\partial}{\partial m} \mathcal{L}(gB, m) \bigg|_{m \to 0} \to 0, \tag{2.108}$$

  where Φ stands for all fields to be integrated over. Hence, the one-loop effective action does not predict chiral symmetry breaking, in contrast to phenomenology and lattice simulations.

2.5 Leading-log model of confinement

Despite the obvious deficiencies of the one-loop calculation, consider the one-loop effective action $\Gamma[A]$ as the simplest example of a possible complete effective action of QCD. The true action will, of course, depend on many more invariant operators of more complicated color and Lorentz structures. But already the present simple approximation features a nontrivial aspect: the *gluon condensate*.

Therefore, it is worthwhile to study the resulting quantum equations of motion; for this, we need to go over to Minkowski space:

$$\mathcal{L}_{\mathrm{E}} = -\mathcal{L}_{\mathrm{M}}, \quad \mathcal{F} = \frac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2). \tag{2.109}$$

We furthermore rescale the coupling into the field strength for convenience,

$$g^2\mathcal{F} \to \mathcal{F}. \tag{2.110}$$

The Minkowski space Lagrangian then reads

$$\mathcal{L} = -\frac{1}{4} b_0 \mathcal{F} \ln \frac{2\mathcal{F}}{e\kappa^2} - A^a_\mu J^{a\mu}, \tag{2.111}$$

where we have included a source term which we choose to be provided by a static quark antiquark pair at a distance R:

$$J^{a0} = Q\hat{n}^a[\delta^{(3)}(\mathbf{x} - \mathbf{x}_1) - \delta^{(3)}(\mathbf{x} - \mathbf{x}_2)], \quad |\mathbf{x}_1 - \mathbf{x}_2| = R. \tag{2.112}$$

The quantum equations of motion read

$$J^{a\nu} = -\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^a_\nu)} = \partial_\mu \left(-\frac{\partial \mathcal{L}}{\partial \mathcal{F}} F^{a\mu\nu} \right) \equiv \partial_\mu \left(\epsilon(\mathcal{F}) F^{a\mu\nu} \right), \tag{2.113}$$

where we have introduced the vacuum dielectric permittivity

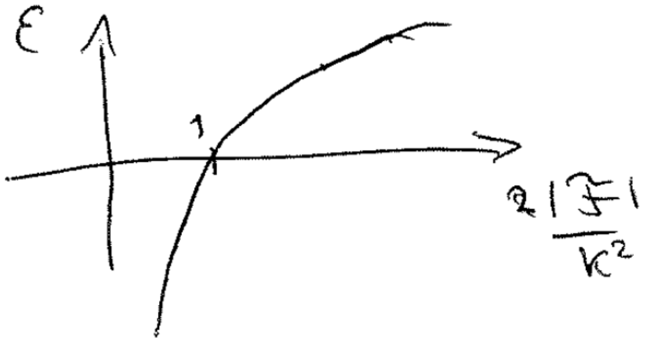
$$\epsilon(\mathcal{F}) = -\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = \frac{1}{4} b_0 \ln \frac{2\mathcal{F}}{\kappa^2}. \tag{2.114}$$

The source-free quantum equations of motion can be satisfied by

$$\begin{aligned} &(a) \qquad F^a_{\mu\nu} = 0 \quad \text{(unstable)} \\ &(b) \qquad 2|\mathcal{F}| = \kappa^2 \quad \Rightarrow \quad \epsilon(\mathcal{F}) = 0 \quad \text{(stable)} \end{aligned} \tag{2.115}$$

For pseudo-abelian sources, there is a pseudo-abelian solution, $F^a_{\mu\nu} = \hat{n}^a F_{\mu\nu}$, which has to satisfy (in non-covariant notation):

$$\begin{aligned} \nabla \cdot \mathbf{D} = J^0, \quad \nabla \times \mathbf{E} = 0, \\ \nabla \times \mathbf{H} = 0, \quad \nabla \cdot \mathbf{B} = 0, \end{aligned} \tag{2.116}$$



supplemented by the material equations

$$\mathbf{D} = \epsilon(\mathcal{F})\mathbf{E}, \quad \mathbf{H} = \epsilon(\mathcal{F})\mathbf{B}. \tag{2.117}$$

To summarize, we have mapped the QCD vacuum onto *nonlinear* electrodynamics. From $\nabla \times \mathbf{H} = 0$, it follows that

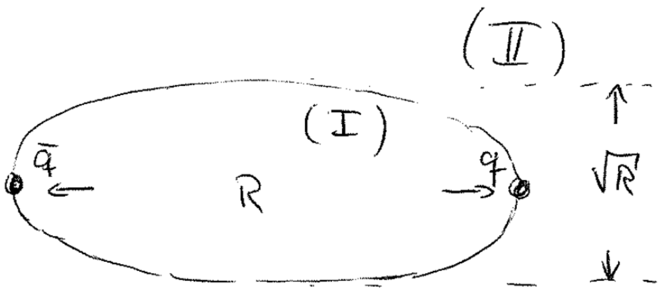
$$0 = \int d^3x\, \mathbf{A} \cdot (\nabla \times (\epsilon \mathbf{B})) = \int d^3x\, \epsilon \mathbf{B}^2 - \oint d\mathbf{f} \cdot \epsilon \mathbf{A} \times \mathbf{B}, \tag{2.118}$$

where the last term vanishes for a solution that approaches Eq. (2.115) (b) at infinity. For a solution with $\epsilon \geq 0$, we obtain

$$\mathbf{B}(\mathbf{x}) = 0 \quad \Rightarrow \quad \begin{cases} \text{(I)} & \mathbf{B} = 0, \quad \mathbf{E}^2 > \kappa^2 \\ \text{(II)} & \epsilon = 0, \quad 2\mathcal{F} = \kappa^2 \end{cases} . \tag{2.119}$$

Region (I) is expected to be observed near the sources where the electric field should show some resemblance with the Coulomb field. Region (II) is the vacuum solution.

In fact, the quantum equations of motion can be solved in an analytic expansion using a **D**-flux formulation, resulting in a quasi-linear partial differential equation (PDE) for the flux potential. We will not review this (interesting) technical step here, but just list a few of the final results: The static quarks are surrounded by a *bag* of region (I) of ellipsoidal shape. Outside the bag, region (II) extends to infinity: The thickness of the bag scales with $R^{1/2}$. Regions (I) and (II) are causally disconnected: the source distribution inside cannot exert any influence on the field configuration outside; technically, the PDE turns from elliptic (I) to parabolic (II) on the boundary. There, the normal second derivative, ∂_n^2, vanishes from the PDE.



The static quark-antiquark potential for long and short distances R reads

$$V_{\text{static}} = -\int d^3x\, \mathcal{L}(A|_{\text{QEoM}})$$
$$= \begin{cases} \kappa QR + \frac{2}{3}Q^{\frac{3}{2}}\sqrt{\frac{2\kappa}{\pi b_0}}\ln(\sqrt{\kappa}R), & \sqrt{\kappa}R \gg 1 \\ -\frac{Q^2}{4\pi R}\frac{1}{b_0(\ln\frac{1}{\sqrt{\kappa}R} + \text{const})}, & \sqrt{\kappa}R \ll 1 \end{cases} \tag{2.120}$$

We observe a *linear confinement* for large R and a log-modified Coulomb potential for small R.

