3. Sources of Cosmological Spacetime Dynamics

An essence of Einstein's theory of gravity – general relativity – is that matter (and energy) tells the spacetime "how to curve", and spacetime tells matter "how to move".

Ignoring the geometric aspects of the theory for a moment, Einstein's GR is a classical field theory with similarities to Maxwell's electrodynamics. For instance, Maxwell's equations in special relativistic notation read

$$\partial_\mu F^{\mu\nu} = J^\nu \tag{3.1}$$

field strength tensor describing $\vec{E}$ & $\vec{B}$ fields

$J^\nu = (\rho, \vec{j})$
4-current describing charge and current densities

Hence, in a vague analogy, the current sources the field, and the field tells the charges where to move (cf. Lorentz force).

Now, the matter and energy distribution in gravity cannot be covered by a 4-vector current, but is described by a 2-nd rank tensor, the energy-momentum tensor $T_{\mu\nu}$.

Einstein's field equation reads:

$$\underbrace{G_{\mu\nu}}_{\substack{\text{Einstein tensor:}\\ \text{properties of spacetime}\\ \text{curvature}}} = \frac{8\pi G}{c^4} \underbrace{T_{\mu\nu}}_{\substack{\text{distribution of}\\ \text{matter and energy}}} \qquad (3.2)$$

In the present section, we take a detailed look on the right-hand side for cosmology, the left-hand side is subject to the next section. Also, we keep explicit track of the constants $G, c, \ldots$ in order to see which terms arise from gravity, and which terms are relativistic.

Before we study the full energy-momentum tensor, let us consider a simpler object: the <u>number density</u> and <u>number current density</u> of propagating particles on spacetime. Both can be combined into a 4-vector N^μ. The $\mu=0$ component, N^0, corresponds to the number density of particles (in cosmological applications, it is useful to think of a galaxy being a particle). The spatial components N^i denote the flux of particles in the x^i direction.

Let us apply the Cosmological Principle to N^μ: for a comoving observer, isotropy requires N^i to vanish as no preferred direction should exist $N^i = 0$ (of course, this should be understood as a mean value if averaged over sufficiently large scales). Homogeneity implies that the density is the same at each point in space $N^0 = N^0(t)$. For the following, it is useful to introduce the 4-velocity $U^\mu = \frac{dx^\mu}{d\tau}$ between the particles and the observer.

For a comoving observer (being at rest relative to the neighboring particles) we have $U^\mu = (c, 0, 0, 0)$. Accounting for the factor of c in the zero-component (converting time into length), it is also useful to write

$$N^0 = c\, n(t) \tag{3.3}$$

where $n(t)$ is the number of particles/galaxies per proper volume. Then, we can write

$$N^\mu = n(t)\, U^\mu \tag{3.4}$$

for the comoving observer. Now, boosting the observer to a different coordinate system at relative speed $\vec{v}$, we would get from Lorentz transformations

$$U^\mu = \gamma\,(c, \vec{v})\,, \quad \gamma = \frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}\,. \tag{3.5}$$

In fact, Eq. (4) also holds for this boosted observer: e.g. $N^0 = \gamma\, c\, n(t)$, where the Lorentz factor γ accounts for the Lorentz contraction of the volume hosting the particles.

If the number of particles is conserved, the rate of change of the density must equal the divergence of the flux, as is expressed by a continuity equation:

$$\partial_0 N^0 = - \partial_i N^i \qquad (3.6a)$$

$$\Rightarrow \qquad \partial_\mu N^\mu = 0 \qquad (3.6b)$$

So far, we have not yet accounted for spacetime curvature. In general spacetimes we have seen that the particles follow geodesics, and that – for 4-vector quantities such as the momentum in (2.16a&b) – this is taken care of by the covariant derivative. In fact, (3.6b) generalizes on curved spacetimes to

$$\nabla_\mu N^\mu = 0 \,. \qquad (3.7)$$

In this form, the continuity equation is valid independently of the choice of coordinates.

Using (2.17), we get

$$\partial_\mu N^\mu = - \Gamma^\mu_{\mu\lambda} \, N^\lambda \,. \qquad (3.8)$$

Specializing to comoving observes on a FLRW spacetime, where $\Gamma^\mu_{\mu}{}_\lambda = \frac{3}{c}\frac{\dot{a}}{a}$, we get

$$\frac{1}{c}\frac{dn}{dt} = -\frac{3}{c}\frac{\dot{a}}{a}n \qquad (3.9)$$

$$\Rightarrow \quad \frac{\dot{n}}{n} = -3\frac{\dot{a}}{a} \Rightarrow \underline{\underline{n(t) \sim \frac{1}{a^3(t)}}} \, . \qquad (3.10)$$

This result is plausible if not expected, since the number density decreases proportional to the increase of a comoving volume $V \sim a^3$.

