# 6 Thermodynamics of the Hot Big Bang

From the scaling of radiation and matter densities $\rho_m \sim \frac{1}{a^3}$ and $\rho_r \sim \frac{1}{a^4}$, early phases of the universe at small $a(t)$ must have been very dense. From the Stefan-Boltzmann law, $u_\gamma = \rho_\gamma c^2 \sim T^4$, c.f. Eq. (5.3), we know that high energy densities necessarily go along with high temperatures. The same holds also for other particles: e.g. for fermions, Pauli blocking requires that higher densities must go along with the population of increasingly high energy levels corresponding to higher temperatures. Strictly speaking, temperature is only a well-defined concept in thermal equilibrium. But the universe is evolving with time and thus never in equilibrium.

The fact that equilibrium thermodynamics is en gross still a useful description can be understood from the relevant time scales:

The reaction rates to bring a state into thermal equilibrium are typically much larger than the expansion rate. In other words, the universe has been slow enough for the particles to reach a state of approximate equilibrium.

In addition to the idealization of equilibrium, the ideal (relativistic) gas limit is often justified, since the relevant interactions are generically small enough or sufficiently elastic in order to justify an ideal gas limit.

Times (or temperatures) when this idealization does not hold typically mark an interesting transition in the cosmological evolution.

Some of these transitions are collected in the following table, going backward in time and upward in temperature:

| EVENT | TIME | REDSHIFT | TEMPERATURE |
|---|---|---|---|
| Photon decoupling | 380 kyr | 1100 | 0.25 eV |

the temperature becomes low enough for $e^- + p^+ \rightarrow H + \gamma$, i.e. for electrons and protons to form stable atoms. At this point, photons start to propagate freely, the universe becomes transparent for light. These photons today are observed as cosmic microwave background (CMB).

| | | | |
|---|---|---|---|
| Recombination | 260–380 kyr | 1100–1400 | 0.25–0.30 eV |

electrons and protons start to form atoms, photons still have enough energy to ionize atoms; the photon mean-free path is still short but becomes longer, the universe undergoes a transition from opaqueness to transparency.

| | | | |
|---|---|---|---|
| Matter radiation equality | 60 kyr | 3400 | 0.75 eV |

almost all energy density is in the form of $e^-$, light nuclei, photons and neutrinos. Only the neutrinos are freely streaming with very weak interactions.

| | | | |
|---|---|---|---|
| Big bang nucleosynthesis | 3 min | $4 \cdot 10^8$ | 100 keV |

the light nuclei, mostly deuterium, helium, and Lithium are formed (heavier elements are formed much later inside stars)

| | | | |
|---|---|---|---|
| Electron-positron annihilation | 6 s | $2 \cdot 10^9$ | 500 keV |

above this energy scale $e^+$ and $e^-$ are in equilibrium in the primordial plasma through $e^+ + e^- \leftrightarrow \gamma$'s; below almost all annihilate into $\gamma$'s, heating up the photon temperature

| | | | |
|---|---|---|---|
| Neutrino decoupling | 1s | $6 \cdot 10^9$ | 1 MeV |

The weak interaction rate of the $\nu$'s with the plasma drops below the expansion rate; the $\nu$'s decouple.

| | | | |
|---|---|---|---|
| QCD phase transition | 20$\mu$s | $10^{12}$ | 150 MeV |

quarks and gluons having formed a plasma at higher energies combine into baryons (protons, neutrons) and mesons (pions, kaons, ...), the latter decay ultimately into $e^\pm$, $\gamma$'s. The detailed properties of the QCD phase transition are still matter of ongoing research.

| | | | |
|---|---|---|---|
| Electroweak phase transition | 2ps | $10^{15}$ | 100 GeV |

The expectation value of the Higgs field becomes nonzero, this generates masses for electrons, quarks and the Z and W bosons.

Though not all of these transitions can already be studied using cosmological data, we have quite some knowledge about the existence of these transitions on the basis of laboratory experiments, simulations and theory.