For heavy-quarkonium spectroscopy, the model shows reasonable agreement with experimental data for $\sqrt{\kappa} \simeq 229\,\text{MeV}$. However, the string tension comes out somewhat too small,

$$\sqrt{\sigma} = \sqrt{Q\kappa} \simeq 246\,\text{MeV}, \quad \text{with } Q = \sqrt{\frac{4}{3}} \quad \text{for SU(3)}. \tag{2.121}$$

which should be compared to the phenomenological value $\sqrt{\sigma} \simeq 420\,\text{MeV}$.

The predictions for the string thickness (*string roughening*) $\sim R^{1/2}$ and the leading correction to linear confinement $\sim \ln R$ should be compared with, e.g., the *bosonic string model*, where the roughening scales like $\sim \ln R$ (not yet measured precisely on the lattice), and the static potential is

$$V_{\text{static, BS}} = \sigma R - \frac{\pi(d-2)}{24R} + \ldots, \quad \text{for } \sqrt{\sigma}R \gg 1, \tag{2.122}$$

where d is the number of spacetime dimensions and the subleading term is a universal prediction of the bosonic string model (*Lüscher term*). The latter has been confirmed quantitatively by lattice simulations.

To summarize, the leading-log model is a first and simple confinement model. The mechanism for confinement arises in this model from the dielectric properties of the quantum vacuum. Due to its perturbative origin, the model is, however, not well founded and various quantitative details are in contradiction with other methods.

Nevertheless, the possibility remains that a nonperturbative computation results in an effective action (of more complicated structure) that supports a dielectric confinement mechanism of the type described here.

3 Gauge fields on loops and lattices

NOTE: In order to conform with the literature, we use a different notation in this chapter. Gauge transformations are denoted by $\Omega(x)$, e.g., $A_\mu^\Omega = \Omega A_\mu \Omega^{-1} - \frac{i}{g}(\partial_\mu \Omega)\Omega^{-1}$ instead of U as in the preceding sections. This is because U will be used for the link variables introduced below, as is common in lattice gauge theory.

3.1 Wegner-Wilson loop

In view of the confinement problem, we would like to study a quark-antiquark pair at large spatial separation R. Here, we encounter a problem: e.g., the correlations between a quark field $\psi(x)$ at position x and a conjugate quark field $\bar{\psi}(y)$ at y are gauge dependent, since the fields at x and y in general transform differently under a local gauge transformation $\Omega(x)$,

$$\psi(x) \to \Omega(x)\psi(x), \quad \bar{\psi}(y) \to \bar{\psi}(y)\Omega^{-1}(y). \tag{3.1}$$

In order to compare quark fields at different points in a meaningful gauge-invariant way with each other, we need to transport the color information of, say, $\bar{\psi}(y)$ to the point x in a gauge-covariant manner.

Technically speaking, we are looking for a bi-local object $U(y, x)$ which transforms as

$$U(y, x) \to \Omega(y)U(y, x)\Omega^{-1}(x), \tag{3.2}$$

and which can be used to form gauge-invariant operators, e.g., $\bar{\psi}(y)U(y, x)\psi(x)$. As a normalization, we require

$$U(x, x) = \mathbb{1}. \tag{3.3}$$

For an infinitesimal distance $y = x + dx$, we find

$$\begin{aligned}
&\Omega(x + dx)U(x + dx, x)\Omega^{-1}(x) \\
&\quad = \Omega(x)U(x, x)\Omega^{-1}(x) + dx^\mu(\partial_\mu\Omega(x))U(x, x)\Omega^{-1}(x) \\
&\qquad + \Omega(x)dx^\mu\partial_\mu^y U(y, x)\big|_{y=x}\Omega^{-1}(x) + \mathcal{O}(dx^2) \\
&\quad = \mathbb{1} + dx^\mu\left[(\partial_\mu\Omega(x))\Omega^{-1}(x) + \Omega(x)\partial_\mu^y U(y, x)\big|_{y=x}\Omega^{-1}(x)\right] + \mathcal{O}(dx^2).
\end{aligned} \tag{3.4}$$

The term in square brackets looks similar to the gauge-transformed gauge potential, if we identify $\partial_\mu^y U(y,x)\big|_{y=x} = igA_\mu(x)$. This observation suggests that $U(x+dx,x)$ can be represented by

$$U(x+dx,x) = \mathbb{1} + igA_\mu(x)dx^\mu, \tag{3.5}$$

and Eq. (4.4) read from right to left shows that this choice has the desired transformation properties. For finite separations $y-x$, $U(y,x)$ can be constructed from a product of $U(x+dx,x)$'s,

$$U(y,x) = \lim_{N\to\infty} \prod_{n=1}^{N} U(y_n, y_{n-1}), \quad y_n = x + n\frac{y-x}{N}, \quad n = 0,\ldots,N. \tag{3.6}$$

If A_μ was a constant number and $dx_\mu = (y-x)_\mu/N$, we would conclude that $U(y,x) = \lim_{N\to\infty}(1+ig\frac{A_\mu(y-x)_\mu}{N})^N = e^{igA_\mu(y-x)^\mu}$. For a number-valued function $A_\mu(x)$ as in U(1) gauge theories, we find $U(y,x) = e^{ig\int_x^y A_\mu(z)dz_\mu}$.

However, for a matrix-valued $A_\mu(x) = A_\mu^a\tau^a$, we have to take care of the non-commuting nature of two $A_\mu(x)$'s at neighboring positions. The result in this case can formally be written as

$$U(y,x) = \mathcal{P}\exp\left(ig\int_x^y A_\mu(z)dz_\mu\right). \tag{3.7}$$

The symbol $\mathcal{P}$ means *path ordering*. For instance, in a Taylor expansion of Eq. (4.7), matrices $A_\mu(z)$ which are attached to a certain point z are ordered from later (left) to earlier (right) positions along the path from x to y, e.g.,

$$\begin{aligned}\mathcal{P}\left[\int_x^y A_\mu(z)dz_\mu\right]^2 &= \mathcal{P}\int_x^y dz_\mu \int_x^y dw_\nu A_\mu(z)A_\nu(w) \qquad (3.8)\\ &= \int_x^y dz_\mu \int_z^y dw_\nu A_\mu(w)A_\nu(z) + \int_x^y dz_\nu \int_x^z dw_\mu A_\mu(z)A_\nu(w).\end{aligned}$$

Equation (4.6) is, of course, path-ordered by construction. Also by construction, U transforms as

$$U(y,x)[A^\Omega] \to \Omega(y)U(y,x)[A]\Omega^{-1}(x). \tag{3.9}$$

The object $U(y,x)[A]$ constitutes a mapping of paths in coordinate space into the gauge group. In Eq. (4.6), we have used a specific straight line path. Whereas $U(y,x)[A]$ is generally path dependent, the gauge transformation property (4.9) is only sensitive to the end points x and y; any other path in Eq. (4.6) would also lead to the desired transformation property Eq. (4.9).

