Real eigenmodes of the non-Hermitian Wilson-Dirac operator and simulations of supersymmetric Yang-Mills theory

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1. Supersymmetric Yang-Mills theory

2. SYM on the lattice

3. Eigenvalues of the Wilson-Dirac operator in SYM simulations

4. Some results for the mass spectrum

5. Conclusions

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Super Yang-Mills theory (SYM)

Supersymmetric Yang-Mills theory:

\[ \mathcal{L} = \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\psi} \slashed{D} \psi - \frac{m_g}{2} \bar{\psi} \psi \right] \]

- gauge sector of supersymmetric extensions of the standard model
- \( \psi \) Majorana fermion in the adjoint representation
- confinement: bound states at low energies
- symmetries: specific form of low energy effective actions
Symmetries

SUSY \((m_g = 0)\)

- supersymmetry predicts paring of bosonic and fermionic states
- no spontaneous breaking / anomaly of SUSY expected

\(U_R(1)\) symmetry: \(\psi \rightarrow e^{-i\theta \gamma_5} \psi\)

- \(U_R(1)\) anomaly: \(\theta = \frac{k\pi}{N_c}\), \(U_R(1) \rightarrow \mathbb{Z}_{2N_c}\)

- \(U_R(1)\) spontaneous breaking: \(\mathbb{Z}_{2N_c} \xrightarrow{\langle \bar{\psi} \psi \rangle \neq 0} \mathbb{Z}_2\)
Quantized continuum SYM

- value of $\langle \bar{\psi} \psi \rangle$ is known
- exact beta function is known

Low energy effective actions:

- susy multiplets (degenerate masses)
  - 1. multiplet$^1$:
    - mesons: $a - f_0$: $\bar{\psi} \psi$ and $a - \eta'$: $\bar{\psi} \gamma_5 \psi$
    - fermionic gluino-glue ($\sigma_{\mu\nu} F_{\mu\nu} \psi$)
  - 2. multiplet$^2$:
    - glueballs: $0^{++}$ and $0^{--}$
    - fermionic gluino-glue

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$^1$[Veneziano, Yankielowicz, Phys. Lett. B113 (1982)]

Supersymmetric Yang-Mills theory on the lattice

Lattice action:

\[ S_L = \beta \sum_P \left( 1 - \frac{1}{N_c} \Re U_P \right) + \frac{1}{2} \sum_{xy} \bar{\psi}_x (D_w (m_g))_{xy} \psi_y \]

- “brute force” discretization: Wilson fermions

\[ D_w = 1 - \kappa \sum_{\mu=1}^{4} \left[ (1 - \gamma_\mu)_{\alpha,\beta} \hat{T}_\mu + (1 + \gamma_\mu)_{\alpha,\beta} \hat{T}^\dagger_\mu \right] \]

\[ \hat{T}_\mu \psi(x) = V_\mu \psi(x + \hat{\mu}); \quad \kappa = \frac{1}{2(m_g + 4)} \]

- links in adjoint representation: \((V_\mu)_{ab} = 2 \text{Tr} [U^\dagger_\mu T^a U_\mu T^b]\)
  
  gauge group SU(2)
Symmetries of lattice SYM

- supersymmetry always broken in local lattice theory\(^1\)
- Wilson mass spoils mass degeneracy
- chiral symmetry \((U_R(1))\) broken by the Wilson-Dirac operator
- no controlled breaking (Ginsparg-Wilson relation)

\[\Rightarrow\] need fine tuning!

Ward identities on the lattice

Ward identities of supersymmetry and chiral symmetry:

\[
\langle \nabla_\mu J^\mu_S(x) \mathcal{O}(y) \rangle = m_g \langle D_S(x) \mathcal{O}(y) \rangle + \langle X_S(x) \mathcal{O}(y) \rangle \\
\langle \nabla_\mu J^\mu_A(x) \mathcal{O}(y) \rangle = m_g \langle D_A(x) \mathcal{O}(y) \rangle + \langle X_A(x) \mathcal{O}(y) \rangle + \propto \langle F \tilde{F} \mathcal{O} \rangle
\]

