4. Homogeneous dynamics of Cosmological Spacetimes

Having discussed now the possible ingredients of matter, radiation and energy in the cosmological fluids appearing on the RHS of Einstein's equation (3.2), let us now study the LHS in order to determine the dynamics of the scale factor $a(t)$. For this, we need the explicit form of the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} \qquad (4.1)$$

in terms of curvature quantities, with $R_{\mu\nu}$ being the Ricci tensor and $R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}$ the Ricci scalar.

(NB: (4.1) follows straightforwardly from an action principle, assuming that the action has the simplest possible form respecting all symmetries and being of lowest nontrivial order (linear) in the curvature.)

Setting all geometric beauty of GR aside, all we need for the moment is the definition of the Ricci tensor:

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} \quad (4.2)$$

(NB: this may vaguely remind you of the field strength tensor in electromagnetism, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In fact, the last two terms which are not present in electromagnetism induce nonlinearities. While photons cannot (directly) interact with photons giving rise to the superposition principle, there is no such principle in GR. The theory is inherently nonlinear.)

Knowing the FLRW metric as well as the corresponding Christoffel symbols explicitly, cf. (2.19), it is straightforward, though tedious, to compute the Ricci tensor. Nowadays, this can straightforwardly be done by suitable computer algebra packages for any given metric $g_{\mu\nu}$. Let us first note that $R_{0i} = R_{i0} = 0$, due to isotropy requiring 3-vectors to vanish.

As an example, let us explicitly compute

$$R_{00} = \partial_\lambda \Gamma^\lambda_{00} - \partial_0 \Gamma_0{}^\lambda{}_\lambda + \Gamma^\lambda_{\lambda\rho} \Gamma^\rho_{00} - \Gamma^\rho_{0\lambda} \Gamma^\lambda_{0\rho} .$$

Since Γ's with two indices $= 0$ vanish, this reduces to

$$R_{00} = - \partial_0 \Gamma_0{}^i{}_i - \Gamma_0{}^i{}_j \Gamma^j_{0i}$$

Using $\Gamma_0{}^i{}_j = \frac{1}{c}\frac{\dot{a}}{a}\delta^i_j$ (reinstating c in comparison to (2.19) where $c=1$), we get

$$\underline{\underline{R_{00}}} = - \frac{1}{c^2}\frac{d}{dt}\left(3\frac{\dot{a}}{a}\right) - \frac{3}{c^2}\left(\frac{\dot{a}}{a}\right)^2 = \underline{- \frac{3}{c^2}\frac{\ddot{a}}{a}} .$$

(4.3)

Without any further detailed computation, we give the result for the remaining components:

$$\underline{\underline{R_{ij}}} = \underline{\underline{\frac{1}{c^2}\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{kc^2}{a^2 R_0^2}\right] g_{ij}}} \qquad (4.4)$$

The proportionality to $\sim g_{ij}$ is in fact dictated by homogeneity and isotropy. Using this fact makes it easier to compute the prefactor by projecting onto the 3-d trace $g^{ij} R_{ij}$ (you may do this as an exercise).

From (4.3) & (4.4), we can compute the Ricci Scalar:

$$\underline{\underline{R}} = g^{\mu\nu} R_{\mu\nu} = -R_{00} + g^{ij} R_{ij}$$

$$= \frac{3}{c^2} \frac{\ddot{a}}{a} + \frac{3}{c^2} \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2 \frac{kc^2}{a^2 R_0^2} \right]$$

$$\underline{\underline{= \frac{6}{c^2} \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2 R_0^2} \right]}} \tag{4.5}$$

This leads us via (4.1) to the nontrivial components of the Einstein tensor $G^\mu{}_\nu = g^{\mu\lambda} G_{\lambda\nu}$

$$G^0{}_0 = -\frac{3}{c^2} \left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2 R_0^2} \right], \tag{4.6}$$

$$G^i{}_j = -\frac{1}{c^2} \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2 R_0^2} \right] \delta^i_j, \tag{4.7}$$

as can be straightforwardly verified.