Consider now the important case, where the path is a closed contour $\mathcal{C}$:

$$\mathcal{C} : \{x_\mu(s)|x_\mu(0) = x_\mu(1), \quad s \in [0,1]\}. \tag{3.10}$$

A fully gauge-invariant object is then given by

$$W(\mathcal{C}) = \operatorname{tr} U(\mathcal{C}) = \operatorname{tr} \mathcal{P} \exp\left(ig \oint_{\mathcal{C}} A_\mu(z) dz_\mu\right), \tag{3.11}$$

since

$$W^\Omega(\mathcal{C}) = \operatorname{tr}\left[\Omega(x(1)) U(\mathcal{C}) \Omega^{-1}(x(0))\right] = \operatorname{tr} U(\mathcal{C}) = W(\mathcal{C}). \tag{3.12}$$

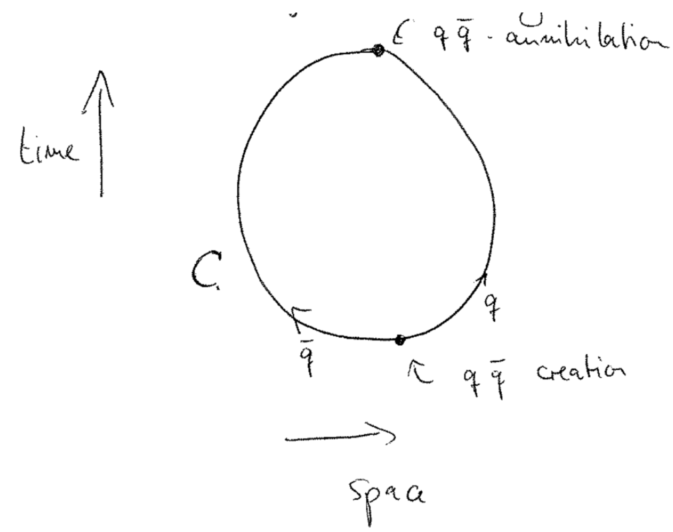
This is the Wegner-Wilson loop which plays a key role in confining gauge theories. Note that the exponent can be written as

$$ig \oint_{\mathcal{C}} A_\mu(z) dz_\mu = \int d^4z \, \mathcal{J}^a_\mu(z) A^a_\mu(z), \tag{3.13}$$

where

$$\mathcal{J}^a_\mu(x) = ig \oint_{\mathcal{C}} dz_\mu \, \delta^{(4)}(x - z) \tau^a \tag{3.14}$$

can be viewed as a source term of a charged particle in fundamental representation propagating along the closed contour $\mathcal{C}$ in spacetime (NB: the "i" is due to our Euclidean conventions). Alternatively, $\mathcal{J}^a_\mu(x)$ can be interpreted as a source term for a quark-antiquark pair being created at some initial time, then propagating some distance and then annihilating again at a later time. The Wegner-Wilson



loop expectation value therefore is nothing but the generating functional for a special source $\mathcal{J}^a_\mu(x)$:

$$\langle W(\mathcal{C}) \rangle = \frac{1}{Z[0]} Z[\mathcal{J}] = \frac{1}{Z[0]} \int \mathcal{D}A \, \Delta_{\mathrm{FP}}[A] \, e^{-S_{\mathrm{YM}} - S_{\mathrm{gf}} + \int \mathcal{J}^a_\mu A^a_\mu}, \tag{3.15}$$

where the path-ordering prescription is implicitly understood.

The meaning of the Wegner-Wilson loop can heuristically be understood in terms of its quantum mechanical analogue; here, the connection between the functional integral (path integral) and the Hamiltonian formulation of quantum mechanics is most immediate. In Euclidean quantum mechanics, we get for the partition function of a particle moving in $d = 3$ for a given time T in the presence of some interaction with j:

$$\begin{aligned} Z[j] &= \int d^3x_i \langle \mathbf{x}_i | e^{-H(j)T} | \mathbf{x}_i \rangle \\ &= \int d^3x_i \int_{\mathbf{x}(0)=\mathbf{x}_i}^{\mathbf{x}(T)=\mathbf{x}_i} \mathcal{D}\mathbf{x}\, e^{-S[\mathbf{x},j]} \\ &= \mathrm{Tr}\, e^{-H(j)T}, \\ &= \sum_{n=0}^{\infty} \langle n | e^{-H(j)T} | n \rangle = \sum_{n=0}^{\infty} e^{-E_n(j)T} \\ &= e^{-E_0(j)T} \left[1 + \sum_{n=1}^{\infty} e^{-(E_n - E_0)T} \right]. \end{aligned} \tag{3.16}$$

Since $E_{n>0} > E_0$, we find for large times T that the partition function is dominated by the ground state energy $E_0(j)$,

$$E_0(j) = - \lim_{T\to\infty} \frac{1}{T} \ln Z[j]. \tag{3.17}$$

Transfering this reasoning to quantum gauge theory suggests that the Wegner-Wilson loop expectation value (3.15) is related to the energy associated with the creation and annihilation of a quark-antiquark pair. Choosing a contour as in the figure, corresponding to a quark-antiquark pair that remains static at a distance R for a time T, we expect the ground-state energy to dominate $\langle W(\mathcal{C}) \rangle$ in the limit $T \to \infty$, and to correspond to the static potential $V(R)$ between the quark and antiquark:

$$V(R) = - \lim_{T\to\infty} \frac{1}{T} \ln \langle W(\mathcal{C}) \rangle. \tag{3.18}$$

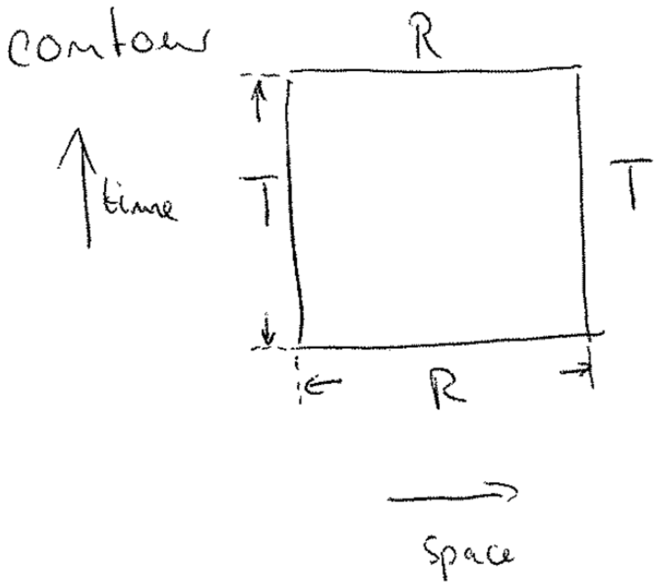
(NB: The connection between $V(R)$ and $\langle W(\mathcal{C}) \rangle$ can indeed more rigorously be shown in QFT with the aid of the transfer matrix formalism.)

Confinement in gauge theories is therefore signaled by

$$\begin{aligned} V(R) &= \sigma R \quad \text{for large } R, \\ &\Rightarrow \ \langle W(\mathcal{C}) \rangle \sim e^{-\sigma R T} = e^{-\sigma A}, \end{aligned} \tag{3.19}$$

where $A = RT$ is the area encircled by the contour $\mathcal{C}$: $\mathcal{C} = \partial A$.

Equation (3.19) expresses the famous *area law* of the Wegner-Wilson loop which serves as an important criterion for confinement.



