13 Modeling Inflation

As seen in the previous section, an inflationary phase with $\frac{d}{dt}\frac{1}{(aH)} < 0$ (shrinking Hubble radius) can be established with a fluid satisfying $(1+3w) < 0$ (violating the strong energy condition). Before we come to concrete models satisfying this property, let us try to derive general properties independent of a specific model. Consider

$$0 > \frac{d}{dt}\frac{1}{aH} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} =: -\frac{1}{a}(1-\varepsilon), \tag{13.1}$$

where we have defined the slow-roll parameter

$$\begin{aligned}
\varepsilon &= -\frac{\dot{H}}{H^2} = -\frac{1}{H^2}\frac{d}{dt}H \\
&= -\frac{1}{H^2}\frac{da}{dt}\frac{dH}{da} = -\frac{1}{H^2}\underbrace{\frac{\dot{a}}{a}}_{H}a\frac{dH}{da} \\
&= -\frac{1}{H}\frac{dH}{d\ln a} = -\frac{d\ln H}{dN}
\end{aligned} \tag{13.2}$$

where $N = \ln a$. Eq. (13.1) shows that a shrinking Hubble radius requires $\varepsilon < 1$.

Eq (13.2) in turn shows that $\varepsilon < 1$ requires that the relative change of the Hubble parameter, $d\ln H = \frac{1}{H} dH$ per e-fold dN is small. In this sense, inflation is a phase where physical quantities related to the Hubble parameter vary slowly. If ε satisfies $\varepsilon \ll 1$, one speaks of <u>slow-roll inflation</u>. In that case, the assumption $H_i \simeq H_e$ made in the previous section is fully justified. In the limit $\varepsilon \to 0$, we have $H = \text{const}$, which — according to the Friedmann equation corresponds to a solution where dark energy dominates and spacetime is approximately a <u>de Sitter space</u>

$$ds^2 = -dt^2 + e^{2Ht} d\vec{x}^2 \tag{13.3}$$

Of course, inflation must end at some point. Therefore, also spacetime must become different from de Sitter space such that (13.3) can only be approximation during the inflationary phase. Inflation is thus also called a quasi de Sitter phase.

In order to have a sufficiently long inflationary phase (~ 60 efolds), $\varepsilon < 1$ has to hold for a sufficiently long time. This condition is measured by a second slow-roll parameter (often called also η)

$$\kappa = \frac{d \ln \varepsilon}{dN} = \frac{\dot{\varepsilon}}{H\varepsilon} . \qquad (13.4)$$

For $|\kappa| < 1$, the relative change of ε per e-fold is small and inflation pesists.

Models of inflation are typically formulated using scalar fields. They constitute the simplest examples. Such a scalar field should be thought of as an effective maybe even emergent degree of freedom that can be treated as classical. It may have a quantum origin (maybe as a matter composite, maybe as a part of (quantum) gravity), but such details are not important.

A generic scalar field in Minkowski space has an action of the type

$$S = \int dt\, d^3x \left[\underbrace{\tfrac{1}{2}\dot\phi^2}_{\text{kinetic energy}} - \tfrac{1}{2}(\vec\nabla\phi)^2 - \underbrace{V(\phi)}_{\text{potential}} \right] \qquad (13.5)$$

The equation of motion follows from the action principle

$$\delta S = \int dt\, d^3x \left[\dot\phi\, \delta\dot\phi - \vec\nabla\phi\, \vec\nabla\delta\phi - \frac{\partial V}{\partial\phi}\delta\phi \right]$$

$$\overset{\text{i.b.p}}{=} \int dt\, d^3x \left[\left(-\ddot\phi + \vec\nabla^2\phi - \frac{\partial V}{\partial \phi} \right) \delta\phi \right] \qquad (13.6)$$

where we assume that the variations $\delta\phi$ are chosen to vanish on the boundary. Since $\delta\phi$ is otherwise arbitrary, we get from $\delta S = 0$,

$$\ddot\phi - \nabla^2\phi \equiv -\Box\phi = -\frac{\partial V}{\partial\phi}, \qquad (13.7)$$

which is the Klein-Gordon equation with a general potential. The standard form is obtained for a purely quadratic potential $V(\phi) = \frac{1}{2}m^2\phi^2$, where m is a mass parameter.

Let us generalize this derivation to FLRW spacetimes. For this, all we need to know is that infinitesimal spatial coordinate elements acquire a scale factor, $dx_i \to a\, dx_i$.