Now, using Einstein's equation $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$, say in comoving coordinates, we get, in principle, two different equations, one from the 00 component, and one from the spatial components.

Let us begin with the 00 component for which $\frac{8\pi G}{c^4} T^0{}_0 = -\frac{8\pi G}{c^2} \rho$, cf. (3.13)

$$\Rightarrow \quad \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2 R_0^2} \tag{4.8}$$

This is the (first) Friedmann equation.

Here, ρ denotes the sum of all contributions to the energy density of the universe, e.g. including ρ_r for radiation (or more specifically ρ_γ for photons and ρ_ν for neutrinos), ρ_m for matter (or ρ_c for cold dark matter and ρ_b for baryons), and ρ_Λ for a dark energy contribution.

Eq. (4.8) governs the evolution of the scale factor, once we specify ρ.

The second Friedmann equation which we obtain from the spatial components can be brought into the form

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right). \qquad (4.9)$$

It is also known as the Raychaudhuri equation. However, as is discussed in the exercises, (4.9) follows from the (first) Friedmann equation together with the continuity equation (3.20). It thus doesn't give any new piece of information.

Using the Hubble parameter $H = \frac{\dot{a}}{a}$, the (first) Friedmann equation is often written as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2 R_0^2} \qquad (4.10)$$

As a first result, we can read off that – for a flat universe $k=0$ – the Hubble parameter is directly linked with the energy density. Evaluating this relation today at $t=t_0$, we define the so-called critical density

$$\rho_{cr,0} = \frac{3H_0^2}{8\pi G} = 1,9 \cdot 10^{-26}\, h^2 \,\frac{kg}{m^3}$$

$$= 2.8 \cdot 10^{11}\, h^2\, M_\odot\, Mpc^{-3} \qquad (4.11)$$

$$= 11\, h^2 \,\frac{\text{protons}}{m^3}$$

(with $h \simeq 0.7$, this corresponds to a handful of protons per cubic meter or to a galaxy per cubic Megaparsec.)

It has become standard to measure all densities relative to the critical density:

$$\Omega_i := \frac{\rho_{i,0}}{\rho_{cr,0}}, \quad i = r, m, \Lambda \quad (\text{or } \gamma, \nu, c, b, \Lambda) \qquad (4.12)$$

This suggests to rewrite the Friedmann equation (4.10) as

$$\frac{H^2}{H_0^2} = \frac{\rho}{\rho_{cr,0}} - \frac{1}{a^2}\frac{kc^2}{H_0^2 R_0^2}. \qquad (4.13)$$

Let us define the curvature "density" parameter

$$\Omega_k := -\frac{kc^2}{H_0^2 R_0^2}, \qquad (4.14)$$

(Note: positive curvature $k>0$ implies negative Ω_k and vice versa.)

Also, decomposing ρ into its matter/radiation/dark energy contributions,

$$\rho = \rho_r + \rho_m + \rho_\Lambda , \qquad (4.15)$$

we know from the continuity equation that each part scales differently with a,

$$\rho_r \sim \frac{1}{a^4} , \quad \rho_m \sim \frac{1}{a^3} , \quad \rho_\Lambda \sim a^0 . \qquad (4.16)$$

With our normalization $a(t_0) = 1$, we can thus relate each of these contributions to their density today:

$$\rho_r = \frac{\rho_{r,0}}{a^4} , \quad \rho_m = \frac{\rho_{m,0}}{a^3} , \quad \rho_\Lambda = \rho_{\Lambda,0} \qquad (4.17)$$

Inserting (4.17) into (4.13) and using the definitions (4.12) & (4.14), the Friedmann equation reads

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda \qquad (4.18)$$

This form is subject to the constraint at $t=t_0$

$$1 = \underbrace{\Omega_r + \Omega_m + \Omega_\Lambda}_{=:\,\Omega_0} + \Omega_k \tag{4.19}$$