3.2 Wegner-Wilson loop in QED

As an illustration, let us compute the Wegner-Wilson expectation value in U(1) gauge theory in Feynman gauge:

$$\langle W(\mathcal{C})\rangle = \frac{1}{Z}\int \mathcal{D}A\, e^{-\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}\int(\partial_\mu A^\mu)^2 + \int \mathcal{J}_\mu A^\mu}, \tag{3.20}$$

Using $\alpha = 1$ (Feynman gauge) and

$$\begin{aligned} \frac{1}{4}\int F_{\mu\nu}F^{\mu\nu} &= \frac{1}{2}\int A_\mu(-\partial^2 + \partial_\mu\partial_\nu)A^\nu, \\ \frac{1}{2\alpha}\int(\partial_\mu A^\mu)^2 &= \frac{1}{2}\int A_\mu(-\partial^\mu\partial^\nu)A_\nu, \end{aligned} \tag{3.21}$$

we get

$$\begin{aligned} \langle W(\mathcal{C})\rangle &= \frac{1}{Z}\int \mathcal{D}A\, e^{-\frac{1}{2}\int A_\mu(-\partial^2)A^\nu + \int \mathcal{J}_\mu A^\mu} \\ &= \frac{1}{Z}\int \mathcal{D}A\, e^{-\frac{1}{2}\int (A_\mu - A_{0\mu})(-\partial^2)(A^\nu - A_0^\nu)} e^{\frac{1}{2}\int \mathcal{J}_\nu \frac{1}{(-\partial^2)}\mathcal{J}^\nu}, \\ &\qquad \text{where } A_{0\mu} = \frac{1}{(-\partial^2)}\mathcal{J}^\nu. \end{aligned} \tag{3.22}$$

The seeming source dependence in the first exponential drops out by shifting the integration variable $A_\mu \to A_\mu + A_{0\mu}$. The integral is thus exactly equal to the normalization factor Z, and we obtain

$$\langle W(\mathcal{C})\rangle = e^{\frac{1}{2}\int \mathcal{J}_\nu \frac{1}{(-\partial^2)}\mathcal{J}^\nu}. \tag{3.23}$$

The symbol $\dfrac{1}{(-\partial^2)}$ denotes nothing but the Green's function of the 4-dimensional Laplacian,

$$(-\partial^2)G = \mathbb{1}, \quad \text{i.e.,} \; -\partial_x^2 G(x-y) = \delta^{(4)}(x-y), \tag{3.24}$$

which can be determined as

$$G(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2} = \frac{1}{4\pi^2(x-y)^2}. \tag{3.25}$$

This is the Euclidean photon propagator in spacetime in Feynman gauge. In U(1) gauge theory, the source term for a static e^+e^- pair reads (cf. Eq. (3.14)):

$$\mathcal{J}_\mu(x) = ie\oint_{\mathcal{C}} dz_\mu\, \delta^{(4)}(x-z), \tag{3.26}$$

where $\mathcal{C}$ denotes the rectangular contour in a previous figure. The exponent in Eq. (3.23) thus reads

$$\frac{1}{2}\int \mathcal{J}_\nu(x)\frac{1}{(-\partial^2)}\mathcal{J}^\nu(x) = -\frac{e^2}{2}\frac{1}{4\pi^2}\oint_{\mathcal{C}} dz_\nu \oint_{\mathcal{C}} dz'_\nu\, \frac{1}{(z-z')^2}. \tag{3.27}$$

Now, $dz_\mu dz'_\nu$ is only nonzero if both differentials are parallel for our contour $\mathcal{C}$, i.e., if z_μ and z'_ν lie on the same or opposite sides. Representing the photon exchange by a wavy line, there are four types of contributions, cf. Fig. 3.1: The latter two

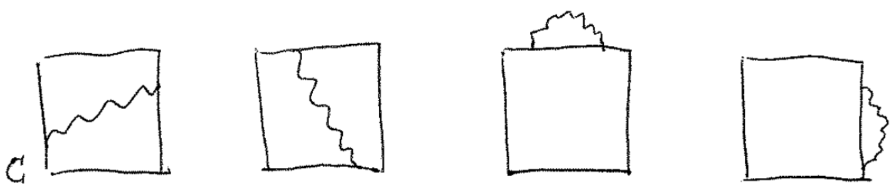


Abbildung 3.1: Different contributions to the exponent of the Wegner-Wilson loop expectation value in QED due to photon exchange between different segments of the rectangular contour $\mathcal{C}$.

describe the electromagnetic self-interactions of a particle, contributing to the (naively divergent) self-energy. For the interactions between the e^+e^- pair, these terms are irrelevant, and we drop them.

The remaining integrals can straightforwardly be performed. For the relevant limit $T/R \to \infty$, only the first diagram type contributes by a finite amount, cf. exercises. For this integral, we obtain:

$$\frac{1}{2}\int \mathcal{J}_\nu(x) \frac{1}{(-\partial^2)} \mathcal{J}^\nu(x) = -\frac{e^2}{2\pi^2}\left(-\frac{T}{R}\arctan\frac{T}{R} + \frac{1}{2}\ln\left(1+\frac{T^2}{R^2}\right)\right). \tag{3.28}$$

This implies for the static potential (3.18):

$$\begin{aligned} V(R) &= -\lim_{T\to\infty}\frac{1}{T}\ln e^{-\frac{e^2}{2\pi^2}\left(-\frac{T}{R}\underbrace{\arctan\frac{T}{R}}_{\to\pi/2}+\frac{1}{2}\ln\left(1+\frac{T^2}{R^2}\right)\right)} \\ &= -\lim_{T\to\infty}\frac{1}{T}\frac{e^2}{2\pi^2}\frac{T}{R}\frac{\pi}{2} \\ &= -\frac{e^2}{4\pi R}, \end{aligned} \tag{3.29}$$

which exactly corresponds to the Coulomb potential!

3.3 Gauge fields on the lattice

In section 3.1, we have constructed $U(y,x)$, which acts as a *parallel transporter* of color information, from a sequence of infinitesimal steps, see Eq. (4.6). Note, however, that the desired gauge-transformation property, $U(y,x) \to \Omega(y)U(y,x)\Omega^{-1}(x)$, is already present for the infinitesimal step. Since there is an infinitesimal one-to-one correspondence between $U(x+dx,x)$ (for arbitrary dx) and $A_\mu(x)$, cf. Eq. (4.5),

$$U(x+dx,x) = \mathbb{1} + igA_\mu(x)dx_\mu, \tag{3.30}$$

this suggests that a gauge theory can fully be formulated in a discrete fashion on a spacetime lattice in terms of the variables $U(x+dx,x)$ with full gauge symmetry.

Consider a hypercubic lattice with lattice spacing a. Let us denote the *sites* by x, and a unit vector pointing into the μ direction by $\hat{\mu}$. A neighboring site to x in μ direction is then denoted by $x + a\hat{\mu}$. To every *link* between two neighboring sites, we associate a parallel transporter

$$U_{x\mu} \equiv U(x, x+a\hat{\mu}) \in \mathrm{SU}(N_\mathrm{c}), \tag{3.31}$$

The inverse is given by

$$U_{x\mu}^{-1} \equiv U(x+a\hat{\mu}, x) = U_{x\mu}^\dagger. \tag{3.32}$$

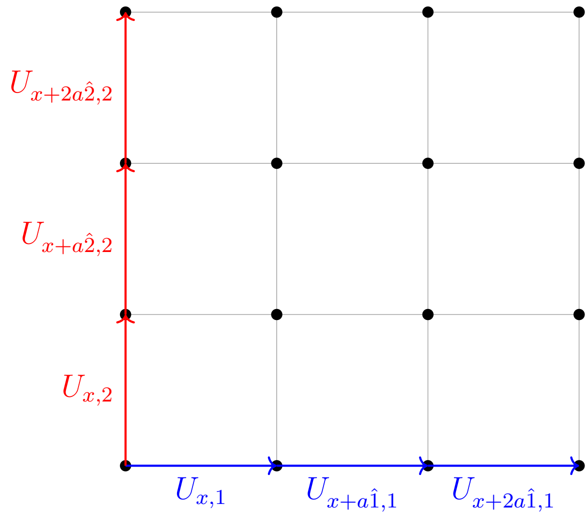


Abbildung 3.2: A 2D lattice with link variables $U_{x\mu} \equiv U_{x,x+a\hat{\mu}}$ connecting neighboring sites.

