2. Particle kinematics in expanding spacetimes

Before we study cosmological dynamical models in terms of the behavior of the scale factor $a(t)$, let us discuss how particles or light move on FLRW spacetimes. After all, these are the carriers of information that tell us about the evolution of the universe.

2.1 Geodesics

In Newtonian mechanics, free particles in Cartesian coordinates simply satisfy $\ddot{\vec{x}} = 0$, implying that each coordinate satisfies $\ddot{x}^i = 0$.

Already in curvilinear coordinates in flat Euclidean space, this is no longer true.

For instance, for a particle moving freely in a plane, the Lagrangian in polar coordinates reads

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2 \dot{\phi}^2\right) \qquad (2.1)$$

The equations of motion are obtain via the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0 \quad , x^i = (r, \phi), \qquad (2.2)$$

yielding, for instance

$$0 = \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m r \dot{\phi}^2 - m \ddot{r} \Rightarrow \ddot{r} = r \dot{\phi}^2 \quad (2.3)$$

$\neq 0$ in general

Hence, we cannot expect $\ddot{x}^i = 0$ to hold in a generic coordinate system in a dynamically evolving space-time.

Let us, for simplicity, study this problem in Euclidean space for general coordinates with metric $g_{ij} = g_{ij}(x)$ using a Lagrangian for a massive particle of the form

$$L = \frac{m}{2} g_{ij}(x) \dot{x}^i \dot{x}^j \quad (2.4)$$

Since the mass will drop out as in (2.3), we set $m = 1$ in the following. For the Euler-Lagrange equations (2.2) for general coordinates x^i, we need

$$\frac{\partial L}{\partial x^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j \equiv \frac{1}{2} \partial_k g_{ij} \dot{x}^i \dot{x}^j \quad (2.5)$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} = \frac{d}{dt}\left(g_{ik} \dot{x}^i \right)$$

$$\Rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^k} = g_{ik}\,\ddot{x}^i + \underbrace{\partial_j g_{ik}\, \dot{x}^j}_{=\dot{g}_{ik}}\, \dot{x}^i$$

$$= g_{ik}\,\ddot{x}^i + \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik}\right)\dot{x}^j\dot{x}^i \qquad (2.6)$$

In the last step, we have symmetrized $\partial_j g_{ik}$ in the indices j,i, since the term is multiplied by the symmetric product $\dot{x}^j\dot{x}^i$.

Read together, we obtain

$$g_{ik}\,\ddot{x}^i = -\frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)\dot{x}^i\dot{x}^j \qquad (2.7)$$

Multiplying both sides with the inverse of the metric g^{-1}, which we write as g^{lk}, $g^{lk} g_{ki} = \delta^l_i$, we get

$$\ddot{x}^l = -\frac{1}{2} g^{lk}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)\dot{x}^i\dot{x}^j$$

$$=: -\Gamma^l_{ij}\,\dot{x}^i\dot{x}^j \,. \qquad (2.8)$$

Here, we have defined the Christoffel symbol

$$\Gamma^l_{ij} := \frac{1}{2} g^{lk}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right) \qquad (2.9)$$

which parametrizes the "deviations" from

Cartesian coordinates. In summary, we have

$$\frac{d^2 x^i}{dt^2} = - \Gamma^i_{kl} \frac{dx^k}{dt} \frac{dx^l}{dt} \qquad (2.10)$$

Note that the Christoffel symbol is symmetric in the lower indices.

While this was just a Euclidean warm-up, it turns out (cf. exercises) that the path of a massive particle in spacetime $x^\mu(\tau)$, parametrized by its proper time τ, satisfies an equation of the same form:

$$\underline{\underline{\frac{d^2 x^\mu}{d\tau^2} = - \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}}} \qquad (2.11)$$

This is the geodesic equation, where

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right). \qquad (2.12)$$

(Propertime is the time of a clock in the instantaneous rest frame of the particle, i.e. $ds^2 = -d\tau^2$.)