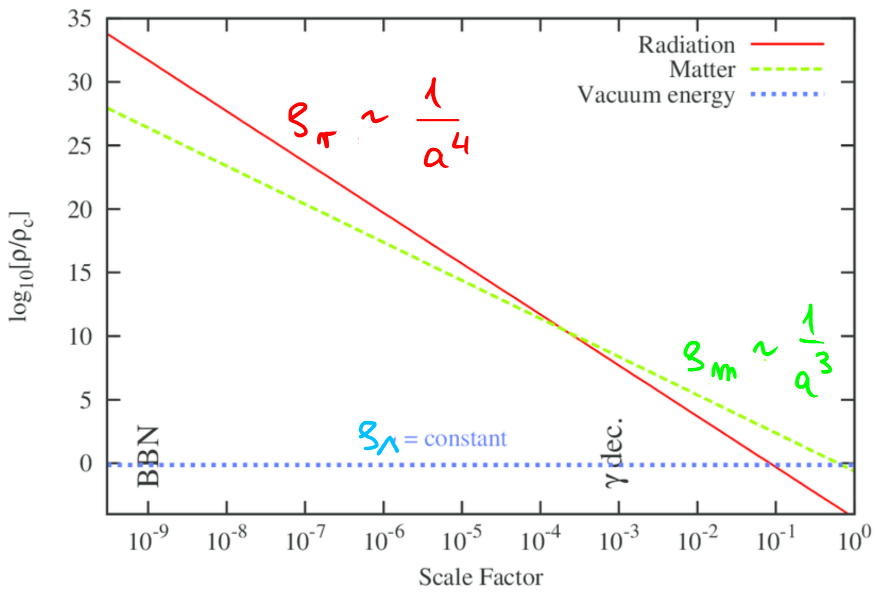
Summarizing all fluid contributions by Ω_0, the curvature parameter reads

$$\Omega_k = -\frac{k c^2}{(R_0 H_0)^2} = 1 - \Omega_0 \,. \tag{4.20}$$

Observational cosmology thus has the task to determine the Ω_i's and thereby the composition of the universe.

In the following, we study solutions of the Friedmann equation for various matter content. Let us start with a flat universe $k=0$ containing a single fluid component. Though this is a simplification, the solutions are relevant as the universe appears to be flat and – during most of its evolution –

was dominated by one of the components. This is visible in the following plot, where the energy density components as measured today are scaled backwards according to their scaling with a:



Away from the crossing points, a single fluid component dominated typically by a few orders of magnitude.

Using the parameter w_i for the i'th component, cf. (3.21), the Friedmann equation reads

$$\frac{\dot a}{a} = H_0 \sqrt{\Omega_i}\; a^{-\frac{3}{2}(1+w_i)} \,, \quad i = r, m, \Lambda, \qquad (4.21)$$

where we assume $\Omega_i > 0$ and all other Ω's $= 0$.

The time-dependence of the scale factor in each case is then given by

$$a(t) \sim \begin{cases} t^{\frac{2}{3(1+w_i)}} \,, & w_i \neq -1 \quad = \begin{cases} t^{2/3} \,, & i = m \\ t^{1/2} \,, & i = r \end{cases} \\ e^{H_0 \sqrt{\Omega_\Lambda}\, t} \,, & w_\Lambda = -1 \qquad\qquad i = \Lambda \end{cases}$$

$$(4.22)$$

For a matter or radiation dominated universe, $a(t)$ grows according to a characteristic power law, for a dark energy dominated universe even exponentially.

Sometimes, also the scaling in conformal time η is useful, see (1.25) $d\eta = \frac{dt}{a(t)}$:

$$a(\eta) \sim \begin{cases} \eta^{\frac{2}{(1+3w_i)}} \,, & w_i \neq -1 \quad = \begin{cases} \eta^2 \,, & i = m \\ \eta \,, & i = r \end{cases} \\ (-\eta)^{-1} \,, & w_i = -1 \qquad\qquad i = \Lambda \end{cases} \tag{4.23}$$

The proportionality constant could in each case be fixed, e.g., by the choice $a(t_0) = 1$.