Considering a as an infinitesimal distance, the link variable can be related to the gauge field A_μ as in Eq. (3.30),

$$U_{x\mu} = \mathbb{1} - igaA_\mu(x). \tag{3.33}$$

Gauge transformations are defined on the sites, Ω_x, and the links transform as

$$U_{x\mu} \to \Omega_x U_{x\mu} \Omega^{-1}_{x+a\hat{\mu}}. \tag{3.34}$$

The links encode all gauge-field information and can thus be viewed as the true gauge-field degrees of freedom. The relation between the links and continuum gauge fields is not unique; e.g., equally valid definitions to order a are (cf. Eq. (3.30)):

$$U_{x\mu} = \mathbb{1} - igaA_\mu(x + \tfrac{1}{2}a\hat{\mu}), \quad \text{or} \quad U_{x\mu} = \exp\left(-igaA_\mu(x + \tfrac{1}{2}a\hat{\mu})\right). \tag{3.35}$$

Here, we have associated the gauge-field coordinate with the link "center of mass", i.e., lying halfway between the two neighboring sites. Let us now introduce the *plaquette* variable

$$\begin{aligned} U_{\mu\nu} &= U_{x,x+a\hat{\mu}} U_{x+a\hat{\mu},x+a\hat{\mu}+a\hat{\nu}} U_{x+a\hat{\mu}+a\hat{\nu},x+a\hat{\nu}} U_{x+a\hat{\nu},x} \\ &= U_{x\mu} U_{x+a\hat{\mu},\nu} U^\dagger_{x+a\hat{\nu},\mu} U^\dagger_{x\nu}, \end{aligned} \tag{3.36}$$

To order a^2 and using the 2nd definition of Eq. (3.35) for this purpose, $U_{\mu\nu}$ is given by

$$U_{\mu\nu} = e^{-igaA_\mu(x+\frac{a}{2}\hat{\mu})} e^{-igaA_\nu(x+a\hat{\mu}+\frac{a}{2}\hat{\nu})} e^{igaA_\mu(x+a\hat{\nu}+\frac{a}{2}\hat{\mu})} e^{igaA_\nu(x+\frac{a}{2}\hat{\nu})}. \tag{3.37}$$

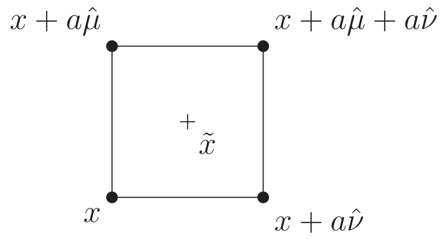


Abbildung 3.3: Plaquette associated with the site x, or $\tilde{x}$ on the dual lattice.

Denoting the center of the plaquette by $\tilde{x} = x + \frac{a}{2}\hat{\mu} + \frac{a}{2}\hat{\nu}$, the last line reads:

$$\begin{aligned}
U_{\mu\nu} &= e^{-igaA_\mu(\tilde{x}-\frac{a}{2}\hat{\nu})} e^{-igaA_\nu(\tilde{x}+\frac{a}{2}\hat{\mu})} e^{igaA_\mu(\tilde{x}+\frac{a}{2}\hat{\nu})} e^{igaA_\nu(\tilde{x}-\frac{a}{2}\hat{\mu})} \\
&= \Big[\mathbb{1} - g^2a^2A_\mu A_\nu + g^2a^2A_\mu A_\nu + g^2a^2A_\nu A_\mu - g^2a^2A_\mu A_\nu \\
&\quad + \frac{i}{2}ga^2\partial_\nu A_\mu - \frac{i}{2}ga^2\partial_\mu A_\nu + \frac{i}{2}ga^2\partial_\nu A_\mu - \frac{i}{2}ga^2\partial_\mu A_\nu + \mathcal{O}(a^3)\Big]_{\tilde{x}} \\
&= \big[\mathbb{1} - iga^2\left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]\right)\big] + O(a^3) \\
&= \big[\mathbb{1} - iga^2F_{\mu\nu}\big] + O(a^3). \\
&= e^{-iga^2F_{\mu\nu}} + O(a^3).
\end{aligned} \tag{3.38}$$

In the first step, we have already made use of the fact that all terms linear in a and those $\sim A_\mu A_\mu$ cancel. Now consider the quantity (where β denotes a normalization constant)

$$\begin{aligned}
S_p &= \beta\left(1 - \frac{1}{N_\mathrm{c}}\mathrm{Re}\,\mathrm{tr}\,U_{\mu\nu}\right) = \beta\left(1 - \frac{1}{N_\mathrm{c}}\mathrm{Re}\,\mathrm{tr}\,e^{-iga^2F_{\mu\nu}}\dots\right) \\
&= \beta\left(1 - \frac{1}{N_\mathrm{c}}\underbrace{\mathrm{tr}\,\mathbb{1}}_{=N_\mathrm{c}} + \frac{ga^2}{N_\mathrm{c}}\mathrm{Re}\underbrace{\mathrm{tr}\,iF_{\mu\nu}}_{=0} + \frac{g^2a^4}{2N_\mathrm{c}}\mathrm{Re}\,\mathrm{tr}\,F_{\mu\nu}F_{\mu\nu} + O(a^6)\right)
\end{aligned} \tag{3.39}$$

where there is no summation over μ, ν implied. In summary, we have

$$S_p = \frac{\beta g^2a^4}{2N_\mathrm{c}}\,\mathrm{tr}\,F_{\mu\nu}F_{\mu\nu} + O(a^6). \tag{3.40}$$

Summing over all possible plaquettes, we define the *Wilson action* on a lattice

$$\begin{aligned}
S_W &= \sum_p S_p = \beta\sum_{\tilde{x}}\sum_{\mu<\nu}(1 - \frac{1}{N_\mathrm{c}}\mathrm{Re}\,\mathrm{tr}\,U_{\mu\nu}) \\
&= \sum_{\tilde{x}}\frac{1}{2}\sum_{\mu,\nu}\frac{\beta g^2a^4}{2N_\mathrm{c}}\,\mathrm{tr}\,F_{\mu\nu}(x)F_{\mu\nu}(x) + O(a^6) \\
&\xrightarrow[a\to 0]{} \beta\frac{g^2}{2N_\mathrm{c}}\int d^4x\frac{1}{2}\,\mathrm{tr}\,F_{\mu\nu}(x)F_{\mu\nu}(x),
\end{aligned} \tag{3.41}$$

where the last line uses the standard Einstein sum convention again. The continuum limit of the Wilson action coincides with the Yang-Mills action provided we choose

$$\beta = \frac{2N_\mathrm{c}}{g^2}. \tag{3.42}$$

(The real part Re in Eq. (3.39) is introduced in order to keep the action real also to higher orders in a.)

