Lectures on Symmetries

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Abstract: Notes for lectures that introduce students of physics to symmetries and groups, with an emphasis on the Lie groups relevant to particle physics. Prepared for a course at Heidelberg university in the summer term 2020.

Contents

1 Introduction and motivation

1.1 Plan of the course

These lectures are intended for Master students of physics. The implications of symmetry in physics are ubiquitous and very interesting. Mathematically, they are described by group theory. The lectures start with finite groups and then discuss the most important Lie groups and Lie algebras, in particular SU(2), SU(3), the Lorentz and Poincaré groups, the conformal group and the gauge groups of the standard model and of grand unified theories.

- Introduction and overview
- Symmetries and conservation laws
- Finite groups
- Lie groups and Lie algebras
- SU(2)
- SU(3)
- Classification of compact simple Lie algebras
- Lorentz and Poincaré groups
- Conformal group
- Consequences of symmetries for effective actions
- Non-abelian gauge theories and the standard model
- Grand unification

1.2 Suggested literature

The application of group theory in physics is a well established mathematical subject and there are many good books available. A selection of references that will be particularly useful for this course is as follows.

- M. Fecko, *Differential Geometry and Lie Groups for Physicists*
- P. Ramond, *Group Theory, A Physicist's Survey*
- A. Zee, *Group Theory in a Nutshell for Physicists*
- J. Fuchs and C. Schweigert, *Symmetries, Lie algebras and Representations*
- H. Georgi, *Lie algebras in Particle Physics*
- H. F. Jones, *Groups, Representations and Physics*

1.3 Symmetry transformations

Studying symmetries and their consequences is one of the most fruitful ideas in physics. This holds especially in high energy and particle physics but by far not only there. To get started, we first define the notion of a symmetry transformation and relate it to the mathematical concept of a group.

It is natural to characterize a symmetry transformation by the following properties

- One symmetry transformation followed by another should be a symmetry transformation itself.
- There should be a unique (trivial) symmetry transformation doing nothing.
- For each symmetry transformation there needs to be a unique symmetry transformation reversing it.

With these properties, the set of all symmetry transformations G forms a group in the mathematical sense. More formally, a group G has the following properties.

i) *Closure*: For all elements $f, g \in G$ the composition $g \cdot f \in G$.
(We use most of the time transformations acting to the right so that $g \cdot f$ should be read as a transformation where we apply first f and then g.)

ii) *Associativity*: $h \cdot (g \cdot f) = (h \cdot g) \cdot f$.

iii) *Identity element*: There exists a unique unit element $\mathbb{1}$ or sometimes called e in the group $e \in G$, such that $e \cdot f = f \cdot e = f$ for all $f \in G$.

iv) *Inverse element*: For all elements $f \in G$ there is a unique inverse $f^{-1} \in G$ such that $f \cdot f^{-1} = f^{-1} \cdot f = e$.

These basic properties characterize many possible symmetry groups, both of continuous and discrete kind. We will mainly be concerned with the former but also mention briefly a few of the latter.

1.4 Continuous symmetries and infinitesimal transformations

In physics the group elements $g \in G$ which describe a symmetry can often be parametrised by a continuous parameter on which the group elements depend in a differentiable way, for example

$$\begin{aligned} \mathbb{R} &\to G, \\ \alpha &\to g(\alpha). \end{aligned}$$

In this situation it is possible to study infinitesimal symmetries which are characterised by their action close to the identity element $\mathbb{1}$ of the group. Without loss of generality we can choose $g(0) = \mathbb{1}$. Moreover, if the parametrisation obeys

$$g(\alpha_1) \cdot g(\alpha_2) = g(\alpha_1 + \alpha_2),$$

for any two group elements $g(\alpha_1)$ and $g(\alpha_2)$, one speaks of a *one-parameter subgroup*. Let us now restrict to very small parameters α_1, α_2 such that the corresponding group elements are close to the identity element. In this case one speaks of an *infinitesimal symmetry transformation* and as a one-parameter subgroup in the limit $\alpha \to 0$ it is in fact fully characterised by the derivative at $\alpha = 0$,

$$\frac{d}{d\alpha} g(\alpha)\Big|_{\alpha=0}.$$

The idea is to decompose finite transformations into many infinitesimal transformations. This idea will lead to the most important properties of Lie groups and in particular to Lie algebras.

2 Symmetries and conservation laws

Before diving further into mathematical properties of symmetry groups, let us recall the deep connection between continuous symmetries and conservation laws in classical and quantum mechanics. The relation has first been understood and formulated as a theorem by *Emmy Noether* in 1918.

2.1 Classical mechanics in Lagrangian description

In classical mechanics, the equations of motions of a physical system with action

$$S = \int dt\ L(q(t), \dot{q}(t), t),$$

where the Lagrangian L is a function of position $q(t)$ and velocity $\dot{q}(t)$, as well as possibly time t, can be derived by the *variational principle of least action*

$$\delta S = \int dt \left\{ \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right\} = \int dt \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \right\} \delta q = 0.$$

This yields the Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0.$$

Suppose now that there is a map (actually a one-parameter (sub)group of symmetry transformations)

$$q \to h_s(q), \tag{2.1}$$

which depends on a parameter $s \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$ such that

$$h_{s=0}(q) = q,$$

for any position q, and an induced map for velocities

$$\dot{q} \to \hat{h}_s(\dot{q}) = \frac{\partial h_s(q)}{\partial q}\dot{q}, \tag{2.2}$$

for the corresponding velocity $\dot{q}$. The Lagrangian is then said to be invariant under (2.1) and (2.2) if there is a differentiable function $F_s(q, \dot{q}, t)$ such that

$$L(h_s(q), \hat{h}_s(\dot{q}), t) = L(q, \dot{q}, t) + \frac{d}{dt}F_s(q, \dot{q}, t). \tag{2.3}$$

The action S changes then only by a boundary term so that one can speak of a (continuous) symmetry of the action. The invariance (2.3) gives rise to a conservation law.

To see this, consider a solution to the equations of motion

$$q(t) = \phi(t).$$

We define then

$$\Phi(s, t) = h_s(\phi(t)),$$

and find from (2.3)

$$
\begin{aligned}
0 &= \frac{\partial}{\partial s}\left(L(\Phi, \partial_t\Phi) - \frac{d}{dt}F_s\right) \\
&= \frac{\partial L}{\partial q}\frac{\partial \Phi}{\partial s} + \frac{\partial L}{\partial \dot{q}}\frac{\partial^2 \Phi}{\partial s \partial t} - \frac{d}{dt}\frac{\partial F_s}{\partial s} \\
&= \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{q}}\frac{\partial \Phi}{\partial s} - \frac{\partial F_s}{\partial s}\right)}_{=Q} = \frac{d}{dt}Q,
\end{aligned}
$$

where Q is a conserved quantity called Noether charge. In the last line we have used the equation of motion.

Consider for example a point particle of mass m with the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x}),$$

where $\mathbf{x} \in \mathbb{R}^3$ is the position. Suppose the potential $V(\mathbf{x})$ is translational invariant such that L is invariant under the map

$$\mathbf{x} \to h_s(\mathbf{x}) = \mathbf{x} + s\mathbf{a},$$

for $\mathbf{a} \in \mathbb{R}^3$ and $s \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$ while the induced map for velocities $\hat{h}_s$ is the identity map and $F_s = 0$. Then

$$\frac{\partial \Phi}{\partial s} = \frac{\partial h_s(\mathbf{x})}{\partial s} = \mathbf{a},$$

and therefore

$$Q = m\dot{\mathbf{x}} \cdot \mathbf{a}.$$

We have found momentum conservation in direction $\mathbf{a}$.

The close connection between symmetries and conservation laws exposed by the Noether theorem is truly remarkable, in particular if one takes into account how large the role of conservation laws is in theoretical physics. A few examples for symmetries and the corresponding conservation laws are

- translations in time $\to$ energy conservation
- translations in space $\to$ momentum conservation
- rotations $\to$ angular momentum conservation
- Galilean or Lorentz boosts $\to$ center of mass conservation
- global $U(1)$ in non-relativistic theories $\to$ particle number conservation
- local $U(1)$ $\to$ charge conservation
- general coordinate invariance $\to$ covariant energy-momentum conservation

2.2 Classical mechanics in Hamiltonian description

In the Hamiltonian formulation of classical mechanics the physical information of a system in encoded in the Hamiltonian H which is a function of position q, momenta p and time t. The equations of motion are

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}.$$

For systems which can also be described in the Lagrangian framework, the Hamiltonian is given by the *Legendre transformation*

$$H(q, p, t) = p\dot{q} - L(q, \dot{q}, t),$$

where $\dot{q}$ is defined implicitly by

$$p = \frac{\partial L}{\partial \dot{q}}.$$

The fundamental tool in Hamiltonian mechanics is the Poisson bracket. It is a map from the space of pairs of differentiable functions of dynamical variables q and p to a single differential function. For two such functions f, g we define a new function by

$$\{\cdot, \cdot\} : \Big(f(q,p), g(q,p)\Big) \to \{f, g\}(q, p),$$

where

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}.$$

The Poisson bracket has three properties [Exercise: Show these.]

i) *Bilinearity*: $\{\lambda f + \mu g, h\} = \lambda\{f, h\} + \mu\{g, h\}$,

i) *Antisymmetry*: $\{f, g\} = -\{g, f\}$,

iii) *Jacobi identity*: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$,

for differentiable functions f, g, h and for $\lambda, \mu \in \mathbb{R}$. By these properties the Poisson bracket turns the space of functions of q and p into a *Lie algebra*. We will discuss much more about Lie algebras later on.

If H does not explicitly depend on time, and for any differentiable function $f(q(t), p(t))$ which is evaluated on a trajectory that is a solution to the equations of motion, the time derivative can be written

$$\frac{d}{dt} f(q(t), p(t)) = \{H, f\}.$$

One can formally solve this through an operator $\exp[\Delta t\{H, \cdot\}]$ for time translation,

$$f(q(t + \Delta t), p(t + \Delta t)) = \exp[\Delta t\{H, \cdot\}]\, f(q(t), p(t)).$$

One says that H generates time translations in this sense. [Exercise: Work this out for an infinitesimally small Δt so that only linear terms need to be kept.]

Consider now again some map $(q(t), p(t)) \to (q_s(t), p_s(t))$ with a continuous parameter s and assume that this transformation is generated by Q in the sense that

$$\frac{d}{ds} q_s(t) = \{Q, q_s(t)\}, \qquad \frac{d}{ds} p_s(t) = \{Q, p_s(t)\},$$

so that similar as for time translations one has for translations in s

$$f(q_{s+\Delta s}(t), p_{s+\Delta s}(t)) = \exp\left[\Delta s\{Q, \cdot\}\right] f(q_s(t), p_s(t).$$

This transformation connects different solutions to the equations of motion if the operators for translations in t and s commute,

$$\exp\left[\Delta s\{Q, \cdot\}\right] \exp\left[\Delta t\{H, \cdot\}\right] = \exp\left[\Delta t\{H, \cdot\}\right] \exp\left[\Delta s\{Q, \cdot\}\right].$$

As we will show later in general terms, this is the case if

$$\{H, Q\} = 0.$$

But this actually means that Q is conserved, $dQ/dt = 0$.

We note here that the connection between symmetries and conservation laws is a little less direct in the Hamiltonian formalism than in the Lagrangian one, but in any case the connection is close.

2.3 Quantum mechanics

Starting from the Hamiltonian description it is most convenient to work in the Heisenberg picture of quantum mechanics. Canonical quantisation maps the Poisson bracket of differentiable functions in q and p to the commutator of the associated operators $\hat{q}$ and $\hat{p}$ in some suitable Hilbert space $\mathcal{H}$,

$$\{\cdot, \cdot\} \to \frac{i}{\hbar}[\cdot, \cdot].$$

Here i is the imaginary unit, $\hbar$ denotes Planck's constant and the commutator is defined by

$$[A, B] = AB - BA.$$

In particular one has

$$\{p, q\} = 1 \to \frac{i}{\hbar}[\hat{p}, \hat{q}] = 1,$$

which is Heisenbergs commutation relation. The commutator equips $\mathcal{H}$ with a Lie algebra in the same way the Poisson bracket does in the Hamiltonian description of classical mechanics. In particular we have the properties

i) *Bilinearity*: $[\lambda A + \mu B, C] = \lambda[A, C] + \mu[B, C]$,

ii) *Antisymmetry*: $[A, B] = -[B, A]$,

iii) *Jacobi identity*: $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$, for operators A, B, C and $\lambda, \mu \in \mathbb{R}$. Analogous to before the time dependance of the operators is now described by

$$\frac{d}{dt}A(t) = \frac{i}{\hbar}[H, A].$$

Observables which describe conservations laws are characterised by

$$[H, Q] = 0.$$

By the Jacobi identity, the space of all conserved quantities forms a closed Lie subalgebra.

As an example we consider angular momentum conservation of the electron in the quantised hydrogen atom. Let the eigenstates of the Hamiltonian H be labelled by the principal quantum number n, total angular momentum l and the quantum number for the projection on the z-axis, m. Because angular momentum is conserved, the operator commutes with the Hamiltonian,

$$[L_z, H] = 0,$$

and therefore for two eigenstates $|n', l', m'\rangle$, $|n, l, m\rangle$,

$$0 = \langle n', l', m'|[L_z, H]|n, l, m\rangle = (m' - m) \langle n', l', m'|H|n, l, m\rangle .$$

Accordingly,

$$\langle n', l', m'|H|n, l, m\rangle$$

can only be non-zero for $m = m'$, which is a selection rule.

One may also start from a conserved operator and construct the corresponding symmetry transformation. For a self-adjoint operator $A \in \mathcal{H}$ with

$$[H, A] = 0,$$

we define the unitary operator

$$U_A(s) = e^{isA} = \sum_{n=0}^{\infty} \frac{(isA)^n}{n!},$$

for $s \in \mathbb{R}$. They obey the multiplication law

$$U_A(s_1) \cdot U_A(s_2) = U_A(s_1 + s_2),$$

for $s_1, s_2 \in \mathbb{R}$ and form a unitary one-parameter group. As an example consider the momentum operator in position space defined as

$$\hat{p} = \frac{\hbar}{i}\frac{d}{dx},$$

which generates infinitesimal translations. For $a \in \mathbb{R}$ the operator

$$\exp\left(i\frac{a}{\hbar}\hat{p}\right) = e^{a\frac{d}{dx}}$$

generates finite translations, e. g.

$$e^{a\frac{d}{dx}}f(x) = f(x + a),$$

for a suitable function f.

3 Finite groups

Now that we discussed the close relation between symmetries and the mathematical concept of a group it seems natural to start our analysis with simple cases, i.e. finite groups of low order. The order ord(G) of a group G is defined to be the number of group elements.

3.1 Order 1

There is only one trivial group with a single element, namle $\{e\}$.

3.2 Order 2

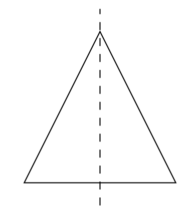
The simplest symmetry transformation is a reflection,

$$P : x \to -x, \tag{3.1}$$

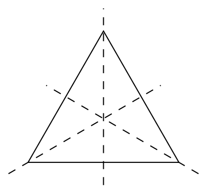
which applied twice is the identity, i. e. $PP = e$. The parity transformation (3.1) has the simplest finite group structure involving only two elements. The group is known as the cyclic group Z_2 with the following multiplication table.

Z_2	e	P
e	e	P
P	P	e

The group Z_2 has many manifestations for example in terms of reflections about the symmetry axis of an isosceles triangle.



Studying higher order groups will naturally involve more elements which can manifest in more symmetries, e.g. the equilateral triangle is symmetric under reflections about any of its three medians as well as rotations of $\frac{2\pi}{3}$ around its center,



which is described by a group of order six. In general we denote the elements of a group of order n by $\{e, a_1, a_2, ..., a_{n-1}\}$ where e denotes the identity element.

3.3 Order 3

There is only one group of order three which is Z_3 with the following multiplication table.

Z_3	e	a_1	a_2
e	$e \cdot e = e$	$e \cdot a_1 = a_1$	$e \cdot a_2 = a_2$
a_1	$a_1 \cdot e = a_1$	$a_1 \cdot a_1 = a_2$	$a_1 \cdot a_2 = e$
a_2	$a_2 \cdot e = a_2$	$a_2 \cdot a_1 = e$	$a_2 \cdot a_2 = a_1$

Sometimes one writes this compactly as follows.

Z_3	e	a_1	a_2
e	e	a_1	a_2
a_1	a_1	a_2	e
a_2	a_2	e	a_1

The group elements can be written as $\{e, a_1 = a, a_2 = a^2\}$, with the law that $a^3 = e$. One says that a is an order three element. The structure of the group is completely fixed by the multiplication table. However, this gets rather cumbersome, in particular for larger groups and it is convenient to use another characterisation of the group elements and their multiplication laws, the so-called *presentation*. For example Z_3 has a presentation

$$\langle a \mid a^3 = e \rangle \,.$$

The first part of the presentation lists the group elements from which all others can be constructed – the so-called generators – while the second part gives the rules needed to construct the multiplication table.

