9. Boltzmann equation and Dark Matter Freeze Out

While the assumption of thermal equilibrium is surprisingly powerful and can be justified for large periods of the cosmological evolution, essential features of the universe as we see it cannot be addressed. For instance, particle densities in equilibrium follow a Boltzmann-suppression $n \sim \left(\frac{m}{T}\right)^{3/2} e^{-\frac{m}{T}}$ when T drops below m. Still, we observe many particles around us directly, and even more dark matter appears to be out there.

In order for the corresponding particles to survive until today, they must drop out of equilibrium before T drops much below m. One possibility for such a mechanism is <u>decoupling</u> (as we discuss in the exercises for neutrinos): this means that the interaction rate of the particles with respect to the plasma drops below the expansion rate of the universe. Since interactions (including possible annihilation processes) no longer occur then (or only very rarely), the number of particles per comoving volume can stay (almost) constant. This is often called "freeze out".

Such non-equilibrium processes can be described with the <u>Boltzmann</u> equation which is an equation for the phase space distribution function of the particles. For our purposes, it is sufficient to consider the <u>integrated</u> <u>Boltzmann</u> <u>equation</u> which describes the evolution of particle densities, $n = \int_p f$.

In Eq. (3.9), we have already derived the evolution equation for a conserved number of particles in an FLRW spacetime

$$\frac{dn}{dt} + 3\,\frac{\dot{a}}{a}\,n = 0. \qquad (9.1)$$

This equation is equivalent to a dilution of the particle density with the expansion of the spatial volume $\frac{d(na^3)}{dt} = 0 \Rightarrow n \sim \frac{1}{a^3}$.

Let us generalize (9.1) to several particle species n_i. In order to include possible interactions among the particles, we include a <u>collision</u> term on the RHS:

$$\frac{1}{a^3}\,\frac{d}{dt}(n_i a^3) = \underline{\underline{C_i[\{n_j\}]}} \qquad (9.2)$$

The collision term depends on the details of the interaction. For our purposes, it suffices to consider 2-body interactions (classically, it is much less probable that 3 particles meet and interact than 2 particle collisions):

$$p_1 + p_2 \longleftrightarrow p_3 + p_4 \,. \qquad (9.3)$$

Here we allow for the production of new particle species p_3, p_4 in a collision of particles p_1 and p_2 (and vice versa).

Let us concentrate on particle species p_1.

The rate of change of particle density n_1 in a comoving volume then depends on the difference between eliminating a particle p_1 upon collision with p_2 and producing a particle p_1 upon collision of p_3 and p_4. As an ansatz for this difference, we write

$$\frac{1}{a^3}\frac{d}{dt}\left(n_1 a^3\right) = \underbrace{-\alpha\, n_1 n_2}_{\text{elimination}} + \underbrace{\beta\, n_3 n_4}_{\text{production}} \qquad (9.4)$$

Microscopically, the parameter α can be computed from $\alpha = \langle \sigma v \rangle$, where

σ is the cross section, v the relative velocity, and $\langle \ldots \rangle$ denotes the ensemble average. The parameter β would follow similarly, but can also be inferred from the observation that the collision term in (9.4) has to vanish if all n_i's have acquired their equilibrium values

$$\Rightarrow \quad \beta = \left(\frac{n_1 n_2}{n_3 n_4} \right)_{eq} \alpha \,. \tag{9.5}$$

(This is often called "detailed balance".)

We thus obtain

$$\frac{1}{a^3} \frac{d}{dt} \left(n_1 a^3 \right) = - \langle \sigma v \rangle \left[n_1 n_2 - \left(\frac{n_1 n_2}{n_3 n_4} \right)_{eq} n_3 n_4 \right] \tag{9.6}$$

Eq. (9.6) can be rewritten in a form that makes the dependence of the non-equilibrium process on the interaction rate of species 1,

$$\Gamma_1 = n_2 \langle \sigma v \rangle \tag{9.7}$$

in comparison to the Hubble rate $H = \frac{\dot{a}}{a}$

explicitly visible:

$$\frac{1}{n_1 a^3}\frac{d}{dt}(n_1 a^3) = -\Gamma_1\left[1 - \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq}\frac{n_3 n_4}{n_1 n_2}\right]. \quad (9.8)$$