This was just a useful warm-up. For the energy-momentum tensor, we need not only consider particles or particle densities, but energy densities ρ that can account for particles as well as radiation, etc. The physical meaning of $T_{\mu\nu}$ is

$$T_{\mu\nu} = \left(\begin{array}{c|c} T_{00} & T_{0i} \\ \hline T_{i0} & T_{ij} \end{array}\right) = \left(\begin{array}{c|c} \text{energy density} & \text{momentum density} \\ \hline \text{energy flux} & \text{stress tensor} \end{array}\right) . \qquad (3.11)$$

Again, for a comoving observer, the form of $T_{\mu\nu}$ is constrained by the Cosmological Principle:

$T_{00} = \rho(t)\, c^2$ must be position independent (homogeneity)

and the 3-vectors $T_{i0}, T_{0j} = 0$ must vanish (isotropy).

Moreover, since isotropy has to hold also at the spatial origin $\vec{x} = 0$, the only 3-tensor compatible with isotropy and 2 indices is δ_{ij}

$$\Rightarrow T_{ij}(x=0) \sim \delta_{ij} \sim \overset{a^2\delta_{ij}}{\overset{\downarrow}{g_{ij}}}(x=0) \qquad (3.12)$$

In order to bring this information into a relativistic form where the Kronecker δ is written as $\delta^{\mu}{}_{\nu}$, let us raise one index:

$$T^{\mu}{}_{\nu} = g^{\mu\lambda} T_{\lambda\nu} = \begin{pmatrix} -\rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}. \qquad (3.13)$$

Since the physical meaning of a stress is a force per area, the spatial diagonal components correspond to a pressure-like quantity P. We verify below that

P indeed denotes the thermodynamic pressure. Similar to the number density, we can use the 4-velocity of an observer relative to the comoving coordinates U^μ, to write (3.13) in an explicitly covariant form:

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U_\mu U_\nu + P g_{\mu\nu} . \qquad (3.14)$$

This is the energy-momentum tensor of a "perfect fluid".

In Minkowski space, energy and momentum are conserved. Introducing the momentum density $\pi^i = \frac{1}{c} T^i{}_0$, energy conservation can again be written as a continuity equation

$\frac{1}{c}\frac{\partial T^0{}_0}{\partial x^0} = \frac{d\rho}{dt} = \dot{\rho} = -\partial_i \pi^i$, and the dynamics of the momentum density is driven by the pressure:

$\dot{\pi}_i = \partial_i P$ (Euler equation). Taken together, this reads $\quad \partial_\mu T^{\mu\nu} = 0 . \quad (3.15)$

On curved spacetimes, this is promoted to the covariant conservation law

$$\nabla_\mu T^\mu{}_\nu = 0 \tag{3.16}$$

In order to work out (3.16) explicitly, we need the action of the covariant derivative on 2nd-rank tensors with mixed upper and lower indices. This is studied in the exercises. Here, we take over the result:

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\lambda} T^\lambda{}_\nu - \Gamma^\lambda_{\mu\nu} T^\mu{}_\lambda = 0 \tag{3.17}$$

We are interested here in the generalization of the equation for energy conservation given by the $\nu = 0$ component:

$$\partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu\lambda} T^\lambda{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda = 0 \tag{3.18}$$

In comoving coordinates $T^i{}_0$ vanishes by isotropy

$$\Rightarrow \frac{1}{c}\frac{d(\rho c^2)}{dt} + \Gamma^\mu_{\mu 0}\, \rho c^2 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda = 0 \tag{3.19}$$

Here, $\Gamma^\mu_{\mu 0} = \Gamma^i_{i0} = \frac{3}{c}\frac{\dot a}{a}$, and also $\Gamma^\lambda_{\mu 0}$

vanishes unless $\mu = \lambda = i$ are spatial indices

$$\Rightarrow \quad \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \qquad (3.20)$$

This is the generalization of "energy conservation" to the cosmological context. Note that there is no time translation invariance in an expanding universe, and hence energy is <u>not</u> a conserved Noether charge. In fact, in general it is not conserved at all; hence, (3.20) is as close as we can get to "energy conservation" in an expanding universe.