Hence, the action reads

$$S = \int dt\, d^3x\; a^3(t) \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2(t)} (\vec{\nabla}\phi)^2 - V(\phi) \right] \tag{13.8}$$

For simplicity, it suffices to consider homogeneous fields (cosmological principle!) such that

$$S = \int dt\, d^3x\; a^3(t) \left[\frac{1}{2}\dot{\phi}^2 - V(\phi) \right] \tag{13.9}$$

The variational principle yields

$$0 = \delta S = \int dt\, d^3x\; a^3(t) \left[\dot{\phi}\, \delta\dot{\phi} - \frac{\partial V}{\partial \phi}\delta\phi \right]$$

$$= \int dt\, d^3x \left[-\frac{d}{dt}\left(a^3 \dot{\phi}\right) - a^3 \frac{\partial V}{\partial \phi} \right] \delta\phi, \tag{13.10}$$

from which we get

$$a^3 \ddot{\phi} + 3a^2 \frac{da}{dt}\dot{\phi} = -a^3 \frac{\partial V}{\partial \phi}$$

$$\Rightarrow \qquad \ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial \phi} \qquad (13.11)$$

The second term is reminiscent to a friction term in classical mechanics. This so-called <u>Hubble friction</u> plays a crucial role in inflationary dynamics.

Let us now assume that the scalar field dominates the cosmological evolution in between η_i and η_e (before the hot big bang). For the Friedmann equation, we need the energy density and the pressure associated with the scalar field. This, we have already worked out in the exercises, yielding

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \,, \quad P_\phi = \frac{1}{2}\dot{\phi} - V(\phi) \,. \qquad (13.12)$$

The change of the energy density can be related to the Hubble friction, because

$$\dot{\rho}_\phi = \left(\ddot{\phi} + \frac{\partial V}{\partial \phi}\right)\dot{\phi} = -3H\dot{\phi}^2 \,. \qquad (13.13)$$

As also discussed in the exercises, the equation of state

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)} \qquad (13.14)$$

is in general not constant and ranges in $-1 \leq w_\phi \leq 1$. In the case of a small kinetic energy $\frac{1}{2}\dot{\phi} \ll V(\phi)$, i.e. a slow motion of the amplitude, we have $w_\phi \simeq -1$ similar to a cosmological constant and thus implying an exponential expansion and a shrinking Hubble radius.

The complete dynamics of FLRW spacetime and the scalar field is given by (13.11) together with the Friedmann equation

$$H^2 \overset{(4.10)}{=} \frac{8\pi G}{3} \rho_\phi = \frac{8\pi}{3 m_P^2} \rho_\phi \qquad (13.15)$$

where we have used the Planck mass as defined in Eq. (0.2). For reasons of convenience let us introduce the "reduced Planck mass"

$$M_P^2 = \frac{m_P^2}{8\pi} \quad \Rightarrow \quad \underline{H^2 = \frac{1}{3M_P^2}\rho_\phi} \qquad (13.16)$$

Let us study the evolution of the Hubble parameter

$$\frac{d}{dt}H^2 = 2H\dot{H} \overset{(13.16)}{=} \frac{1}{3M_P^2}\dot{\rho}_\phi \overset{(13.13)}{=} \frac{1}{3M_P^2}(-3)H\dot{\phi}^2$$

$$\Rightarrow \quad \dot{H} = -\frac{1}{2M_P^2}\dot{\phi}^2 \qquad (13.17)$$

This allows us to determine the slow-roll parameter

$$\varepsilon \overset{(13.2)}{=} -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}^2}{2M_P^2 H^2} \underset{(13.12)}{\overset{(13.16)}{=}} \frac{\frac{3}{2}\dot{\phi}^2}{\frac{1}{2}\dot{\phi} + V} \qquad (13.18)$$

Inflation requires $(1+3w_\phi) < 0$ i.e. $w_\phi < -\frac{1}{3}$ which necessitates according to (13.14):

$$\frac{1}{2}\dot{\phi}^2 < \frac{1}{2}V \qquad (13.19)$$

The case that $\frac{1}{2}\dot{\phi}^2 \ll V$ where $w_\phi \simeq -1$ goes along with $\varepsilon \ll 1$, cf. (13.18). This case is called <u>slow-roll</u> inflation.

