7 Entropy & Expansion History

In all branches of physics, conserved quantities are extremely useful for formulating efficient descriptions as well as for a conceptual understanding of physical systems. Often, the total energy plays this informative role. However in cosmology, energy is not conserved.
In the context of thermodynamics, let us therefore take a closer look at entropy S or the corresponding entropy density $s = \frac{S}{V}$.
Let us start with the first law of thermodynamics in the form

$$\delta Q = T\, \underline{\underline{d(sV)}} = T\,dS = dU + \delta W = \underline{\underline{d(\rho V) + P\,dV}}$$

$$\Rightarrow \quad T s\,dV + TV\,ds = \rho\,dV + V\,d\rho + P\,dV \tag{7.1}$$

The densities s and ρ do not depend on the volume, but they do depend on temperature, hence

$$(Ts - \rho - P)\,dV + V\left(T\frac{ds}{dT} - \frac{d\rho}{dT}\right)dT = 0 \tag{7.2}$$

As (7.2) has to be satisfied for independent variations of T and V, we get

$$s = \frac{\rho + P}{T} \tag{7.3}$$

and

$$\frac{ds}{dT} = \frac{1}{T}\frac{d\rho}{dT} . \tag{7.4}$$

In the context of cosmology, we can now combine this with the continuity equation on FLRW spacetimes,

$$\frac{d\rho}{dt} = -3H(\rho+P) \overset{(7.3)}{=} -3HTs . \tag{7.5}$$

As the universe evolves in time and generically cools down, we can rewrite (7.4) also as

$$0 = \frac{ds}{dT} - \frac{1}{T}\frac{d\rho}{dT} = \left(\frac{ds}{dt} - \frac{1}{T}\frac{d\rho}{dt}\right)\left(\frac{dT}{dt}\right)^{-1} \tag{7.6}$$

where $\left(\frac{dT}{dt}\right)^{-1}$ is in general nonzero, so that the prefactor has to vanish. With this, let us study

$$\frac{d}{dt}(sa^3) = \frac{ds}{dt}a^3 + s\,3a^2\frac{da}{dt}$$

$$\Rightarrow \underline{\underline{\frac{d}{dt}(sa^3)}} = a^3 \left(\frac{ds}{dt} + 3s \underbrace{\frac{1}{a}\frac{da}{dt}}_{=H} \right)$$

$$\overset{(7.6)}{=} a^3 \left(\frac{1}{T}\frac{d\rho}{dt} + 3sH \right)$$

$$\overset{(7.5)}{=} a^3 \left(\frac{1}{T}(-3HTs) + 3sH \right) \underline{\underline{= 0}} \quad (7.7)$$

Since the entropy density dilutes with the scale factor as a density, $s \sim \frac{1}{a^3}$, this implies that the total entropy is a conserved quantity in equilibrium.

Let us again focus on relativistic species.

Integrating (7.4), we get

$$s(T) = \int_0^T \frac{dT'}{T'} \frac{d\rho}{dT'} = \frac{\rho(T)}{T} + \int_0^T \frac{dT'}{T'^2} \rho(T') \quad (7.8)$$

where we have integrated by parts and used $\rho(T=0) = 0$ (ignoring the small non-relativistic contributions).

Comparing (7.8) to (7.3), we see

that the last integral term must correspond to $\frac{P}{T}$. Hence, we can study the equation of state in the form

$$\underline{\underline{w(T)}} = \frac{P(T)}{\rho(T)} = T\int_0^T \frac{\rho(T')}{\rho(T)} \frac{dT'}{T'^2}$$

$$\overset{(6.71)}{=} T\int_0^T \frac{\frac{\pi}{30} g_*(T') T'^4}{\frac{\pi}{30} g_*(T) T^4} \frac{dT'}{T'^2}$$

$T' = yT$

$$= \underline{\underline{\int_0^1 \frac{g_*(yT)}{g_*(T)}}} y^2 \, dy \,. \qquad (7.9)$$

If all particles are relativistic, we have $g_* = \text{const}$ and thus $w = \int_0^1 dy^2 dy = \frac{1}{3}$, as expected for radiation. Eq. (7.9) parametrizes the deviations from the relativistic limit $w = \frac{1}{3}$ due to the decoupling of degrees of freedom.

Similarly to the energy density, also the entropy density can be brought into a form that relies on the counting of degrees of freedom. Starting from Eq. (7.3) and accounting for different species, we write

$$s = \sum_i \frac{\rho_i + P_i}{T_i} =: \frac{2\pi^2}{45} g_{*S}(T)\, T^3, \quad (7.10)$$

where $g_{S*}(T)$ denotes the effective number of degrees of freedom in entropy:

$$g_{*S}(T) \overset{\text{relativistic}}{\approx} \sum_{i \in \text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i=f} g_i \left(\frac{T_i}{T}\right)^3 . \quad (7.11)$$