The behavior of the energy density depends on the pressure P and thus on the equation of state of the matter or substance under consideration. In many relevant cases, cosmological "fluids" can be parametrized by

$$w = \frac{P}{\rho c^2} \qquad (3.21)$$

where $w \simeq$ const.

In this case, (3.20) yields

$$\frac{\dot{\varrho}}{\varrho} = -3(1+w)\frac{\dot{a}}{a} \Rightarrow \varrho \sim \frac{1}{a^{3(1+w)}} \tag{3.22}$$

In the following, we classify cosmological fluids in terms of w:

1) Nonrelativistic particles (aka "matter")

Consider an ideal gas for simplicity. The equation of state reads

$$PV = N k_B T$$

with $k_B T \sim \langle \frac{1}{2} m v^2 \rangle$ being of the order of the kinetic energy.

However, the energy density ϱ accounts for both rest mass as well as kinetic energy densities,

$$\frac{E}{N} = \frac{(\varrho c^2) V}{N} = mc^2 + \frac{1}{2} m v^2 + \mathcal{O}(v^4)$$

where $\frac{1}{2} m v^2 \ll m c^2$ for non-relativistic particles. Therefore, we have $|P| \ll \varrho c^2$

(3.23)

for the ideal gas. More generally, we use (3.23) as a criterion for what we call a matter fluid, for which

$$w \simeq 0 \tag{3.24}$$

According to (3.22) we get

$$\rho \sim \frac{1}{a^3} \tag{3.25}$$

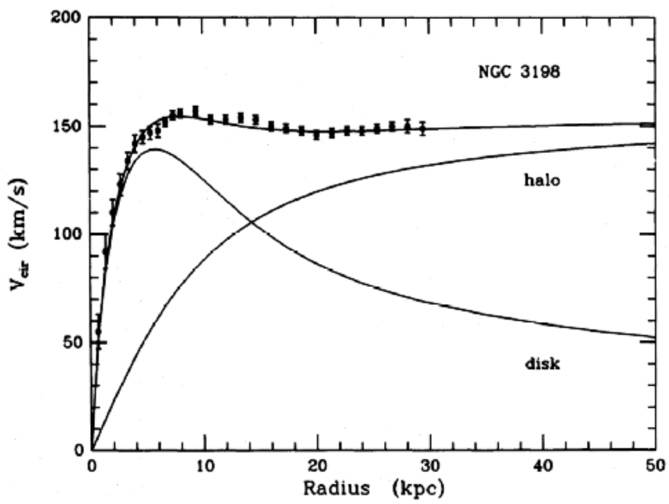
reflecting a dilution of the energy density by the growth of the comoving volume $V \sim a^3$. The energy per comoving volume remains constant for matter.

In cosmology, one distinguishes between two sorts of matter:

- baryons : summarizing (in an awful abuse of terminology) ordinary matter consisting of nuclei and electrons (of course, electrons are leptons, but cosmologists have a coarse-grained perspective also in terms of terminology).

- <u>dark matter</u>: while baryonic matter interacts with photons, we have lots of direct, visible information about baryons in the universe. However, a variety of observations (galaxy rotation curves, weak and strong gravitational lensing effects, properties of the CMB) seem to require more matter than is observed in terms of baryons.

DISTRIBUTION OF DARK MATTER IN NGC 3198



typical galaxy rotation curve

"disk": expected curve from visible matter distribution

"halo": contribution from dark matter halo

For cosmology, it is not so relevant what dark matter actually <u>is</u>, as long as sufficient amounts of dark matter is around.

There are many proposals and dark matter candidates such as weakly interacting massive particles (WIMPs), stable supersymmetric partner particles (e.g. gravitinos), axions on the particles-physics side, or massive compact halo objects (MACHOS) or primordial black holes on the astrophysical side.

For cosmological purposes, it is only relevant that dark matter behaves as a pressureless fluid (cf. (3.23)) for large parts of the cosmological evolution. This is the <u>cold dark matter</u> paradigm (CDM) which is an integral part of the current standard model of cosmology.

(NB: There are also serious and partly successful attempts to find alternative explanations: instead of adding dark matter to the RHS of Einstein's equation, some people try to modify the LHS, i.e. modify the theory of gravity. While this appears to work well for galaxy rotation curves and partly for lensing, there is no convincing alternative that explains the precision CMB data. Through the goggles of cosmology, cold dark matter is preferred by Ockham's razor).