It is useful to rewrite the geodesic equation in terms of the particles 4-momentum

$$P^\mu = m \frac{dx^\mu}{d\tau} \qquad (2.13)$$

which we consider as a function of the coordinates $P^\mu = P^\mu(x)$. We have

$$m \frac{d^2 x^\mu}{d\tau^2} = \frac{d}{d\tau} P^\mu = \frac{\partial P^\mu}{\partial x^\alpha} \frac{dx^\alpha}{d\tau} = \partial_\alpha P^\mu \frac{P^\alpha}{m} \qquad (2.14)$$

Using the geodesic equation, we get

$$P^\alpha \partial_\alpha P^\mu = m^2 \frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta} \left(m \frac{dx^\alpha}{d\tau} \right) \left(m \frac{dx^\beta}{d\tau} \right)$$

$$= -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta \qquad (2.15)$$

This, we can also write as

$$P^\alpha \underline{\underline{\left(\partial_\alpha P^\mu + \Gamma^\mu_{\alpha\beta} P^\beta \right)}} = 0, \qquad (2.16a)$$

or as $\quad P^\alpha \nabla_\alpha P^\mu = 0, \qquad (2.16b)$

where we have introduced the <u>covariant</u> <u>derivative</u>

$$\nabla_\alpha P^\mu = \partial_\alpha P^\mu + \Gamma^\mu_{\alpha\beta} P^\beta \qquad (2.17)$$

(NB: this concept is similar to the covariant derivative in gauge theories $D_\mu(A) = \partial_\mu + ie A_\mu$, with the Christoffel symbol playing the role of the gauge field. In GR, the covariant derivative has a geometric meaning of comparing the momentum at infinitesimally neighboring points by parallel transporting the vector.)

The form (2.16) of the geodesic equation is useful for us, as it is also applicable to massless particles, i.e. photons, with a 4-momentum $P^\mu = (\omega, \vec{k})$, where ω is the photon frequency and $\vec{k}$ the wave vector.

2.2 Photon propagation on FLRW spacetimes

In order to make use of the geodesic equation, we need to work out the Christoffel symbols for the FLRW metric. Here, we use the form

$$ds^2 = -dt^2 + a^2(t)\,\gamma_{ij}\,dx^i dx^j \qquad (2.18)$$

with the spatial metric γ_{ij} for some useful set of spatial coordinates. The block diagonal form of the FLRW metric $g = \begin{pmatrix} -1 & 0 \\ 0 & a^2\gamma_{ij} \end{pmatrix}$ simplifies the computation of the Christoffel symbols.

For instance, all Christoffel symbols with two time indices vanish, e.g.

$$\Gamma^{\mu}_{00} = \frac{1}{2} g^{\mu\lambda} \Big(\underbrace{\partial_0 g_{0\lambda}}_{\substack{=0,\ \text{since} \\ g_{00} = \text{const} \\ g_{0i} = 0}} + \underbrace{\partial_0 g_{0\lambda}}_{=0} \underbrace{- \partial_\lambda \underbrace{g_{00}}_{=\text{const}}}_{=0} \Big) = 0$$

The only non-trivial components are

$$\begin{aligned} \Gamma^{0}_{ij} &= a\,\dot{a}\,\gamma_{ij} \\ \Gamma^{i}_{0j} &= \frac{\dot{a}}{a}\,\delta^{i}_{j} \\ \Gamma^{i}_{jk} &= \gamma^{il}\left(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk}\right) \end{aligned} \tag{2.19}$$

(as will be computed in the exercises.)

Also, the Cosmological Principle comes to our help: Because of homogeneity of the FLRW spacetime, a particle does not accelerate in some direction in comoving coordinates, i.e. $\partial_i P^\mu = 0$

Thus, we need to study (2.15) in the form

$$P^0 \frac{dP^\mu}{dx^0} = -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta \qquad (2.20)$$

These are 4 equations for $\mu = 0,1,2,3$. Consider $\mu = 0$:

$$P^0 \frac{dP^0}{dx^0} = -\Gamma^0_{\alpha\beta} P^\alpha P^\beta \overset{\Gamma^0_{0\beta} = \Gamma^0_{\alpha 0} = 0}{=} -\Gamma^0_{ij} P^i P^j \qquad (2.21)$$

