Programming exercise: SU(2) lattice gauge theory

Georg Bergner

TPI FSU Jena
WWU Münster

Topics covered in the exercise.

Main exercises

1. Main idea of the pure gauge part on the lattice.
2. Gauge links on the lattice.
3. Metropolis algorithm for the pure gauge ensemble.
4. Measurement of plaquette expectation values as a function of gauge coupling.

Advanced tasks

1. Polyakov line and deconfinement transition.
2. Localize deconfinement transition.

Topics will be further explained in the lecture on Thursday; we will extend these investigations on Friday, please ask for any further aspects you are interested in.

Pure gauge theory in the continuum

- gauge field A_μ, generalization of Maxwell field in electrodynamics (traceless hermitian matrices)
- gauge invariance

$$\text{electrodynamics: } A_\mu \to A_\mu(x) + \partial_\mu \alpha(x)$$

$$\text{Yang-Mills: } A_\mu \to \Omega(x) A_\mu(x) \Omega^\dagger(x) + i(\partial_\mu \Omega(x))\Omega^\dagger(x)$$

$$\text{covariant derivative: } D_\mu = \partial_\mu + iA_\mu(x) \to \Omega(x) D_\mu \Omega^\dagger(x)$$

- gauge field in Algebra of the gauge group

$$A_\mu = \sum_i A_\mu^i T_i; \quad [T_i, T_j] = if_{ijk} T_k; \quad \mathrm{tr}[T_i T_j] = \frac{1}{2}\delta_{ij}$$

$$\mathrm{SU}(2): \quad T_i = \frac{1}{2}\sigma_i; \quad \mathrm{SU}(3): \quad T_i = \frac{1}{2}\lambda_i;$$

Pure gauge theory in the continuum

- pure gauge action of Yang-Mills theory

$$S_g = \frac{1}{2g^2} \int d^4x \quad \mathrm{tr}[F_{\mu\nu} F_{\mu\nu}]$$

$$F_{\mu\nu} = -i[D_\mu(x), D_\nu(x)] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i[A_\mu, A_\nu]$$

- important difference to electrodynamics: self interactions of gauge field
- electrodynamics: gauge group U(1), $[A_\mu, A_\nu] = 0$
- Yang-Mills: gauge group SU(2) or SU(3), $[A_\mu, A_\nu] \neq 0$

Gauge transporter and link variables

- lattice discretization without loosing gauge invariance by gauge transporter along path C between x and y

$$G(x,y) = \mathcal{P}\exp(i\int_{C_{xy}} A\cdot ds); \quad G(x,y) \to \Omega(x)G(x,y)\Omega^\dagger(y)$$

- consequently the trace of $(G(x,y)$ along a closed path is gauge invariant
- gauge fields on the lattice: $G(x,y)$ links connecting points separated by lattice spacing a

$$U_\mu(x) = \exp(iaA_\mu(x)) = G(x,x+\hat{\mu}) + O(a^2)$$

Backward link: $U_{-\mu}(x) = U_\mu^\dagger(x-\hat{\mu})$

Gauge invariant observables and lattice gauge action

- gauge invariant observables: trace of transport around closed paths
- on the lattice: closed loops of links
- simplest choice: Plaquette $U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_{-\mu}(x+\hat{\mu}+\hat{\nu})U_{-\nu}(x+\hat{\nu})$ related to $F_{\mu\nu}$
- Wilson gauge action (SU(N_c) gauge group)

$$S_G[U] = \frac{\beta}{N_c}\sum_x \sum_{\mu<\nu} \Re\mathrm{tr}[1 - U_{\mu\nu}(x)]$$

- inverse gauge coupling $\beta = \frac{2N_c}{g^2}$

Path integral of lattice gauge theory

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[U] \, e^{-S_G[U]} \, O[U]; \quad Z = \int \mathcal{D}[U] \, e^{-S_G[U]}$$

Integration of group manifold: $\int \mathcal{D}[U] = \prod_x \prod_\mu \int dU_\mu(x)$

- gauge transformation: $U_\mu \to \Omega(x) U_\mu(x) \Omega^\dagger(x + \hat{\mu})$
- gauge invariance: $dU = d(UV) = d(VU)$ for all $V \in$ gauge group
- Haar measure

Numerical approximation by importance sampling

$$\langle O \rangle \approx \sum_n O[U_n]$$

configuration U_n with probability $e^{-S_G[U_n]}$

Metropolis update

- propose new configuration U' with trial probability $T(U', U)$
- accept with acceptance probability

$$A(\{s_x\}_i, \{s_x\}_j) = \min\left(\frac{e^{-S_G[U']}T(U, U')}{e^{-S_G[U]}T(U', U)}, 1\right)$$

- if rejected next configuration same as U

Trial probability and SU(2) representation

SU(2): $U = x_0 1 + i\mathbf{x} \cdot \sigma$ with $\det U = \sum_i x_i^2 = 1$

- need to cover complete group manifold with the trial updates
- $U'_\mu = XU_\mu$ for some random SU(2) matrix X
- random direction on sphere
- random angle $\phi \in [0, 2\pi)$
- random matrix: $1 \cos(\phi) + i \sin(\phi)\mathbf{x} \cdot \sigma$

Accept reject step

- acceptance depends on difference of the action $e^{-S[U']+S[U]} = e^{-\Delta S}$
- local Metropolis: propose a change of single link $U_\mu(x)$

Change of the action:

$$\Delta S_{\text{loc}} = -\frac{\beta}{N_c} \sum_{\text{plaquettes with } U_\mu(x)} \Re\mathrm{tr}(U'_{\mu\nu} - U_{\mu\nu})$$

$$= -\frac{\beta}{N_c} \Re\mathrm{tr}\left[(U'_\mu(x) - U_\mu(x))V\right]$$

Staple:

$$V = \sum_{\mu\neq\nu}(U_\nu(x+\hat{\mu})U_{-\mu}(x+\hat{\mu}+\hat{\nu})U_{-\nu}(x+\hat{\nu})+$$
$$U_{-\nu}(x+\hat{\mu})U_{-\mu}(x+\hat{\mu}-\hat{\nu})U_\nu(x-\hat{\nu}))$$

Observables of the simulations

Basic observable, related to the action density:

$$O_1 = \frac{1}{24\text{Volume}} \sum_{\mu \neq \nu, x} \mathrm{tr}\, U_{\mu\nu}(x)$$

More advanced observable: Polyakov line

- periodic lattice: product of all links in one direction is gauge transport back to same point
- trace of this product leads to gauge invariant observable

$$O_2 = \frac{1}{2\text{Volume}} \sum_{\mathbf{x}} \mathrm{tr} \left[\prod_t U_4(x) \right]$$

$\Rightarrow$ observable related to deconfinement (see lecture)

Code development

`www.tpi.uni-jena.de/~gbergner/compmeth.html`

C++ object oriented approach:

- predefined program with separate header files
- based on Eigen matrix library
- functions like random SU(2) matrix predefined

C or Fortran style:

- function calls for matrix multiplication like matmul(A, B)

Or use other code base ...