Übungen Monte-Carlo-simulation II: SU(2) Yang-Mills-Theorie

Georg Bergner

TPI FSU Jena
WWU Münster

Aufgabenstellung.

Aufgaben

1. Implementieren Sie die Plaquette und Gitter-Eich-Wirkung.
2. Simulieren Sie SU(2) Yang-Mills-Theorie Simulationen mit dem Metropolis-Algorithmus.
3. Bestimmen Sie den Erwartungswert der Plaquette als Funktion der Kopplungskonstante β.

Zusatz-Aufgaben

1. Implementieren Sie die Polyakov-Schleife.
2. Bestimmen Sie den Deconfinement-Phasenübergang mit Hilfe der Polyakov-Schleife.

Im Folgenden sind eine kurze Zusammenfassung aus den Vorlesungen und Hinweise zur Lösung in englischer Sprache angefügt. Eine Hilfestellung in Form von Code-Bausteinen finden Sie unter

`www.tpi.uni-jena.de/~gbergner/compmethws2324.html`.

Gauge invariant observables and lattice gauge action

- gauge invariant observables: trace of transport around closed paths
- on the lattice: closed loops of links
- simplest choice: Plaquette $U_{\mu\nu}(x) = U_\mu(x) U_\nu(x+\hat{\mu}) U_{-\mu}(x+\hat{\mu}+\hat{\nu}) U_{-\nu}(x+\hat{\nu})$ related to $F_{\mu\nu}$
- Wilson gauge action (SU(N_c) gauge group)

$$S_G[U] = \frac{\beta}{N_c} \sum_x \sum_{\mu<\nu} \Re \mathrm{tr}[1 - U_{\mu\nu}(x)]$$

- inverse gauge coupling $\beta = \frac{2N_c}{g^2}$

Path integral of lattice gauge theory

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U] \, e^{-S_G[U]} \, O[U]; \quad Z = \int \mathcal{D}[U] \, e^{-S_G[U]}$$

Integration of group manifold: $\int \mathcal{D}[U] = \prod_x \prod_\mu \int dU_\mu(x)$

- gauge transformation: $U_\mu \to \Omega(x) U_\mu(x) \Omega^\dagger(x + \hat{\mu})$
- gauge invariance: $dU = d(UV) = d(VU)$ for all $V \in$ gauge group
- Haar measure

Numerical approximation by importance sampling

$$\langle O \rangle \approx \frac{1}{N_{conf}} \sum_{n=1}^{N_{conf}} O[U_n]$$

configuration U_n with probability $e^{-S_G[U_n]}$

Metropolis update

- propose new configuration U' with trial probability $T(U', U)$
- accept with acceptance probability

$$A(U, U') = \min\left(\frac{e^{-S_G[U']}T(U, U')}{e^{-S_G[U]}T(U', U)}, 1\right)$$

- if rejected next configuration same as U
- local update $T(U, U') = T(U', U)$: $U'_\mu = XU_\mu$ for some random SU(2) matrix X

Trial probability and SU(2) representation

SU(2): $X = x_0 1 + i\mathbf{x}\cdot\sigma$ with $\det(X) = \sum_i x_i^2 = 1$

- random matrix X:
  1. random direction on sphere
  2. random angle $\phi \in [0, 2\pi)$
  3. random matrix: $1\cos(\phi) + i\sin(\phi)\mathbf{x}\cdot\sigma$
- need to cover complete group manifold with the trial updates, satisfied by $U'_\mu = XU_\mu$

Accept reject step

- acceptance depends on difference of the action $e^{-S[U']+S[U]} = e^{-\Delta S}$
- local Metropolis: propose a change of single link $U_\mu(x)$

Change of the action:

$$\Delta S_{\text{loc}} = -\frac{\beta}{N_c} \sum_{\text{plaquettes with } U_\mu(x)} \Re\mathrm{tr}(U'_{\mu\nu} - U_{\mu\nu})$$
$$= -\frac{\beta}{N_c} \Re\mathrm{tr}\left[(U'_\mu(x) - U_\mu(x))V\right]$$

Staple:

$$V = \sum_{\mu\neq\nu} (U_\nu(x+\hat{\mu})U_{-\mu}(x+\hat{\mu}+\hat{\nu})U_{-\nu}(x+\hat{\nu}) + U_{-\nu}(x+\hat{\mu})U_{-\mu}(x+\hat{\mu}-\hat{\nu})U_\nu(x-\hat{\nu}))$$

Observables of the simulations

Basic observable, related to the action density:

$$O_1 = \frac{1}{24\text{Volume}} \sum_{\mu \neq \nu, x} \mathrm{tr}\, U_{\mu\nu}(x)$$

More advanced observable: Polyakov line

- periodic lattice: product of all links in one direction is gauge transport back to same point
- trace of this product leads to gauge invariant observable

$$O_2 = \frac{1}{2\text{Volume}} \sum_{\mathbf{x}} \mathrm{tr} \left[\prod_t U_4(x) \right]$$

$\Rightarrow$ observable related to deconfinement (see lecture)

Code development

C++ object oriented approach:

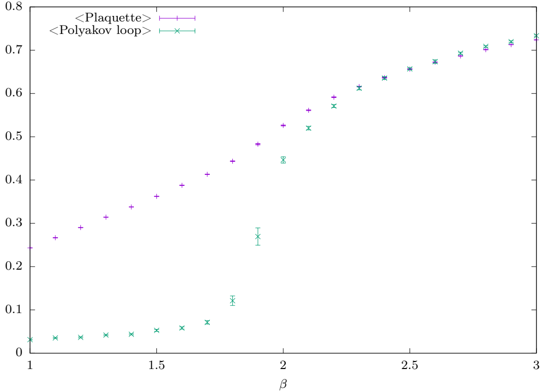
- predefined program with separate header files
- based on Eigen matrix library
- functions like random SU(2) matrix predefined

C or Fortran style:

- function calls for matrix multiplication like matmul(A, B)

Or use other code base ...

Example solutions



Simulation result on $6^3 \times 2$ lattice