With $P^0 = E$ being the energy of a particle and (2.19), we get

$$E \frac{dE}{dt} = -a\dot{a}\,\gamma_{ij} P^i P^j . \qquad (2.22)$$

Specializing to the case of a photon $P^\mu = (\omega, \vec{k})$, that satisfies

$$0 = P^2 = P_\mu P^\mu = g_{\mu\nu} P^\mu P^\nu = -\omega^2 + a^2 \gamma_{ij} k^i k^j , \qquad (2.23)$$

Eq. (2.22) becomes

$$\omega \frac{d\omega}{dt} = -a\dot{a} \frac{1}{a^2} \underbrace{a^2 \gamma_{ij} k^i k^j}_{\omega^2} = -\frac{\dot{a}}{a} \omega^2$$

$$\Rightarrow \qquad \frac{1}{\omega} \dot{\omega} = -\frac{1}{a} \dot{a} \qquad (2.24)$$

$$\Rightarrow \qquad (\ln \omega)^{\dot{}} = -(\ln a)^{\dot{}}$$

which implies that the frequency of a propagating photon on a FLRW spacetime scales as

$$\omega \sim \frac{1}{a} \tag{2.25}$$

For an expanding spacetime, the frequency of a propagating photon hence is redshifted to smaller values.

Using $\omega = \frac{1}{\lambda}$, a light signal of wavelength λ_1 emitted at time t_1 will be observed at a later time t_0 (say "today") at wavelength

$$\lambda_0 = \frac{a(t_0)}{a(t_1)} \lambda_1 \tag{2.26}$$

For an expanding universe with $a(t_0) > a(t_1)$, Eq. (2.26) describes an increase of the wavelength called the cosmological redshift. One can think of the wavelength being stretched while propagating on an expanding spacetime. (This is the general relativistic analogue of the Doppler effect.)

(NB: the argument leading to (2.25) can also be generalized to massive particles. In this case, one finds that the spatial momentum scales as $|\vec{p}| \sim \frac{1}{a}$ which implies that the peculiar velocities of particles on an expanding universe tend to zero, i.e. particles / stars / galaxies "slow down" in comoving coordinates in the absence of forces.)

In astronomy, the redshift is defined as the relative change of the wavelength

$$z := \frac{\lambda_0 - \lambda_1}{\lambda_1} \qquad (2.27)$$

The redshift z is directly observable from the measurement of the shift of absorption lines in the spectrum of stars / galaxies such as the famous 21cm line of hydrogen upon the absorption of light by a spin-flip of the electron relative to the proton spin.

Using (2.26) and our convention $a(t_0) = 1$, we find

$$1+z = \frac{1}{a(t_1)} \ . \qquad (2.28)$$

For instance, when we observe a galaxy at redshift $z = 1$, the observed light was emitted at a scale factor $a(t_1) = \frac{1}{2}$, i.e. at a time when the universe was "half as big" as today.

It is common to label events in the history of the universe by their redshift (instead of time or scale factor). For instance, the cosmic microwave background was created at redshift $z = 1100$, first galaxies were formed around $z \sim 10$. The current record observation of the oldest observed galaxy (published in April 2022) is a redshift of $z = 13.3$.

For sufficiently close sources ($z \lesssim 1$), we can expand the scale factor in a Taylor series about $t = t_0$ ("today")

$$a(t_1) = a(t_0) + (t_1 - t_0)\,\dot{a}(t_0) + \dots$$

$$= a(t_0)\left[1 + (t_1 - t_0) H_0 + \dots\right], \qquad (2.29)$$

where $t_0 - t_1 > 0$ is the look-back time and

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \qquad (2.30)$$

is the Hubble constant.

Using (2.28), we get a direct relation between the redshift and the look-back time

$$z = H_0 (t_0 - t_1) + \dots \qquad (2.31)$$

For near-by objects, $t_0 - t_1$ is simply equal to the distance d (since $c = 1$), and we get a linear relation between the redshift and the distance, $z = H_0 d + \dots$

In terms of the "recession speed" v which is the speed inferred from the redshift according to the Doppler effect, $v = z \cdot c$, we can write this linear relation as

$$\underline{\underline{v}} = z \cdot c \simeq \underline{\underline{H_0 \cdot d}} \qquad (2.32)$$

This is the celebrated Hubble – Lemaitre law (predicted by Lemaitre and observed (independently) by Hubble).