Example 1: let us for the moment assume that the universe consisted only of matter ("Einstein-de Sitter universe"). The scale factor would then be

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}} . \tag{4.24}$$

From this, we get for the Hubble parameter

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3}\,\frac{1}{t_0}\,\frac{\left(\frac{t}{t_0}\right)^{-\frac{1}{3}}}{\left(\frac{t}{t_0}\right)^{2/3}} = \frac{2}{3}\,\frac{1}{t_0}\left(\frac{t}{t_0}\right)^{-1} . \tag{4.25}$$

Correspondingly, we get a direct relation between the Hubble constant and the age of the universe:

$$H_0 = H(t = t_0) = \frac{2}{3} \frac{1}{t_0} \,. \tag{4.26}$$

Using the observed value of approximately $H_0 \simeq 70 \frac{\text{km}}{\text{s Mpc}}$, we get

$$t_0 = \frac{2 \quad \text{s Mpc}}{3 \cdot 70 \quad \text{km}} \simeq \frac{2\,\text{s} \cdot \cancel{3} \cdot 10^{19}\,\cancel{\text{km}}}{\cancel{3} \cdot 70 \quad \cancel{\text{km}}} \simeq 9 \text{ billion years} \,. \tag{4.27}$$

This constitutes a puzzle, since the age of a purely matter dominated universe would be smaller than that of the oldest astronomically observed stars.

Example 2: The solution for $w_\Lambda = -1$ corresponding to "dark energy only" is also known as de Sitter space. As is visible from the previous figure, it can be expected to be a good approximation to the future universe, since ϱ_Λ has "recently" started to dominate and thus will dominate the future evolution with ρ_r and ρ_m becoming less and less relevant.

In fact, there is some indication that the de Sitter solution also approximates an early phase of cosmology known as the period of inflation to be discussed below.

Except for the de Sitter solution, the radiation as well as the matter dominated solution have a singularity at $t=0$: here, the scale factor goes to zero, $a \to 0$, such that the energy density goes like

$$\rho \sim \left\{ \begin{array}{lll} \text{radiation} & \frac{1}{a^4} \sim \left(\frac{1}{t^{1/2}}\right)^4 \\ \text{matter} & \frac{1}{a^3} \sim \frac{1}{(t^{2/3})^3} \end{array} \right\} \sim \frac{1}{t^2} \qquad (4.28)$$

and thus diverges. One may wonder whether the singularity can be evaded by a suitable choice of matter / radiation mixture. However, as proven by Hawking & Penrose ($\sim$1970), this singularity is a generic feature of big bang cosmologies. Of course, any proof is only as strong as its assumptions. The Hawking–Penrose singularity theorem relies on the "strong energy condition" (satisfied by generic classical matter and radiation):

$$\rho c^2 + 3P \geq 0 \,. \qquad (4.29)$$

While the singularity theorem is much more general, we confine ourselves to spacetimes obeying the Cosmological Principle, and use the second Friedmann equation (Raychaudhuri eq.):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right). \qquad (4.30)$$

For physical scale factors $a>0$, the strong-energy condition (4.29) implies $\ddot{a} < 0$ at all times. This agrees with intuition that gravity is attractive for matter and radiation and hence slows down the cosmological expansion.

Now, let us start from the universe today:

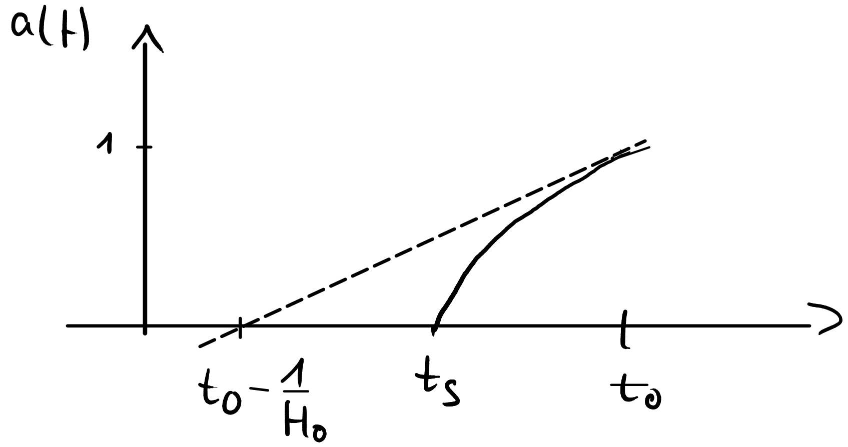
$$a(t_0) = 1 \quad , \quad \dot{a}(t_0) = H_0 > 0 \qquad (4.31)$$

Now, if $\ddot{a} = 0$, the solution would be

$$a(t) = 1 + H_0 (t - t_0) \qquad (4.32)$$

which has a singularity at $t = t_0 - \frac{1}{H_0}$. (4.33)

Since $\ddot{a} < 0$, (4.32) is tangent to the true solution at $t = t_0$ which is concave for all t. Hence there must be a singularity in between $t = t_0 - \frac{1}{H_0} < t_s < t_0$. (4.34)



From a classical viewpoint, i.e. classical GR & classical matter, a singularity at a finite look-back time in terms of a diverging radiation / matter density thus is unavoidable. It can be expected that both quantum matter as well as (some form of) quantum gravity is needed in order to approach or maybe even resolve this singularity.

Let us finally study two relevant cases of <u>two-component universes</u>

1) <u>Radiation</u> + <u>Matter</u> <u>universe</u>

This case is relevant for the early universe, when part of the matter became non-relativistic. It turns out that the solution is easier to construct using conformal time $d\eta = \frac{dt}{a}$.

Then we need:

$$\dot{a} = \frac{da}{dt} = \frac{da}{d\eta}\frac{d\eta}{dt} = \frac{da}{d\eta}\frac{1}{a} =: \frac{a'}{a} \qquad (4.35a)$$

and

$$\ddot{a} = \frac{d}{d\eta}\left(\frac{a'}{a}\right) = \frac{a''}{a^2} - \frac{(a')^2}{a^3} \, . \qquad (4.35b)$$

Insertion into the two Friedman equations yields

$$(a')^2 + \frac{kc^2}{R_0^2} a^2 = \frac{8\pi G}{3} \rho a^4 \qquad (4.36a)$$

$$a'' + \frac{kc^2}{R_0^2} a = \frac{4\pi G}{3} \left(\rho - \frac{3P}{c^2} \right) a^3 \qquad (4.36b)$$

While (4.36 a&b) are valid for arbitrary ρ and P, let us now concentrate on a universe filled with a mixture of matter and radiation:

$$\rho = \rho_m + \rho_r \qquad (4.37)$$

Since $\rho_r \sim \frac{1}{a^4}$ and $\rho_m \sim \frac{1}{a^3}$ and $\rho_r > \rho_m$ in the very early universe (any matter will have been ultrarelativistic at early times), there must be a point in time t_{eq} with a corresponding scale factor $a_{eq} = a(t=t_{eq})$, when both energy densities are equal:

$$\rho(t_{eq}) = \underbrace{\rho_m(t_{eq})}_{=: \frac{\rho_{eq}}{2}} + \underbrace{\rho_r(t_{eq})}_{=: \frac{\rho_{eq}}{2}} = \rho_{eq} \qquad (4.38)$$

This allows us to parametrize the density as a simple function of the scale factor:

$$\rho = \frac{\rho_{eq}}{2}\left[\overset{\text{matter}}{\left(\frac{a_{eq}}{a}\right)^3} + \overset{\text{radiation}}{\left(\frac{a_{eq}}{a}\right)^4}\right]. \qquad (4.39)$$

The second Friedmann equation (4.36b) contains the combination $\rho c^2 - 3P$ which vanishes for radiation according to (3.26): $\rho_r c^2 - 3P_r = 0$.