This generalises to the cyclic group Z_n of order $n \in \mathbb{N}$ which has the group elements $\{e, a, a^2, ..., a^{n-1}\}$ and a presentation is given by

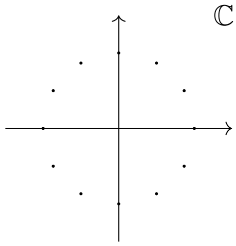
$$\langle a \mid a^n = e \rangle \,. \tag{3.2}$$

One distinguishes between a group G which is an abstract entity defined by the set of its group elements and their multiplication table (or, equivalently, a presentation) and a *representation* of a group.

Representation. A representation can be seen as a manifestation of the group multiplication laws in a concrete system or, in other words, a kind of incarnation of the group. For example, the group Z_3 has a representation in terms of rotations by $\frac{2\pi}{3}$ around the center of an equilateral triangle. Z_n has also a representation

$$\{1, e^{i\frac{2\pi}{n}}, e^{2i\frac{2\pi}{n}}, ..., e^{(n-1)i\frac{2\pi}{n}}\}, \tag{3.3}$$

where the group elements are represented equidistantly on the unit circle in the complex plane and the action of the generator a is represented by a rotation of $\frac{2\pi}{n}$ around the centre.



Often one works with representations as operations in a vector space.

A matrix representation $\mathcal{R}$ of a group G on a vector space V is a group homomorphism

$$\mathcal{R} : G \to GL(V)$$

onto the general linear group $GL(V)$ on V, i.e. a map from a group element g to a matrix $\mathcal{R}(g)$ such that

$$\mathcal{R}(g_1 \cdot g_2) = \mathcal{R}(g_1) \cdot \mathcal{R}(g_2)$$

for all $g_1, g_2 \in G$. We define the dimension of the representation $\mathcal{R}$ by

$$\dim(\mathcal{R}) = \dim(V)$$

and denote the identity element of a group by e and of a representation by $\mathbb{1}$.

3.4 Order 4

At order four there are two different groups. Again there is Z_4 which in the representations (3.3) reads $\{1, i, -1, -i\}$. The other group is the dihedral group D_2 with the multiplication table

D_2	e	a_1	a_2	a_3
e	e	a_1	a_2	a_3
a_1	a_1	e	a_3	a_2
a_2	a_2	a_3	e	a_1
a_3	a_3	a_2	a_1	e

The fact that the multiplication table is symmetric shows that D_2 is abelian, $a_i \cdot a_j = a_j \cdot a_i$. Two representations of D_2 are

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

and

$$\left\{ f_1(x) = x,\ f_2(x) = -x,\ f_3(x) = \frac{1}{x},\ f_4(x) = -\frac{1}{x} \right\}.$$

For both groups of order four we can form a subgroup, namely Z_2. It is straight forward to see from the presentation (3.2) for Z_4,

$$\langle a \mid a^4 = e \rangle,$$

that the group generated by a^2 is Z_2. The same goes for the group generated by $a_i \in D_2$ for any choice $i \in \{1, 2, 3\}$. For D_2 one can actually go on and write it as a direct product of two factors Z_2,

$$D_2 = Z_2 \otimes Z_2.$$

We use here the notion of a *direct product*.

Direct product. Let $(G, \circ)$ and $(K, *)$ be two groups with elements $g_a \in G$, $a \in \{1, ..., n_G\}$ and $k_i \in K$, $i \in \{1, ..., n_K\}$ respectively. The direct product is a group $(G \otimes K, \star)$ of order $n_G n_K$ with elements (g_a, k_i) and the multiplication rule

$$(g_a, k_i) \star (g_b, k_j) = (g_a \circ g_b, k_i * k_j).$$

Since $\circ$ and $*$ act on different sets, they can be taken to be commuting subgroups and we simply write $g_a k_i = k_i g_a$ for the elements of $G \otimes K$.

As long as it is clear we denote for notational simplicity the group $(G, \circ)$ by G and the group operation as $g_1 \circ g_2 = g_1 g_2$. [Exercise: Show that $D_2 = Z_2 \otimes Z_2$ but $Z_4 \neq Z_2 \otimes Z_2$.]

3.5 Lagrange's theorem

If a group G of order N has a subgroup H of order n, then N is necessarily an integer multiple of n.

Consider a group $G = \{g_a \mid a \in \{1, ..., N\}\}$ with subgroup $H = \{h_i \mid i \in \{1, ..., n\}\} \subset G$. Take a group element $g_a \in G$ but not in the subgroup, $g_a \notin H$. Then it follows that also $g_a h_i$ is not in H, $g_a h_i \notin H$, because if there would exist $h_j \in H$ such that $g_a h_i = h_j$ then $g_a = h_j h_i^{-1} \in H$ which would object our assumption. Continuing in this way we can construct the disjoint cosets

$$g_a H = \{g_a h_i \mid i \in \{1, ..., n\}\}$$

and write G as a (right) coset decomposition

$$G = H \cup g_1 H \cup ... \cup g_k H,$$

with $k \in \mathbb{N}$. Therefore the order of G can only be a multiple of its subgroup's order, $N = nk$.

As an immediate consequence, groups of prime order cannot have subgroups of smaller order. Hence for a group of prime order p all elements are of order one or order p and thus $G = Z_p$.

3.6 Order 5

By Lagrange's theorem there is only Z_5.

3.7 Order 6

We now know that there are at least two groups of order six: Z_6 and $Z_2 \otimes Z_3$ but the question is whether they actually differ. Let us consider $Z_2 \otimes Z_3$. The generator a of Z_3 is an order-three element, by which we mean $a^3 = e$. In contrast, the generator b of Z_2 is an order-two element and thus $a^3 = e = b^2$ and we also have $ab = ba$. Now $(ab)^6 = a^6 b^6 = e$ and ab is in fact an order-six element in $Z_2 \otimes Z_3$ and one can infer

$$Z_6 = Z_2 \otimes Z_3.$$

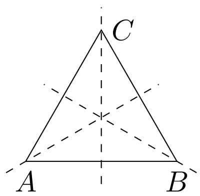
To construct another order six group we again take an order-three element a and an order-two element b. The set

$$\{e, a, a^2, b, ab, a^2 b\},$$

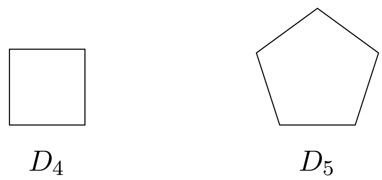
together with the relation $ba = a^2 b \neq ab$ forms the dihedral group D_3 which is non-abelian. A presentation is given by

$$\langle a, b \mid a^3 = e, b^2 = e, bab^{-1} = a^{-1} \rangle \, .$$

D_3 is the symmetry group of the equilateral triangle



where a is a rotation by $\frac{2\pi}{3}$ $(ABC \to BCA)$ and b is a reflection $(ABC \to BAC)$. Higher dihedral groups have higher polygon symmetries.



The group can also be represented by permutations as indicated above. The element a corresponds to a three-cycle $A \to B \to C \to A$

$$\begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \tag{3.4}$$

whereas the element b is a two-cycle: $A \to B \to A$

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}. \tag{3.5}$$

In general the $n!$ permutations of n objects form the symmetric group S_n. From (3.4) and (3.5) we see

$$D_3 = S_3.$$

The group operation for a k-cycle can be represented by $k \times k$ matrices, e.g. a three-cycle can be represented by (3×3) matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} B \\ C \\ A \end{pmatrix}.$$

3.8 Order 8

From what we know already we can immediately write down four order eight groups which are not isomorphic to each other since they contain elements of different order: Z_8, $Z_2 \otimes Z_2 \otimes Z_2$, $Z_4 \otimes Z_2$. Additionally there is the dihedral group D_4 and a new group Q called the quaternion group.

Let us directly focus on the latter. An element q of the deduced quaternion vector space $\mathbb{H}$ generalises the complex numbers,

$$\begin{aligned} q &= x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, \\ \bar{q} &= x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}, \end{aligned}$$

with $x_i \in \mathbb{R}$ and the relation

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,$$

for the imaginary units as well as

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},$$

and cyclic permutations thereof. For $q \in \mathbb{H}$

$$N(q) = \sqrt{q\bar{q}},$$

defines a norm with

$$N(qq') = N(q)N(q').$$

A finite group of order eight can now be taken to be the set

$$\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}, -1, -\mathbf{i}, -\mathbf{j}, -\mathbf{k}\},$$

and is called the quaternion group Q. [Exercise: Convince yourself that Q is a closed group, indeed.]

A matrix representation of the quaternion group Q is given by the Pauli matrices

$$\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}, -1, -\mathbf{i}, -\mathbf{j}, -\mathbf{k}\} = \{1, -i\sigma_1, -i\sigma_2, -i\sigma_3, -1, i\sigma_1, i\sigma_2, i\sigma_3\}$$

with the property

$$\sigma_j \sigma_k = \delta_{jk} + i\varepsilon_{jkl}\sigma_l,$$

and the concrete expressions

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3.9 Permutations

The permutations of n objects can be represented by the symbol

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix}$$

and all possible such permutations form a group (of order $n!$), the *symmetric group* S_n. Every permutation can uniquely be resolved into cycles, e.g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \sim (1234),$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \sim (12)(3)(4),$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \sim (132)(4).$$

Any permutation can also be decomposed into a product of two-cycles,

$$(a_1 a_2 a_3 \dots a_n) \sim (a_1 a_2)(a_1 a_3) \dots (a_1 a_n).$$

The symmetric group S_n has a subgroup of *even* permutations $A_n \subset S_n$ of order $n!/2$. Notice that the odd permutations do not form a subgroup since there is no unit element.

3.10 Cayley's theorem

Every group of finite order n is isomorphic to a subgroup of S_n.

Let $G = \{g_a \mid a \in \{1, \dots, n\}\}$ be a group and associate the group elements with the permutations

$$g_a \to P_a = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_1 g_a & g_2 g_a & \cdots & g_n g_a \end{pmatrix}.$$

This construction leaves the multiplication table invariant,

$$g_a g_b = g_c \to P_a P_b = P_c,$$

and $\{P_a \mid a \in \{1, \dots, n\}\}$ is called the regular representation of G and it is by construction a subgroup of S_n.

3.11 Some notations and concepts

Conjugate with respect to a group element. For a group $G = \{g_a \mid a \in \{1, \dots, n\}\}$ we define the conjugate to an element $g_a \in G$ with respect to the element $g \in G$ as

$$\tilde{g}_a = g \, g_a \, g^{-1}.$$

With this definition, multiplication rules remain unmodified under conjugation,

$$g_a g_b = g_c \to \tilde{g}_a \tilde{g}_b = \tilde{g}_c$$

since

$$g g_a \underbrace{g^{-1} g}_{e} g_b g^{-1} = g g_a g_b g^{-1} = g g_c g^{-1}.$$

Note that if we have $[g, g_a] = 0$, then the conjugate of g_a with respect to g is just g_a again.

Equivalence classes of group elements. For a group G and an element g_a we define the *equivalence class* C_a to consist of all $\tilde{g}_a$ that are conjugate to g_a with respect to *some* element g,

$$C_a = \{\tilde{g}_a = g\, g_a\, g^{-1} \mid g \in G\}.$$

Now take another group element $g_b \in G$ but such that $g_b \notin C_a$. The set of all conjugates forms the class C_b,

$$C_b = \{\tilde{g}_b = g\, g_b\, g^{-1} \mid g \in G\},$$

and so we can go on. Note that the classes are disjoint, i. e. there can be no group elements that are simultaneously in two classes. [Exercise: Show this!] In this way we can decompose G into a set of equivalence classes

$$G = C_1 \cup C_2 \cup ... \cup C_k.$$

Note that the unit element e is always forming an equivalence class of it's own and traditionally it is denoted by C_1. Also, if the group is abelian, each group element forms its own equivalence class so the concepts is only really interesting for non-abelian groups.

Normal or invariant subgroup. Let G be a group and $H \subset G$ a subgroup. Assume now that for all elements of the subgroup $h_i \in H$ the conjugates with respect to *any* group element $g \in G$ stay in H, that is

$$g\, h_i\, g^{-1} \in H.$$

In this case H is called an *invariant* or *normal subgroup*. One can also formulate this by saying that for for a normal subgroup, all the equivalence classes of its elements must be entirely part of the subgroup. Normal subgroups are rather special subgroups and one uses the notation $H \triangleleft G$ as a stronger version of $H \subset G$.

As a first example, consider a direct product group $G = H_1 \otimes H_2$. In that case H_1 and H_2 are invariant or normal subgroups of G. As a second example consider $Z_4 = \{1, -1, i, -i\}$. It has the invariant subgroup $Z_2 = \{1, -1\}$, for example $i\{1, -1\}(-i) = \{1, -1\}$. As a third example, consider the quaternion group

$$Q = \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}, -1, -\mathbf{i}, -\mathbf{j}, -\mathbf{k}\},$$

which has an invariant subgroup $Z_4 = \{1, -1, \mathbf{i}, -\mathbf{i}\}$, for example $\mathbf{j}\{1, -1, \mathbf{i}, -\mathbf{i}\}\mathbf{j}^{-1} = \{1, -1, -\mathbf{i}, \mathbf{i}\}$.

Quotient group. Let us consider a map from a group G to a smaller group K,

$$\begin{aligned} G &\to K, \\ g_a &\to k_a, \end{aligned}$$

such that $g_a g_b = g_c \to k_a k_b = k_c$. Now we take H to be the kernel of this map, i. e. the set of all group elements of G that are mapped to the unit element of K,

$$H = \{g_a \in G \mid g_a \to e\} \subset G.$$

The so-constructed H is then a normal subgroup because for all $h_i \in H$

$$g h_i g^{-1} \to k e k^{-1} = e.$$

Now let us go the other way. Let us assume that G has a normal subgroup H. We will now use H to build sets of group elements and to define a multiplication between these sets. More specific, consider the set

$$gH = \{g\, h \mid h \in H\}.$$

This is known as a *left coset*. Now take two cosets $g_a H$ and $g_b H$ and let $H = eH$ be the trivial coset. We define the multiplication of cosets through the multiplication of their elements,

$$\underbrace{(g_a h_i)}_{\in g_a H}\underbrace{(g_b h_j)}_{\in g_b H} = g_a g_b \underbrace{g_b^{-1} h_i g_b}_{\tilde{h}_i} h_j = g_a g_b \underbrace{\tilde{h}_i h_j}_{\in H} \in g_a g_b H.$$

Note that we have assumed that H is a normal subgroup. This shows that the cosets can be multiplied,

$$(g_a H)(g_b H) = (g_a g_b H),$$

in the same way as the original group elements g_a and g_b.

In other words, the set of cosets $g_a H$ has a group structure itself where the identity element is simply the trivial coset H. This group is called the *quotient group* G/H. The quotient group is a homomorphic image of the original group G, but it is in general *not* a subgroup.

As an example consider the quaternionic group $Q = \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}, -1, -\mathbf{i}, -\mathbf{j}, -\mathbf{k}\}$ which, as we have shown before, has a normal or invariant subgroup $Z_4 = \{1, -1, \mathbf{i}, -\mathbf{i}\}$. The latter is the also the unit element of the quotient group Q/Z_4, which is of order $8/4 = 2$. Its other element besides the unit element is $\mathbf{j}Z_4 = \mathbf{j}\{1, -1, \mathbf{i}, -\mathbf{i}\} = \{\mathbf{j}, -\mathbf{j}, -\mathbf{k}, \mathbf{k}\}$. One sees easily that $(\mathbf{j}Z_4)(\mathbf{j}Z_4) = Z_4$ so that $Q/Z_4 = Z_2$ as expected.

To summarize, if G contains a non-trivial normal subgroup H, we can construct a smaller group, the quotient group G/H, essentially by "dividing out" H. For example starting from a direct product group $H_1 \otimes H_2$ and "dividing out" H_2 leads to $H_1 \otimes H_2 / H_2 = H_1$.

However, not all groups have normal subgroups. For example, one can show that the quotient group G/H has no normal subgroup if H is the maximal normal subgroup.

Simple group. A simple group is a group without (non-trivial) normal subgroups. The simple groups are particularly interesting for mathematicians because they are elementary in the sense that other groups can be reduced to them.

Relatively recently (in 2004), all finite simple groups have been classified with the result that and there are

- the cyclic groups Z_p for p prime
- the alternating groups A_n with $n \geq 5$
- 16 infinite families of groups of Lie type (not to be confused with continuous Lie groups)
- 26 sporadic groups

However, a detailed discussion of this classification goes beyond the scope of these lectures.