We emphasize that this holds for "near-by" objects with $z < 1$; for larger z, not only higher orders can become relevant, but also the meaning of "distance" has to be taken care of.

Hubble's first measurements of the constant H_0 in 1929/30 led to $H_0 \approx 500 \frac{km/s}{Mpc}$ which overestimated the constant substantially.

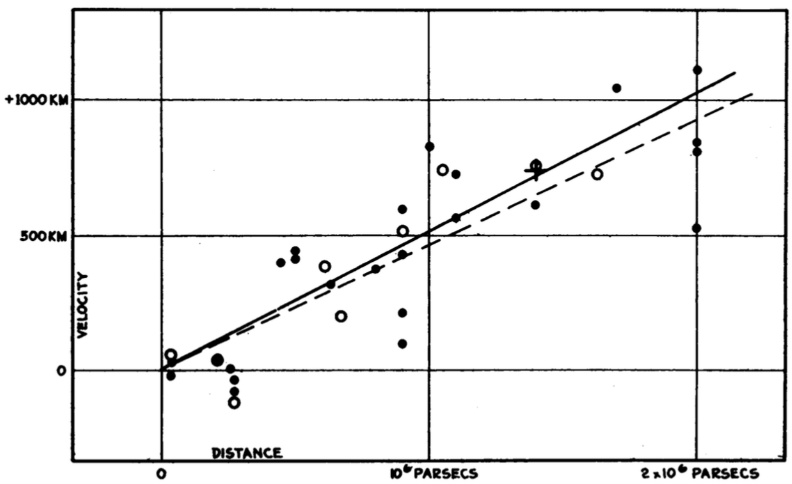


FIGURE 1
Velocity-Distance Relation among Extra-Galactic Nebulae.

Hubble's data from 1929

A direct measurement of the Hubble constant requires accurate data for both the recession speed (directly obtainable from

the redshift, "easy"), preferably corrected by peculiar velocities (requires modeling of local gravitational dynamics, "not so easy"), and the distance ("notoriously difficult").
As we have no direct means to determine the distance, we must infer it from the observational data, typically in the form of light. Various distance measures are used, an important one already used by Hubble is the luminosity distance: if the absolute intrinsic brightness of an object is known, the distance can be inferred from the observed brightness. Again the intrinsic brightness can not be measured directly: a star can look faint because it is far away, or because it is faint. Hubble used Cepheids which are stars with a periodically varying brightness. From observation it turns out that the period is linked to the brightness, such that the intrinsic brightness can be obtained from measurements of the variation period.
For larger distances, the "standard candles" of choice are Type Ia supernovae. These are explosions of white dwarfs that accrete matter from a companion star. White dwarfs are comparatively simple objects, stabilized

by the Fermi pressure of the degenerate electron Fermi gas. The explosion occurs precisely when the mass of the white dwarf exceeds the Chandrasekhar limit beyond which the Fermi pressure can no longer balance the gravitational attraction. This (together with some phenomenological modelling) yields a fairly well understood absolute intrinsic brightness.

Type Ia supernovae occur at a rate of a few per 100 years per galaxy. The observation of many galaxies can yield a remarkable statistics that has been and still is an invaluable resource for quantitative cosmology (Nobel prize 2011: Perlmutter, Schmidt, Riess for the discovery of the accelerated expansion of the universe).

Now, let us assume we observe an astronomical object with known intrinsic luminosity (energy emitted per unit time) L. In our telescopes, we ultimately observe the flux (energy per unit time per receiving area). Assuming isotropic radiation, we can deduce the total flux F through a sphere surrounding the object.