In order to remain in a slow-role state, the acceleration of the scalar field amplitude has to be small. A useful quantity to consider is

$$\delta := -\frac{\ddot{\phi}}{H\dot{\phi}} \qquad (13.20)$$

which is dimensionless. For small δ, the Hubble friction dominates in the Klein-Gordon equation. The kinetic energy $\frac{1}{2}\dot{\phi}^2$ stays small and inflation continues. This can also be seen using

$$\dot{\varepsilon} = \frac{\dot{\phi}\ddot{\phi}}{M_p^2 H^2} - \frac{\dot{\phi}^2 \dot{H}}{M_p^2 H^3} \qquad (13.21)$$

and inserting it into (13.4):

$$K = \frac{\dot{\varepsilon}}{H\varepsilon} = \frac{1}{H}\left(\frac{M_p^2 H^2}{\frac{1}{2}\dot{\phi}^2}\right)\left(\frac{\dot{\phi}\ddot{\phi}}{M_p^2 H^2} - \frac{\dot{\phi}^2 \dot{H}}{M_p^2 H^3}\right)$$

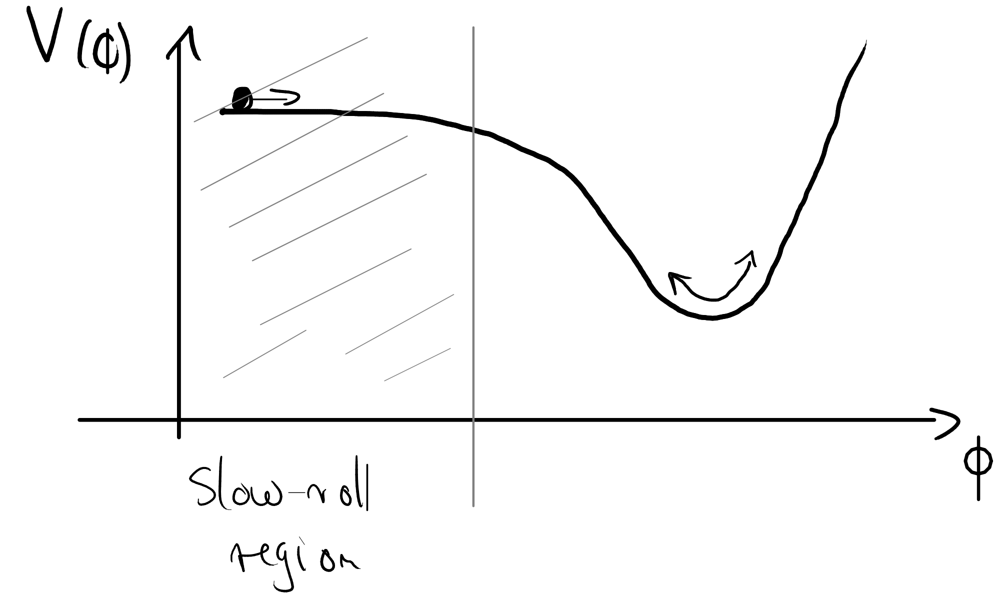
$$= 2\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H^2} = 2(\varepsilon - \delta) \qquad (13.22)$$

We conclude that $\varepsilon, \delta \ll 1$ also entail $|K| \ll 1$.

If the amplitude velocity and acceleration of the inflaton are small, inflation will last for a long time.

Of course, the dynamics of the amplitude is governed by the shape of the inflaton potential.

A generic potential may look like this:



So far, all equations are exact. However, in the case when $\epsilon, \delta, |\kappa| \ll 1$, we can simplify the analysis using the slow-roll approximation:

$\epsilon \ll 1$ implies $\dot{\phi}^2 \ll V$ such that the Friedmann equation simplifies to

$$H^2 \simeq \frac{V}{3M_p^2} \qquad (13.23)$$

such that the Hubble rate is determined by the potential. The condition $\delta \ll 1$ implies that the Klein-Gordon equation simplifies to

$$3H\dot{\phi} \simeq -\frac{\partial V}{\partial \phi} \qquad (13.24)$$

linking the slope of the potential to the speed of the amplitude. Consequently, we have

$$\underline{\underline{\varepsilon}} = \frac{\frac{1}{2}\dot{\phi}^2}{M_p^2 H^2} = \frac{\frac{1}{2}\left(\frac{\partial V}{\partial \phi}\frac{1}{3H}\right)^2}{M_p^2 \; H^2} = \frac{\frac{1}{2}\left(\frac{\partial V}{\partial \phi}\right)^2}{9 M_p^2 \; H^4}$$