(Some open questions, e.g. include: does the QCD phase transition leave an imprint in cosmological data?)

In addition, we expect that further transitions have taken place, the details of which are largely unknown:

Dark matter freeze out ? ? ?

Such a transition is expected, if dark matter is a new kind of elementary particle; in this case, time, redshift and temperature of the transition will depend on the particle parameters.

Baryogenesis ? ? ?

The origin of the matter-antimatter asymmetry (avoiding our annihilation into a radiation-only universe) is unknown and constitutes a major puzzle in cosmology / particle physics.

The corresponding scales are unknown but must be beyond what has been tested so far.

Inflation ? — —

As will be discussed in later sections, there are reasons to believe that the early universe underwent a period of accelerated expansion which is called inflation.

## Elements of relativistic thermodynamics

Consider an ensemble of particles. If you pick a random particle, it will have a momentum $\vec{p}$ (and a corresponding energy $\mathcal{E}(p)$). In a general case, the probability distribution for picking a certain value $\vec{p}$ at a given time $t$, $f(\vec{p},t)$, will be a complicated function.

If we wait long enough, the system will approach a state of maximum entropy which characterizes the equilibrium state. Here, the distribution function (in the grand-canonical formalism) for a given temperature $T$ and chemical potential $\mu$ is given by the Fermi-Dirac distribution $f_F$ for fermions or the Bose-Einstein distribution $f_B$ for bosons:

$$f_{F,B}(\vec{p}) = \frac{1}{e^{\frac{1}{T}(\varepsilon(p)-\mu)} \pm 1} . \quad (6.1)$$

Since the parameters $T$ and $\mu$ referring to a local thermal equilibrium change with time in an expanding universe, also $f_{F,B}$ changes upon the cosmological expansion.

Note: in (6.1), we have used the convention $k_B = 1$; in the following, we will also use $\hbar = c = 1$ in addition.

From the distribution function, we obtain the total particle density by integrating over all momenta and possibly summing over internal degrees of freedom such as spin:

$$n = g \int \frac{d^3p}{(2\pi)^3} \; f(\vec{p}) \qquad (6.2)$$

where $g$ counts the internal degrees of freedom. Similarly, we obtain the energy density by weighting with the single-particle energy

$$u \overset{c=1}{=} \rho = g \int \frac{d^3p}{(2\pi)^3} \; f(p) \; \varepsilon(p) \qquad (6.3)$$

From thermodynamics, we know that the pressure follows accordingly:

$$P = g \int \frac{d^3p}{(2\pi)^3} \; f(p) \; \frac{p^2}{3\,\varepsilon(p)} \qquad (6.4)$$

In the following, we consider relativistic particles with (mostly) negligible interactions such that

$$\varepsilon(p) = \sqrt{\vec{p}^{\,2} + m^2} \qquad (6.5)$$

Different particle species $i$ have different distribution functions $f_i$ and correspondingly different densities $n_i$, $\rho_i$ and pressures $P_i$. However, in equilibrium, they have the same temperature $T$.

Photons have a vanishing chemical potential $\mu_\gamma = 0$. In the early universe, the chemical potential $\mu$ for electrons and protons is small compared to the temperature

$\mu_{e^\pm, p^+} \ll T$ (can be measured in the lab). For all other particles, we believe that $\mu_i \ll T$ should also hold; hence we drop $\mu$ in the following discussion for the moment.