Let us also write $\frac{d}{dt} = \frac{da}{dt}\frac{d}{da} = \frac{da}{dt}\frac{1}{a}\frac{d}{d\ln a} = H\frac{d}{d\ln a}$

and introduce a measure for the number of particles in a comoving volume. In principle, $n_1 a^3$ has precisely the desired properties. For reasons of conventions, one often uses

$$N_1 := \frac{n_1}{s} \quad (9.9)$$

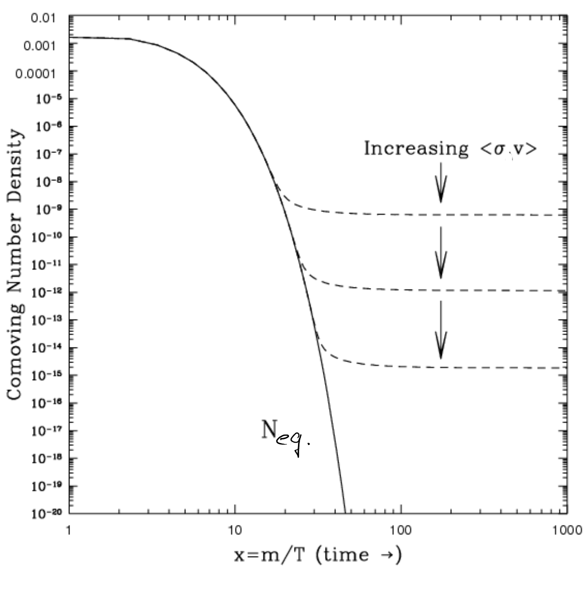
involving the entropy density. Since entropy conservation implies $s \sim \frac{1}{a^3}$, we have $N_1 \sim n_1 a^3$ as intended. In summary Eq. (9.8) can thus be written as

$$\underline{\underline{\frac{d\ln N_1}{d\ln a}}} = -\frac{\Gamma_1}{H}\left[1 - \underline{\underline{\left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq}}}\frac{N_3 N_4}{N_1 N_2}\right] \quad (9.10)$$

On the RHS, the $[1 - ..]$ term describes the deviation from equilibrium, and $\frac{\Gamma_1}{H}$ is a measure for the interaction efficiency.

As long as $\Gamma_1 \gg H$, the system relaxes quickly to equilibrium: e.g. let's assume that $N_{2,3,4} \simeq N_{2,3,4\,eq.}$ but $N_1 > N_{1\,eq.}$ Then, the RHS is negative, implying that particles get destroyed until N_1 reduces to $N_{1\,eq.}$ Analogous conclusions hold for $N_1 < N_{1\,eq.}$ or for all other particle species deviating from their equilibrium values.

However, as soon as Γ_1 drops below the expansion rate H, the RHS gets suppressed even if the N_i's have not approached their eq. values. For $\frac{\Gamma_1}{H} \to 0$, the comoving density of particles approaches a constant <u>relic density</u>, $N_1 \to$ const (different from $N_{1,eq.}$).



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Application to Dark Matter Freeze-Out

Let us apply the Boltzmann equation to the freeze-out of massive particles, which may be applicable to dark-matter models based on a new particle.

Let us assume that some heavy particle X exists. (For concreteness, let us consider a fermion as almost all matter fields are fermions.)

We also assume that X can decay into (much lighter) standard model particles "l" by annihilation with its antiparticle $\bar{X}$:

$$X + \bar{X} \leftrightarrow l + \bar{l} \,. \tag{9.11}$$

For simplicity, we also assume that $l, \bar{l}$ are tightly coupled to the primordial plasma such that their densities obey the equilibrium formulas at any time $n_l = n_{l,eq.}$. Assuming the mass M_X of X to be much heavier than l, the standard model particles participating in (9.11) will be ultra-relativistic and can be considered as massless.

We also assume no initial asymmetry between X and $\bar{X}$ such that $n_X = n_{\bar{X}}$.

Finally, we assume no other characteristic particle threshold to be crossed during this freeze-out such that we can take $T \sim \frac{1}{a}$ for the relevant times here.