3.12 Representations

Let V be a N dimensional vector space with an orthonormal basis $|i\rangle$,

$$\sum_{i=1}^{N} |i\rangle \langle i| = \mathbb{1}, \qquad \langle i|j\rangle = \delta_{ij},$$

where $\mathbb{1}$ is the identity element in the space of N-dimensional matrices $GL(V)$. Let G be a group of order n with representation $\mathcal{R}$ on V such that the action of $g \in G$ is represented by

$$|i\rangle \to |i(g)\rangle = M_{ij}(g)\,|j\rangle\,,$$

where $M(g) \in GL(V)$. Then for $g_a, g_b, g_c, g \in G$ and $g_a g_b = g_c$,

$$\begin{aligned} M(g_a) \cdot M(g_b) &= M(g_c), \\ M(g^{-1})_{ij} &= (M(g)^{-1})_{ij}. \end{aligned}$$

The representation is called trivial if $M(g) = \mathbb{1}$ for all $g \in G$. The representation $\mathcal{R}$ is called reducible if one can arrange $M(g)$ in the form

$$M(g) = \begin{pmatrix} M^{[1]}(g) & 0 \\ N(g) & M^{[\perp]}(g) \end{pmatrix},$$

where $M^{[1]}(g) \in GL(V_1)$ acts on the subspace $V_1 \subset V$ of dimension d_1, $M^{[\perp]}(g) \in GL(V_1^{\perp})$ on its orthogonal complement $V_1^{\perp} \subset V$ of dimension $N - d_1$ and $N(g)$ is a $(N - d_1) \times d_1$ matrix. Then for $g, g' \in G$

$$\begin{aligned} M^{[1]}(gg') &= M^{[1]}(g) \cdot M^{[1]}(g'), \\ M^{[\perp]}(gg') &= M^{[\perp]}(g) \cdot M^{[\perp]}(g'), \end{aligned}$$

as well as

$$N(gg') = N(g) \cdot M^{[1]}(g') + M^{[\perp]}(g) \cdot N(g').$$

One can in fact simplify this further. With

$$W = \frac{1}{\text{ord}(G)} \sum_{g \in G} M^{[\perp]}(g^{-1}) \cdot N(g),$$

we diagonalise the representation

$$\begin{pmatrix} 1 & 0 \\ W & 1 \end{pmatrix} \cdot \begin{pmatrix} M^{[1]} & 0 \\ N & M^{[\perp]} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -W & 1 \end{pmatrix} = \begin{pmatrix} M^{[1]} & 0 \\ 0 & M^{[\perp]} \end{pmatrix},$$

using

$$\begin{aligned} W \cdot M^{[1]}(g_1) &= \frac{1}{\text{ord}(G)} \sum_{g \in G} M^{[\perp]}(g^{-1}) \cdot N(g) \cdot M^{[1]}(g_1) \\ &= -N(g_1) + \frac{1}{\text{ord}(G)} \sum_{g' \in G} M^{[\perp]}(g_1 g'^{-1}) \cdot N(g') \\ &= -N(g_1) + M^{[\perp]}(g_1) \cdot W, \end{aligned}$$

with $g' = gg_1$. In this basis we say $\mathcal{R}$ is completely reducible,

$$\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_1^{\perp}.$$

If $\mathcal{R}_1^{\perp}$ is reducible we can further reduce until there are no subspaces left,

$$\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \ldots \oplus \mathcal{R}_k,$$

with $k \in \mathbb{N}$.

3.13 Schur's lemmas

From the notation established before we write the action of a $g \in G$ in the subspace $V_1 \subset V$ as

$$|a\rangle \to |a(g)\rangle = M_{ab}^{[1]}(g) \, |b\rangle \, ,$$

where $|a\rangle$ is a orthonormal basis of V_1. Since V_1 is a subspace of V we can write

$$|a\rangle = S_{ai} \, |i\rangle \, ,$$

with S a $d_1 \times N$ matrix. Now we write the group action

$$|a\rangle \to |a(g)\rangle = M_{ab}^{[1]}(g) \, |b\rangle = M_{ab}^{[1]}(g) \; S_{bi} \, |i\rangle \, ,$$

or equivalently

$$|a\rangle = S_{ai} \, |i\rangle \to S_{ai} \, |i(g)\rangle = S_{ai} \; M_{ij}(g) \, |j\rangle \, .$$

Therefore,

$$\mathcal{R} \text{ reducible} \quad \Rightarrow \quad S_{ai} \; M_{ij}(g) = M_{ab}^{[1]}(g) \; S_{bj} \quad \text{for all} \quad g \in G. \tag{3.6}$$

Schur's first lemma: *If matrices of two irreducible representations of different dimension can be related as in (3.6), then $S = 0$.*

If now $d_1 = N$, S is an $N \times N$ matrix and if $|a\rangle$ and $|j\rangle$ span the same space the representations $\mathcal{R}$ and $\mathcal{R}_1$ are related by a similarity transformation

$$M^{[1]} = S \cdot M \cdot S^{-1}.$$

Thus for $S \neq 0$ we conclude that either $\mathcal{R}$ is reducible or there is a similarity relation.

Schur's second lemma: *Let $\mathcal{R}$ be an irreducible representation. Any matrix S with*

$$M(g) \cdot S = S \cdot M(g)$$

for all $g \in G$ is proportional to $\mathbb{1}$.

If $|i\rangle$ is an eigenket of S then $|i(g)\rangle$ is also an eigenket,

$$\underbrace{M(g)\,\underbrace{S\,|i\rangle}_{\lambda|i\rangle}}_{\lambda|i(g)\rangle} = S\,\underbrace{M(g)\,|i\rangle}_{|i(g)\rangle}$$

$$\Rightarrow S = \lambda\mathbb{1}.$$

Let $\mathcal{R}_a, \mathcal{R}_b$ be two irreducible representations of dimension d_a and d_b respectively. Construct the map

$$S = \sum_{g \in G} M^{[a]}(g) \cdot N \cdot M^{[b]}(g^{-1}),$$

where N is *any* $d_a \times d_b$ matrix. Then for $g \in G$

$$M^{[a]}(g) \cdot S = S \cdot M^{[b]}(g).$$

For $\mathcal{R}_a \neq \mathcal{R}_b$ by Schur's first lemma

$$\frac{1}{\mathrm{ord}(G)} \sum_{g \in G} M^{[a]}_{ij}(g)\ M^{[b]}_{pq}(g^{-1}) = 0.$$

On the other hand for $\mathcal{R}_a = \mathcal{R}_b$ by Schur's second lemma

$$\frac{1}{\mathrm{ord}(G)} \sum_{g \in G} M^{[a]}_{ij}(g)\ M^{[a]}_{pq}(g^{-1}) = \frac{1}{d_a}\delta_{iq}\delta_{jp}.$$

Combining these two results leaves us with

$$\frac{1}{\mathrm{ord}(G)} \sum_{g \in G} M^{[a]}_{ij}(g)\ M^{[b]}_{pq}(g^{-1}) = \frac{1}{d_a}\delta_{iq}\delta_{jp}\delta^{ab}.$$

Let us define the scalar product

$$(i, j) = \frac{1}{\mathrm{ord}(G)} \sum_{g \in G} \langle i(g)|j(g)\rangle\,, \tag{3.7}$$

where $|j(g)\rangle = M(g)\,|j\rangle$. It is invariant under the group action

$$\begin{aligned}(i(g_a), j(g_a)) &= \frac{1}{\mathrm{ord}(G)} \sum_{g\in G} \langle i(g_a g)|j(g_a g)\rangle \\ &= \frac{1}{\mathrm{ord}(G)} \sum_{g'\in G} \langle i(g')|j(g')\rangle \\ &= (i,j),\end{aligned}$$

where $g' = g_a g$ runs over all group elements. Therefore for $\mathbf{v}, \mathbf{u} \in V$

$$(M(g^{-1})\mathbf{v}, \mathbf{u}) = (\mathbf{v}, M(g)\mathbf{u}).$$

On the other hand

$$(M^\dagger(g)\mathbf{v}, \mathbf{u}) = (\mathbf{v}, M(g)\mathbf{u}),$$

where $M^\dagger$ is the Hermitian conjugate of M and therefore

$$M^\dagger(g) = M(g^{-1}) = M^{-1}(g).$$

Therefore all representations of finite groups are unitary with respect to the scalar product $(\cdot,\cdot)$ that we have constructed as an average over the group elements in (3.7).

[Exercise: Use the scalar product $(\cdot,\cdot)$ to show that every reducible representation is necessarily also completely reducible, which means that it can be brought to block diagonal form. Hint: use that there is an invariant sub-vector-space and consider the orthogonal complement.]

3.14 Crystals

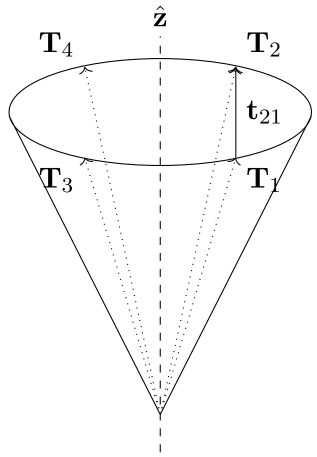
Crystals live in between finite and infinite groups; they have symmetries of both kinds. A crystal in three dimensions is defined as a lattice which is invariant under discrete translations of the form

$$\mathbf{T} = n_1\mathbf{u}_1 + n_2\mathbf{u}_2 + n_3\mathbf{u}_3, \tag{3.8}$$

with three linearly independent $\mathbf{u}_i \in \mathbb{R}^3$ and $n_i \in \mathbb{Z}$. The entire symmetry group of a crystal (the so-called *space group*) contains besides translations also rotations, reflections, and combinations thereof. A subgroup that consists of rotations and reflections (and their combinations of course) that leave one point invariant, is know as the smaller *point group*. We will not go into further detail here but mention that in three dimensions there are 230 crystallographic space groups and 32 crystallographic point groups. The latter determine macroscopic properties of crystals. Point groups are also used to classify molecules, but because they do not need to have translational symmetries, there are more point groups.

A crystal in two dimensions may also be called an *ornament* and the corresponding symmetry group is sometimes called a *wallpaper group*. In two dimensions there are 17 space groups and only 9 crystallographic point groups: the cyclic groups Z_2, Z_3, Z_4 and Z_6 as well as the dihedral groups D_2, D_3, D_4 and D_6. In addition to this there is the trivial group $Z_1 = D_1$ with only one element. It is remarkable that there are no 5-fold, 7-fold or higher n-fold rotation symmetries possible. This is subject to the following theorem.

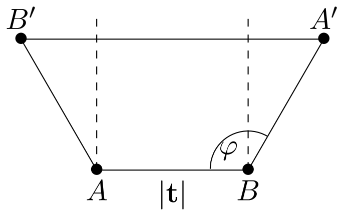
Crystallographic restriction theorem: *Consider a crystal invariant under rotations through $2\pi/n$ around an axis. Then n is restricted to $n \in \{1, 2, 3, 4, 6\}$.*



First, the case $n = 1$ is trivial and we can exclude it and concentrate on the other cases. Consider a translation vector $\mathbf{T}_1$. By rotations it gets transformed to $\mathbf{T}_2, \mathbf{T}_3, ..., \mathbf{T}_n$. By the group property of the space group, also the differences $\mathbf{t}_{ij} = \mathbf{T}_i - \mathbf{T}_j$ are translation vectors and by construction they are orthogonal to the rotations axis, which we may take to coincide with the z-axis. Take now the minimum of the differences,

$$|\mathbf{t}| = \min_{i,j}(|\mathbf{t}_{ij}|),$$

and without loss of generality normalise $|\mathbf{t}| = 1$. Take now a point A on the rotational symmetry axis which by $\mathbf{t}$ gets translated to another symmetry point B. Rotation by an angle φ around A brings B to B'. Similar, rotation by φ around B brings A to A'. Because A' and B' are also symmetry points, the difference $A'B'$ must be a translation vector parallel to AB, and so we can infer that $\overline{A'B'} = p \in \mathbb{N}_0$.



With

$$\overline{A'B'} = 1 + 2\sin\left(\varphi - \frac{\pi}{2}\right) = 1 - 2\cos(\varphi),$$

we conclude

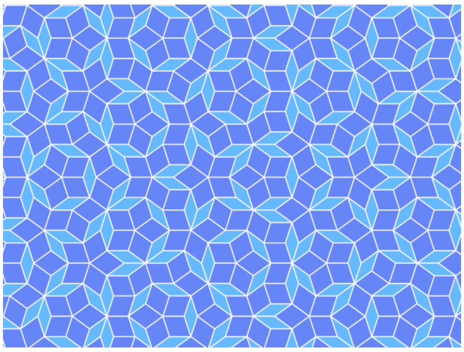
$$\cos(\varphi) = \frac{1-p}{2},$$

and thus the possible values are

$$\begin{aligned} &p \in \{0, 1, 2, 3\}, \\ \Rightarrow\ &\varphi \in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi\right\}, \\ \Rightarrow\ &n \in \{6, 4, 3, 2\}. \end{aligned}$$

This closes the proof of the restriction theorem above.

Quasicrystals may have elements or subregions with five-fold or other symmetries but have in fact no periodic structure (3.8), for example a Penrose tiling.



4 Lie groups and Lie algebras

Lie groups can be defined as *differentiable manifolds* with a group structure. In contrast to the finite groups we have discussed before they have now an infinite number of elements. Let us start with a few examples.

4.1 Examples for Lie groups

- $G = \mathbb{R}$, the *additive* group of real numbers. The group "product" is here the addition, the inverse of an element is its negative and the neutral or unit element is zero. This is clearly an abelian group.

- $G = \mathbb{R}_+^*$, the *multiplicative* group of positive real numbers. Also an abelian group.

- $G = \mathrm{GL}(n, \mathbb{R})$, the general linear group of real $n \times n$ matrices g with $\det(g) \neq 0$ (such that they are invertible). Similarly, $G = \mathrm{GL}(n, \mathbb{C})$, the general linear group of complex $n \times n$ matrices. These groups are non-abelian for $n > 1$.

- $G = \mathrm{SL}(n, \mathbb{R})$ the special linear group is a subgroup of $\mathrm{GL}(n, \mathbb{R})$ with $\det(g) = 1$. This is a more general notion, the S for *special* usually means $\det(g) = 1$.

- $G = \mathrm{O}(n)$, the *orthogonal* group of real $n \times n$ matrices R with $R^T R = \mathbb{1}$. This immediately implies $\det(R) = \pm 1$. Again this is a subgroup of $\mathrm{GL}(n, \mathbb{R})$. As a manifold, $\mathrm{O}(n)$ is not connected. One component is the subgroup $\mathrm{SO}(n)$ with $\det(R) = 1$, the other is a separate submanifold where $\det(R) = -1$. One can understand $\mathrm{O}(n)$ as the group of *rotations* and *reflections* in the n-dimensional Euclidean space. The simplest non-trivial case is for $n = 2$ where $\mathrm{SO}(2)$ consists of elements of the form

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} .$$

  This is clearly isomorphic to the group $\mathrm{U}(1)$ of complex phases $e^{i\theta}$. $\mathrm{SO}(n)$ is non-abelian for $n > 2$.

- $G = \mathrm{U}(n)$, the *unitary* group of complex $n \times n$ matrices U with $U^\dagger U = \mathbb{1}$. Now we immediately infer that $\det(U)$ is a complex number with absolute value 1. $\mathrm{U}(n)$ is non-abelian for $n > 1$.

- $G = \mathrm{SU}(n)$, the special unitary group with unit determinant. Plays an important role in physics and we will discuss in particular $\mathrm{SU}(2)$ and $\mathrm{SU}(3)$ in more detail.

- $G = \mathrm{O}(r, n - r)$ the indefinite orthogonal group of $n \times n$ matrices R that leaves the metric $\eta = \mathrm{diag}(-1, \ldots, -1, +1, \ldots, +1)$ with r entries -1 and $n - r$ entries $+1$ invariant, in the sense that $R^T \eta R = \eta$. An example is $\mathrm{O}(1, 3)$, the group of Lorentz transformations in $d = 1 + 3$ dimensions.

- $G = \mathrm{Sp}(2n, \mathbb{R})$ is the symplectic group of $2n \times 2n$ matrices M that leaves a symplectic bilinear form
$$\Omega = \begin{pmatrix} 0 & +\mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \tag{4.1}$$
invariant in the sense that $M^T \Omega M = \Omega$. Here $\mathbb{1}_n$ is the n dimensional unit matrix and similarly 0. This is obviously a subgroup of $\mathrm{GL}(2n, \mathbb{R})$. There is also a complex version $\mathrm{Sp}(2n, \mathbb{C})$.

[Exercise: Convince yourself that all these examples are groups and also differentiable manifolds.]

Lie groups have very nice features and a rich mathematical structure because they are both, *groups* and *manifolds*. We will now first introduce Lie groups and Lie algebras from an algebraic point of view, and subsequently also briefly introduce a differential-geometric characterization.