The quantum gauge theory is finally defined by integrating over all possible values for the gauge variables $U_{x\mu}$,

$$Z = \int \mathcal{D}U\, e^{-S_W[U]}, \quad \text{where } \mathcal{D}U = \prod_{x,\mu} dU_{x\mu}, \tag{3.43}$$

and $dU_{x\mu}$ denotes the Haar measure on $\mathrm{SU}(N_\mathrm{c})$. Given a parametrization of U in terms of coordinates ω^a on group space, e.g., $U = e^{-i\omega^a\tau^a}$, the Haar measure corresponds to the reparametrization invariant measure with respect to coordinate transformations,

$$dU = \nu\sqrt{\det g}\prod_a d\omega^a, \tag{3.44}$$

where

$$g_{ab} = (2)\,\mathrm{tr}\left(\frac{\partial U}{\partial\omega^a}\frac{\partial U^\dagger}{\partial\omega^b}\right). \tag{3.45}$$

Here, the factor "(2)" holds for all $\mathrm{SU}(N_\mathrm{c} > 1)$ but is replaced by "(1)" for U(1). Equation (3.45) denotes the induced metric on group space, and the normalization ν can be chosen such that

$$\int dU = 1. \tag{3.46}$$

The metric transforms covariantly under coordinate transformations, $\omega'^a = f^a(\omega)$,

$$g'_{ab} = \frac{\partial\omega^c}{\partial\omega'^a}\frac{\partial\omega^d}{\partial\omega'^b}g_{cd}, \tag{3.47}$$

implying that Jacobian factors from coordinate transformations cancel explicitly in Eq. (3.44),

$$dU' = \nu\sqrt{\det g'}\prod_a d\omega'^a = \nu\sqrt{\det g}\prod_a d\omega^a = dU. \tag{3.48}$$

A special case of coordinate transformations is given by left and right translations in group space,

$$U \to U' = \Omega U, \quad \text{or} \quad U' = U\Omega, \tag{3.49}$$

(i.e., $U'(\omega')) = \Omega U(\omega)$ can be viewed as a coordinate transformation $\omega' = f(\omega)$). Hence, the Haar measure is simultaneously left- and right-invariant,

$$dU' = d(\Omega U) = d(U\Omega) = dU. \tag{3.50}$$

A simple illustration of all this is given by the Haar measure on U(1):

$$U = e^{-i\omega}, \quad \omega \in [0, 2\pi) \quad \Rightarrow \quad dU = \frac{d\omega}{2\pi}, \quad \int dU = 1. \tag{3.51}$$

Consider $\Omega = e^{i\alpha}$ (e.g. left translation). Then,

$$U'(\omega') = e^{-i\omega'} = \Omega U(\omega) = e^{-i(\omega-\alpha)} \quad \Rightarrow \quad \omega' = \omega - \alpha. \tag{3.52}$$

Of course, explicit representations can be worked out for the Haar measure of $SU(N_\mathrm{c})$, but this will not be a matter of concern here.

Let us finally remark that the partition function (3.43) is finite for finite lattices and compact gauge groups; hence, correlators and observables can immediately be computed,

$$\langle \mathcal{O}(U) \rangle = \frac{1}{Z} \int \mathcal{D}U \, \mathcal{O}(U) e^{-S_W[U]}. \tag{3.53}$$

In particular, gauge fixing is not necessary for non-perturbative lattice computations.

3.4 Wegner-Wilson loop in strong-coupling expansion

The Wegner-Wilson loop on the lattice is simply given by the (trace over the) product of link variables along the contour $\mathcal{C}$,

$$W(\mathcal{C}) = \mathrm{tr} \prod_{\mathcal{C}} U, \tag{3.54}$$

and its expectation value reads

$$\langle W(\mathcal{C}) \rangle = \frac{1}{Z} \int \mathcal{D}U \;\mathrm{tr} \prod_{\mathcal{C}} U \, e^{-S[U]}, \tag{3.55}$$

with S being the Wilson action $\sim \frac{1}{g^2}$, cf. Eq. (3.41),

$$S[U] = \frac{2N_\mathrm{c}}{g^2} \sum_{\tilde{x}} \sum_{\mu\nu} \left(1 - \frac{1}{N_\mathrm{c}} \mathrm{Re}\,\mathrm{tr}\, U_{\mu\nu} \right). \tag{3.56}$$

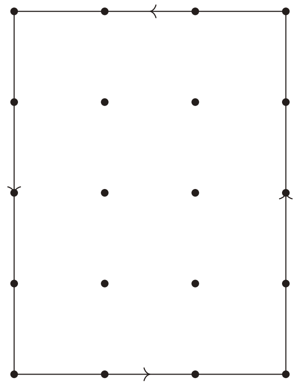


Abbildung 3.4: Rectangular lattice enclosed by a Wilson line (contour) $\mathcal{C}$.

We may try to compute Eq. (3.56) in a strong-coupling expansion, $\frac{1}{g^2} \to 0$. This is reminiscent to a high-temperature expansion in statistical mechanics, where the Boltzmann weight $e^{-\beta H}$ is expanded for small $\beta = \frac{1}{k_B T}$. For this, we need some elementary integrals over group space; by construction, we have

$$\int dU = 1. \tag{3.57}$$

Furthermore, we have the following basic integrals:

$$\int dU\, U_{ij}(\ell_{x\mu}) = 0, \tag{3.58}$$

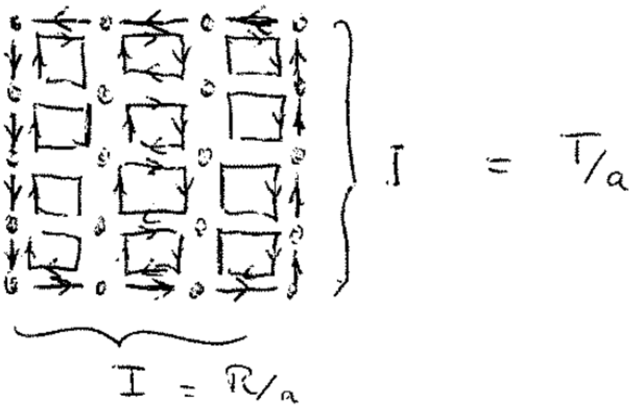
$$\int dU\, U_{ij}(\ell_{x\mu}) U^\dagger_{kl}(\ell_{y\nu}) = \frac{1}{N_\mathrm{c}} \delta_{ik}\delta_{jl}\delta_{\ell_{x\mu}\ell_{y\nu}}, \tag{3.59}$$

where $\ell_{x\mu}$ denotes the link starting at site x in direction μ. These integrals can be derived using an explicit parametrization.

A strong-coupling expansion of Eq. (3.55) corresponds to a Taylor expansion of the exponential $e^{-S[U]} \sim e^{-\frac{1}{g^2}\sum \mathrm{tr}\, U_{\mu\nu}}$. To zeroth order, we get

$$\langle W(\mathcal{C})\rangle^{(0)} = \frac{1}{Z}\int \mathcal{D}U\ \mathrm{tr} \prod_{\mathcal{C}} U = 0, \tag{3.60}$$

since at any link ℓ, we have Eq. (3.58). We observe that any link of the Wegner-Wilson loop can contribute only, if there is another link “on top of it”, such that there product has a singlet component according to Eq. (3.59). This happens to be the case when a link on the contour is multiplied by a conjugate link being



part of a plaquette. To lowest non-vanishing order in $\frac{1}{g^2}$, this implies that the Wegner-Wilson loop area has to be tiled completely by plaquettes