- classical (tree level): \( X_S(x) = O(a), \ X_A(x) = O(a) \)
- renormalization, operator mixing\(^1,2\):

\[
\langle \nabla_\mu Z_A J^\mu_A(x) \mathcal{O} \rangle = (m_g - \bar{m}_g) \langle D_A(x) \mathcal{O} \rangle + \propto \langle F \tilde{F} \mathcal{O} \rangle + O(a) \\
\langle \nabla_\mu (Z_S J^\mu_S(x) + \tilde{Z}_S \tilde{J}^\mu_S(x)) \mathcal{O} \rangle = (m_g - \bar{m}_g) \langle D_S(x) \mathcal{O} \rangle + O(a)
\]

\[\Rightarrow \text{tuning of } m_g: \text{chiral limit } = \text{SUSY limit } + O(a)\]

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\(^1\) [Bochicchio et al., Nucl.Phys.B262 (1985)]

\(^2\) [Veneziano, Curci, Nucl.Phys.B292 (1987)]
Supersymmetric chiral limit

practical problems:
- noisy signal of supersymmetric Ward identities
- chiral Ward identities contain anomaly

\[
\langle \bar{\psi}(x) \gamma_5 \psi(x) \bar{\psi}(y) \gamma_5 \psi(y) \rangle = \langle \begin{array}{c} \bigcirc \bigcirc \end{array} \rangle - 2 \langle x \begin{array}{c} \bigcirc \end{array} y \rangle
\]

define connected part as adjoint pion \((a - \pi)\)
- disconnected part contains anomaly (OZI approximation)
- chiral limit: \(m_{a-\pi}\) vanishes
  \[\Rightarrow\] possible definition of gluino mass: \(\propto (m_{a-\pi})^2\)

At the end the consistency with the SUSY Ward identities is checked!
Simulations of SYM

- simulating Majorana fermions:

\[
\int \mathcal{D}\psi e^{-\frac{1}{2} \int \bar{\psi} D\psi} = \text{Pf}(CD) = \text{sign}(\text{Pf}(CD)) \sqrt{\text{det} D}
\]

\[
= \text{sign}(\text{Pf}(CD)) \int \mathcal{D}\bar{\phi} \mathcal{D}\phi e^{-\int \bar{\phi}(D^\dagger D)^{-1/4}\phi}
\]

- reweighting with Pfaffian (Pf) sign
- PHMC algorithm: \( x^{-1/4} \approx P(x) \)
- improvement of the polynomial approximation: reweighting with exact contribution of smallest eigenvalues
The sign of the Pfaffian

- $\gamma_5 D \gamma_5 = D^\dagger \Rightarrow$ paring $\lambda, \lambda^*$
- $C D C^T = D^T \Rightarrow$ degenerate eigenvalues
- $|\text{Pf}(C D)| = \sqrt{\det(D)} = \prod_{i=1}^{N/2} |\lambda_i|$
- $|\text{Pf}(C(D - \sigma \mathbb{1}))| = \prod_{i=1}^{N/2} |\lambda_i - \sigma|$
- Pfaffian polynomial in $\sigma \Rightarrow \text{Pf}(C D) = \prod_{i=1}^{N/2} \lambda_i$
- number of negative paired real eigenvalues of $D$ even / odd $\Rightarrow$ positive / negative Pfaffian
- on small lattices: checked with exact Pfaffian
- same problem (apart from degeneracy): determinant sign in $N_f = 1$ QCD
Obtaining the lowest real eigenvalues of $D_w$

Focus the Arnoldi algorithm on small stripe around real axis!
Polynomial transformation of $D_w$

- reflected spectrum: largest real eigenvalues should be computed
- Arnoldi algorithm calculates eigenvalues with real part above certain value
- computed region contains large number of unwanted eigenvalues

Two effects of transformation $D_w \rightarrow P(D_w)$:

1. focusing: better overlap of transformed wanted region with region computed by Arnoldi
2. acceleration, if eigenvalues not computed by Arnoldi compressed in a small region

- eigenvalues of $D_w$ obtained from eigenvectors of $P(D_w)$
Simple transformation\(^1\) \( P(D_w) = (D_w + \sigma_0 1\mathbb{I})^{n_0} \)

- complex eigenvalues “rotated away” from real axis:
  \[ \lambda_i = \rho_i e^{i\theta} : \theta \rightarrow n_0 \theta \]
- computed regions in original spectrum:

![Graph showing computed regions](image)

- saturation at higher orders, broad outer part of computed region

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\(^1\) [H. Neff, Nucl. Phys. Proc. Suppl. 106 (2002)]
The iterated transformation\(^1\)

- \( P(D_w) = \ldots ((D_w + \sigma_0 \mathbb{1})^{n_0} + \sigma_1 \mathbb{1})^{n_1} \ldots \) 
  - optimization at each step 
- computed regions in original spectrum:

narrow outer part of computed region!