4.2 Algebraic approach to Lie groups and Lie algebras

Because a Lie group is also a manifold, group elements can be labeled by a (usually multi-dimensional) parameter or coordinate $\xi = (\xi^1, \ldots, \xi^m)$, i.e. we can write them as $g(\xi)$. Without loss of generality we can assume that $\xi = 0$ corresponds to the unit element, $g(0) = \mathbb{1}$. Let us now consider infinitesimal transformations. We can write them as
$$g(d\xi) = \mathbb{1} + i d\xi^j T_j + \ldots, \tag{4.2}$$
where we use Einsteins summation convention and the ellipses stand for terms of quadratic and higher order in $d\xi$. Note that we can write
$$iT_j = \frac{\partial}{\partial \xi^j} g(\xi)\Big|_{\xi=0}. \tag{4.3}$$
Formally, the objects iT_j constitute a basis of the tangent space of the Lie group manifold at the position of the unit element $g(\xi) = \mathbb{1}$, which is at $\xi = 0$. The factor i is conventional and used by physicists, while mathematicians usually work in a convention without it. The T_j are also known as the generators of the *Lie algebra*, to which we turn in a moment. The generators constitute a basis such that any element of the Lie algebra can be written as a linear superposition $v^j T_j$.

A very important idea is now that one can compose finite group elements, at least in some region around the unit element, out of very many infinitesimal transformations. In other words one writes
$$g(\xi) = \lim_{N \to \infty} \left(\mathbb{1} + \frac{i\xi^j T_j}{N} \right)^N = \exp\left(i\xi^j T_j\right). \tag{4.4}$$
One recognizes here that the limit in (4.4) would give the exponential if T_j were just numbers, and one can essentially use this limit to define also the exponential of Lie algebra elements. Alternatively, the exponential may also be evaluated as the usual power series
$$\exp\left(i\xi^j T_j\right) = \mathbb{1} + i\xi^j T_j + \frac{1}{2}\left(i\xi^j T_j\right)^2 + \frac{1}{3!}\left(i\xi^j T_j\right)^3 + \ldots$$

Note that for $\alpha, \beta \in \mathbb{R}$ one can combine

$$\exp\left(i\alpha\xi^j T_j\right)\exp\left(i\beta\xi^j T_j\right) = \exp\left(i(\alpha+\beta)\xi^j T_j\right). \quad (4.5)$$

Such transformations (for fixed ξ) form a one-parameter subgroup. [Exercise: Verify this.]

However, it is more difficult to combine transformations $\exp\left(i\xi^j T_j\right)$ and $\exp\left(i\zeta^j T_j\right)$ when ξ is not parallel to ζ. The reason is that $\xi^j T_j$ and $\zeta^j T_j$ can not be assumed to commute. To combine two transformations, one needs to use the Baker-Campbell-Hausdorff formula

$$\exp(X)\exp(Y) = \exp(Z(X,Y)), \quad (4.6)$$

with

$$Z(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \ldots \quad (4.7)$$

This shows that it is crucial to know how to calculate commutators between the Lie algebra elements.

For transformations close to the identity element we can write using (4.6) and (4.7)

$$\exp\left(i\xi^j T_j\right)\exp\left(i\zeta^j T_j\right) = \exp\left(i\omega^j T_j\right), \quad (4.8)$$

with

$$\omega^l = \xi^l + \zeta^l - \frac{1}{2}\xi^j\zeta^k f_{jk}{}^l + \ldots,$$

where the ellipses stand now for terms of quadratic and higher order in ξ and / or in ζ. We are using here the *structure constants* $f_{jk}{}^l$ of the Lie algebra defined through the commutator

$$[T_j, T_k] = i\, f_{jk}{}^l\, T_l. \quad (4.9)$$

The structure constants are obviously anti-symmetric,

$$f_{jk}{}^l = -f_{kj}{}^l.$$

Equation (4.9) tells that the commutator of two generators can itself be expressed as a linear combination of generators. Together with eqs. (4.6) and (4.7) this makes sure that the group elements (4.4) can be multiplied and indeed form a group. In other words, if eq. (4.9) holds, we can multiply group elements as in eq. (4.8) to yield another term of the same structure such that they form a group. On the other side, one could also start from the group multiplication law and demand that the left hand side of (4.8) can be written as on the right hand side. At order $\sim \xi\zeta$ this implies then a relation of the form (4.9).

But (4.9) also makes sure that linear combinations of generators, which obviously form a vector space, constitute a *Lie algebra*. As usual, the Lie bracket $[\cdot,\cdot]$ has the properties

i) *Bilinearity*: $[\lambda A + \mu B, C] = \lambda[A,C] + \mu[B,C]$,

ii) *Antisymmetry*: $[A,B] = -[B,A]$,

iii) *Jacobi identity*: $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$.

From the Jacobi identity for the generators

$$[T_j, [T_k, T_l]] + [T_k, [T_l, T_j]] + [T_l, [T_j, T_k]] = 0, \tag{4.10}$$

one infers for the structure constants

$$f_{jn}{}^m f_{kl}{}^n + f_{kn}{}^m f_{lj}{}^n + f_{ln}{}^m f_{jk}{}^n = 0. \tag{4.11}$$

For unitary Lie groups where $g^\dagger = g^{-1}$ one has $T_j = T_j^\dagger$ and that in that case the structure constants are real,

$$f_{jk}{}^l = \left(f_{jk}{}^l\right)^*.$$

[Exercise: Check that.]

Representations. The commutation relation (4.9) and in particular the structure constants define a Lie algebra similar as the multiplication rules do for a group. Again one distinguishes between a particular Lie algebra as an abstract entity and a concrete incarnation or *representation* of it.

A first example is the *fundamental representation*

$$(T_j^{(F)})^m{}_n = (t_j)^m{}_n. \tag{4.12}$$

For SU(N), the generators in the fundamental representation t_j are hermitian and traceless $N \times N$ matrices, e.g. the three Pauli matrices for SU(2) and the eight Gell-Mann matrices for SU(3).

From the Jacobi identity (4.11), one can see that the structure constants can actually be used to construct another representation, the so-called *adjoint representation*. Here one sets the matrices to

$$(T_j^{(A)})^m{}_n = i f_{jn}{}^m. \tag{4.13}$$

Indeed, one can check that this fulfills the commutation relation (4.9). [Exercise: Do it!]

The dimension of the adjoint representation equals the number of generators of the Lie algebra. For example, the Lie algebra of SU(3) has 8 generators and accordingly the adjoint representation is given by 8×8 matrices.

The fundamental and the adjoint representation are the most important representations of a *Lie algebra* and we will need them often in the following. However, there are many more and they all induce corresponding representations of the *Lie group* through the exponential map (4.4).

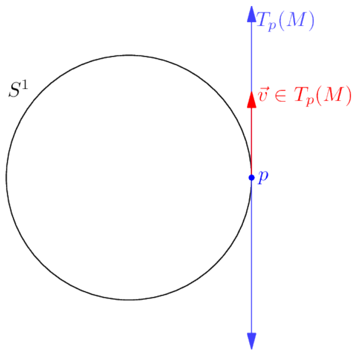
One remark of caution should be made here: Group elements in some neighborhood of the unit elements can be written in the exponential form (4.4). However, it is not at all clear that *all* group elements can be written is this way. For a group where the manifold has two disconnected components, such as O(N), this can only be the case for the part that is connected to the identity element. Only for groups that are *connected* and *compact* one can show that the exponential map (4.4) reaches indeed all elements.

4.3 Differential geometric approach to Lie groups and Lie algebras

Now that we understand already some of the properties of Lie groups and Lie algebras, let us discuss them also from a geometric point of view. It is interesting to consider the group multiplication as a map on the group manifold,

$$L_h : G \to G, \qquad L_h(g) = hg. \tag{4.14}$$

This is the so-called *left translation*.



Recall that in (4.3) we have introduced the generators T_j as a basis for the *tangent space* of the group manifold G at the identity $g = \mathbb{1}$ or $\xi = 0$.

More formally, one can construct the tangent space of a manifold as a basis for vectors, which are in turn defined through curves. Let us review this construction. We first consider a curve in the group manifold parametrized by some parameter $\alpha \in \mathbb{R}$ and we assume that it goes through the unit element $g(\alpha_0) = \mathbb{1}$. We can write the curve as $g(\alpha)$, or in terms of coordinates ξ on the group manifold as $\xi(\alpha)$ such that $g(\alpha) = g(\xi(\alpha))$ and $\xi(\alpha_0) = 0$. Now consider the derivative

$$\frac{d}{d\alpha} g(\alpha)\Big|_{\alpha=\alpha_0} = \frac{\partial}{\partial \xi^j} g(\xi)\Big|_{\xi=0} \frac{d\xi^j}{d\alpha} = iT_j \frac{d\xi^j}{d\alpha}.$$

This is now an element of the tangent space $T_{\mathbb{1}}(G)$ of the group manifold at the point where $g(\xi) = \mathbb{1}$. Any element of this vector space can be written as a linear combination of the basis elements

$$iT_j = \frac{\partial}{\partial \xi^j} g(\xi)\Big|_{\xi=0}.$$

Interestingly, this basis for $T_{\mathbb{1}}(G)$ can be extended to a basis for the tangent spaces at other positions of the group manifold. To that end we can use the left translation (4.14). Specifically, from the curve $g(\alpha)$ we can construct another curve through the left translation (4.14)

$$\tilde{g}(\alpha) = L_h(g(\alpha)) = hg(\alpha).$$

The derivative at the point α_0 is now

$$\frac{d}{d\alpha} \tilde{g}(\alpha)\Big|_{\alpha=\alpha_0} = \frac{\partial}{\partial \xi^j} hg(\xi)\Big|_{\xi=0} \frac{d\xi}{d\alpha}.$$

One observes that a basis for the tangent space $T_h(G)$ is given by

$$iT_j(h) = \frac{\partial}{\partial \xi^j} hg(\xi)\Big|_{\xi=0} = h\frac{\partial}{\partial \xi^j} g(\xi)\Big|_{\xi=0} = i\, hT_j. \tag{4.15}$$

In this way we can actually get a basis for the tangent spaces everywhere in the entire group manifold. It is quite non-trivial that the tangent spaces can be parametrized by a single set of basis functions $iT_j(h)$. One says that the manifold G is *parallelizable*.

Formally, the map (4.15) between the tangent spaces $T_{\mathbb{1}}(G)$ and $T_h(G)$ is an example for a *pushforward*, induced by the map (4.14) on the manifold itself. One also writes this as

$$T_j(h) = L_{h*}T_j(e) = L_{h*}T_j.$$

One may now construct *vector fields* on the entire manifold as linear combinations,

$$V(h) = v^j(h)T_j(h). \tag{4.16}$$

Such a vector field is called *left invariant* if

$$L_{g*}V(h) = V(gh).$$

Because the basis $T_j(h)$ is left-invariant by construction, the vector field (4.16) is left-invariant when the coefficients $v^j(h)$ are independent of the position on the manifold, i. e. independent of h. [Exercise: Check that $T_j(h)$ is left-invariant.]

In summary, we may say that the generators of the Lie algebra T_j induce actually a *left-invariant* basis for *vector fields* on the entire group manifolds. One may even understand the Lie algebra itself as an algebra of left-invariant vector fields. The Lie bracket is then introduced as the *Lie derivative* of vector fields.

4.4 Examples for matrix Lie algebras

Let us end this section with a few examples for Lie algebras corresponding to matrix Lie groups introduced previously.

- $\mathfrak{su}(n)$ is the Lie algebra corresponding to the group SU(n). We write the group elements as $U = \exp(it)$. From $U^\dagger U = \mathbb{1}$ one infers $t^\dagger = t$. Writing this in components, the real part is symmetric, $\mathrm{Re}(t_{nm}) = \mathrm{Re}(t_{mn})$, and the imaginary part is anti-symmetric, $\mathrm{Im}(t_{nm}) = -\mathrm{Im}(t_{mn})$. Moreover, we have the condition $\det(U) = 1$. The latter can be rewritten as

$$\det(U) = \exp(\ln(\det(U))) = \exp(\mathrm{Tr}\{\ln(U)\}) = \exp(i\mathrm{Tr}\{t\}) = 1, \tag{4.17}$$

  so that we need $\mathrm{Tr}\{t\} = 0$. These arguments show that the Lie algebra $\mathfrak{su}(n)$ as a real vector space has $n^2 - 1$ linearly independent generators T_j.

- $\mathfrak{so}(n)$ is the Lie algebra corresponding to the group SO(n). Here we write the group elements as $R = \exp(it)$ and they are real matrices such that $R^T R = \mathbb{1}$. For the Lie algebra elements we have again $t = t^\dagger$. In order for an infinitesimal transformation $R = \mathbb{1} + it$ to be real, the components t_{mn} must be imaginary, and therefore also anti-symmetric. The condition $\mathrm{Tr}\{t\} = 0$ is then automatically fulfilled. These arguments show that the Lie algebra $\mathfrak{so}(n)$ has $n(n-1)/2$ linearly independent generators T_j.

- $\mathfrak{sp}(2n)$ is the Lie algebra corresponding to the group $\mathrm{Sp}(2n)$. The group elements $R = \exp(it)$ are real matrices that satisfy $R^T \Omega R = \Omega$ with $\Omega = -\Omega^T$ given in (4.1). For an infinitesimal transformation $R = \mathbb{1} + it$ one finds the condition

$$\Omega t + t^T \Omega = \Omega t - t^T \Omega^T = \Omega t - (\Omega t)^T = 0. \tag{4.18}$$

  In other words, Ωt must be symmetric. These arguments show that the Lie algebra $\mathfrak{sp}(2n)$ has $n(2n+1)$ linearly independent generators T_j.

[Exercise: Consider the three dimensional rotation group SO(3). Find convenient generators T_j of the Lie algebra $\mathfrak{so}(3)$ such that $\mathrm{Tr}\{T_j T_k\} = 2\delta_{jk}$, and work out the structure constants $f_{jk}{}^l$. For rotations around a particular axis (e.g. the z-axis), work out the exponential map explicitly. Show also that SO(3) is non-abelian.

Consider also the Lie algebra $\mathfrak{su}(2)$, find convenient generators normalized now such that $\mathrm{Tr}\{T_j T_k\} = \frac{1}{2}\delta_{jk}$, and show that the structure constants are the same as for $\mathfrak{so}(3)$. In this sense the two Lie algebras actually agree, $\mathfrak{su}(2) = \mathfrak{so}(3)$, even though the Lie groups SU(2) and SO(3) differ.]

5 SU(2)

5.1 The Lie algebra and its representations

The Lie algebra $\mathfrak{su}(2)$ of SU(2) is one of the smallest non-trivial Lie algebras. It plays an important role in physics not only because it is isomorphic to the Lie algebra of rotations SO(3) but also because there are many direct applications of SU(2) such as for two-state systems in quantum mechanics. We will study representations of the Lie algebra of SU(2) in Hilbert spaces because a lot of physics can be described there.

In the simplest non-trivial case the Lie algebra of SU(2) is represented in a two-dimensional Hilbert space with orthonormal basis $|j\rangle$, where $j \in \{1, 2\}$. There are three linearly independent generators which can be taken to be

$$T_+ = |1\rangle \langle 2| , \qquad T_- = |2\rangle \langle 1| , \qquad T_3 = \frac{1}{2} \left(|1\rangle \langle 1| - |2\rangle \langle 2| \right) .$$

They satisfy the commutation relations

$$[T_+, T_-] = 2T_3, \qquad [T_3, T_\pm] = \pm T_\pm. \tag{5.1}$$

We see from the second relation that T_+ raises the value of T_3 by one unit, while T_- lowers it by one unit. One can also define the hermitian operators

$$T_1 = \frac{1}{2} \left(T_- + T_+ \right) , \qquad T_2 = \frac{i}{2} \left(T_- - T_+ \right) ,$$

such that the commutator algebra reads

$$[T_j, T_k] = i\epsilon_{jkl}\, T_l, \tag{5.2}$$

for $j, k, l \in \{1, 2, 3\}$. The algebra (5.2) is closed under commutation and satisfies the Jacobi identity and is therefore a Lie algebra. This explicit construction gives an irreducible representation of the Lie algebra $\mathfrak{su}(2)$ of dimension two, denoted **2**. It is called the fundamental or spinor representation.

To study other representations it is useful to introduce *Casimir operators*. These are operators that *commute* with the generators of the Lie algebra and in the case of $\mathfrak{su}(2)$ there is only one quadratic Casimir,

$$C_2 = T_1^2 + T_2^2 + T_3^2, \qquad [C_2, T_j] = 0.$$

Because $T_j = T_j^\dagger$, by construction we also have $C_2 = C_2^\dagger$.

The states of the Hilbert space can be labelled by the eigenvalues of a maximal number of commuting operators (recall that commuting operators can de diagonalized simultaneously). Here we can take C_2 and one of the generators. We choose T_3 and thus the algebra will be represented on eigenstates of these two operators,

$$C_2 \,|c, m\rangle = c \,|c, m\rangle , \qquad T_3 \,|c, m\rangle = m \,|c, m\rangle ,$$

with $c, m \in \mathbb{R}$ because C_2 and T_3 are hermitian.