As we consider light, conformal coordinates are particularly useful. Because of homogeneity, we can choose the spatial origin of our coordinates in the center of the observed object. Then, the light moves exactly radially towards us. The relevant part of the line element (1.26) thus reads:

$$ds^2 = a^2(\eta)\left(-d^2\eta + d^2\chi\right) \tag{2.33}$$

Since light propagates precisely with $ds^2 = 0$, we have

$$d\chi = d\eta \overset{(1.25)}{=} \frac{dt}{a(t)} \tag{2.34}$$

and thus

$$\chi = \chi(t_0, t_1) = \int_{t_1}^{t_0} \frac{dt}{a(t)} \tag{2.35}$$

In a static Euclidean space, we would have a simple luminosity-flux relation, since $\chi = r$ and thus

$$F = \frac{L}{4\pi r^2} = \frac{L}{4\pi\chi^2} \quad \text{(static space)} \tag{2.36}$$

In an expanding spacetime, this result is modified for three reasons:

1) When the light reaches our telescope at time $t_0 > t_1$, the sphere with radius χ has an area of (c.f (1.26, 1.24))

$$4\pi a^2(t_0)\, S_k^2(\chi) =: 4\pi \underbrace{a^2(t_0)}_{=1}\, d_m^2 \qquad (2.37)$$

where $d_m = S_k(\chi)$ is the so-called metric distance. Note that d_m differs from χ in curved space.

2) The arrival rate of the photons is smaller than the emission rate, since the coordinate time t has also expanded. The rate scales like $\sim \frac{a(t_1)}{a(t_0)} = \frac{1}{1+z}$ and so does the flux.

3) Due to the redshift, the photon energy $E = \omega$ decreases by the same factor $\frac{1}{1+z}$

Hence, the analogue of (2.36) in an expanding cosmological space time is

$$F = \frac{L}{4\pi d_m^2 (1+z)^2} =: \frac{L}{4\pi d_L^2} \quad \text{(expanding space)} \qquad (2.38)$$

Here, we have defined the <u>luminosity distance</u>

$$\underline{\underline{d_L = (1+z)\, d_m}} \qquad (2.39)$$

Knowing the intrinsic brightness / luminosity L of an object, and measuring the flux at a telescope, we can have measure d_L via (2.38). We also measure the redshift z, such that we get information about the metric distance d_m. The latter encodes the information about the cosmological evolution.

Let us work this out more specifically. For this, we expand the scale factor to second order in the look-back time, cf. (2.29):

$$a(t) = 1 + H_0 (t-t_0) + \frac{1}{2} \ddot{a}(t_0) (t-t_0)^2$$

$$=: 1 + H_0 (t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 \qquad (2.40)$$

where we have introduced the deceleration parameter

$$q_0 := - \frac{\ddot{a}}{a H^2} \Big|_{t=t_0} \qquad (2.41)$$

which is a measure for the deceleration of the scale factor $a(t)$. (The naming is somewhat unfortunate, since we know that $q_0 \simeq -0.5$, i.e. the universe is, in fact, accelerating its expansion today.)

Substituting (2.40) at time t_1 into (2.28), we get

$$z = \frac{1}{a(t_1)} - 1 = \left(1 + H_0 (t_1 - t_0) - \frac{1}{2} q_0 H_0^2 (t_1 - t_0)^2 \dots \right)^{-1} - 1$$

$$\Rightarrow \quad z = H_0(t_0-t_1) + \frac{1}{2}(2+q_0)H_0^2(t_0-t_1)^2 + \dots \qquad (2.42)$$

As we are working in the small redshift regime $z < 1$, we can invert (2.42) as

$$H_0(t_0-t_1) = z - \frac{1}{2}(2+q_0)z^2 + \dots \qquad (2.43)$$

At this point, let us go back to the comoving distance (2.35)

$$\chi = \int_{t_1}^{t_0} \frac{dt}{a(t)} \overset{(2.40)}{=} \int_{t_1}^{t_0} dt \left(1 - H_0(t-t_0) + \dots\right)$$

$$= (t_0-t_1) + \frac{1}{2}H_0(t_0-t_1)^2 + \dots$$

$$\overset{(2.43)}{=} \frac{z}{H_0} - \frac{1}{2}(2+q_0)\frac{z^2}{H_0} + \frac{1}{2}H_0\frac{z^2}{H_0^2} + \dots$$

$$= \frac{1}{H_0}\left(z - \frac{1}{2}(1+q_0)z^2 + \dots\right) \qquad (2.44)$$

With (1.24), χ is directly linked to the metric distance $d_m = S_k(\chi)$ such that we can understand the metric distance as a function of redshift through (2.44): $d_m = d_m(z)$

Using the luminosity distance (2.39), we get d_L as a function of z as well with H_0, q_0 and the curvature as explicit parameters.

