12 The Inflationary Paradigm

The cosmological naturalness problems suggest that there was an epoch "before" the hot big bang (i.e. the phases discussed so far) that provided for the unnatural initial conditions and correlations in a natural way. This phase is called the inflationary phase. Often, inflation is introduced in terms of a concrete model involving scalar fields. We will also use this description further below.

But first, we wish to emphasize that inflation is mostly a paradigm rather than a concrete theory. In fact, there is a current discussion (even involving some of the founders of inflation on both sides of the discussion) whether inflation really is a "theory", because it is rather unspecific

(many different ideas work well), there are many parameters but only few observations to single out "the true" inflationary model.
Therefore, I call it a paradigm in the sense that whatever the true model of the universe is, it has to reflect the features that all the models of inflation share. Whether it does so in the way a generic model does, or whether it works in a different way in detail, remains to be seen.

It's useful to think of inflation as a phase in the early universe in which the Hubble radius decreased

$$\frac{d}{dt}\frac{1}{(aH)} < 0 . \qquad (12.1)$$

Of course, since $aH = \dot{a}$, this condition implies

$$\frac{d}{dt}\frac{1}{\dot{a}} = -\frac{\ddot{a}}{(\dot{a})^2} < 0 \Rightarrow \underline{\underline{\ddot{a} > 0}} \qquad (12.2)$$

which demonstrates that a phase with decreasing Hubble radius is synonymous to a phase of accelerated expansion.

Let's see how this property "solves" the various naturalness problems, starting with the horizon problem. Consider again the particle horizon

$$d_H(\eta) \overset{(11.2)}{=} \int_{\ln a_i}^{\ln a} d\ln a \; \frac{1}{aH} \qquad (12.3)$$

We have argued below (11.3) that the integral is dominated by the maximum region of $\frac{1}{aH}$. For a standard big bang cosmology, this is at late times such that $d_H \sim \frac{1}{aH}$. However, if $\frac{1}{aH}$ had a decreasing phase at early times, d_H can now be dominated by the maximum at early times $d_H \sim \frac{1}{a_i H_i}$. If parameters are suitably adjusted, we may even have

$d_H \sim \frac{1}{a_i H_i} \gg \frac{1}{aH}$.

Let us make this more explicit. Consider a single-fluid approximation with equation of state $w = \frac{P}{\rho}$. Then, the Friedmann equation is solved by

$$\frac{1}{aH} = \frac{1}{H_0} a^{\frac{1}{2}(1+3w)} . \qquad (12.4)$$

Upon insertion into (12.3), we get

$$d_H = \int_{a_i}^{a} \frac{da}{a} \frac{1}{H_0} a^{\frac{1}{2}(1+3w)}$$

$$= \frac{2}{(1+3w)} \frac{1}{H_0} \left[a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)} \right]$$

$$= \eta - \eta_i . \qquad (12.5)$$

For ordinary matter satisfying the strong energy condition $1+3w>0$, the Hubble radius grows and the particle horizon d_H is dominated by late times where $a \gg a_i$. E.g. for $\eta_i = 0$ and $a_i = 0$ (initial singularity), we get $d_H = \eta$ as before.

Let us now take $1+3w<0$. The Hubble

radius in (12.4) now shrinks and the particle horizon may receive large contributions from early times. Considering the contribution from the lower bound, we now have

$$\eta_i = \frac{2}{(1+3w)} \frac{1}{H_0} a_i^{\frac{1}{2}(1+3w)} \qquad (12.6)$$