The Casimir operator C_2 is quadratic in T_3, and for a finite value c the spectrum of T_3 needs to be bounded with minimal and maximal values roughly between $-\sqrt{c}$ and $\sqrt{c}$. From the commutation relations (5.1) we infer

$$T_+ |c, m\rangle = d_{m(+)} |c, m+1\rangle .$$

for some number $d_{m(+)}$. Then

$$\begin{aligned} T_3 T_+ |c, m\rangle &= (m+1) T_+ |c, m\rangle \\ &= (m+1) d_{m(+)} |c, m+1\rangle , \end{aligned}$$

and

$$\begin{aligned} C_2 T_+ |c, m\rangle &= c T_+ |c, m\rangle \\ &= c d_{m(+)} |c, m+1\rangle . \end{aligned}$$

Because T_3 is bounded, there needs to be a so-called *highest weight state* $|c, j\rangle$ of the representation for which $j \in \mathbb{R}$ is the *maximal value* of T_3. It cannot be raised further,

$$T_+ |c, j\rangle = 0, \qquad T_3 |c, j\rangle = j |c, j\rangle .$$

Similarly we can infer from the commutation relations (5.1)

$$T_- |c, m\rangle = d_{m(-)} |c, m-1\rangle ,$$

for some number $d_{m(-)}$ and we can find a *lowest weight state* $|c, k\rangle$ for some $k \in \mathbb{R}$,

$$T_- |c, k\rangle = 0, \qquad T_3 |c, k\rangle = k |c, k\rangle .$$

Then the Casimir operator is actually determined by the highest weight state,

$$\begin{aligned} C_2 |c, j\rangle &= \left(T_3^2 + \tfrac{1}{2} (T_+ T_- + T_- T_+)\right) |c, j\rangle \\ &= \left(T_3^2 + \tfrac{1}{2} [T_+, T_-]\right) |c, j\rangle \\ &= \left(T_3^2 + T_3\right) |c, j\rangle \\ &= (j^2 + j) |c, j\rangle . \end{aligned}$$

Analogously we can proceed for the lowest weight state

$$C_2 |c, k\rangle = (k^2 - k) |c, k\rangle ,$$

and thus

$$c = j(j+1) = k(k-1). \tag{5.3}$$

Taking into account that j is the maximal value of T_3 there is only the solution $k = -j$ to (5.3). We conclude that there are $(2j+1)$ states,

$$|j, m\rangle , \qquad m \in \{-j, ..., j\},$$

such that $2j \in \mathbb{N}$. We have replaced the label c by j for convenience. We denote the representation by the number of states, written as $\mathbf{2j+1}$. Mathematicians usually use $2j$ for the classification, the so-called Dynkin label.

Let us now also determine the numbers $d_{m(\pm)}$. First of all, at the end of the chain for $m = \pm j$ we have $d_{j(+)} = d_{-j(-)} = 0$. Using

$$T_\pm \left|j, m\right\rangle = d_{m(\pm)} \left|j, m \pm 1\right\rangle,$$

we find from

$$[T_+, T_-] \left|j, m\right\rangle = 2T_3 \left|j, m\right\rangle,$$

that

$$d_{m-1(+)}d_{m(-)} - d_{m(+)}d_{m+1(-)} = 2m.$$

This is solved by

$$d_{m(+)} = \sqrt{(j-m)(j+m+1)}, \qquad d_{m(-)} = \sqrt{(j+m)(j-m+1)}.$$

The eigenstates for fixed j form an orthonormal and complete set,

$$\langle j, m | j, m' \rangle = \delta_{mm'}, \qquad \sum_{m=-j}^{j} \left|j, m\right\rangle \left\langle j, m\right| = \mathbb{1}.$$

In other words, these states form an orthonormal basis for the Hilbert space associated with the representation $\mathbf{2j+1}$.

Fundamental representation 2. In the smallest representation **2** with $j = 1/2$ the generators are represented by the Pauli matrices,

$$T_j = \frac{\sigma_j}{2}, \tag{5.4}$$

which satisfy

$$\sigma_j \sigma_k = \delta_{jk} \mathbb{1} + i\,\epsilon_{jkl}\,\sigma_l, \qquad \sigma_j^* = \sigma_j^T = -\sigma_2 \sigma_j \sigma_2. \tag{5.5}$$

Explicit expressions are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.6}$$

This representation **2** is also known as duplet, spin-1/2 or fundamental spinor representation.

Adjoint representation 3. For $j = 1$ the generators can be represented by the three hermitian matrices,

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.7}$$

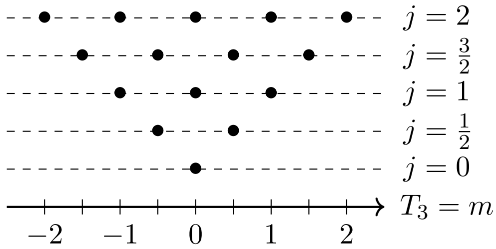
They can also be written using the Levi-Civita-symbol,

$$(T_j)_{mn} = -i\epsilon_{jmn}.$$

This is indeed the adjoint representation introduced in (4.13). It is also known as the triplet, spin-1 or vector representation.

Trivial representation 1. There is also a trivial representation **1** with $j = 0$ where the generators simply vanish, $T_j = 0$. This satisfies the commutation relation (5.2) in a trivial way. This representation is known as the singlet, spin-0 or scalar representation.

The construction generalises to an infinite number of irreducible representations such that the possible eigenvalues of $T_3 = m$ are half integer or integer numbers, lying on an axis from $-j$ to j.



In this diagram, a point represents an element of a representation, with different representations aligned parallel to the T_3-axis. As shown above, the Lie algebra of SU(2) has one Casimir operator $C_2 = j(j+1)$ and the states for given j are uniquely determined by one label m. In general Lie algebras will have as many labels as Casimir operators. The number of Casimirs is called the *rank of the Lie algebra* and hence the Lie algebra of SU(2) has rank one.

Another way of generating representations is by taking direct products, starting for example from of the smallest non-trivial representation. Let us suppose $T_j^{(1)}$ and $T_j^{(2)}$ are two copies of the Lie algebra $\mathfrak{su}(2)$ each satisfying

$$\left[T_j^{(a)}, T_k^{(a)}\right] = i\,\epsilon_{jkl}\,T_l^{(a)},$$

for $a \in \{1, 2\}$, $j, k, l \in \{1, 2, 3\}$ and they commute with each other,

$$\left[T_j^{(1)}, T_k^{(2)}\right] = 0.$$

Then the sum of the generators

$$\left[T_j^{(1)} + T_j^{(2)}, T_k^{(1)} + T_k^{(2)}\right] = i\,\epsilon_{jkl}\left(T_l^{(1)} + T_l^{(2)}\right),$$

fulfills the same algebra. They act now on the direct product states, denoted by $|\cdot\rangle_{(1)}\,|\cdot\rangle_{(2)}$. (Below we drop the indices (1) and (2) when no confusion can arise.)

One can represent the action of the sums of generators in terms of the two original representations where $T_j^{(a)}$ and $T_j^{(b)}$ act. For concreteness, let us consider the direct product of representations $\mathbf{2j+1} \otimes \mathbf{2k+1}$. We expect that this direct product contains different irreducible representations which we aim to find.

The highest weight state $|j, j\rangle\,|k, k\rangle$ is uniquely determined by its values of $T_3^{(a)}$, $a \in \{1, 2\}$ and we have

$$T_3\,|j, j\rangle\,|k, k\rangle = \left(T_3^{(1)} + T_3^{(2)}\right)|j, j\rangle\,|k, k\rangle = (j + k)\,|j, j\rangle\,|k, k\rangle\,.$$

This must also be the highest weight state of an irreducible representation $\mathbf{2(j+k)+1}$. To generate the remaining states of $\mathbf{2(j+k)+1}$ we apply the combined lowering operator $T_- = T_-^{(1)} + T_-^{(2)}$ to the highest weight state,

$$T_- |j,j\rangle |k,k\rangle = d_{j(-)} |j,j-1\rangle |k,k\rangle + d_{k(-)} |j,j\rangle |k,k-1\rangle \, , \tag{5.8}$$

and in this way we can go. However, there is also a combination of states orthogonal to (5.8). This is (up to overall normalization)

$$d_{k(-)} |j,j-1\rangle |k,k\rangle - d_{j(-)} |j,j\rangle |k,k-1\rangle \, ,$$

and it must be the highest weight state of a $\mathbf{2(j+k-1)+1}$ representation because T_3 acting on it gives $(j+k-1)$ and applying a raising operator leads to zero,

$$T_+ \left(d_{k(-)} |j,j-1\rangle |k,k\rangle - d_{j(-)} |j,j\rangle |k,k-1\rangle \right) = 0.$$

Continuing this way by applying lowering operators and searching for orthogonal states leads to the following decomposition of the direct product representation,

$$\left[\mathbf{2j+1}\right] \otimes \left[\mathbf{2k+1}\right] = \left[\mathbf{2(j+k)+1}\right] \oplus \left[\mathbf{2(j+k-1)+1}\right] \oplus ... \oplus \left[\mathbf{2(j-k)+1}\right],$$

where we assumed without loss of generality $j \geq k$.

As an example consider the direct product $\mathbf{2} \otimes \mathbf{2}$ of two spinor representations of the Lie algebra $\mathfrak{su}(2)$. We denote the highest weight state by

$$|\uparrow\uparrow\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle \, |\tfrac{1}{2}, \tfrac{1}{2}\rangle \, ,$$

and generate the state

$$|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \, |\tfrac{1}{2}, \tfrac{1}{2}\rangle + |\tfrac{1}{2}, \tfrac{1}{2}\rangle \, |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \, ,$$

by applying the combined lowering operator $T_- = T_-^{(1)} + T_-^{(2)}$. Doing so again will give the lowest weight state

$$|\downarrow\downarrow\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \, |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \, .$$

These three states form the three-dimensional representation $\mathbf{3}$ of the Lie algebra. The linear combination

$$|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \, |\tfrac{1}{2}, \tfrac{1}{2}\rangle - |\tfrac{1}{2}, \tfrac{1}{2}\rangle \, |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

is a singlet state as it is annihilated by either the combined lowering or raising operator and therefore forms the scalar representation $\mathbf{1}$. In summary we have confirmed $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$.

Similarly consider the direct product $\mathbf{2} \otimes \mathbf{3}$ of the a spinor and adjoint representation of the Lie algebra $\mathfrak{su}(2)$. The highest weight state

$$|\tfrac{3}{2}, \tfrac{3}{2}\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle \, |1,1\rangle \, ,$$

is uniquely determined by its values of T_3. Applying the combined lowering operator generates the other states, e. g.

$$\begin{aligned}|\tfrac{3}{2},\tfrac{1}{2}\rangle &= \frac{1}{\sqrt{3}}\,T^-\,|\tfrac{3}{2},\tfrac{3}{2}\rangle \\ &= \sqrt{\frac{1}{3}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\,|1,1\rangle + \sqrt{\frac{2}{3}}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle\,|1,0\rangle\,,\end{aligned}$$

where the coefficients of the states are called *Clebsch-Gordan coefficients*. Continuing analogous to before we decompose the direct product representation to the sum of the representation **4** and **2**, i.e. $\mathbf{2}\otimes\mathbf{3}=\mathbf{4}\oplus\mathbf{2}$.

5.2 From the Lie algebra representation to the group

Let T_j with $j\in\{1,2,3\}$ denote the hermitian generators of the Lie algebra $\mathfrak{su}(2)$. We now study the exponential map (4.4) in order to go from the Lie algebra to the Lie group,

$$U(\boldsymbol{\theta}) = e^{i\theta^j T_j}, \qquad \boldsymbol{\theta} = \begin{pmatrix}\theta^1\\ \theta^2\\ \theta^3\end{pmatrix} \in \mathbb{R}^3. \tag{5.9}$$

Fundamental spinor representation. The group generated by (5.9) with T_j in the fundamental representation is just SU(2). The group elements read

$$U(\boldsymbol{\theta}) = \exp\left(i\theta^j\frac{\sigma_j}{2}\right) = \cos\left(\frac{\theta}{2}\right)\mathbb{1}_2 + i\sin\left(\frac{\theta}{2}\right)\hat{\theta}^j\sigma_j, \tag{5.10}$$

where $\theta=|\boldsymbol{\theta}|$ is a magnitude and $\hat{\theta}^j=\theta^j/\theta$ denotes a direction. From (5.10) it is easy to see that the group elements transform under $\theta\to\theta+2\pi$ like

$$U \to -U.$$

Because the group elements are unitary, and have unit determinant, their general form is given by

$$U = \begin{pmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix},$$

where $\alpha,\beta\in\mathbb{C}$ and $\det(U)=|\alpha|^2+|\beta|^2=1$. Writing

$$\alpha=\alpha_1+i\alpha_2, \qquad \beta=\beta_1+i\beta_2,$$

such that $\alpha_i,\beta_i\in\mathbb{R}$ we can represent the group elements as points on the surface of the three-sphere S^3,

$$\alpha_1^2+\alpha_2^2+\beta_1^2+\beta_2^2=1.$$

In other words, as a manifold, SU(2) is isomorphic to S^3 and therefore the group manifold of SU(2) is path connected (there is a path from the unit element to every other element) and it is also simply-connected or 1-connected: every closed loop in the manifold can be contracted to a point.

Adjoint or vector representation. If we insert the adjoint representation of T_j in (5.7) into the exponential map (5.9) one obtains the group elements of SO(3). (Recall that the Lie algebras $\mathfrak{so}(3) = \mathfrak{su}(2)$ agree.) Infinitesimal transformation matrices read

$$R_{mn} = \delta_{mn} + i\, d\theta^j (T_j)_{mn} = \delta_{mn} + d\theta^j \epsilon_{jmn}. \tag{5.11}$$

These generate rotations in three-dimensional Euclidean space such that a vector $\boldsymbol{v} \in \mathbb{R}^3$ transforms under the group action as

$$v_m \to v'_m = R(\boldsymbol{\theta})_{mn} v_n.$$

From the infinitesimal transformation (5.11) one can easily check that indeed

$$R(\boldsymbol{\theta})^T R(\boldsymbol{\theta}) = \mathbb{1}_3, \qquad \det(R(\boldsymbol{\theta})) = 1.$$

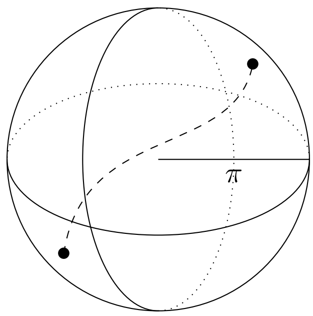
Finite versions of the group elements (5.11) can be written as

$$R_{mn} = \cos(\theta)\delta_{mn} + \sin(\theta)\hat{\theta}^j \epsilon_{jmn} + (1 - \cos(\theta))\hat{\theta}_m \hat{\theta}_n.$$

One can now check that this is *symmetric* under $\theta \to \theta + 2\pi$,

$$R \to R.$$

Limiting the range to $\theta \in (-\pi, \pi)$ we can represent the group elements as points *inside* of the closed ball S^2 where the antipodal points of the surface are to be identified because $\theta = \pi$ and $\theta = -\pi$ represent the same group element.



From this consideration we can infer that the group manifold of SO(3) is connected (there is a path from the unit element to every other element), but it is not simply connected or 1-connected. A loop in the group manifold extending to the boundary that is closed by going through the antipodal point can *not* be contracted to a single point. We see here the even though the Lie algebras of SU(2) and of SO(3) agree, the groups do not. In particular the topological properties of the group manifolds are actually different.