With these assumptions, the Boltzmann eq. (9.6) reduces to

$$\frac{1}{a^3}\frac{d(n_X a^3)}{dt} = -\langle \bar{\sigma v} \rangle \left[n_X^2 - (n_{X,eq})^2 \right] \quad (9.12)$$

It is convenient to introduce

$$Y_X := \frac{n_X}{T^3} \quad . \qquad (9.13)$$

As long as $\frac{1}{T} \sim a$, we have $N_X = \frac{n_X}{s} \sim n_X a^3 \sim Y_X$ and we can use N_X and Y_X synonymously, and the LHS of (9.12) becomes $\sim T^3 \frac{dY_X}{dt}$. Since the relevant part of freeze-out happens near $T \sim M_X$, let us introduce

$$x := \frac{M_X}{T} \quad , \quad \frac{dx}{dt} = M_X \left(-\frac{1}{T^2}\right) \frac{dT}{dt} = \frac{M_X}{T} \frac{1}{a}\frac{da}{dt}$$

$$= H x , \qquad (9.14)$$

such that we can use x as a new "time" variable, $dt = \frac{dx}{Hx}$.

The Boltzmann equation then takes the form

$$\frac{dY_X}{dt} = -T^3 \langle \sigma v \rangle \left[Y_X^2 - Y_{X,eq}^2 \right]$$

$$\frac{dY_X}{dx} = -\frac{T^3 \langle \sigma v \rangle}{Hx} \left[Y_X^2 - Y_{X,eq}^2 \right] \quad (9.15)$$

For weakly interacting particles, the decoupling happens very early when the universe was still radiation-dominated, i.e. $a \sim t^{1/2}$, $H = \frac{\dot{a}}{a} \sim \frac{1}{t} \sim \frac{1}{a^2}$.

This allows us to rewrite the Hubble parameter (considered in (9.15) as a function of time and thus of temperature $H(T) \sim \frac{1}{a^2} \sim T^2$) as

$$H = H(T) = H(M_X) \left(\frac{T}{M_X}\right)^2, \quad (9.16)$$

where $H(M_X)$ denotes the Hubble rate at the scale set by the heavy mass.

Let us take a closer look at the prefactor on the RHS in (9.15):

$$\frac{T^3 \langle \sigma v \rangle}{Hx} \overset{(9.16)}{=} \frac{T M_X^2 \langle \sigma v \rangle}{H(M_X)\, x} = \frac{M_X^3 \langle \sigma v \rangle}{H(M_X)\, x^2} =: \frac{\lambda}{x^2} \quad (9.17)$$

Here, we have introduced the parameter λ

$$\lambda := \frac{M_X^3 \langle \sigma v \rangle}{H(M_X)} =: \frac{\Gamma(M_X)}{H(M_X)}, \quad (9.18)$$

where we have introduced the interaction rate $\Gamma(M_X)$ at the scale M_X by $\Gamma(M_X) = M_X^3 \langle \sigma v \rangle$ (9.19)

This is an estimate justified by dimensional analysis and the fact that M_X is the relevant largest scale in the system. We arrive at

$$\frac{dY_X}{dx} = -\frac{\lambda}{x^2}\left[Y_X^2 - Y_{X,eq}^2\right], \tag{9.20}$$

which is a differential equation of <u>Riccati</u> type.

For simplicity, we consider λ as a constant. Still, this does not allow to construct analytic solutions. Numerical solutions can nevertheless be straightforwardly obtained.

As we have scaled out the temperature $Y_X = \frac{n_X}{T^3}$ in (9.13), the initial condition at high temperatures