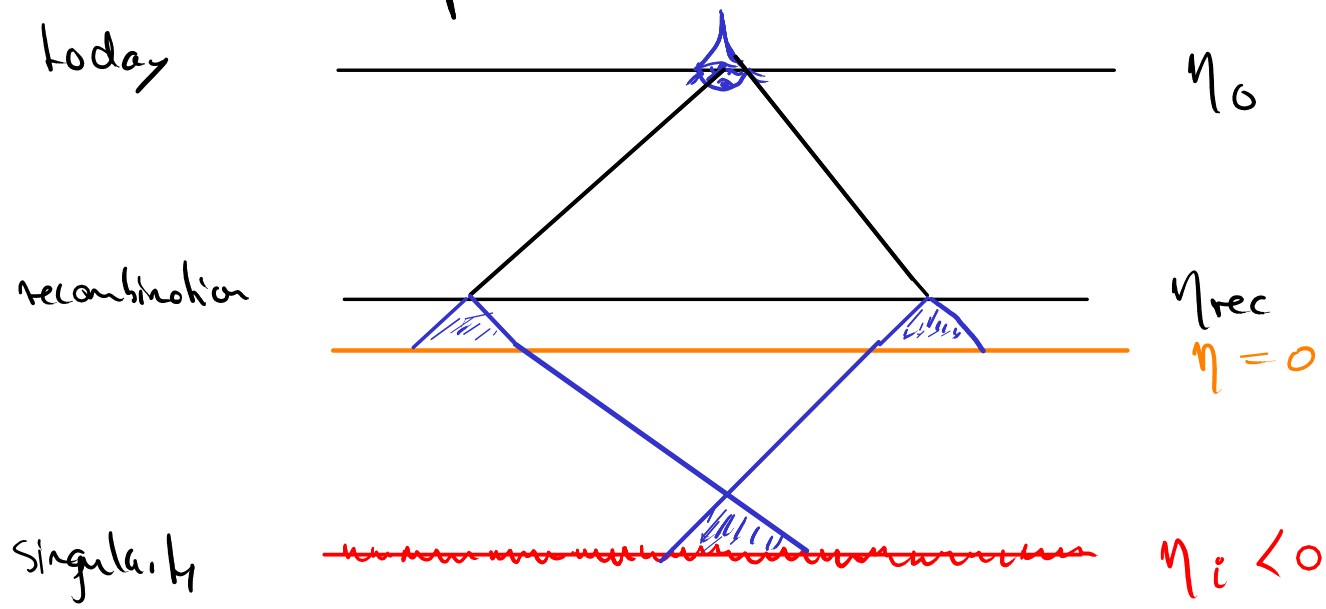
For $a_i \to 0$ and $w < -\frac{1}{3}$, we have

$$\eta_i \to -\infty . \qquad (12.7)$$

The hypothesized phase of shrinking Hubble radius pushes the big bang singularity to negative conformal time. In this way, there can be much more conformal time between the initial singularity and, say, the time of recombination / photon-decoupling.

In an inflationary cosmology, $\eta = 0$ is not a singularity, but a transition time between inflation and the hot big-bang phase. (Still, the big bang singularity is at $t = 0$.)

This suggests the following graphical resolution of the horizon problem:



Now, the backward light cones of two opposite points where the CMB was formed, can have an overlap at the singularity. The common temperature at η_{rec} can have a causally connected common origin.

The new ingredient (in addition to $1+3w < 0$) is a so-called reheating surface replacing the singularity at $\eta = 0$. I.e. an inflationary model must entail a mechanism that heats up the universe at about $\eta = 0$.

The decreasing Hubble radius also solves the flatness problem. To see this, let us first note that the curvature contribution

Ω_k to the Friedmann equation scales with $\frac{1}{a^2}$, see (4.18). Given some initial curvature contribution $\Omega_k(t_i)$, at any other time we have

$$\Omega_k(t) = \frac{(a_i H_i)^2}{(a H)^2} \, \Omega_k(t_i) \qquad (12.8)$$

For given initial parameters, $\Omega_k(t)$ decreases for a decreasing Hubble radius $\frac{1}{aH}$. This can solve the flatness problem.

More explicitly, consider again a single-fluid approximation with equation of state w and curvature. The Friedmann equation then reads, cf. (4.18)

$$\frac{H^2}{H_i^2} = \left[\Omega_{f,i} \, a^{-(3+3w)} + \Omega_{k,i} \frac{a_i^2}{a^2} \right] \qquad (12.9)$$

where we have used the time t_i as normalization point. At $t = t_i$, we have

$$1 = \Omega_{f,i} \, a_i^{-(3+3w)} + \Omega_{k,i}$$

$$\Rightarrow \quad \Omega_{f,i} = \frac{1 - \Omega_{k,i}}{a_i^{-(3+3w)}} \qquad (12.10)$$

such that the Friedmann equation can be written

$$\overset{(12.9)}{\Longrightarrow} \quad (aH)^2 = (a_i H_i)^2 \left[(1-\Omega_{k,i}) \left(\frac{a}{a_i}\right)^{-(1+3w)} + \Omega_{k,i} \right] \tag{12.11}$$

Upon insertion into (12.8), we obtain

$$\Omega_k(t) = \frac{\Omega_{k,i}}{(1-\Omega_{k,i})\left(\frac{a}{a_i}\right)^{-(1+3w)} + \Omega_{k,i}} \ . \tag{12.12}$$