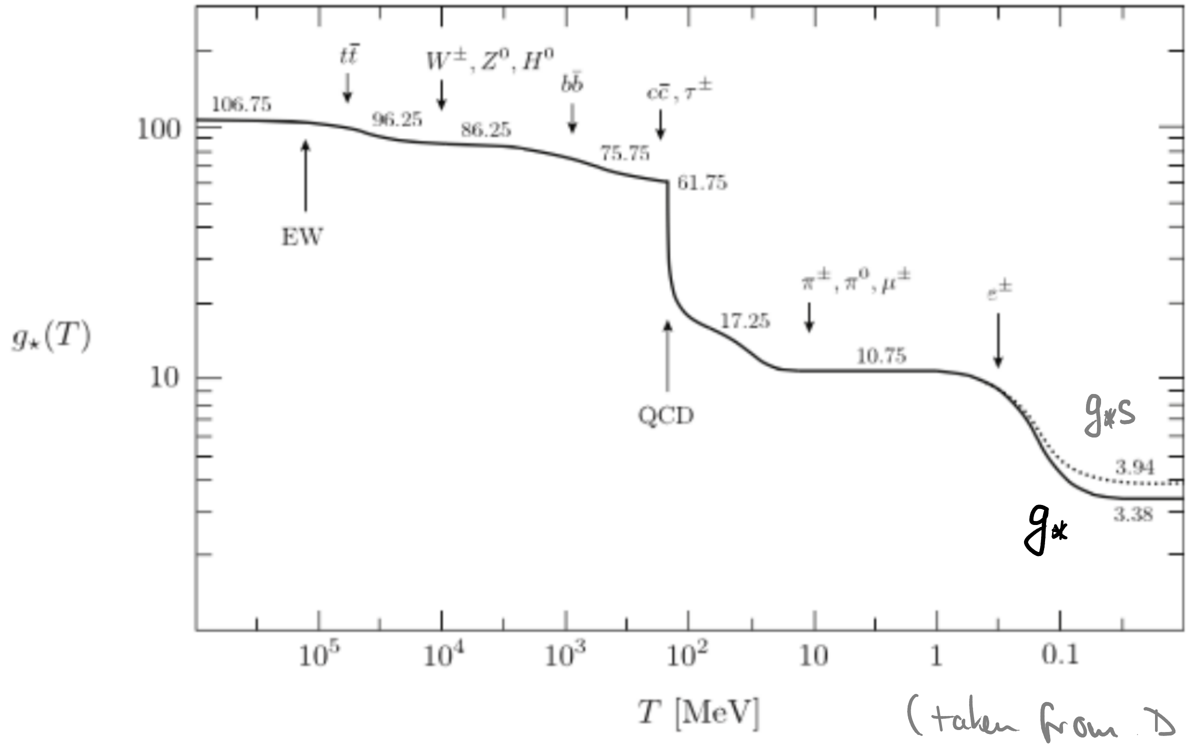
If all particles are in equilibrium at the same temperature $T_i = T$, we simply have

$g_{*S}(T) = g_*(T)$. In turn, the two differ if some particle species acquire a different temperature.

Since the photon density satisfies $n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3$, the entropy density (7.10) can also be written as

$$\underline{\underline{s}} = \frac{2\pi^2}{45} g_{*S}(T) \frac{\pi^2}{2\zeta(3)} n_\gamma \simeq 1.8\, \underline{\underline{g_{*S}(T)\, n_\gamma}} . \quad (7.12)$$

As already mentioned (and as will be discussed in the exercizes) the temperature associated with the cosmic neutrino background is smaller than the CMB temperature $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$ which is a consequence of the neutrino decoupling happening before e^+, e^- annihilation. As a consequence, g_* and g_{*S} differ slightly after e^+, e^- annihilation.



(taken from D. Baumann's lecture notes as quoted by M. Bauer, T. Plehn arXiv: 1705.01987)

With $g_{*S} \simeq 3.94$ below e^+, e^- annihilation, Eq. (7.12) can be written

$$s \simeq 7\, n_\gamma \,. \tag{7.13}$$

Entropy conservation also helps us to link the temperature to the scale factor:

$$\frac{d(sa^3)}{dt} = 0 \implies g_{*s}(T)\,T^3 a^3 = \text{const.}$$

$$\Rightarrow \quad T \sim \frac{1}{g_{*s}^{1/3}} \cdot \frac{1}{a} \,. \qquad (7.14)$$

Away from mass thresholds, g_{*s} is $\simeq$ const, hence the temperature decreases inversely propotional to the scale factor $T \sim \frac{1}{a}$.

At a mass threshold $g_{*s}(T)$ decreases when a particle species becomes non-relativistic. This implies that the temperature of the remaining particles increases / (decreases slightly more slowly than $\frac{1}{a}$).

As entropy is conserved, we can interpret this temperature increase as a transfer of entropy from the particles that undergo decoupling to the relativistic species in the thermal plasma.

Let us finally take a closer look at

the cosmic thermal history of the early universe. For this, we can ignore curvature and dark energy, such that the Friedmann equation can be written as

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\,\rho \simeq \frac{4\pi^3 G}{45}\, g_* \, T^4 . \qquad (7.15)$$

This equation relates $a(t)$ with the temperature $T(t)$. To solve it, we need another equation. (Previously, we used the equation of state w as another input to solve the Friedmann equation.)

For a proportionality estimate, we can here use Eq. (7.14), $T \sim \frac{1}{g_{*s}^{1/3}} \frac{1}{a}$, away from thresholds, where $g_{*s} \simeq$ const, such that

$$\left(\frac{\dot a}{a}\right)^2 \sim \frac{g_*}{g_{*s}^{4/3}} \frac{1}{a^4} \quad \Rightarrow \quad \dot a \sim \frac{1}{a}$$

which is solved by $a \sim t^{1/2}$ as is familiar from a radiation-dominated universe.

This implies $H = \frac{\dot a}{a} = \frac{1}{2t}$, which, upon insertion into (7.15) gives

$$\frac{T}{1\,\mathrm{MeV}} \simeq 1.5\, g_*^{-1/4} \left(\frac{1\,\mathrm{sec}}{t}\right)^{1/2} . \qquad (7.16)$$

This rule of thumb which we have already briefly discussed on one of the first exercize sheets represents an acceptable estimate for the behavior of the temperature of the universe for the time $t \leq 1\,\mathrm{sec}$.