$$\overset{(13.23)}{=} \underline{\underline{\frac{M_p^2}{2}\left(\frac{\frac{\partial V}{\partial \phi}}{V}\right)^2}} \tag{13.25}$$

which expresses ε in terms of the potential. Similarly, using (13.24), $3\dot{H}\dot{\phi} + 3H\ddot{\phi} \simeq -\frac{\partial^2 V}{\partial \phi^2}\dot{\phi}$, we can express $\delta+\varepsilon$ as

$$\delta + \varepsilon = -\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} = -\frac{1}{H\dot{\phi}}\left(-\frac{\partial^2 V}{\partial \phi^2}\frac{\dot{\phi}}{3H} - \cancel{\frac{\dot{H}\dot{\phi}}{H}}\right) \cancel{-\frac{\dot{H}}{H^2}}$$

$$\simeq M_p^2 \frac{\frac{\partial^2 V}{\partial \phi^2}}{V} \tag{13.26}$$

Hence, potentials can be classified according to their potential slow-roll parameters

$$\varepsilon_V := \frac{M_p^2}{2}\left(\frac{\frac{\partial V}{\partial \phi}}{V}\right)^2, \quad \eta_V := M_p^2 \frac{\frac{\partial^2 V}{\partial \phi^2}}{V} \tag{13.27}$$

They are linked to the (Hubble) slow-roll parameters in the slow-roll approximation by

$$\varepsilon_V \simeq \varepsilon \quad , \quad \eta_V \simeq 2\varepsilon - \frac{1}{2}\kappa \qquad (13.28)$$

Since the slow-roll behavior is useful for a sufficiently long period of inflation, we can also relate the number of e-folds to the potential in slow-roll approximation

$$N = \int_{a_i}^{a_e} d\ln a = \int_{t_i}^{t_e} H(t)\, dt = \int_{\phi_i}^{\phi_e} \frac{H}{\dot{\phi}}\, d\phi \qquad (13.29)$$

Here, t_i and t_e are defined as the times of entrance and exit of the inflationary phase. Since $\frac{d}{dt}\frac{1}{aH} = -\frac{1}{a}(1-\varepsilon)$, we have $\varepsilon(t_i) = 1 = \varepsilon(t_e)$.

Using $\varepsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_p^2 H^2}$, we have in slow-roll approximation

$$N \simeq \int_{\varphi_i}^{\varphi_e} \frac{1}{\sqrt{2\varepsilon_V}} \frac{|d\phi|}{M_p} \qquad (13.30)$$

Here, $|d\phi|$ indicates that the sign is chosen such that N is positive (inflation can correspond to a roll inwards or outwards, both is acceptable).

For a solution of the horizon problem, we need $N \gtrsim 60$ which provides a constraint on the possible models of inflation.

Let us exemplify these findings using a simple example: quadratic inflation

$$V(\phi) = \frac{1}{2} m^2 \phi^2 \tag{13.31}$$

(The model has by now been ruled out by CMB data, but it is still an illustrative example.)

The potential slow-roll parameters are

$$\varepsilon_V = \varepsilon_V(\phi) = \frac{M_P^2}{2} \left(\frac{\frac{\partial V}{\partial \phi}}{V}\right)^2 = 2\left(\frac{M_P}{\phi}\right)^2 \tag{13.32}$$

$$\eta_V = \eta_V(\phi) = M_P^2 \frac{\frac{\partial^2 V}{\partial \phi^2}}{V} = 2\left(\frac{M_P}{\phi}\right)^2 \equiv \varepsilon_V .$$

In order to meet the slow-roll criteria, we at least need $\varepsilon_V, \eta_V < 1$. Hence, the field amplitude needs to acquire super-Planckian values during inflation

$$\phi > \sqrt{2}\, M_P . \tag{13.33}$$

Since the minimum is at $\phi = 0$, (13.33) defines

the field amplitude where the universe exists inflation:
$$\phi_e = \sqrt{2}\, M_p \qquad (13.34)$$