2) Relativistic particles (aka "radiation")

satisfy, as we know from thermodynamics,

$$PV = \frac{1}{3} U \Rightarrow P = \frac{1}{3} \frac{U}{V} = \frac{1}{3} \rho c^2 \qquad (3.26)$$

(in the case of an ideal gas).

In this case, we have

$$w = \frac{P}{\rho c^2} = \frac{1}{3}, \qquad (3.27)$$

and thus from (3.22):

$$\rho \sim \frac{1}{a^4}. \qquad (3.28)$$

The dilution of the energy density hence includes the volume increase $V \sim a^3$ and a redshift $\sim \frac{1}{a}$ of the energy.

As is clear from (3.26), this behavior does not only apply to photon radiation, but to all ultra-relativistic matter components — still summarized as "radiation" in the cosmologists nomenclature.

More specifically, we consider

- photons which are always relativistic as they are massless. Today, we observe cosmological photons in the form of the CMB.

- light particles: here, the mass scale of a particle species should be compared to the temperature. For $T \gg m$, any gas of particle behaves ultra-relativistically and hence (3.28) applies and the particles are counted as "radiation". At later stages in the cosmological evolution, the temperature can drop below a mass threshold, and hence the corresponding particles behave as "matter".

- neutrinos are, in principle, light particles, but are often counted separately, since they behaved like radiation until rather recently.
- gravitons may have been produced in the early universe, albeit at a presumably small density.

Both the cosmic neutrino as well as graviton background have not yet been observed, but might be in reach within the next years/decades.

It is by now an observational fact that the universe is (since recently) in a state of accelerated expansion. None of the above fluid components can drive this property. For this, a new component is needed which is summarized as

3) dark energy, characterized by an equation of state

$$P = -\rho c^2 \tag{3.29}$$

$$\Rightarrow \quad \underline{\underline{w = -1}} \tag{3.30}$$

Such that (3.22) gives,

$$\rho \sim a^0 = \text{const}, \tag{3.31}$$

a constant energy density irrespective of the cosmological expansion.

Dark energy doesn't dilute, it is created as the universe expands.

An essential difference to dark matter is that dark matter clusters under the force of gravity, whereas dark energy doesn't.

Dark energy summarizes various perspectives of trying to understand its origin:

- Vacuum energy: as is familiar from

the harmonic oscillator in QM, oscillating quantum fields come along with a zero-point energy (which does not take any influence on particle-physics experiments), the value of which is essentially a free parameter. Computations of the vacuum expectation value of the energy-momentum tensor yield the form

$$\overline{T_{\mu\nu}}^{\,vac} = - \rho_{vac}\, c^2\, g_{\mu\nu} \quad , \qquad (3.32)$$

where the form $\sim g_{\mu\nu}$ is dictated by Lorentz invariance. A comparison with (3.14) yields, indeed,

$$P_{vac} = - \rho_{vac}\, c^2 \, . \qquad (3.33)$$

(Very) naive estimates of ρ_{vac} within QFT yield "natural" results, which are a factor of 10^{120} off the value measured in cosmology.

- <u>cosmological constant</u>

The left-hand side of the Einstein equation is not unique in the sense that a term linear in the metric can be added

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \qquad (3.34)$$

since $\nabla^\mu g_{\mu\nu} = 0$ by the definition of the covariant derivative ("metric compatibility")

(NB: $G_{\mu\nu}$ also satisfies $\nabla^\mu G_{\mu\nu} = 0$).

Λ is the famous <u>cosmological</u> <u>constant</u> introduced by Einstein to allow for a static solution (which has turned out to be unstable).

This term can, of course, also be brought to the RHS in the form:

$$T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda c^4}{8\pi G} g_{\mu\nu} =: -\rho_\Lambda c^2 g_{\mu\nu} \qquad (3.35)$$

which is of the same form as (3.32).

- <u>dark energy</u>: in addition to ideas of vacuum energy or a cosmological constant, also dynamical fields have been suggested in the late 80's (Wetterich '87: cosmon, Ratra, Peebles '88: quintessence) that dynamically yield

$$w \simeq -1 \tag{3.36}$$

but may also vary in time. Many models have been invented since then.

Since current data is compatible with $w = -1$, all these ideas (vacuum energy, cosm. constant, etc.) are nowadays summarized as <u>dark energy</u>, using the symbol Λ as a subscript for the corresponding contributions.

As dark energy as well as dark matter are decisive ingredients of the current standard model of cosmology, the latter is often called <u>ΛCDM model</u>.