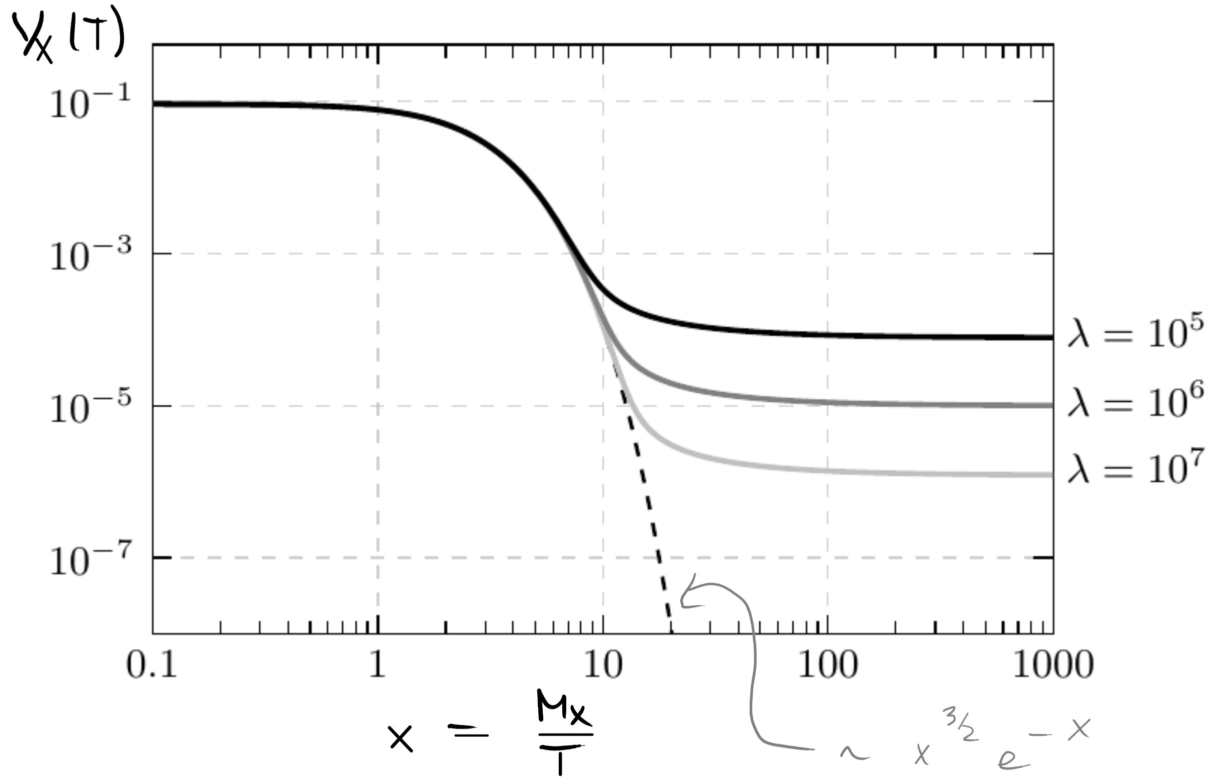
$$x = \frac{M_X}{T} \to 0 \quad : \quad Y_X \simeq Y_{eq} = \frac{3\,\zeta(3)}{4\pi^2} \simeq 0.1 \tag{9.21}$$

(for a fermionic particle).

For decreasing temperatures across the threshold, Y_X will pass over to the non-relativistic limit $Y_X \sim \left(\frac{x}{2\pi}\right)^{3/2} e^{-x}$ for $x \gtrsim 1$.

For increasing x (and depending on λ), the RHS of (9.20) will approach zero, such that Y_X no longer follows the equilibrium curve but freezes out, i.e. approaches a constant. For a wide range of λ values, this occurs near $x_f \approx 10$.

Heuristically, we may interpret this as a decrease of the particle density such that the particles become so rare (and/or the interaction so weak) that collisions and thus annihilations practically no longer occur. The density per comoving volume therefore approaches a constant.



The final relic abundance

$$Y_{X,\infty} := \lim_{x \to \infty} Y_X(x) \tag{9.22}$$

then is a measure for the residual particle number per comoving volume after freeze-out. Let us try to estimate this number. As is visible in the plot Y_X is much larger than the equilibrium value $Y_{X,eq}$ for $x > x_f$. Hence, in this late-time regime. The Riccati equation simplifies to

$$\frac{dY_X}{dx} \simeq -\frac{\lambda}{x^2} Y_X^2 . \tag{9.23}$$

Integrating this from the freeze-out time x_f to $x \to \infty$, we get by separation of variables

$$\frac{1}{Y_{X,\infty}} - \frac{1}{Y_{X,f}} = \frac{\lambda}{x_f} , \tag{9.24}$$

where $Y_{X,f} = Y_X(x_f)$. Typically, we have $Y_{X,f} \gg Y_{X,\infty}$, such that we arrive at the simple formula for the relic abundance

$$\underline{\underline{Y_{X,\infty} \simeq \frac{x_f}{\lambda}}} \tag{9.25}$$

In principle, the freeze-out time x_f depends on λ; however, this dependence is rather weak, and the value $x_f \simeq 10$ gives a good estimate, cf. the plot above.