5.3 Three-dimensional harmonic oscillator

The quantum harmonic oscillator is described by the Hamiltonian

$$\begin{aligned} H &= \frac{\mathbf{p}^2}{2M} + \frac{1}{2} M\omega^2 \mathbf{x}^2 \\ &= \hbar\omega \left(a_j^\dagger a_j + \frac{3}{2} \right), \end{aligned}$$

where $\mathbf{x}$, $\mathbf{p}$ are position and momentum operators respectively, M is the particle's mass and ω the angular frequency of the oscillator. The Hamiltonian is then rewritten in terms of the creation and annihilation operators $a_j^\dagger$ and a_j respectively which obey the commutation relations

$$[a_j, a_k^\dagger] = \delta_{jk}, \qquad [a_j^\dagger, a_k^\dagger] = [a_j, a_k] = 0.$$

Then the energy eigenstates are

$$H\,|n_1, n_2, n_3\rangle = \hbar\omega\left(n_1 + n_2 + n_3 + \frac{3}{2}\right)|n_1, n_2, n_3\rangle \ ,$$

for $n_j \in \mathbb{N}_0$ where

$$|n_1, n_2, n_3\rangle = \prod_{j=1}^{3} \frac{(a_j^\dagger)^{n_j}}{\sqrt{n_j!}}\,|0\rangle \, .$$

Define transition operators for the first exited states such that

$$\begin{aligned} P_{12}\,|1,0,0\rangle &= i\,|0,1,0\rangle \, , \\ P_{23}\,|0,1,0\rangle &= i\,|0,0,1\rangle \, , \\ P_{31}\,|0,0,1\rangle &= i\,|1,0,0\rangle \, . \end{aligned}$$

These states have all the same energy and the operators should P_{jk} commute with the Hamiltonian,

$$[H, P_{ij}] = 0. \tag{5.12}$$

One can realize hermitian operators that behave as we want in terms of creation and annihilation operators,

$$P_{jk} = i(a_k^\dagger a_j - a_j^\dagger a_k).$$

In fact, they can be identified with the angular momentum operators

$$\begin{aligned} L_1 &= P_{23} = x_2 p_3 - x_3 p_2, \\ L_2 &= P_{31} = x_3 p_1 - x_1 p_3, \\ L_3 &= P_{12} = x_1 p_2 - x_2 p_1, \end{aligned}$$

which fulfill the Lie algebra of SU(2),

$$[L_j, L_k] = i\epsilon_{jkl} L_l.$$

Equation (5.12) expresses actually angular momentum conservation, $[H, L_j] = 0$. Also one finds

$$[L_j, a_k^\dagger] = i\,\epsilon_{jkl}\,a_l^\dagger.$$

This tells that the operators $a_j^\dagger$ transform like vectors and thus span the **3** representation of the Lie algebra of SU(2). One can now also construct higher dimensional representations by acting several times with the creation operators. Of course, the creation operators commute, so the result will always be symmetric under the exchange of two of them. The N'th excited state transforms as the N-fold symmetric direct product

$$(\underbrace{\mathbf{3}\otimes ... \otimes \mathbf{3}}_{N \text{ times}})_{\text{sym}}.$$

We can now decompose any exited level, e.g. the second excited state,

$$(\mathbf{3}\otimes\mathbf{3})_{\text{sym}} = \mathbf{5}\oplus\mathbf{1},$$

where the **5** or spin-2 representation is a symmetric and traceless tensor (using Einsteins summation convention),

$$\left[\frac{1}{2}a_j^\dagger a_k^\dagger + \frac{1}{2}a_k^\dagger a_j^\dagger - \frac{1}{3}\delta_{jk}a_l^\dagger a_l^\dagger\right]|0\rangle\,,$$

and similarly the singlet **1** is

$$\left[a_j^\dagger a_j^\dagger\right]|0\rangle\,.$$

The tensor decomposition we are using here is part of a more general decomposition of a tensor of rank two into three irreducible representations (of the rotation group), namely a symmetric and trace-less tensor with five independent components, an anti-symmetric tensor with three independent components, and a trace corresponding to one component,

$$T_{ij} = T_{ij}^{\text{sym, trace-less}} + \underbrace{T_{ij}^{\text{anti-sym}}}_{\epsilon_{ijk}t_k} + \delta_{ij}t.$$

This realizes the tensor product decomposition $\mathbf{3}\otimes\mathbf{3} = \mathbf{5}\oplus\mathbf{3}\oplus\mathbf{1}$.

5.4 Bohr atom

As another application we can consider Bohr's Hamiltonian for the relative motion of the electron and the proton in a simple hydrogen atom,

$$H = \frac{\mathbf{p}^2}{2M} - \frac{e^2}{r}.$$

The spectrum of bound states is given by

$$E_n = -\frac{\text{Ry}}{n^2},$$

where n is the principal quantum number, with some degeneracy in the angular momentum quantum number $l = 0, 1, ..., n-1$ and azimuthal quantum number $m = -l, ..., 0, ..., l$. For example, the states for $n = 2$ are in the representations

$$\text{states with } n = 2: \begin{cases} l = 0 : \text{singlet } \mathbf{0} \\ l = 1 : \text{triplet } \mathbf{3} \end{cases}$$

and have the same energy. The fact that these two states have the same energy poses the question whether there is a (hidden) symmetry that connects them. We would need a transition operator from **0** to **3** that commutes with the Hamiltonian. To be in agreement with rotation symmetry, this transition operator must be vector-like. In classical mechanics there is the Laplace-Runge-Lenz vector which precisely meets these requirements,

$$A_i^{\text{classical}} = \epsilon_{ijk} p_j L_k - M e^2 \frac{x_i}{r}.$$

For quantum theory we define it in the hermitian form

$$A_i = \frac{1}{2}\epsilon_{ijk}(p_j L_k + L_k p_j) - M e^2 \frac{x_i}{r},$$

which obeys

$$L_j A_j = A_j L_j = 0, \qquad [L_j, A_k] = i\epsilon_{jkl} A_l,$$

as well as

$$[H, A_j] = 0.$$

This invariance is a hidden symmetry for $1/r$-potentials. We will now use this to find the spectrum algebraically. We start with the commutation relation

$$[A_j, A_k] = i\epsilon_{jkl} L_l (-2MH),$$

and define the rescaled operator

$$\hat{A}_j = \frac{A_j}{\sqrt{-2MH}},$$

to give the simple commutation relation

$$[\hat{A}_j, \hat{A}_k] = i\epsilon_{jkl} L_l,$$

as well as

$$[L_j, \hat{A}_k] = i\epsilon_{jkl} \hat{A}_l.$$

Additionally we know

$$[L_j, L_k] = i\epsilon_{jkl} L_l.$$

We define now the linear combinations

$$X_j^{(+)} = \frac{1}{2}(L_j + \hat{A}_j),$$
$$X_j^{(-)} = \frac{1}{2}(L_j - \hat{A}_j),$$

which commute

$$[X_i^{(+)}, X_j^{(-)}] = 0.$$

Interestingly, we are now left with two independent copies of the Lie algebra $\mathfrak{su}(2)$,

$$[X_j^{(+)}, X_k^{(+)}] = i\epsilon_{jkl} X_l^{(+)},$$
$$[X_j^{(-)}, X_k^{(-)}] = i\epsilon_{jkl} X_l^{(-)}.$$

We also observe that the quadratic Casimir operators are the same

$$C_2^{(+)} = \frac{1}{4}(L_k + \hat{A}_k)(L_k + \hat{A}_k) = \frac{1}{4}(L_k - \hat{A}_k)(L_k - \hat{A}_k) = C_2^{(-)},$$

because of $\hat{A}_k L_k = L_k \hat{A}_k = 0$ and therefore

$$C_2^{(+)} = C_2^{(-)} = j(j+1),$$

where $j_1 = j_2 = j$. To express the Hamiltonian in terms of the Casimir operator we calculate (somewhat lengthy)

$$A_k A_k = (-2MH)\hat{A}_k \hat{A}_k = \underbrace{\left(\mathbf{p}^2 - \frac{2Me^2}{r}\right)}_{-2MH}(L_k L_k + 1) + M^2 e^4,$$

and arrive at

$$H = -\frac{Me^4/2}{L_k L_k + \hat{A}_k \hat{A}_k + 1} = -\frac{Me^4/2}{4C_2^{(+)} + 1} = -\frac{Me^4/2}{(2j+1)^2}.$$

In other words, we get the well-known result for the energy spectrum! The principle quantum number is related to the Casimir by

$$n = 2j + 1.$$

To find the degeneracy of these states we note that the angular momentum operator is given by

$$L_k = X_k^{(+)} + X_k^{(-)}.$$

For given value of j we have therefore states with different values of angular momentum as in the direct product representation $(\mathbf{2j+1}) \times (\mathbf{2j+1})$. The possible values for l are

$$l = 2j = n-1, \qquad l = 2j-1 = n-2, \qquad \ldots, \qquad l = 0,$$

with the usual degeneracy in the quantum number $m = -l, ..., l$. We have solved the Bohr atom without a single differential equation!

5.5 Isospin

Fermi-Yang model. Even though protons (mass $m_p = 938$ MeV) carry electromagnetic charge and neutrons (mass $m_n = 939$ MeV) do not, the small mass difference of the two particles led to the assumption that this is due to symmetry of the strong interaction. Assume that nucleons and antinucleons are fermionic states

$$|p\rangle = b_1^\dagger |0\rangle, \qquad |n\rangle = b_2^\dagger |0\rangle,$$
$$|\bar{p}\rangle = \bar{b}_1^\dagger |0\rangle, \qquad |\bar{n}\rangle = \bar{b}_2^\dagger |0\rangle,$$

where $|0\rangle$ denotes the vacuum state and the creation and annihilation operators satisfy the anticommutation relations

$$\{b_i, b_j^\dagger\} = \{\bar{b}_i, \bar{b}_j^\dagger\} = \delta_{ij},$$

with all other anticommutators vanishing. From these operators one can construct three generators for an $\mathfrak{su}(2)$ algebra,

$$I_j = \frac{1}{2} b^\dagger_\alpha (\sigma_j)_{\alpha\beta} b_\beta - \frac{1}{2} \bar{b}^\dagger_\alpha (\sigma^*_j)_{\alpha\beta} \bar{b}_\beta,$$

as well as one generator for the (rather trivial) Lie algebra of U(1),

$$I_0 = \frac{1}{2}(b^\dagger_\alpha b_\alpha - \bar{b}^\dagger_\alpha \bar{b}_\alpha).$$

The direct product of the two isospin representations **2**

$$\begin{pmatrix} p \\ n \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix}$$

then decomposes according to $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$ such that the **3** representation furnishes three particles,

$$|\pi^+\rangle = b^\dagger_1 \bar{b}^\dagger_2 |0\rangle\,, \qquad |\pi^0\rangle = \frac{1}{2}(b^\dagger_1 \bar{b}^\dagger_1 - b^\dagger_2 \bar{b}^\dagger_2) |0\rangle\,, \qquad |\pi^-\rangle = b^\dagger_2 \bar{b}^\dagger_1 |0\rangle\,.$$

They turn out to have the masses $m_{\pi^\pm} = 139$ MeV and $m_{\pi^0} = 135$ MeV. The pions have here the same quantum numbers as nucleon-anti-nucleon bound states. As for the proton and neutron the mass difference is explained by the symmetry breaking when weak and electromagnetic forces are taken into account. The electro-magnetic charge is given by the combination

$$Q = I_3 + I_0,$$

where I_0 equals 1/2 for nucleons and $-1/2$ for anti-nucleons. In fact, $2I_0$ is known as baryon number, and it is a globally conserved quantum number in QCD. Pions have vanishing baryon number, $I_0 = 0$.

One may also ask here where the singlet **1** went; one may identify it here with the η-meson (with mass 548 MeV).

Wigner supermultiplet model. Extending the isospin symmetry by combining it with spin,

$$\mathrm{SU}(4) \supset \mathrm{SU}(2)_{\text{isospin}} \otimes \mathrm{SU}(2)_{\text{spin}}\,,$$

leads to nucleons N and antinucleons $\bar{N}$ transforming as isospin and spin spinors,

$$|N\rangle \sim \mathbf{4} = (\mathbf{2}_{\text{isospin}}, \mathbf{2}_{\text{spin}}),$$
$$|\bar{N}\rangle \sim \bar{\mathbf{4}} = (\mathbf{2}_{\text{isospin}}, \mathbf{2}_{\text{spin}}).$$

We can now decompose

$$\mathbf{4} \otimes \bar{\mathbf{4}} = \mathbf{15} \oplus \mathbf{1},$$

and the pions belong to the **15** which now includes also other mesons,

$$\mathbf{15} = \underbrace{(\mathbf{3}_{\text{isospin}}, \mathbf{3}_{\text{spin}})}_{\substack{\rho^+,\rho^0,\rho^- \\ \rho\text{-meson (vector)} \\ 1^-,\ 149\ \text{MeV}}} \oplus \underbrace{(\mathbf{1}_{\text{isospin}}, \mathbf{3}_{\text{spin}})}_{\substack{\omega \\ \omega\text{-meson (vector)} \\ 1^-,\ 783\ \text{MeV}}} \oplus \underbrace{(\mathbf{3}_{\text{isospin}}, \mathbf{1}_{\text{spin}})}_{\substack{\pi^+,\pi^0,\pi^- \\ \text{pion (scalar)} \\ 0^-}}\,.$$

The singlet state **1** corresponds again to the scalar η-meson (0^-, 548 MeV).

The modern understanding of mesons is not as nucleon-anti-nucleon bound states but as quark-anti-quark bound states, for example $p\bar{n} \sim uud\bar{d}\bar{d}\bar{u} \ \sim u\bar{d} \sim \pi^+$.

5.6 From SU(2) to the real symplectic group and Bogoliubov transformations

We move now away from the Lie algebra of SU(2) or SO(3) and broaden the view somewhat. Consider the quantum mechanical commutation relation between position x and momentum p,

$$[x, p] = i,$$

In terms of the combined variable

$$q = \begin{pmatrix} x \\ p \end{pmatrix},$$

the commutation relation can be rewritten as

$$[q_j, q_k] = i\,\Omega_{jk},$$

with the antisymmetric matrix

$$\Omega = i\sigma_2 = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}.$$

One may consider canonical transformations

$$q_j \to q'_j = q_m S_{mj}, \tag{5.13}$$

such that

$$[q'_j, q'_k] = S_{mj} S_{nk} [q_m, q_n] = i\, S^T_{jm} \Omega_{mn} S_{nk} = i\,\Omega_{jk}.$$

In other words, the canonical commutation relation should remain unmodified. This leads to the condition

$$S^T \Omega S = \Omega, \tag{5.14}$$

which characterizes the symplectic group $\mathrm{Sp}(2, \mathbb{R})$. (We do not allow complex transformations because we want q_j to remain real, or hermitian as an operator.) As usual we write the group element in exponential form,

$$S = \exp\left(i\theta^j T_j\right).$$

The Lie algebra elements of $\mathrm{Sp}(2, \mathbb{R})$ are subject to $\Omega T = (\Omega T)^T$. We can take

$$T_1 = -i\frac{1}{2}\sigma_1, \qquad T_2 = \frac{1}{2}\sigma_2, \qquad T_3 = -i\frac{1}{2}\sigma_3. \tag{5.15}$$

such that $\Omega T_j = i\sigma_2 T_j$ are symmetric and iT_j are real matrices. Note that the basis (5.15) is obtained from the basis (5.4) for $\mathfrak{su}(2)$ by multiplying two generators with $-i$. This has an important consequence for the quadratic Casimir, which becomes now

$$C_2 = -T_1^2 + T_2^2 - T_3^2, \qquad [C_2, T_j] = 0. \tag{5.16}$$

This is not positive semi-definite any more! This means that for a given value of C_2, the spectrum of the T_j does not need to be bounded and indeed, many representations of the Lie algebra (5.15) are actually infinite.

The group matrices themselves can be written as

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

and from (5.14) we find the condition $s_{11}s_{22} - s_{12}s_{21} = 1$. The matrix entries s_{mn} are real but not bounded. The group manifold of $\mathrm{Sp}(2,\mathbb{R})$ is not compact. This is closely related to the fact that C_2 is not positive semi-definite.

One can use symplectic transformations to rewrite quadratic expression of the form (with $\Delta_{12} = \Delta_{21}$)

$$\frac{1}{2}(x,p)\begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix} = \frac{1}{2}q^T\Delta\, q = \frac{1}{2}q'^T S\Delta S^T q' = \frac{1}{2}q'^T\Delta' q', \tag{5.17}$$

with $q = S^T q'$ and

$$\Delta' = S\Delta S^T.$$

Note that this is not a standard similarity transform because $S^T \neq S^{-1}$. Nevertheless, one can make some general statements about such symplectic transformations and in particular according to *Williamsons theorem*[1] one can here find an S such that $\Delta' = \mathrm{diag}(\lambda,\lambda)$ is diagonal with two equal eigenvalues. In that case, the expression on the right hand side of (5.17) becomes just the Hamiltonian of a harmonic oscillator, $\frac{\lambda}{2}(x'^2 + p'^2)$. In other words, symplectic transformations can be used to simplify Hamiltonians that are quadratic in variables and their conjugate momenta and in order to bring them to a simple standard form.

One can also relate the symplectic transformations (5.13) to transformations between creation and annihilation operators

$$\begin{pmatrix} a \\ a^\dagger \end{pmatrix} \to \begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix}\begin{pmatrix} a \\ a^\dagger \end{pmatrix}, \tag{5.18}$$

where $u, v \in \mathbb{C}$. Then

$$[b, b^\dagger] = [ua + va^\dagger, v^*a + u^*a^\dagger] = (|u|^2 - |v|^2)[a, a^\dagger],$$

and thus for $|u|^2 - |v|^2 = 1$ the transformation is canonical (it preserves the canonical commutation relation) and known as a *Bogoliubov transformation*.

Also, the form of the quadratic Casimir in (5.16) suggests a relation to the indefinite orthogonal group $\mathrm{SO}(2,1)$ of Lorentz transformations in $d = 2+1$ dimensions, which is also not compact. This can indeed be established, as well.

[Exercise: Find the relation between real symplectic transformations and Bogoliubov transformations. Also establish a relation between $\mathrm{Sp}(2,\mathbb{R})$ and indefinite orthogonal transformations $\mathrm{SO}(2,1)$.]