The integrals occuring in (6.2) - (6.4) are well-known in thermodynamics. For $\mu = 0$ the expressions can be compactly summarized as

$$n = \frac{g}{2\pi^2} T^3 \, I_{F/B}(x) \quad , \; x = \frac{m}{T}$$

$$\rho = \frac{g}{2\pi^2} T^4 \, J_{F/B}(x) \qquad (6.6)$$

where (using $\xi = p/T$)

$$I_{F,B}(x) = \int_0^\infty d\xi \, \frac{\xi^2}{e^{\sqrt{\xi^2+x^2}} \pm 1}$$

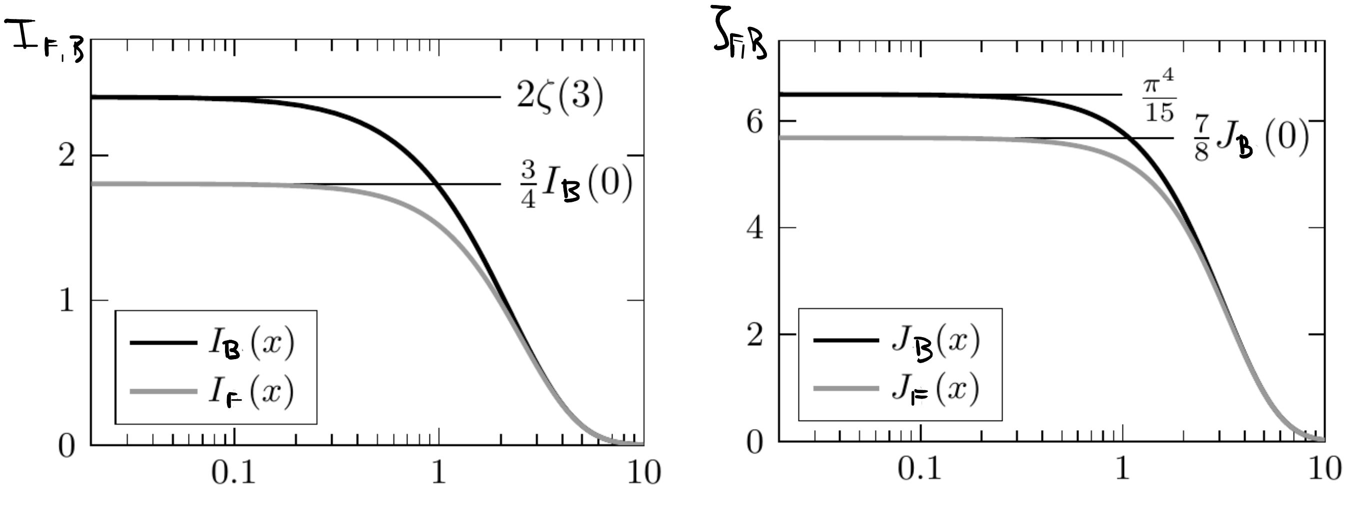
$$J_{F,B}(x) = \int_0^\infty d\xi \, \frac{\xi^2 \sqrt{\xi^2+x^2}}{e^{\sqrt{\xi^2+x^2}} \pm 1} \qquad (6.7)$$

A similar representation also holds for the pressure.

The relativistic limit is obtained by neglecting the mass compared to the temperature $m \ll T$, i.e. $x \to 0$, e.g.

$$I_{F,B}(0) = \int_0^\infty d\xi \, \frac{\xi^2}{e^{\xi} \pm 1} \qquad (6.8)$$

The functions $I$ and $J$ thus become simple constants



Here, $\zeta(3)$ is the Riemann $\zeta$ function

$$\zeta(n) = \sum_{i=1}^{\infty} \frac{1}{i^n} \tag{6.9}$$

with $\zeta(3) \simeq 1.202\ldots$

Correspondingly, we get in the relativistic limit

$$n = \frac{\zeta(3)}{\pi^2}\, g\, T^3 \begin{cases} 1 & \text{bosons} \\ \frac{3}{4} & \text{fermions} \end{cases} \tag{6.10}$$

$$\varrho = \frac{\pi^2}{30}\, g\, T^4 \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases} \tag{6.11}$$

(For photons (= bosons with $g=2$ polarization degrees of freedom, this precisely agrees with (5.2) & (5.3) using $k_B = \hbar = c = 1$.)

For the pressure in the relativistic limit, we can insert $\varepsilon(p) = p$ in (6.4) and straight-forwardly obtain

$$P = \frac{1}{3}\varrho \tag{6.12}$$

as expected for a gas of relativistic particles (a.k.a "radiation").

