1 Geometry & the Cosmological Principle

"Weißt Du wieviel Sternlein stehen...?"

Yes, there are about a hundred billion (10^{11}) stars in a galaxy (as many as neurons in our brains), and about a hundred billion galaxies in the visible universe. This makes $\sim 10^{22}$ stars in the cosmos. How could we ever hope that the evolution of the cosmos can be described by a set of simple equations?

This is possible by taking a sufficiently coarse-grained perspective, very much in the same way as we can describe the motion of a ball without looking at each individual atom. This perspective is phrased as the Cosmological principle.

In physics, a "principle" is essentially an a priory intuitive guess that is ultimately supported by observation.

The Cosmological principle states that the universe is spatially homogeneous (looks the same everywhere) and isotropic (looks the same in any direction).

It is obvious that this principle does not hold on solar system scales (planets are not homogeneously smeared out) nor on galactic scales (galaxies have a preferred axis, their rotation axis).

It turns out that the principle is justified if the universe is averaged over scales of $\sim 100\,\mathrm{Mpc}$ (corresponding to scales of Local superclusters).

This principle leads to a simple description of the universe as spacetime geometry becomes rather simple. However, not too simple.
For instance, the maybe simplest spacetime, namely a static universe, is ruled out by gravity being completely attractive for all matter. A static case must therefore be unstable as all matter would want to clump (and general relativity (GR) connects the properties of spacetime to the matter distribution).
Therefore, spacetime must be dynamical and evolve with time.

As a warm-up, let us take a look at the Euclidean geometry of 3-dimensional space. The physical distance between two infinitesimally close points with cartesian coordinate differences dx, dy, dz is

$$dl^2 = dx^2 + dy^2 + dz^2 = \sum_{i,j=1}^{3} \delta_{ij}\, dx^i\, dx^j \qquad (1.1)$$

where we have used the notation $(x^1, x^2, x^3) = (x, y, z)$. Here, the Kronecker $\delta_{ij} = \mathrm{diag}(1,1,1)$ plays the role of the <u>metric</u>.
The latter can be viewed as an object that turns

coordinate distances into physical distances.

We can parametrize 3-dim Euclidean space equally well with spherical coordinates in which the distance dl would read

$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 = \sum_{i,j=1}^{3} g_{ij} \, dx^i dx^j \, . \qquad (1.2)$$

This time, we have used $(x^1, x^2, x^3) = (r, \theta, \Phi)$ and the metric takes the form

$$g_{ij} = g_{ij}(x) = \mathrm{diag}\,(1, r^2, r^2 \sin^2\theta) \, . \qquad (1.3)$$

Observers using different coordinates do in general not agree on the coordinate values, but they would agree on the physical distance dl which thus is an <u>invariant</u>.

From special relativity, we know that the invariant distance of two infinitesimally close spacetime events is given by

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \, x^\mu x^\nu \overset{\substack{\text{Einsteins sum} \\ \text{convention}}}{\equiv} g_{\mu\nu} \, dx^\mu dx^\nu \qquad (1.4)$$

where we have used the coordinates $x^\mu = (ct, x^i)$ and denote the Minkowski metric $g_{\mu\nu}$ by

$$g_{\mu\nu} = \mathrm{diag}\,(-1, +1, +1, +1) \, , \qquad (1.5)$$

such that

$$ds^2 = -c^2 dt^2 + \delta_{ij} \, dx^i dx^j \qquad (1.6)$$

(from now on, we use $c = 1$, of course).

In general relativity, the metric becomes a dynamical field variable $g_{\mu\nu} \to g_{\mu\nu}(t,\vec{x})$ which is determined by Einstein's field equation (cf. later) that connects the metric to the distribution of energy and matter.

As is obvious from (1.3), not every $(t,\vec{x})$-dependent metric is associated with a non-trivial spacetime. A $(t,\vec{x})$-dependence can simply arise from the choice of coordinates.
However, a spacetime dependent metric $g(t,\vec{x})$ that cannot be transformed back into (1.5) describes a spacetime with non-trivial geometric properties. The metrics in Einstein's GR describe spacetime curvature as sourced by energy and matter.
Now, the Cosmological Principle, i.e. homogeneity and isotropy in space constrain the relevant metric of cosmological spacetime to be of the form

$$ds^2 = -dt^2 + a^2(t)\, dl^2 \qquad (1.7a)$$

$$\text{where } dl^2 = \gamma_{ij}(x)\, dx^i dx^j . \qquad (1.7b)$$

In order to meaningfully apply the Cosmological Principle, we have to assume that spacetime can be understood as a time-ordered sequence of spatial slices Σ_t, because the Cosmological principle applies only to the spatial part Σ_t of spacetime. The <u>scale factor</u> $a(t)$ in front of the spatial part dl^2 cannot depend on x^i as it would violate homogeneity.