Defining the so-called number of "e-folds",

$$N := \ln \frac{a}{a_i} \ , \tag{12.13}$$

being a measure for the magnitude of exponential growth, we obtain

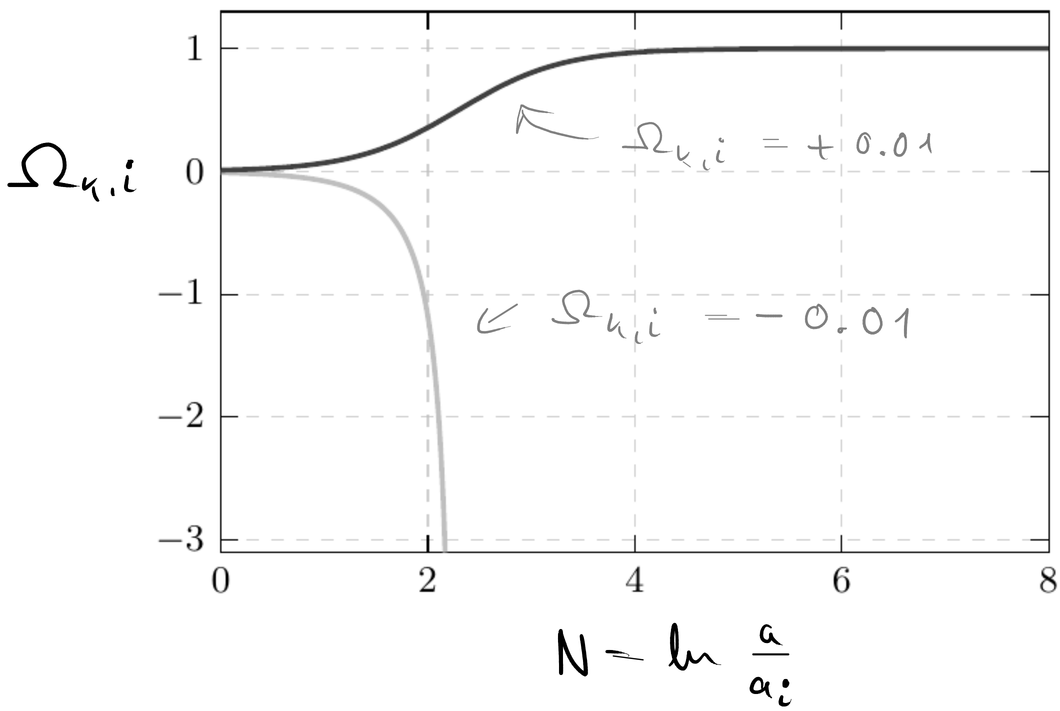
$$\Omega_k(N) = \frac{\Omega_{k,i}\, e^{(1+3w)N}}{(1-\Omega_{k,i}) + \Omega_{k,i}\, e^{(1+3w)N}} \tag{12.14}$$

For an initial condition $\Omega_{k,i}$, (12.14) describes the evolution of the curvature contribution. This evolution has two fixed points:

$\Omega_{k,i} = 0$ (flat universe) and

$\Omega_{k,i} = 1$ (a curved but empty, so called Milne universe).

For $(1+3w) > 0$ such as radiation or matter satisfying the strong energy condition, the fixed point $\Omega_{k,i} = 0$ is unstable, i.e. any deviation grows with time. For instance for $w = \frac{1}{3}$, we have:

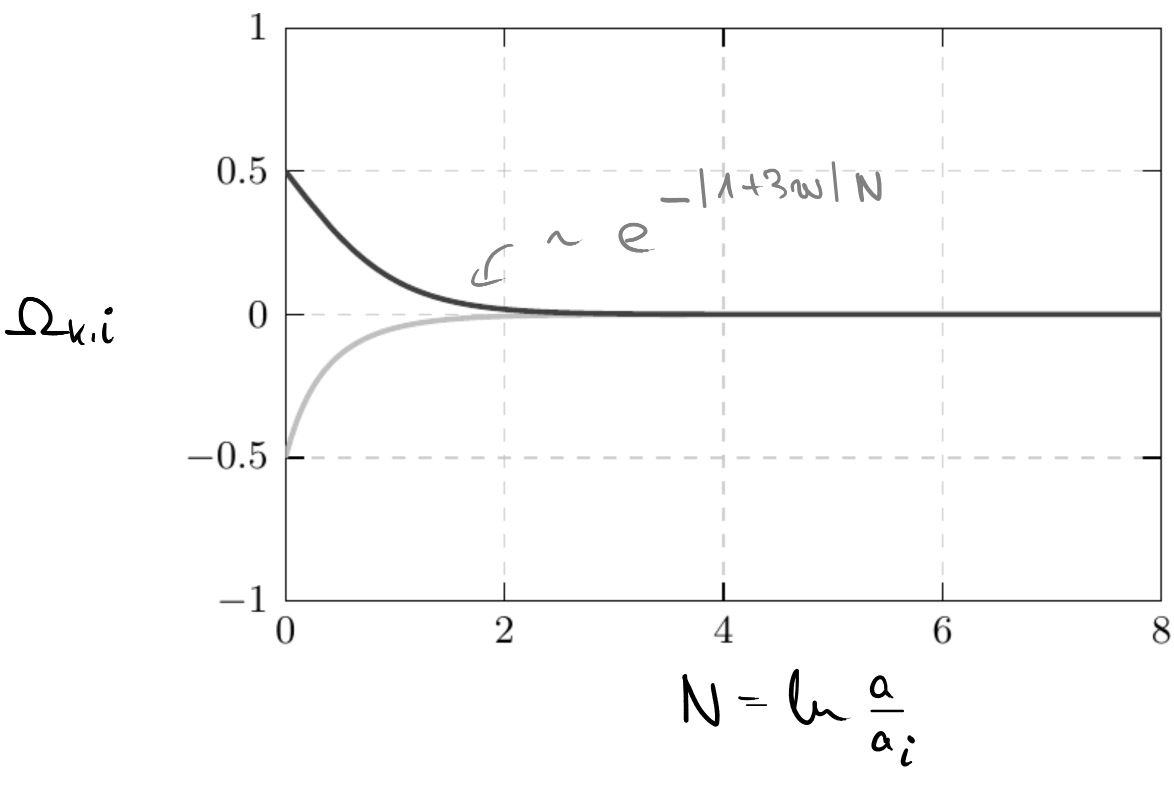


For initial conditions $\Omega_{k,i} > 0$, the universe ends up in a Milne universe, for $\Omega_{k,i} < 0$, the growth accelerates and Ω_k diverges at

$$N = \frac{1}{1+3w} \ln \frac{\Omega_{k,i} - 1}{\Omega_{k,i}} \tag{12.15}$$

where the denominator of (12.14) becomes zero. The fact that we observe $\Omega_k < 0.005$ today implies that the initial conditions $\Omega_{k,i}$ have to be tuned very close to zero for an $(1+3w) > 0$ universe, reflecting the flatness problem.

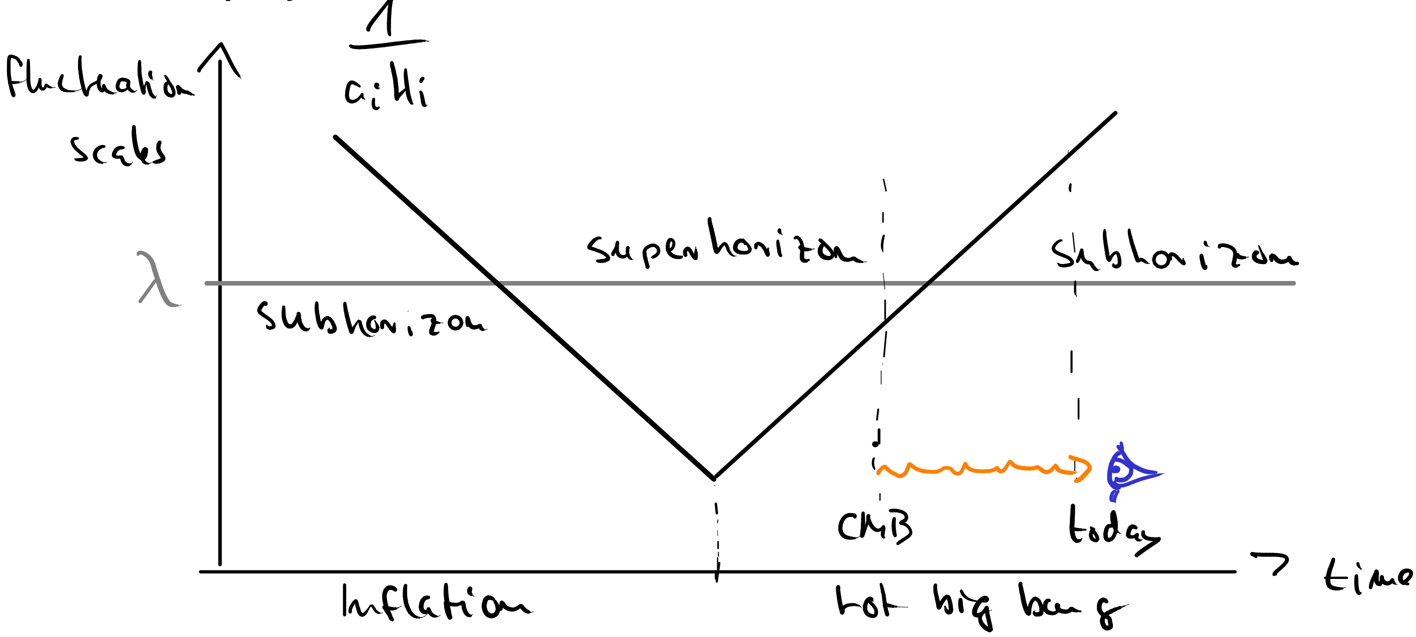
The situation changes, if we assume that $(1+3w) < 0$. Then the fixed point $\Omega_{k,i} = 0$ is an attractive fixed point ("attractor"):