[1] This theorem states that a real symmetric positive definite matrix of even dimension can be brought to diagonal form by a symplectic congruence transformation. The diagonal entries come in pairs and are called symplectic eigenvalues.

6 Matrix Lie groups and tensors

Now that we understood the Lie algebra of the simplest unitary and orthogonal groups up to SU(2) and of SO(3), and before moving on to SU(3), let us pause for a moment and explore what can be said about general matrix Lie groups GL$(N, \mathbb{C})$ and specifically about the groups SO(N) and SU(N) and their tensor representations on general grounds.

6.1 General linear group GL$(N, \mathbb{C})$

Quite generally, a matrix Lie group has a fundamental representation of the form

$$v^j \to M^j{}_k v^k.$$

Here M could be an element of the orthogonal group O(N) for example, but also any other matrix Lie group. In this subsection we make only the minimal assumption $M \in \mathrm{GL}(N, \mathbb{C})$.

One can then immediately build higher rank tensor representations that transform like

$$v^{jn\cdots p} \to M^j{}_k M^n{}_m \cdots M^p{}_q \, v^{km\cdots q}.$$

It is also convenient to introduce representations with lower index, which should transform like

$$w_j \to (M^{-1})^k{}_j w_k. \tag{6.1}$$

Now one can contract upper and lower indices to yield an invariant term, for example,

$$w_j v^j \to (M^{-1})^m{}_j M^j{}_k w_m v^k = \delta^m{}_k w_m v^k = w_k v^k.$$

A general tensor can now have upper and lower indices and transforms accordingly, for example,

$$v^j{}_n \to M^j{}_k \, (M^{-1})^m{}_n v^k{}_m.$$

In other words, upper indices transform with M, lower indices transform with the inverse M^{-1}. The unit tensor with one upper and one lower transforms like

$$\delta^j{}_n \to M^j{}_k \, (M^{-1})^m{}_n \delta^k{}_m = M^j{}_m \, (M^{-1})^m{}_n = \delta^j{}_n.$$

One may say that $\delta^j{}_n$ is an *invariant symbol.* In contrast, δ^{mn} or δ_{mn} are in general *not* invariant.

To find another invariant symbol consider also the N-dimensional Levi-Civita symbol $\epsilon^{ijk\cdots n}$. It is totally antisymmetric, which means anti-symmetric in any pair of indices,

$$\epsilon^{\cdots k \cdots m \cdots} = -\epsilon^{\cdots m \cdots k \cdots}, \tag{6.2}$$

and normalized such that

$$\epsilon^{123\cdots N} = 1. \tag{6.3}$$

Under the matrix group it transforms to

$$\epsilon^{ijk\cdots n} \to M^i{}_p M^j{}_q \cdots M^n{}_s \epsilon^{pq\cdots s} = \det(M) \epsilon^{ij\cdots n}.$$

This shows that for the special linear group $M \in \mathrm{SL}(N, \mathbb{C})$ where $\det(M) = 1$ the Levi-Civita symbol $\epsilon^{ijk\ldots n}$ is also an *invariant symbol.* Similarly, an analogously defined Levi-Civita symbol with lower indices $\epsilon_{ijk\cdots n}$ is also invariant for $\det(M) = 1$.

6.2 Orthogonal group SO(N)

The group of rotations in N-dimensional Euclidean space is $SO(N)$ where the group elements are $N \times N$ matrices R (in the fundamental representation) satisfying

$$R^T R = R R^T = \mathbb{1}, \qquad \text{or} \qquad R^j{}_m R^k{}_n \delta^{mn} = \delta^{jk}, \tag{6.4}$$

and

$$\det(R) = 1. \tag{6.5}$$

This is clearly a subgroup of SL$(2, \mathbb{R})$. A vector in the sense of rotations is defined by its transformation behavior under rotations, namely

$$v^j \to R^j{}_k \, v^k.$$

This constitutes obviously a representation of the group SO(N). Analogously we define tensors be to objects transforming like

$$v^{ij} \to R^i{}_m R^j{}_n v^{mn}, \tag{6.6}$$

and so on for higher rank tensors. To be complete, a scalar transforms in the trivial way, $s \to s$.

Interestingly, SO(N) has more invariant symbols then GL(N) or SL(N). Specifically, the defining property in eq. (6.4) tells that

$$\delta^{jk} \to R^j{}_m R^k{}_n \delta^{mn} = \delta^{jk},$$

and similarly δ_{jk}, are also invariant symbols. These two can now be used to raise and lower indices so that one does not really need to distinguish upper and lower indices for SO(N). For the rest of the discussion in this subsection we will only use lower indices for this reason.

One may ask whether the tensor representation (6.6) can de decomposed into different parts that do not mix under rotations, i. e. whether the representation is *reducible*. This is indeed the case.

The anti-symmetric combination

$$A_{ij} = \frac{1}{2}(M_{ij} - M_{ji}) = -A_{ji},$$

again is a tensor and it also does not mix with other parts under rotations. There are $N(N-1)/2$ independent components for anti-symmetric second-rank tensors. Analogously we can argue for a symmetric second-rank tensor

$$S_{ij} = \frac{1}{2}(M_{ij} + M_{ji}),$$

which has $N(N+1)/2$ independent components. The trace is actually a scalar because

$$S_{jj} \to R_{jk} R_{jl} S_{kl} = (R^T)_{kj} R_{jl} S_{kl} = \delta_{kl} S_{kl} = S_{kk}.$$

Constructing the symmetric and trace-less second-rank tensor

$$\tilde{S}_{ij} = S_{ij} - \delta_{ij}\frac{S_{kk}}{N},$$

we are left with $N(N+1)/2 - 1$ components. Therefore we have decomposed the tensor representation into three irreducible representations,

$$\mathbf{N} \otimes \mathbf{N} = \left[\frac{\mathbf{N(N-1)}}{\mathbf{2}}\right] \oplus \left[\frac{\mathbf{N(N+1)}}{\mathbf{2}} - \mathbf{1}\right] \oplus \mathbf{1}.$$

For example for SO(3) this gives $\mathbf{3} \otimes \mathbf{3} = \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1}$ as we have seen before. For a general decomposition of higher rank tensors $M_{ijk\ldots m}$ the technique of *Young tableaux* was invented to keep track of the emerging patterns. We will not discuss this further here, however.

Dual tensors. Let A_{ij} be an anti-symmetric second-rank tensor. We define its dual to be the totally anti-symmetric $(N-2)$-rank tensor

$$B_{k\ldots n} = \epsilon_{ijk\ldots n} A_{ij}.$$

For example, for $N = 3$ we get in this way a vector

$$B_k = \epsilon_{ijk} A_{ij},$$

and for $N = 2$ a scalar

$$B = \epsilon_{ij} A_{ij}.$$

In the case of $N = 4$ we again get a relation between two anti-symmetric second-rank tensors as it is used in electromagnetism, namely the fields strength tensor and its dual in 4-dimensional Euclidean space (for the treatment in Minkowski space we are dealing with $SO(3,1)$ not SO(4) anymore),

$$B_{kl} = \epsilon_{ijkl} A_{ij}.$$

Self-dual / Anti-self-dual tensors. For an even number of dimensions, i. e. for $SO(2n)$ we can construct two irreducible representations with an additional feature. Consider an totally anti-symmetric n-rank tensor $A_{i_1 i_2 \ldots i_n}$ and its dual,

$$B_{i_1 i_2 \ldots i_n} = \frac{1}{n!}\epsilon_{i_1 i_2 \ldots i_n j_1 j_2 \ldots j_n} A_{j_1 j_2 \ldots j_n},$$

such that

$$A_{i_1 i_2 \ldots i_n} = \frac{1}{n!}\epsilon_{i_1 i_2 \ldots i_n j_1 j_2 \ldots j_n} B_{j_1 j_2 \ldots j_n}.$$

In other words, the two tensors A and B are dual to each other. We can form the linear combinations

$$M^{\pm}_{i_1 \ldots i_n} = \frac{1}{2}(A_{i_1 \ldots i_n} \pm B_{i_1 \ldots i_n}),$$

and they are self-dual and anti-self-dual. This can be seen by writing schematically

$$\epsilon M^{\pm} = \frac{1}{2}(\epsilon A \pm \epsilon B) = \pm\frac{1}{2}(A \pm B) = \pm M^{\pm}.$$

The self-dual and anti-self-dual tensors are also each irreducible representations of the rotation group $SO(N)$.

6.3 Unitary group SU(N)

Now that we have discussed SO(N), let us move to the unitary group SU(N). For SU(N) we write the fundamental representation as

$$\psi^j \to U^j{}_k \psi^k. \tag{6.7}$$

The complex conjugate spinor transforms then formally like

$$\psi^{*j} \to (U^j{}_k)^* \psi^{*k}.$$

However, we also have the transformation law for a lower index spinor,

$$\xi_j \to (U^{-1})^k{}_j \xi_k = (U^\dagger)^k{}_j \xi_k. \tag{6.8}$$

The matrix elements of $(U^j{}_k)^*$ and $(U^\dagger)^k{}_j$ are actually identical so that we can simply and consistently write complex conjugate spinors with a lower index,

$$(\psi^j)^* = \psi^*_j.$$

This has the nice feature that the quadratic form $\xi^\dagger \psi = \xi^*_j \psi^j$ transforms like

$$\xi^*_j \psi^j \to \xi^*_j (U^\dagger)^k{}_j U^j{}_m \psi^m = \xi^*_j \delta^j{}_m \psi^m = \xi^*_j \psi^j,$$

so it is invariant as expected. One can now also consider tensors with several upper and lower indices and transform them consistently, for example

$$\phi^{ij}{}_k \to U^i{}_l U^j{}_m (U^\dagger)^n{}_k \, \phi^{lm}{}_n.$$

It is important to note that δ^{jk} and δ_{mn} are *not* invariant symbols any more. [Exercise: Check this!] They can accordingly also not be used to raise or lower indices. This means that one must indeed be careful to distinguish upper and lower indices for SU(N).

We can sometimes use the fully anti-symmetric Levi-Civita symbol $\epsilon^{ij\cdots n}$ introduced in eqs. (6.2) and (6.3) and its counterpart $\epsilon_{ij\cdots n}$ to raise and lower indices. This works because the elements of SU(N) have unit determinant. For example for SU(4) we can lower indices of a tensor $\varphi_k{}^{ij}$ that is anti-symmetric in i and j and define a new tensor with only lower indices as $\phi_{kpq} = \epsilon_{ijpq} \, \varphi_k{}^{ij}$.

In case of SU(2) there is a special situation because the invariant symbols ϵ_{ij} and ϵ^{ij} allow to pull all indices up or all indices down, e. g.

$$\psi_m = \epsilon_{mj} \psi^j.$$

This also implies that there is no real difference between the fundamental representation and its complex conjugate. This can also be understood directly from the exponential map

$$U = \exp\left(i\theta^j \frac{\sigma_j}{2}\right).$$

The complex conjugate of the Pauli matrices can be written as (recall eq. (5.5))

$$\sigma^*_j = -S^{-1} \sigma_j S,$$

where S is given by

$$S = -i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that the matrix entries of S agree with the Levi-Civita symbol, $S^{jk} = \epsilon^{jk}$. The group element U and its complex conjugate U^* are related through

$$U^* = \exp\left(-i\theta^j \frac{\sigma_j^*}{2}\right) = S^{-1} \exp\left(i\theta^j \frac{\sigma_j}{2}\right) S = S^{-1} U S,$$

which is just a similarity transform. Such a simple relation between a representation and its complex conjugate is not available for $N > 2$, though. In general, for a given representation of SU(N) with multiplication law

$$U_1 U_2 = U_3,$$

we can obtain another representation by taking complex conjugates.

$$U_1^* U_2^* = U_3^*,$$

While the fundamental representation itself is for a spinor ψ^j as in (6.7), the complex conjugate representation is for a lower index spinor ξ_j as in (6.8).

7 SU(3)

7.1 Lie algebra

We now consider the Lie algebra of SU(3). The vector space of hermitian and trace-less 3×3 matrices is spanned by the Gell-Mann matrices. Their role is similar to the role played by the Pauli matrices for SU(2). The Gell-Mann matrices can be taken as

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

They are normalised to $\mathrm{Tr}\{\lambda_a \lambda_b\} = 2\delta_{ab}$. Observe that the first three Gell-Mann matrices contain the Pauli matrices, $(\lambda_1, \lambda_2, \lambda_3) \sim (\sigma_1, \sigma_2, \sigma_3)$, acting on a subspace. They can be used to construct an $\mathfrak{su}(2)$ subalgebra.

We can now investigate the other commutators and because λ_3 and λ_8 are diagonal we immediately find

$$[\lambda_3, \lambda_8] = 0.$$

Moreover we introduce a new matrix $\lambda_{[4,5]}$ by

$$2i\lambda_{[4,5]} = [\lambda_4, \lambda_5] = 2i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i(\lambda_3 + \sqrt{3}\lambda_8),$$

and therefore $\lambda_4, \lambda_5, \lambda_{[4,5]}$ work similarly as the Pauli matrices but now in the 1-3-sector. They can also be used to span an $\mathfrak{su}(2)$ algebra. Something similar can also be done in the 2-3-sector with λ_6, λ_7 and $\lambda_{[6,7]}$ defined through

$$2i\lambda_{[6,7]} = [\lambda_6, \lambda_7] = 2i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i(-\lambda_3 + \sqrt{3}\lambda_8).$$

In total we find three overlaping copies of the $\mathfrak{su}(2)$ algebra.

The Lie algebra $\mathfrak{su}(3)$ is generated in the fundamental representation by

$$T_j = \frac{\lambda_j}{2}.$$

In the light of the above discussion we define six new operators

$$I_\pm = T_1 \pm iT_2, \qquad V_\pm = T_4 \pm iT_5, \qquad U_\pm = T_6 \pm iT_7. \tag{7.1}$$

Note that $I_\pm^\dagger = I_\mp$ and similar for $V_\pm$ and $U_\pm$. The operators in eq. (7.1) form a basis of the Lie algebra together with the commuting diagonal generators T_3 and T_8, $[T_3, T_8] = 0$. One finds the commutation relations

$$\begin{aligned}
&[T_3, I_\pm] = \pm I_\pm, && [T_3, U_\pm] = \mp\frac{1}{2}U_\pm, && [T_3, V_\pm] = \pm\frac{1}{2}V_\pm,\\
&[T_8, I_\pm] = 0, && [T_8, U_\pm] = \pm\frac{\sqrt{3}}{2}U_\pm, && [T_8, V_\pm] = \pm\frac{\sqrt{3}}{2}V_\pm,\\
&[I_+, I_-] = 2T_3, && [U_+, U_-] = \sqrt{3}T_8 - T_3, && [V_+, V_-] = \sqrt{3}T_8 + T_3,\\
&[I_+, V_-] = -U_-, && [I_+, U_+] = V_+, && [U_+, V_-] = I_-,\\
&[I_+, V_+] = 0, && [I_+, U_-] = 0, && [U_+, V_+] = 0.
\end{aligned}$$

For $\mathfrak{su}(3)$ we have two Casimir operators, one quadratic,

$$C_2 = \sum_{j=1}^{8} T_j^2,$$

and one cubic

$$C_3 = \sum_{j,k,l=1}^{8} f_{jkl} T_j T_k T_l.$$

Here f_{jkl} are the structure constants, defined through the commutation relation

$$[T_j, T_k] = i f_{jkl}\, T_l. \tag{7.2}$$

Because there are now two Casimir operators there are also two labels. We can label the states in every irreducible representation by eigenstates of T_3 and T_8,

$$|i_3, i_8\rangle\,,$$

such that

$$T_3\,|i_3, i_8\rangle = i_3\,|i_3, i_8\rangle\,, \qquad T_8\,|i_3, i_8\rangle = i_8\,|i_3, i_8\rangle\,.$$

One can then verify that

$$\begin{aligned}
T_3 I_\pm\,|i_3, i_8\rangle &= (I_\pm T_3 \pm I_\pm)\,|i_3, i_8\rangle = (i_3 \pm 1) I_\pm\,|i_3, i_8\rangle\,,\\
T_8 I_\pm\,|i_3, i_8\rangle &= I_\pm T_8\,|i_3, i_8\rangle = i_8 I_\pm\,|i_3, i_8\rangle\,,
\end{aligned}$$

and thus

$$I_\pm\,|i_3, i_8\rangle \propto |i_3 \pm 1, i_8\rangle\,.$$

Analogously

$$\begin{aligned}
T_3 U_\pm\,|i_3, i_8\rangle &= \left(i_3 \mp \tfrac{1}{2}\right) U_\pm\,|i_3, i_8\rangle\,,\\
T_8 U_\pm\,|i_3, i_8\rangle &= \left(i_8 \pm \tfrac{\sqrt{3}}{2}\right) U_\pm\,|i_3, i_8\rangle\,,
\end{aligned}$$

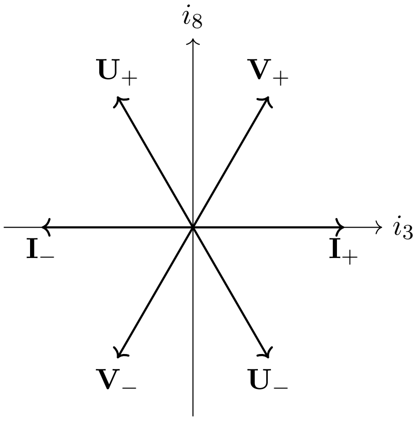
giving

$$U_\pm\,|i_3, i_8\rangle \propto |i_3 \mp \tfrac{1}{2}, i_8 \pm \tfrac{\sqrt{3}}{2}\rangle\,,$$

and finally in the same manner

$$V_\pm\,|i_3, i_8\rangle \propto |i_3 \pm \tfrac{1}{2}, i_8 \pm \tfrac{\sqrt{3}}{2}\rangle\,.$$

Root vectors. We can summarize these relations diagrammatically. On a lattice spanned by i_3 and i_8 the operators $I_\pm$, $V_\pm$ and $U_\pm$ can be represented as follows.