In the non-relativistic limit, we have $x \gg 1$ $(T \ll m)$, such that the exponential factor in the denominator is large and the $\pm 1$ can be neglected, e.g.

$$I_{F,B} \simeq \int_0^\infty d\xi \, \frac{\xi^2}{e^{\sqrt{\xi^2+x^2}}} \simeq \int_0^\infty d\xi \, \frac{\xi^2}{e^{x+\xi^2/2x}}$$

$$= e^{-x} \int_0^\infty d\xi \, \xi^2 \, e^{-\frac{\xi^2}{2x}}$$

$$= \sqrt{\frac{\pi}{2}} \, x^{3/2} \, e^{-x} \, , \quad x = \frac{m}{T} \qquad (6.13)$$

Correspondingly, we find for the density

$$n = g \left( \frac{mT}{2\pi} \right)^{3/2} e^{-\frac{m}{T}} \, , \quad m \gg T \qquad (6.14)$$

We observe that massive particles are exponentially rare at low temperatures. The corresponding result for the energy density could analogously be worked out from $J_{F,B}$, but also follows from observing that

$$\varepsilon(p) = \sqrt{\vec{p}^2 + m^2} \simeq \underbrace{m}_{\text{restmass}} + \underbrace{\frac{\vec{p}^2}{2m}}_{E_{kin} \text{ for NR particles}} \qquad (6.15)$$

and thus (using the equipartition theorem)

$$\rho = m\,n + \frac{3}{2} n T. \qquad (6.16)$$

While also derivable from (6.4), we know that the non-relativistic limit for the pressure must yield the ideal gas equation of state:

$$P = n T \qquad (6.17)$$

(using $k_B = 1$ and $n = \frac{N}{V}$).

Since $T \ll m$, we have $P \ll \rho$ in agreement with our classification as "matter":

$$w = \frac{P}{\rho} \simeq 0 \qquad (6.18)$$

Comparing the relativistic high-temperature $T \gg m$ to the low-temperature $T \ll m$ limit, we see that particle and energy densities as well as pressure are exponentially suppressed ("Boltzmann suppressed") in the non relativistic limit. For particle-antiparticle degrees of freedom, this can be interpreted as an annihilation process. At high $T$, e.g. processes of the type

$$e^+ + e^- \longleftrightarrow 2\gamma \qquad (6.19)$$

goes in both directions since there are many high energy photons with sufficient energy to create a massive pair.

When the temperature drops as a consequence of the cosmological expansion (the $\gamma$'s get also redshifted), the $\gamma$'s at some point do no longer have sufficient energy to produce pairs and the "$\leftarrow$" process does no longer happen – in contrast to the "$\longrightarrow$" process. This leads to the exponential decrease of the particle number density.

In the early universe, the energy density of all particle degrees of freedom is given by a sum over all contributions

$$\rho = \sum_i \frac{g_i}{2\pi^2} T_i^4 \, J_\pm(x_i) \,, \quad x_i = \frac{m_i}{T_i} \,, \qquad (6.20)$$

where, in principle, different species of particles may have different temperatures $T_i$.

Different temperatures can occur as a consequence of decoupling processes.

For a more general perspective it is useful to write the energy density as a function of the "temperature of the universe" $T$ (often identified with the photon temperature):

$$\rho = \frac{\pi^2}{30} g_*(T) \, T^4 \qquad (6.21)$$

where $g_*(T)$ denotes an <u>effective number of degrees of freedom</u> at temperature $T$,

$$g_*(T) = \sum_i g_i \left( \frac{T_i}{T} \right)^4 \frac{J_\pm(x_i)}{J_-(0)} \,. \qquad (6.22)$$

Since the energy density of relativistic species is much larger than that of Boltzmann-suppressed non-relativistic species, it is often sufficient to include only relativistic species in (6.22)

$$\Rightarrow g_*(T) \simeq \sum_{i \in \text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{i \in \text{fermions}} g_i \left(\frac{T_i}{T}\right)^4 \qquad (6.23)$$

For the case that all particles are in equilibrium at a common temperature $T$, the value of $g_*$ can be "counted".