For matter, we have $P \ll \rho_m c^2$ such that the pressure term can be dropped for matter. For the case of zero curvature, (4.36b) reduces to

$$a'' = \frac{4\pi G}{3} \underbrace{\rho_m a^3}_{=\,\text{const}} = \frac{2\pi G}{3} \rho_{eq} a_{eq}^3 \qquad (4.40)$$

where the RHS is constant. The solution thus reads:

$$a(\eta) = \frac{\pi G}{3} \rho_{eq} a_{eq}^3 \eta^2 + C\eta + D \qquad (4.41)$$

with integration constants C & D. Since

$a(\eta = 0) = 0$ due to the singularity theorem, we have $D = 0$. The constant C follows from inserting the solution (4.41) into the first Friedmann equation (4.36a):

$$\left(\frac{2\pi G}{3} \rho_{eq} a_{eq}^3 \eta + C\right)^2 = \frac{8\pi G}{3} \rho a^4$$

$$\overset{(4.39)}{=} \frac{8\pi G}{3} \frac{\rho_{eq}}{2} \left[a_{eq}^3 \, a + a_{eq}^4 \right] \tag{4.42}$$

Evaluating this at $\eta = 0$ gives

$$C^2 = \frac{4\pi G}{3} \rho_{eq} \, a_{eq}^4 \,. \tag{4.43}$$

Defining

$$\eta_* := \left(\frac{\pi G}{3} \rho_{eq} \, a_{eq}^2 \right)^{-1/2} = \frac{\eta_{eq}}{\sqrt{2}-1}, \tag{4.44}$$

The solution can be written as

$$a(\eta) = a_{eq} \left[\left(\frac{\eta}{\eta_*} \right)^2 + 2 \left(\frac{\eta}{\eta_*} \right) \right] . \tag{4.45}$$

The solution has the expected limits: for $\eta \ll \eta_{eq}$, we rediscover the radiation-dominated behavior, $a \sim \eta$, whereas $\eta \gg \eta_{eq}$ leads back to the matter-dominated case $a \sim \eta^2$ at late times, c.f. (4.23).

2) Matter & Dark energy

Since curvature and radiation contributions are small today, a flat universe with only matter and dark energy is a good approximation today. The (first) Friedmann equation for this case reads (cf. (4.18))

$$\frac{H^2}{H_0^2} = \frac{\Omega_m}{a^3} + \Omega_\Lambda , \qquad (4.46)$$

where $\Omega_m + \Omega_\Lambda = 1$ (no curvature!).

It is straightforward to check, that the solution reads

$$a(t) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}\alpha t\right), \qquad (4.47)$$

where $\alpha = \sqrt{\Omega_\Lambda}\, H_0$. Again, we can check the early / late time limits: for small t, we get $a \sim t^{2/3}$ in agreement with the pure-matter case in (4.22); at late times, we get

$$a(t) \sim \left(e^{\frac{3}{2}\alpha t}\right)^{2/3} = e^{\alpha t}, \qquad (4.48)$$

again, as expected. Using $a(t_0) = 1$ as todays boundary condition, we can solve (4.27) for the age of the universe:

$$t_0 = \frac{2}{3\sqrt{\Omega_\Lambda}\,H_0}\,\mathrm{Ar\,sinh}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_m}}\right). \tag{4.49}$$

Using the values $\Omega_m \simeq 0.32$ and $\Omega_\Lambda \simeq 0.68$ observed today (CMB data), we get

$$\underline{\underline{t_0}} \simeq 0.96\,\frac{1}{H_0} \simeq \underline{\underline{14 \text{ billion}}} \text{ years} \tag{4.50}$$

This demonstrates that dark energy is a decisive ingredient to solve the age puzzle of the "matter only" (Einstein - de Sitter) universe in (4.27). The result (4.50) is remarkably close to current best estimates.