One prediction of (9.25) is that the freeze-out abundance $Y_{X,\infty}$ decreases if the interaction λ increases. This is plausible, since the particles stay longer close to the equilibrium curve for large interactions. Having an application to the dark-matter freeze out in mind, this implies that the substantial dark-matter abundance observed in the universe suggests, that the dark-matter particles are weakly interacting. Let us make this more quantitative:

We need to relate the relic abundance after freeze-out to the relic matter density today.

For this, let t_1 be a time late enough for Y_X to have reached almost its freeze-out value $Y_{X,\infty}$. Correspondingly, let T_1 and

a_1 denote the temperature and scale factor at that time. Then, the number density today can be written as

$$n_{X,0} = n_{X,1} \left(\frac{a_1}{a_0}\right)^3 \simeq Y_{X,\infty} T_1^3 \left(\frac{a_1}{a_0}\right)^3$$
$$= Y_{X,\infty} T_0^3 \left(\frac{a_1 T_1}{a_0 T_0}\right)^3 . \quad (9.26)$$

Let t_1 still be a time, when no other thresholds have been crossed yet. Then the last factor in (9.26) knows about all deviations from $a \sim \frac{1}{T}$ due to all particle thresholds to be crossed from T_1 to T_0 today.

Quantitatively, this is captured by the conservation of entropy $sa^3 \sim g_{*S} (aT)^3 = \text{const}$, such that

$$n_{X,0} = Y_{X,\infty} T_0^3 \frac{g_{*S}(T_0)}{g_{*S}(M_X)} . \quad (9.27)$$

The use of $g_{*S}(M_X)$ takes into account that T_1 is a temperature that also includes the entropy transferred to the plasma from the annihilation of the dark matter X particles.

With (9.27), we can estimate the energy density fraction of these particles today,

$$\underline{\underline{\Omega_X}} = \frac{\rho_{X,0}}{\rho_{cr,0}} \overset{(4.11)}{=} \frac{M_X\, n_{X,0}}{\frac{3H_0^2}{8\pi G}} = \frac{8\pi G M_X T_0^3}{3H_0^2} Y_{X,\infty} \frac{g_{*s}(T_0)}{g_{*s}(M_X)}$$

$$\overset{(9.25)}{=} \frac{8\pi G T_0^3}{3H_0^2} \frac{M_X\, x_f}{\lambda} \frac{g_{*s}(T_0)}{g_{*s}(M_X)}$$

$\overset{(9.17)}{\lambda =} \frac{M_X^3 \langle\sigma v\rangle}{H(M_X)}$

$$= \frac{8\pi G T_0^3}{3H_0^2} \frac{H(M_X)}{M_X^2} \frac{x_f}{\langle\sigma v\rangle} \frac{g_{*s}(T_0)}{g_{*s}(M_X)} \quad . \tag{9.28}$$

Now, using Eq. (7.15) for $H(M_X)$, $H(M_X)^2 = \frac{4\pi^3 G}{45} g_*(M_X) M_X^4$

we get

$$\Omega_X = \frac{16\pi^2 \sqrt{\pi}\, G^{3/2}}{9\sqrt{5}\, H_0^2} T_0^3 \frac{x_f}{\langle\sigma v\rangle} \sqrt{g_{*s}(M_X)}\; g_{*s}(T_0) \tag{9.29}$$

Inserting the known values for G, H_0, T_0 and $g_{*s}(T_0) = 3.91$, we arrive at

$$\Omega_X \simeq 0.1 \frac{x_f}{\sqrt{g_{*s}(M_X)}} \frac{10^{-8}\, GeV^{-2}}{\langle\sigma v\rangle} \tag{9.30}$$

The result does not directly depend on the mass of the dark matter particle, but is essentially governed by the cross section. Assuming a particle content of approximately the standard model, $\sqrt{g_{*s}(M_X)} \simeq 10$, using $x_f \simeq 10$ and the known value of $\Omega_X \simeq 0.25$, We get an estimate for the cross section