For the Standard model, the counting goes as follows:

| PARTICLE | | MASS | SPIN | $g$ | |
|---|---|---|---|---|---|
| • photon | $\gamma$ | 0 | 1 | 2 | (transversal polarizations) |
| • electroweak gauge bosons | $W^\pm$ | 80 GeV | 1 | 3 | (transversal + longitudinal polarization) |
| | $Z$ | 91 GeV | 1 | 3 | |
| • gluons | $g$ | 0 | 1 | $2 \times 8$ | (trasv. pol. in 8 color combinations) |
| • Higgs boson | $H$ | 125 GeV | 0 | 1 | |

- quarks

| | | | | | |
|---|---|---|---|---|---|
| top | $t$ | 173 GeV | $\frac{1}{2}$ | $4 \times 3 = 12$ | (2: spin states<br>2: particle–antiparticle<br>3: different colors) |
| bottom | $b$ | 4 GeV | $\frac{1}{2}$ | 12 | |
| charm | $c$ | 1 GeV | $\frac{1}{2}$ | 12 | |
| strange | $s$ | 100 MeV | $\frac{1}{2}$ | 12 | |
| down | $d$ | 5 MeV | $\frac{1}{2}$ | 12 | |
| up | $u$ | 2 MeV | $\frac{1}{2}$ | 12 | |

- leptons

| | | | | | |
|---|---|---|---|---|---|
| tauon | $\tau$ | 1777 MeV | $\frac{1}{2}$ | 4 | (2 spin states,<br>2 particle/antiparticle) |
| muon | $\mu$ | 106 MeV | $\frac{1}{2}$ | 4 | |
| electron | $e$ | 511 keV | $\frac{1}{2}$ | 4 | |
| $\tau$-neutrino | $\nu_\tau$ | $< 0.12$ eV | $\frac{1}{2}$ | 2 | (2 spin states) |
| $\mu$-neutrino | $\nu_\mu$ | $< 0.12$ eV | $\frac{1}{2}$ | 2 | |
| e-neutrino | $\nu_e$ | $< 0.12$ eV | $\frac{1}{2}$ | 2 | |

The precise nature of neutrinos is not yet clear. If they are Dirac spinors, $g = 4$, would also hold for them as for electrons. However, they could also be Majorana spinors which allow for particle – antiparticle conversion and thus $g = 2$.

(NB: the counting for the electroweak gauge bosons and the Higgs has been performed in the Higgs phase; above the electroweak phase transition, each of the $W^\pm$ and $Z$ bosons would give $g=2$ and the Higgs field as a complex two-component scalar would give $g=4$. Thus the total result is the same.)

Above the temperature $T \gtrsim \mathcal{O}(100)\,\text{GeV}$ all particles are approximately relativistic and thus contribute to $g_*$. We have

$g_b = 28$ ( $\gamma$'s (2), $W^\pm$ & $Z$ (3x3), gluons (2x8), and Higgs (1) )