Correspondingly, we need $\phi_i > \phi_e$ large enough to provide for sufficiently many e-folds:
$$N = \int_{\phi_e}^{\phi_i} \frac{d\phi}{M_p} \frac{1}{\sqrt{2\varepsilon_V}} = \int_{\phi_e}^{\phi_i} \frac{d\phi}{M_p} \frac{\phi}{2M_p} = \frac{\phi^2}{4M_p^2}\Big|_{\phi_e}^{\phi_i}$$
$$= \frac{\phi_i^2}{4M_p^2} - \frac{1}{2} \,. \qquad (13.35)$$

For $N \gtrsim 60$, the initial field amplitude must satisfy
$$\phi_i \gtrsim 2\sqrt{60}\, M_p \simeq 15\, M_p \qquad (13.36)$$

This means that the field amplitude is super-Planckian during all of the inflationary phase. Such models are often referred to as large-field inflation. It is a viable question as too whether the model should be trusted for such large amplitudes where we naively expect quantum gravity to be relevant.

As visible in the simple model but generically true in inflationary models, most of the energy density during inflation is in the potential energy of the inflaton field.

Inflation ends, when the field acquires more kinetic energy violating (13.19). This may happen when the amplitude approaches a region where the potential becomes steeper.

The energy in the inflaton sector then has to be transferred to the particle sector of the standard model. This is what is meant by <u>reheating</u>: a process that is difficult to describe accurately but which kicks off the hot big bang.

To see what is needed for this, let us assume that inflation ends near a minimum of the potential that can be approximated by $V(\phi) \simeq \frac{1}{2} m^2 \phi^2$. Then the equation of motion reads

$$\ddot{\phi} + 3H\dot{\phi} = -m^2\phi . \qquad (13.37)$$

The energy density satisfies according to (13.13)

$$\dot{\rho}_\phi = -3H\dot{\phi}^2 = -3H\Big(\underbrace{\tfrac{1}{2}\dot{\phi}^2 + V(\phi)}_{=\rho_\phi} + \underbrace{\tfrac{1}{2}\dot{\phi}^2 - V(\phi)}_{=P_\phi}\Big)$$

$$\Rightarrow \quad \dot{\rho}_\phi + 3H\rho_\phi = -3H P_\phi = \frac{3}{2}H\left(m^2\phi^2 - \dot{\phi}^2\right) \qquad (13.38)$$

Near the minimum, the field will start to oscillate like a damped harmonic oscillator, cf. Eq. (13.37). Assuming that the Hubble rate is approximately constant over an oscillation, the right hand side averages to zero over an oscillation period.

We would be left – on average – with

$$0 = \dot{\rho}_\phi + 3H\rho_\phi = \dot{\rho}_\phi + 3\frac{\dot{a}}{a}\rho_\phi \qquad (13.39)$$

implying that ρ_ϕ behaves like pressureless matter, $\rho_\phi \sim \frac{1}{a^3}$. With the decrease of ρ_ϕ, also the amplitude of the oscillation would decrease and the universe would end up completely empty.

In order to avoid this fate, the inflaton <u>must</u> have a coupling to the standard model fields such that the inflaton can decay into standard model fields. Let us assume that this decay is parametrized by a decay rate Γ_ϕ. Then Eq. (13.39) would read

$$\dot{\rho} + 3H\rho_\phi = -\Gamma_\phi \rho_\phi . \qquad (13.40)$$

The details of the decay and its time scales depends on the coupling between the inflaton and the standard model fields.

A rapid decay is possible if ϕ couples to bosons (e.g. the Higgs field), since then non-perturbative phenomena such as parametric resonance (think of a coupled double pendulum) are possible. Such a rapid decay is called <u>preheating</u>, since the corresponding standard model field would be created far from equilibrium.

This state would subsequently couple to all other standard model fields and eventually reach thermal equilibrium.

This equilibrium state then characterizes the initial state of the hot big bang at the <u>reheating</u> surface. The energy density at reheating satisfies $\rho_R < \rho_{\phi,e}$ depending on how efficient the conversion of $\rho_{\phi,e}$ at the end of inflation is. The reheating temperature is then given by

$$\rho_R = \frac{\pi^2}{30} g_*(T_R)\, T_R^4 \qquad (13.41)$$

From BBN, we know that T_R must have been higher than 1MeV. If we eventually understand baryogenesis within a hot big bang scenario, T_R must exceed the corresponding energy scale. Other than this, we have no direct data or constraints on T_R.

Because of this very limited amount of data, there is a plethora of viable inflationary models — and currently little hope to falsify a significant number of them.