Also, all metric components g_{0i} or g_{i0} are set to zero in (1.7a). This is because of isotropy, if, e.g. $g_{01} \neq 0$, the x direction would be a distinguished direction.

In principle, the g_{00} component could be different from -1. However, in this case, we could introduce a new time coordinate $dt' = \sqrt{g_{00}}\, dt$ and absorb such a factor in a redefinition of time.

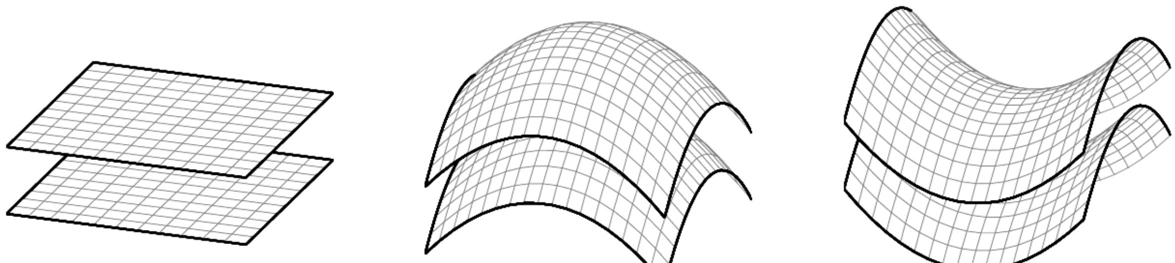
Finally, the spatial metric $\gamma_{ij}(x^i)$ in (1.7b) is not arbitrary, but also has to satisfy the cosmological principle.

Homogeneity demands that – if the spatial slice has nonzero curvature – the curvature must be constant on Σ_t.

Qualitatively, there are 3 different possibilities:

1) Flat space (zero curvature): this simplest possibility corresponds to 3-dim. Euclidean space with line element

$$dl^2 = \delta_{ij}\, x^i x^j \qquad (1.8)$$



2-dim analogues of spatial geometries

from D. Baumann, Cosmology

This space is clearly invariant under translations $x^i \to x^i + a^i$, $a^i = \text{const}$, (homogeneity), and under rotations $x^i \to R^i{}_k x^k$, $R^T R = \mathbb{1}$ (isotropy).

2) Spherical space: these are spaces with constant positive curvature which can be represented as 3-spheres S^3 embedded into 4-dim Euclidean space:

$$dl^2 = d\vec{x}^{\,2} + du^2 \quad , \quad \vec{x}^{\,2} + u^2 = R_0^2 \qquad (1.9)$$

where R_0 is the radius of the sphere.
Homogeneity and Isotropy are obvious from the symmetry of the line element dl^2 under 4-dim rotations.

3) Hyperbolic space: analogously, we can represent spaces of negative constant curvature as hyperboloids H^3 embedded into a 4-dimensional Lorentzian space $\mathbb{R}^{1,3}$:

$$dl^2 = d\vec{x}^{\,2} - du^2 \quad , \quad \vec{x}^{\,2} - u^2 = - R_0^2 \qquad (1.10)$$

where $R_0^2 > 0$ is connected to the constant (negative) curvature

Homogeneity & Isotropy are inherited from the 4-dim. Lorentz transformations with u being a time-like coordinate.

Cases 2) & 3) can be combined into

$$dl^2 = d\vec{x}^2 \pm du^2 \quad , \vec{x}^2 \pm u^2 = \pm R_0^2 \qquad (1.11)$$

The differential of the embedding constraint implies

$$\vec{x}\cdot d\vec{x} = \mp u\,du \quad \Rightarrow \quad du^2 = \frac{(\vec{x}\cdot d\vec{x})^2}{R_0^2 \mp \vec{x}^2} \qquad (1.12)$$

which can be used to eliminate the auxiliary u coordinate

$$dl^2 = d\vec{x}^2 \pm \frac{(\vec{x}\cdot d\vec{x})^2}{R_0^2 \mp \vec{x}^2} \; . \qquad (1.13)$$

All cases 1)–3) can be summarized in the spatial line element

$$dl^2 = d\vec{x}^2 + k\,\frac{(\vec{x}\cdot d\vec{x})^2}{R_0^2 - k\vec{x}^2} \quad , \text{ for } k = \begin{cases} 0 & \mathbb{R}^3 \\ +1 & S^3 \\ -1 & H^3 \end{cases} \qquad (1.14)$$

The symmetries become more visible when using spherical coordinates:

$$\begin{aligned} d\vec{x}^2 &= dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \\ \vec{x}\cdot d\vec{x} &= r\,dr \\ \vec{x}^2 &= r^2 \end{aligned} \qquad (1.15)$$

$$\Rightarrow dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + k\,\frac{r^2 dr^2}{R_0^2 - kr^2}$$

$$\Rightarrow \quad dl^2 = \frac{dr^2}{1 - k\frac{r^2}{R_0^2}} + r^2 d\Omega^2 \qquad (1.16)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$ is the metric on the unit 2-sphere.