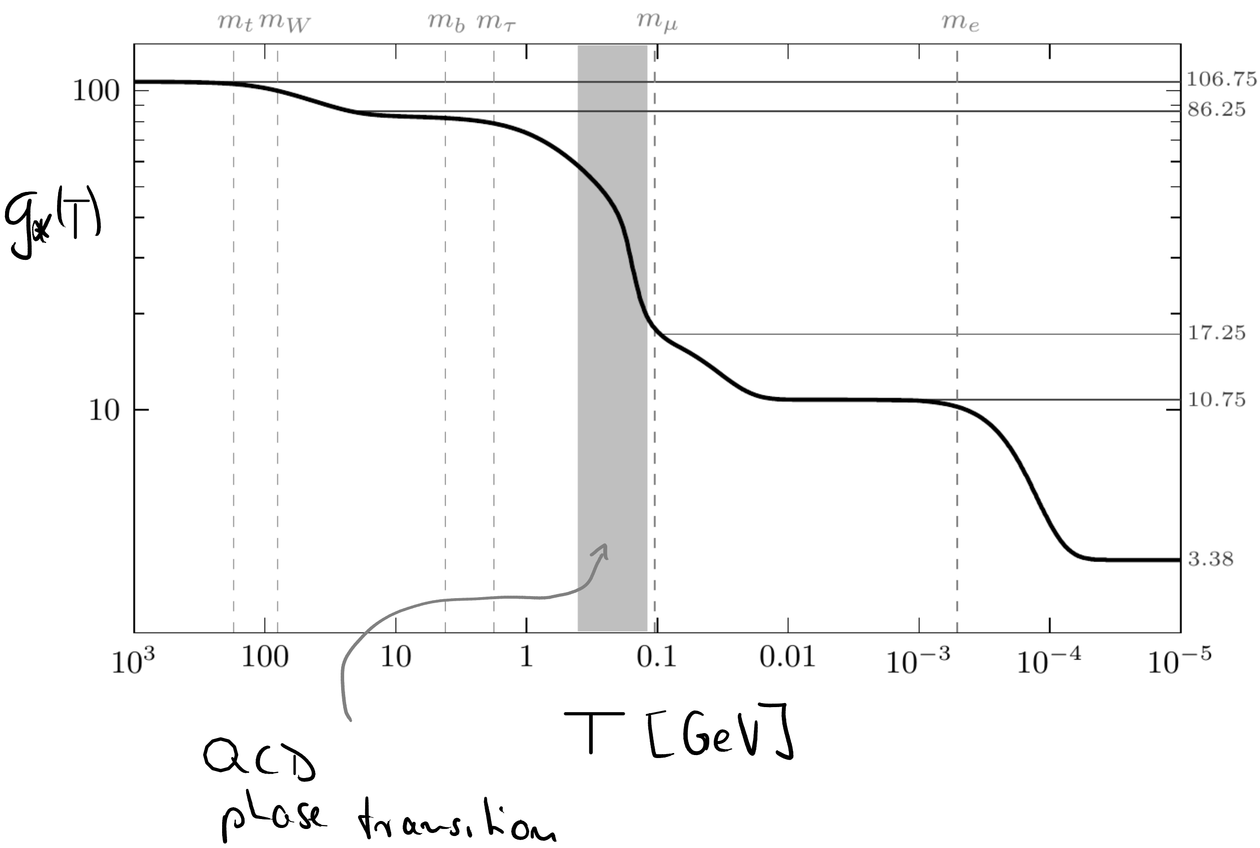
$g_f = 90$ ( quarks (6x12), charged leptons (3x4), and neutrinos (3x2) )

and hence

$$g_* = g_b + \frac{7}{8}\, g_f = 106.75 \,. \tag{6.24}$$

As the temperature drops below a certain mass $m_i$, the particle decouples by exponential suppression as is parametrized by the J functions in (6.22). The largest drop off of a contribution occurs in between $m_i \gtrsim T \geq \frac{1}{10} m_i$.

We obtain the following effective number of relativistic degrees of freedom




The QCD modifies the simple counting slightly, as it produces many baryonic and mesonic bound states. Out of these, only the pions $\pi^0, \pi^\pm$ are light enough to propagate relativistically.

Hence, below the QCD phase transition at $T \simeq 150\,\text{MeV}$, we have

$g_b = 5$ (photons (2), pions (3))

$g_f = 14$ (electrons (4), muons (4), neutrinos (6))

$\Rightarrow \quad g_*(150\,\text{MeV}) = 17.25$

Below the elektron threshold, only photons and neutrinos remain. Here, however, the different temperature of the neutrinos $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$ has to be accounted for, leading to the lowest value shown in the plot.