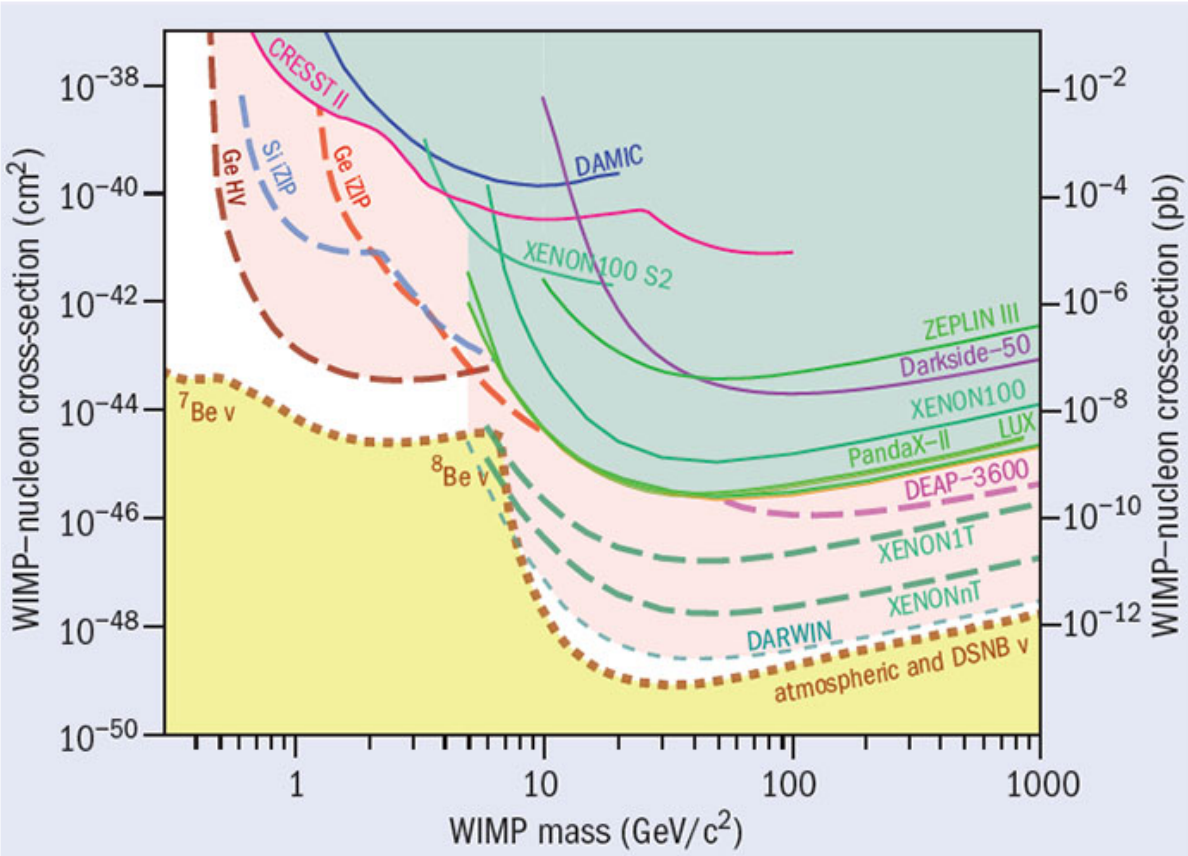
$$\underline{\underline{\sqrt{\langle \sigma v \rangle}}} \sim 10^{-4}\ \mathrm{GeV}^{-1} \sim 0.1\,\underline{\underline{\sqrt{G_F}}} \qquad (9.31)$$

It is interesting to see that a cross section in line with an electro-weakly interacting particle has just the right size to be compatible with the required relic abundance of the observed dark-matter content of the universe.

This has been dubbed the <u>WIMP miracle</u>, as a <u>W</u>eakly <u>I</u>nteracting <u>M</u>assive <u>P</u>article would thus be an excellent dark-matter candidate. It has offered the prospect of searching for these particles in indirect and direct ways. Despite their weak interaction, $X + \bar{X} \to \gamma + \gamma$ annihilations are still expected to occur rarely. Since the photons would have a characteristic frequency of $\omega = M_X$, one can search for pronounced peaks in the cosmic photon spectrum arising from regions with a high density of dark matter (galaxies, clusters) in comparison to voids.

So far, this search has been inconclusive.

Direct searches assume that also our laboratories move through an average density of dark matter and search for collisions by looking for recoil events inside well controlled detectors. Many experiments have been performed or are still in operation / preparation. No undisputed evidence for a WIMP has been discovered so far:



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