For the phenomenologically relevant case of flat space, we have $S_k(\chi) = \chi$ and thus

$$d_L = (1+z)\, d_m \overset{d_m = \chi \text{ (flat space)}}{=} (1+z)\chi$$

$$\overset{(2.44)}{=} (1+z) \frac{1}{H_0} \left(z - \frac{1}{2}(1+q_0) z^2 + \dots \right)$$

$$= \frac{1}{H_0} \left(z + \frac{1}{2}(1-q_0) z^2 + \dots \right) \qquad (2.45)$$

(flat space)

which is the second-order version of the Hubble-Lemaitre law.

Measuring d_L as a function of redshift $d_L = d_L(z)$, the Hubble constant H_0 and the deceleration parameter q_0 can be fitted. As discussed later, q_0 depends on the energy content of the universe.

Since the Hubble constant historically had large uncertainties, it has become conventional to parametrize H_0 by a dimensionless parameter h in the form

$$H_0 = 100\, h \, \frac{\text{km/s}}{\text{Mpc}} \qquad (2.46)$$

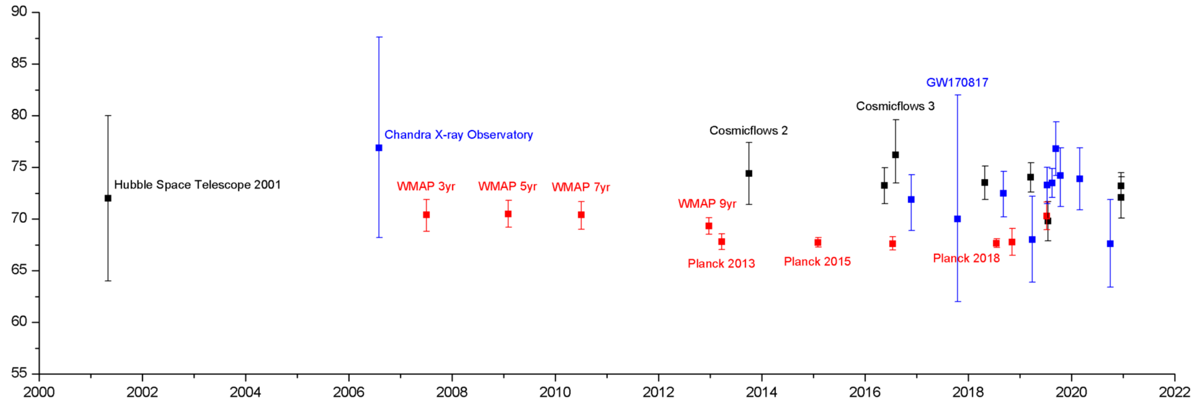
Measurements in recent years have become remarkably precise. Precision data comes from Type Ia supernovae, from measurements of the cosmic microwave background (CMB) (+ cosmological model assumptions) and recently also from gravitational wave measurements (though less precise so far). For instance, the latest supernovae data yield

$$h = 0.7403 \pm 0.0142 \quad \text{(supernovae) (March 2019)} \tag{2.47}$$

whereas the CMB data gives

$$h = 0.6766 \pm 0.0042 \quad \text{(CMB) (Planck 2018)} \tag{2.48}$$

The discrepancy is statistically significant and on the $\sim 4.5\sigma$ level. This puzzle is called the "Hubble tension". It is currently unclear whether it is due to an unknown source of error or whether it provides first evidence that our standard cosmological model is incomplete.



compilation of H_0 measurements (Wikipedia)