The flatness problem can thus be solved if an inflationary period with $(1+3w) < 0$ exists for a sufficiently long time such that Ω_k tunes itself to a value sufficiently close to zero such that the subsequent expansion doesn't lead to a value beyond the current observational bound.

In practice and depending on the model this requires inflation to last for $N = 40 \ldots 60$ e-folds, see below.

Next, let us take a look at superhorizon correlations. This problem is immediately "solved" by a phase with a shrinking Hubble radius:



If inflation lasted long enough the scales on which we observe correlations and that appear as superhorizon correlations in a standard hot big bang cosmology had been subhorizon scales in the early inflationary phase.
This means that correlations could have been build up by causal mechanisms.
Scales that enter the horizon today, say N_0 e-folds after the end of inflation, left the Hubble horizon N_0 e-folds before the end of inflation.

We see in all examples of "solutions" of the naturalness problems that a sufficient amount of inflation is required. This amount can be quantified and serves as a criterion for the viability of specific models.
Quantitatively, all problems are "solved" if today's observable universe is smaller

than the comoving Hubble radius had been at the beginning of inflation. Then, all of today's universe had been in causal contact in the early universe (NB: a somewhat weaker criterion would involve the particle horizon instead of the Hubble radius, but the Hubble radius is easier to deal with):

$$\underbrace{\frac{1}{a_0 H_0}}_{\text{Hubble radius today}} < \underbrace{\frac{1}{a_i H_i}}_{\text{Hubble radius at the beginning of inflation}} \qquad (12.16)$$

From the last figure above, it is clear that inflation has to compensate for the growth of the Hubble radius during the hot big bang. The amount of growth of the Hubble radius can be connected to the initial temperature at the beginning of the hot big bang phase. Once, we know this temperature (and the matter content of particle physics models and their interactions), we can compute the FLRW evolution of the scale factor and determine the Hubble radius today. We call this temperature the <u>reheating temperature</u> <u>T_R</u>. For the purpose of a simple estimate, let us assume that the universe below T_R has always been radiation dominated. In this case, we have $a \sim \sqrt{t}$, $H = \frac{\dot{a}}{a} \sim \frac{1}{t} \sim \frac{1}{a}$,

and thus

$$\frac{a_0 H_0}{a_R H_R} = \frac{a_0}{a_R}\left(\frac{a_R}{a_0}\right)^2 = \frac{a_R}{a_0} \sim \frac{T_0}{T_R} \sim 10^{-28}\,\frac{(10^{15}\,\mathrm{GeV})}{T_R} \tag{12.17}$$

Here, we have introduced a GUT-like scale of 10^{15} GeV as a reference scale for T_R as a convention.

Assuming that the energy density by the end of inflation was converted quickly into heating the temperature of the plasma without changing the Hubble radius much, we have $\frac{1}{a_e H_e} \sim \frac{1}{a_R H_R}$.

Condition (12.16) then implies

$$\frac{1}{a_i H_i} > \frac{1}{(a_0 H_0)} \sim 10^{28} \left(\frac{T_R}{10^{15}\,\mathrm{GeV}} \right) \frac{1}{(a_e H_e)} \qquad (12.18)$$

Assuming that the Hubble rate during inflation was approximately constant, $H_i \simeq H_e$, we obtain

$$N_{inf} = \ln \frac{a_e}{a_i} > 64 + \ln \frac{T_R}{10^{15}\,\mathrm{GeV}} \qquad (12.19)$$

as the required number of e-folds to "solve" the naturalness problems.

Note that lower reheating temperatures need less e-folds.