The spatial line element (1.16) together with (1.7a) yield

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - k\frac{r^2}{R_0^2}} + r^2 d\Omega\right] \qquad (1.17)$$

which is the celebrated Friedmann - Lemaître - Robertson - Walker (FLRW) metric in spherical / polar coordinates.

It is worthwhile to emphasize that the Cosmological Principle has reduced the 10-independent components of the metric (a symmetric 4x4 matrix) to one single function $a(t)$ of time and the constant R_0 setting the curvature scale.

Let us study some aspects of this metric:

The line element has a rescaling symmetry under

$$a \to \lambda a \ , \ r \to \frac{r}{\lambda} \ , \ R_0 \to \frac{R_0}{\lambda} \qquad (1.18)$$

for $\lambda > 0$. Often, this symmetry is used to scale $R_0 \to 1$.

Here we will use (1.18) to set the scale factor today at $t = t_0$

to unity, $a(t_0) = 1$. Then, R_0 is related to today's curvature scale.

Then, it is important to note that r is not a direct measure for some radial distance. Rather, it is a <u>comoving</u> <u>coordinate</u> (think of a coordinate system that you draw on the surface of a small balloon. The tick marks that you have drawn will "comove" when you start inflating the balloon. At later times, the tick marks will have a larger distance). In particular the coordinate distance can be changed using the rescaling (1.18).

The (more) physical coordinate correspondingly is the product

$$r_{phys} = a(t)\, r \tag{1.19}$$

which remains invariant under the rescaling (1.18).

The corresponding velocity of a "particle" (think of a star or a galaxy) then reads

$$\vec{v}_{phys} = \frac{d\vec{r}_{phys}}{dt} = \frac{da}{dt}\vec{r} + a(t)\frac{d\vec{r}}{dt} = H\,\vec{r}_{phys} + \vec{v}_{pec} \tag{1.20}$$

where we have introduced the <u>Hubble parameter</u>

$$H = H(t) := \frac{\dot{a}}{a}, \quad \dot{a} = \frac{da}{dt} \tag{1.21}$$

The first term in (1.20) $\sim H\vec{r}_{phys}$ is called the Hubble flow. It denotes the velocity of the particle (star / galaxy) resulting from the expansion of the space between $\vec{r}_{phys}$ and the origin of the coordinate system. The second term $\vec{v}_{pec} = a\dot{\vec{r}}$ is the peculiar velocity measured by a comoving observer who him-/herself follows the Hubble flow. It describes the motion of the particle relative to the cosmological rest frame (the frame in which the coarse-grained homogeneous matter distribution in the universe at some instance in time is at rest on average). Peculiar velocities are typically generated by gravity attracting a particle / star / galaxy to fall towards a larger mass distribution.

The g_{rr} - component of the FLRW metric (1.17) is non-trivial. In some cases, it is useful to redefine the radial coordinate and introduce

$$d\chi = \frac{dr}{\sqrt{1 - k\frac{r^2}{R_0^2}}} \qquad (1.22)$$

Then the FLRW metric reads

$$ds^2 = -dt^2 + a^2(t)\left(d\chi^2 + S_k^2(\chi)\, d\Omega^2\right) \quad (1.23)$$

where

$$S_k(\chi) = R_0 \begin{cases} \sinh \frac{\chi}{R_0} & \text{for } k=-1 \\ \frac{\chi}{R_0} & \text{for } k=0 \\ \sin \frac{\chi}{R_0} & \text{for } k=1 \end{cases} \quad (1.24)$$

For $k=0$, we have $r=\chi$.

Similarly, it can be useful to redefine the time coordinate such that the metric factorizes. For this, we introduce <u>conformal time</u>

$$d\eta = \frac{dt}{a(t)} \quad (1.25)$$

Then, the metric reads

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\chi^2 + S_k^2(\chi)\, d\Omega^2\right] \quad (1.26)$$

such that the time-dependence occurs only in the prefactor $a^2(\eta)$. This form is convenient when studying light propagation for which $ds^2 = 0$.