\(^1\)[GB, Wuilloud, Comp. Phys. Comm. (2011)]
Eigenvalues obtained from the iterated transformation

- avoids large eigenvalue densities
- increases efficiency
Eigenvalues of $D_w$ ($32^3 \times 64, \kappa = 0.1495, \beta = 1.75$)

- for the determination of spectrum:
  - low contribution with neg. Pfaffian (6 of 2000 configurations)
- additional acceleration: even-odd preconditioning
Masses and particles

considered operators:

- $0^{++}$:
  - reasonable signal only with variational smearing

- fermionic gluino-glue
  - operator $\sigma^{\mu\nu} \text{Tr}[\hat{F}_{\mu\nu}\psi]$
  - APE smearing on gauge fields and Jacobi smearing on $\psi$

- Meson operators $a - f_0, a - \eta'$:
  - disconnected contribution dominant at small gluino masses:
    $$\langle \boxtimes \boxtimes \rangle = \langle D^{-1}(x, x) D^{-1}(y, y) \rangle_{\text{eff}}$$

- technique: SET (dilution, truncated solver method)
- exact contribution of lowest $\gamma_5 D_w$ eigenvalues
Eigenvalues of even-odd preconditioned Hermitian Wilson-Dirac operator

- acceleration of Arnoldi algorithm: Chebyshev polynomial

⇒ improvement: polynomial approximation of update algorithm (reweighting)

⇒ improvement: measurement of disconnected contributions and condensate
disconnected $a - \eta'$ on a $32^3 \times 64$ lattice:

- reasonable improvement at small gluino masses
- acceleration of SET inversions
Details of the simulations

- simulation algorithm: PHMC
- tree level Symanzik improved gauge action
- stout smearing
- Sexton-Weingarten integrator
- determinant breakup

previous simulations:
- lattice sizes: $16^3 \times 32$, $24^3 \times 48$ ($32^3 \times 64$)
- $r_0 \equiv 0.5\text{fm} \rightarrow a \leq 0.088\text{fm}; \; L \approx 1.5 - 2.3\text{fm}$
- $m_{a-\pi} \approx 440\text{MeV}$
No mass degeneracy in chiral limit!
Tuning with SUSY Ward identities compatible with tuning of $m_{a-\pi}$. [Demmouche et al., Eur.Phys.J.C69 (2010)]
New simulations at smaller lattice spacing

Before speculating about new physics: Most likely explanation are lattice artifacts!

new simulations:
- volume fixed, smaller lattice spacing
- increased $\beta$ from 1.6 to 1.75
- simulations on $32^3 \times 64$ lattice
Confinement and physical scale of the new simulations

good agreement with $V(r) = v_0 + c/r + \sigma r$ (confining)

$\Rightarrow a \approx 0.057\text{fm}, \ L \approx 1.8\text{fm}$
Comparison of the mass gap between $a - \eta'$ and gluino-glue

- mass gap considerably reduced
- gluino-glue has much lower mass
Complete spectrum obtained with the new simulations

- indicates mixing of $a - f_0$ and $0^{++}$ glueball
- in contrast to smaller lattice spacing: $a - f_0$, glueball heavier
Conclusions

- In supersymmetric Yang-Mills theory the unavoidable breaking of SUSY on the lattice can be controlled by a fine tuning of the gluino mass ($\kappa$).
- The sign Pfaffian can be determined from the real eigenvalues of the non-Hermitian Wilson-Dirac operator.
- Polynomial acceleration of Arnoldi algorithm leads to efficient determination of lowest real eigenvalues. \(^1\)
- The eigenvalues of the Hermitian even-odd preconditioned matrix are used to improve the algorithm and the observables.
- Possible further uses of the eigenvalue distributions?
- The mass gap between bosonic and fermionic states is considerably reduced at a smaller lattice spacing.
- Further improvements of the action are currently investigated. \(^1\)

\(^1\)Same method has been applied for the determinant sign in $N_f = 1$ QCD.