For an area of side lingth $I \times J$ (in units of the lattice spacing a), this makes $2 \cdot I \cdot J + I + J$ pairs of links, contributing a factor $\left(\frac{1}{N_\mathrm{c}}\right)^{2IJ+I+J}$ to the Wegner-Wilson loop expectation value according to Eq. (3.59). The contraction of the Kronecker deltas from Eq. (3.59) as well as from the prefactor of the action (3.56), we obtain further contributions of this form. The final result reads for $N_\mathrm{c} = 3$:

$$\langle W(\mathcal{C})\rangle^{(1)} = 3\left(\frac{\beta}{18}\right)^{IJ}(1+O(\beta)) = 3\left(\frac{\beta}{18}\right)^{\frac{T}{a}\frac{R}{a}}(1+O(\beta)), \quad \beta = \frac{6}{g^2} \tag{3.61}$$

to lowest order in $\frac{1}{g^2}$. For the static potential, we thus obtain

$$\begin{aligned} V(R) &= -\lim_{T\to\infty}\frac{1}{T}\ln\langle W(\mathcal{C})\rangle \\ &= -\lim_{T\to\infty}\frac{1}{T}\ln\left[3\left(\frac{\beta}{18}\right)^{\frac{T}{a}\frac{R}{a}} + \ldots\right] \\ &\simeq \left(-\ln\frac{\beta}{18}\right)\frac{R}{a^2} \equiv \sigma R, \end{aligned} \tag{3.62}$$

where $\sigma = -a^{-2}\ln\frac{\beta}{18}$ is the string tension.

The strong-coupling expansion therefore produces a linearly confining potential. The strong-coupling expansion hence gives analytical insight into the structure of the theory at large bare coupling. However, it turns out to be difficult to relate the strong-coupling expansion to the parameter region where the renormalized coupling takes on physically relevant values. (NB: the coupling used here corresponds to the bare coupling defined at the lattice cutoff scale a^{-1}. Since we are interested in the continuum limit $a \to 0$, this coupling goes to zero because of asymptotic freedom. This regime is thus far away from the regime where the strong-coupling expansion can be expected to give reliable results.) The result (3.62), though conceptually highly interesting, does therefore not serve as a proof of confinement.

4 Confinement and monopoles

4.1 Prerequisites

A class of popular models of confinement is based on the idea of a dual Meissner effect. In order to understand this duality hypothesis, let us first sketch the Meissner-Ochsenfeld effect which is a characteristic feature of superconductivity.

Type-I superconductivity can be described by a condensation of Cooper pairs which are bosonic electron composites. Cooper pairs occupy the same quantum state. This condensate is associated with a macroscopic wave function

$$\Phi(\mathbf{x}, t) = \sqrt{|q|N} \cdot \chi(\mathbf{x}, t), \tag{4.1}$$

where $q = -2e$ is the charge of a Cooper pair, N denotes the number of Cooper pairs and $\chi(\mathbf{x}, t)$ is a normalized wave function of one Cooper pair.

The macroscopic charge density is

$$\rho(\mathbf{x}, t) = -|\Phi(\mathbf{x}, t)|^2 = qN|\chi(\mathbf{x}, t)|^2. \tag{4.2}$$

The conservation of the number of Cooper pairs implies a continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{4.3}$$

where $\mathbf{j}$ is the current density. From Eq. (4.2) together with the Schrödinger equation, $i\hbar\partial_t\Phi = H\Phi$, with $H = -\frac{\hbar^2}{2m}\mathbf{D}^2$, $\mathbf{D} = \nabla - i\frac{q}{\hbar}\mathbf{A}$, we obtain the Cooper current

$$\mathbf{j} = \frac{\hbar}{2im}\left(\Phi^*\mathbf{D}\Phi - (\mathbf{D}\Phi)^*\Phi\right). \tag{4.4}$$

For a homogeneous superconductor with $\rho(\mathbf{x}) \simeq \rho = \text{const.}$, and

$$\Phi(\mathbf{x}, t) = \sqrt{|\rho|}e^{i\varphi(\mathbf{x},t)}, \tag{4.5}$$

the current simplifies to

$$\mathbf{j} = \frac{\rho\hbar}{m}\left(\nabla\varphi - \frac{q}{\hbar}\mathbf{A}\right). \tag{4.6}$$

The interaction between the Cooper current and a magnetic field, obeying Maxwell's equations,

$$\nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0 c^2}, \quad \nabla \cdot \mathbf{B} = 0, \tag{4.7}$$

implies

$$\begin{aligned} -\nabla^2 \mathbf{B} &= \nabla \times (\nabla \times \mathbf{B}) = \frac{1}{\epsilon_0 c^2} \nabla \times \mathbf{j} = -\frac{\rho q}{m \epsilon_0 c^2} \mathbf{B}, \\ &\Rightarrow \quad \nabla^2 \mathbf{B} = \frac{1}{\lambda_{\mathrm{L}}^2} \mathbf{B}, \end{aligned} \tag{4.8}$$

where $\lambda_{\mathrm{L}} = \sqrt{\frac{m\epsilon_0 c^2}{\rho |q|}}$ is the London penetration depth. Consider, e.g., a magnetic field close to a superconductor,

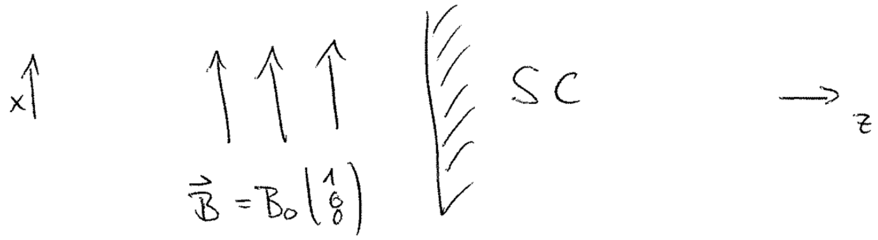


Abbildung 4.1: Magnetic field close to a superconductor exhibiting the Meissner effect.

then the field inside the superconductor has to obey Eq. (4.8), yielding

$$\mathbf{B}(z) = B_0(e^{-z/\lambda_{\mathrm{L}}}, 0, 0), \quad z \geq 0. \tag{4.9}$$

The magnetic field vanishes exponentially inside the superconductor. This is the *Meissner-Ochsenfeld* effect.

For a Type-II superconductor, the magnetic field can penetrate the superconductor in form of thin magnetic flux tubes, called *Abrikosov vortices*. This is also called the Shubnikov phase.

The flux through the vortex yields

$$\Phi_B = \oint_\gamma d\mathbf{x} \cdot \mathbf{A} = \frac{\hbar}{q} \oint_\gamma d\mathbf{x} \cdot \nabla \varphi = \frac{2\pi\hbar}{q} n, \quad n \in \mathbb{Z}, \tag{4.10}$$

where γ is a closed contour around the vortex.

In the second step, we have used Eq. (4.6) together with the fact that the current vanishes far away from the vortex core, since $\mathbf{j} = \epsilon_0 c^2 \nabla \times \mathbf{B}$ and $\mathbf{B} = 0$ inside the

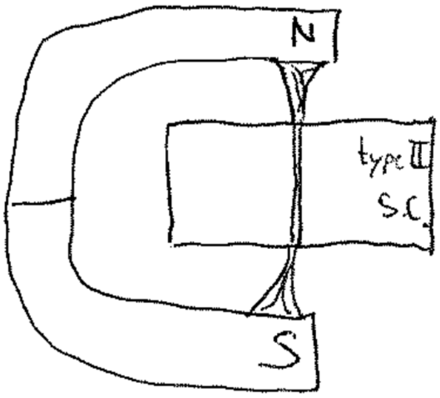


Abbildung 4.2: Abrikosov vortices penetrating a Type-II superconductor.

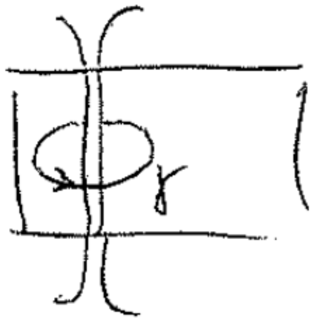


Abbildung 4.3: Contour γ around a magnetic vortex in a superconductor.

superconducting bulk. In Eq. (4.10), we have also used that the phase φ has to be single-valued, i.e., φ can change only by integer multiples of 2π when going around the contour γ. This results in flux quantization, $\Phi_B = n\Phi_0$, where $\Phi_0 = \frac{2\pi\hbar}{|q|}$ is the flux quantum.

Now, imagine that we have two magnetic monopoles of opposite charge "N" and "S" at our disposal. If we bring these monopoles into a Type-II superconductor, the Meissner-Ochsenfeld effect enforces a string-like flux distribution.

The static potential between the monopoles would then be linear inside the superconductor, $V^{\text{NS}}_{\text{s.c.}}(R) \sim \sigma R$ in contrast to the vacuum potential, $V^{\text{NS}}_{\text{vac}}(R) \sim \frac{1}{R}$. (NB: Incidentally, if monopoles existed, the flux distribution inside a Type-I superconductor would also be string-like for two-static monopoles).

This gedanken experiment gives rose to a confinement picture in QCD based on

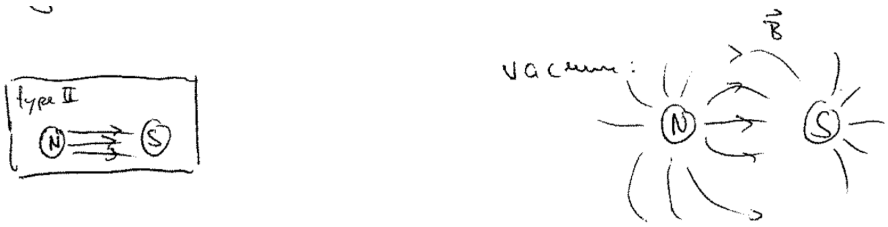
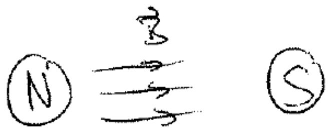
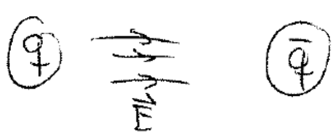


Abbildung 4.4: Magnetic flux string between two monopoles in a Type-II superconductor.

a hypothetical *dual Meissner effect* (t'Hooft, Mandelstam 1976):

superconductor	QCD vacuum
magn. Meissner effect magnetic flux quantization	electric Meissner effect electric flux quantization (OK, since quark charges are quantized)
	
condensation of electric charges (Cooper pairs)	condensation of magnetic charges (magnetic monopole pairs?)

(4.11)

The obvious question is: are there field configurations in Yang-Mills theories with a monopole-like charge content?

4.2 Magnetic monopoles in abelian gauge theory

The source-free Maxwell equations exhibit a duality symmetry between electric and magnetic fields, $\mathbf{E} \leftarrow \mathbf{B}$, $\mathbf{B} \rightarrow -\mathbf{E}$. Promoting this symmetry to hold also in the presence of sources requires the existence of magnetic charges (monopoles) and currents. E.g., electro-magneto statics is described by

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \cdot \mathbf{B} = \rho_{\mathrm{M}}, \tag{4.12}$$

where ρ_{M} denotes the magnetic charge density. The magnetic field of a δ-like magnetic point source at the origin, $\rho_{\mathrm{M}}(\mathbf{r}) = g\delta^{(3)}(\mathbf{r})$, is given by

$$\mathbf{B} = \frac{g}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}, \tag{4.13}$$

which is just the analogue of the electric Coulomb field with magnetic coupling g replacing the electric charge. However, representing this field by a gauge potential

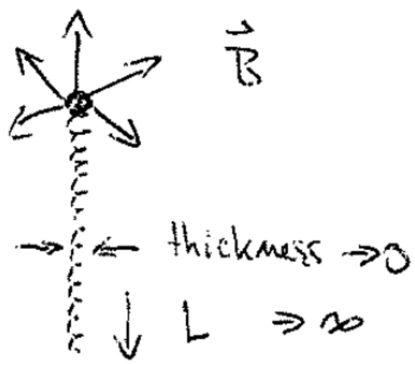


Abbildung 4.5: Magnetic monopole with Dirac string (infinitely long, infinitely thin solenoid) along the negative z-axis.

requires a singular structure,

$$g = \int_V d^3x \rho_{\mathrm{M}} = \int_V d^3x \nabla \cdot \mathbf{B} \stackrel{?}{=} \int_V d^3x \nabla \cdot (\nabla \times \mathbf{A}) = 0. \tag{4.14}$$

If $\mathbf{A}$ is non-singular but regular, the last term vanishes.

The monompole potential can be constructed from that of an idealized infinitely long solenoid:

$$\mathbf{A} = \frac{g}{4\pi r} \frac{1 - \cos\theta}{r \sin\theta} \hat{\boldsymbol{\phi}} \equiv \frac{g}{4\pi}(1 - \cos\theta)\nabla\phi, \tag{4.15}$$

which produces the magnetic field (4.13) everywhere except on the negative z-axis ($\theta = \pi$), where it is singular. This singularity is known as the *Dirac string*:

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{g}{4\pi} \frac{\hat{\mathbf{r}}}{r^2} + g\,\delta(x)\delta(y)\Theta(-z)\hat{\mathbf{z}}, \tag{4.16}$$

where the second term describes the magnetic flux of a solenoid along the negative z axis.

Can the solenoid be detected in the limit of vanishing thickness and infinite length? Classically, the answer would be no. However, quantum mechanically, the answer can be yes because of the Aharonov-Bohm effect:

But even the Aharonov-Bohm effect remains invisible for a solenoid flux being a multiple of the flux quantum:

$$\left.\begin{array}{l} \Phi = \dfrac{2\pi\hbar}{q} n \\ \| \\ \Phi_{\text{solenoid}} = g \end{array}\right\} \Rightarrow \frac{qg}{2\pi\hbar} = n \in \mathbb{Z}, \tag{4.17}$$

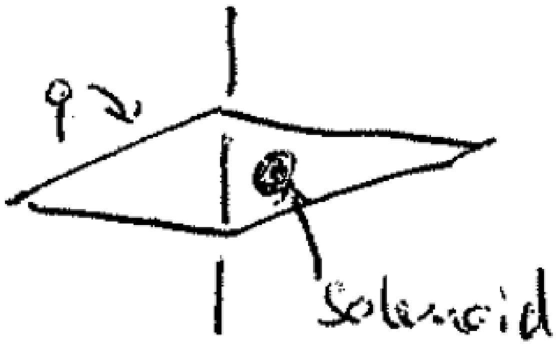


Abbildung 4.6: Aharonov-Bohm effect: Electron wavefunction split into two parts encircling a solenoid with magnetic flux Φ. The interference pattern depends on Φ even if the electrons never pass through a region with non-vanishing magnetic field.

since then the interference pattern is shifted by an integer multiple of 2π. The last equation is the famous *Dirac quantization condition* relating electric charge q and magnetic charge g. It states that the existence of even a single magnetic monopole in the universe requires all electric charges to be quantized.

4.3 Magnetic monopoles and confinement in compact U(1) gauge theory in $d = 3$