The vectors appearing here are called *root vectors*,

$$\mathbf{I}_\pm = (\pm 1, 0)\,, \qquad \mathbf{U}_\pm = \left(\mp\tfrac{1}{2}, \pm\tfrac{\sqrt{3}}{2}\right), \qquad \mathbf{V}_\pm = \left(\pm\tfrac{1}{2}, \pm\tfrac{\sqrt{3}}{2}\right).$$

The root vectors associated to T_3 and T_8 are $(0,0)$ because these operators do not change the labels. We can make a few observations.

- Root vectors have unit lengths or vanish.
- The angles between them are $\frac{\pi}{3}$, $\frac{2\pi}{3}$ and so on.
- In some cases the sum of two root vectors gives another root vector, in some not.
- The sum of a root vector with itself is not a root vector.

Positive roots and simple roots. We define *positive roots* to be characterised by $i_3 > 0$. Therefore V_+, I_+ and U_- are positive while V_-, I_-, and U_+ are negative. Moreover, among the positive roots there is a linear relation

$$\mathbf{I}_+ = \mathbf{V}_+ + \mathbf{U}_-,$$

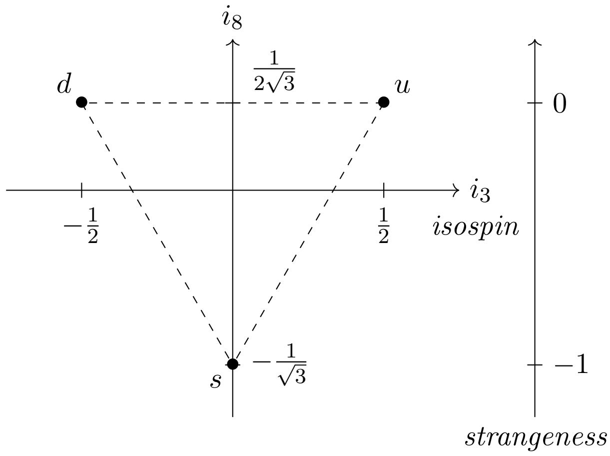
and $\mathbf{V}_+$ as well as $\mathbf{U}_-$ are called the *simple roots.*

7.2 Representations

Weight diagrams and fundamental representation. Let $\mathcal{R}$ be some (irreducible or reducible) representation of $\mathfrak{su}(3)$ and plot the states $|i_3, i_8\rangle$ in the i_3, i_8 plane. For the *fundamental representation* **3** we identify the states with *quark states*,

$$\begin{aligned} |u\rangle &= |\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}\rangle\,, \\ |d\rangle &= |-\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}\rangle\,, \\ |s\rangle &= |0, -\tfrac{1}{\sqrt{3}}\rangle\,. \end{aligned}$$

The weight diagram is as follows.



Using the root vectors we see that

$$\begin{aligned} V_- \left|u\right\rangle &\propto \left|s\right\rangle, & V_+ \left|s\right\rangle &\propto \left|u\right\rangle, \\ I_- \left|u\right\rangle &\propto \left|d\right\rangle, & I_+ \left|d\right\rangle &\propto \left|u\right\rangle, \\ U_+ \left|s\right\rangle &\propto \left|d\right\rangle, & U_- \left|d\right\rangle &\propto \left|s\right\rangle, \end{aligned}$$

as well as (for example)

$$V_+ \left|u\right\rangle = I_+ \left|u\right\rangle = U_+ \left|u\right\rangle = U_- \left|u\right\rangle = 0.$$

Charge or complex conjugation. Let us investigate how the generators behave under charge or complex conjugation,

$$T_j^C = -T_j^* = -T_j^T.$$

Here we used that the generators are hermitian. From the commutation relation (7.2) we find also

$$[T_j^C, T_k^C] = i f_{jkl}\, T_l^C.$$

This is something we have observed before: for every representation of SU(3) there is also a complex conjugate representation. Concretely we find from the Gell-Mann matrices

$$T_j^C = T_j,$$

for $j = 2, 5, 7$ and

$$T_j^C = -T_j^C$$

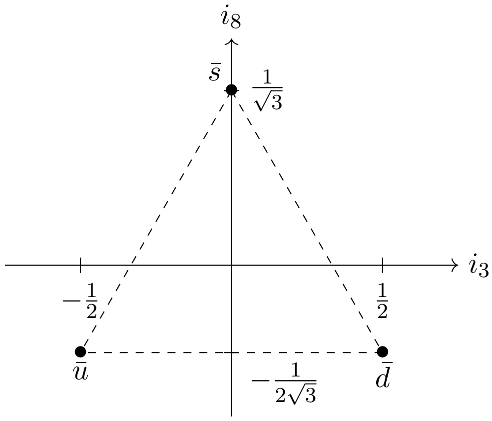
for $j = 1, 3, 4, 6, 8$. Thus we conclude for the charge conjugate

$$\mathsf{C}\left|i_3, i_8\right\rangle = \left|-i_3, -i_8\right\rangle.$$

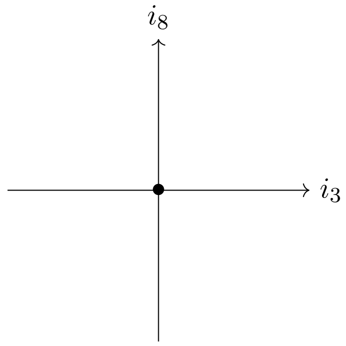
Charge conjugate representation. We are now ready to look at the *charge conjugate representation* $\mathbf{3}^*$ where we identify the states as the antiparticles of the $\mathbf{3}$ states, namely

$$|\bar{u}\rangle = |-\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}\rangle\,,$$
$$|\bar{d}\rangle = |\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}\rangle\,,$$
$$|\bar{s}\rangle = |0, \tfrac{1}{\sqrt{3}}\rangle\,.$$

The weight diagram is now



Singlet representation. For the singlet representation $\mathbf{1}$ the weight diagram is almost trivial with only one state $(i_3, i_8) = (0, 0)$.



Adjoint representation. In the adjoint representation we have

$$\mathrm{adj}(T_j)X = [T_j, X],$$

with $X = X^j T_j$ being a linear combination of generators itself. This satisfies the algebra because

$$[\mathrm{adj}(T_j), \mathrm{adj}(T_k)]X = \mathrm{adj}([T_j, T_k])X$$

is equivalent to

$$[T_j, [T_k, X]] - [T_k, [T_j, X]] = [[T_j, T_k], X]$$

or

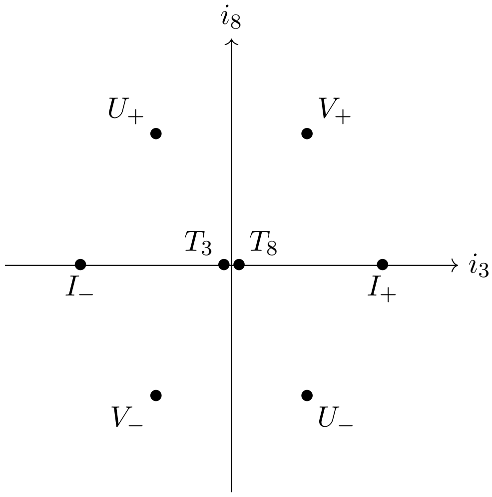
$$[T_j, [T_k, T_l]] + [T_k, [T_l, T_j]] + [T_l, [T_j, T_k]] = 0.$$

This is simply the Jacobi identity.

The weight table for the adjoint representation **8** can be computed using the commutation relations.

state	$[T_3, V]$	$[T_8, V]$	weight (i_3, i_8)
T_3	0	0	$(0, 0)$
T_8	0	0	$(0, 0)$
I_+	$+1$	0	$(+1, 0)$
I_-	-1	0	$(-1, 0)$
U_+	$-\frac{1}{2}$	$+\frac{\sqrt{3}}{2}$	$(-\frac{1}{2}, +\frac{\sqrt{3}}{2})$
U_-	$+\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$(+\frac{1}{2}, -\frac{\sqrt{3}}{2})$
V_+	$+\frac{1}{2}$	$+\frac{\sqrt{3}}{2}$	$(+\frac{1}{2}, +\frac{\sqrt{3}}{2})$
V_-	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

The weight diagram is now as follows.



Tensor product representations. Analogous to what we have done for SU(2) we can consider tensor product representations out of representations $T_k^{(1)}$ and $T_k^{(2)}$ with states $|i_3, i_8\rangle_{(1)}$ and $|j_3, j_8\rangle_{(2)}$ respectively. (We drop the indices (1) and (2) when no confusion can arise.) Define

$$T_k = T_k^{(1)} + T_k^{(2)},$$

and we have for example

$$T_3 |i_3, i_8\rangle |j_3, j_8\rangle = \left(T_3^{(1)} + T_3^{(2)}\right) |i_3, i_8\rangle |j_3, j_8\rangle = (i_3 + j_3) |i_3, i_8\rangle |j_3, j_8\rangle .$$

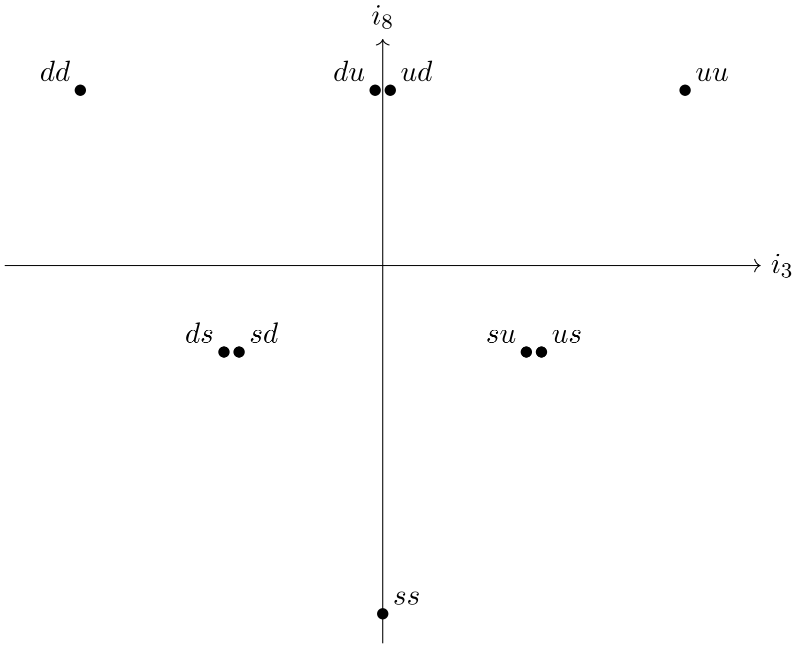
A difference to SU(2) is that we now have two labels for SU(3). The Lie algebra $\mathfrak{su}(3)$ is *rank two.* Let us denote

$$E_+^m = \{I_+, U_+, V_+\},$$
$$E_-^m = \{I_-, U_-, V_-\},$$

where $m = 1, 2, 3$ and study $\mathbf{3} \otimes \mathbf{3}$. The weight table is as follows.

state	weight (i_3, i_8)
$u \otimes u$	$(1, \frac{1}{\sqrt{3}})$
$d \otimes d$	$(-1, \frac{1}{\sqrt{3}})$
$s \otimes s$	$(0, -\frac{2}{\sqrt{3}})$
$u \otimes d, d \otimes u$	$(0, \frac{1}{\sqrt{3}})$
$u \otimes s, s \otimes u$	$(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$
$d \otimes s, s \otimes d$	$(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$

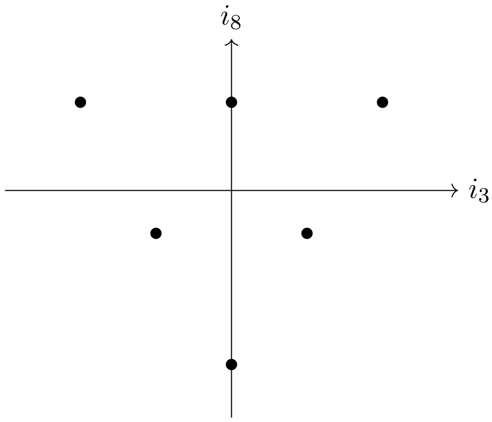
We can represent these in the following weight diagram.



For the highest weight state we need $E_+^m |s\rangle = 0$ and therefore it is $u \otimes u$. Applying $E_-^m |u \otimes u\rangle$ generates the representation **6** with the following weight table.

state	weight (i_3, i_8)
$u \otimes u$	$(1, \frac{1}{\sqrt{3}})$
$d \otimes d$	$(-1, \frac{1}{\sqrt{3}})$
$s \otimes s$	$(0, -\frac{2}{\sqrt{3}})$
$\frac{1}{\sqrt{2}}(u \otimes d + d \otimes u)$	$(0, \frac{1}{\sqrt{3}})$
$\frac{1}{\sqrt{2}}(u \otimes s + s \otimes u)$	$(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$
$\frac{1}{\sqrt{2}}(d \otimes s + s \otimes d)$	$(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$

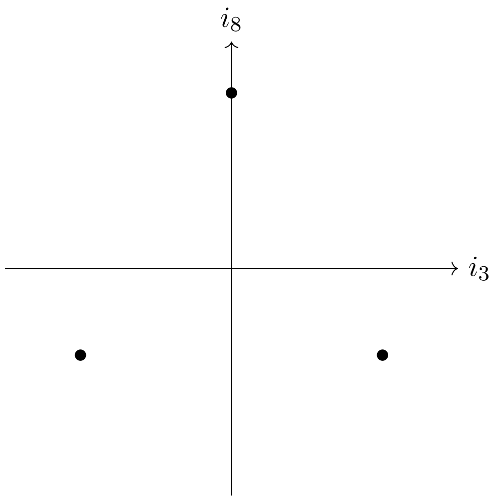
Accordingly this representation **6** has the following weight diagram.



The remainder of the decomposition is $\mathbf{3}^*$ with the weight table

state	weight (i_3, i_8)
$\frac{1}{\sqrt{2}}(d \otimes u - u \otimes d)$	$(0, \frac{1}{\sqrt{3}})$
$\frac{1}{\sqrt{2}}(d \otimes s - s \otimes d)$	$(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$
$\frac{1}{\sqrt{2}}(s \otimes u - u \otimes s)$	$(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$

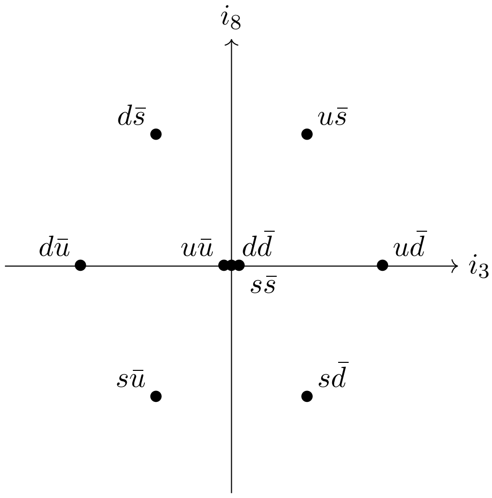
and weight diagram



We have thus decomposed $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \mathbf{3}^*$ where $\mathbf{3}^*$ denotes the charge conjugate. Similar one can derive the weight tables and diagrams for $\mathbf{3}^* \otimes \mathbf{3}^* = \mathbf{6}^* \oplus \mathbf{3}$. Now for $\mathbf{3} \otimes \mathbf{3}^*$. The weight table is

state	weight (i_3, i_8)
$u \otimes \bar{u}, d \otimes \bar{d}, s \otimes \bar{s}$	$(0, 0)$
$u \otimes \bar{s}$	$(\frac{1}{2}, \frac{\sqrt{3}}{2})$
$u \otimes \bar{d}$	$(1, 0)$
$d \otimes \bar{s}$	$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$
$d \otimes \bar{u}$	$(-1, 0)$
$s \otimes \bar{u}$	$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$
$s \otimes \bar{d}$	$(\frac{1}{2}, -\frac{\sqrt{3}}{2})$

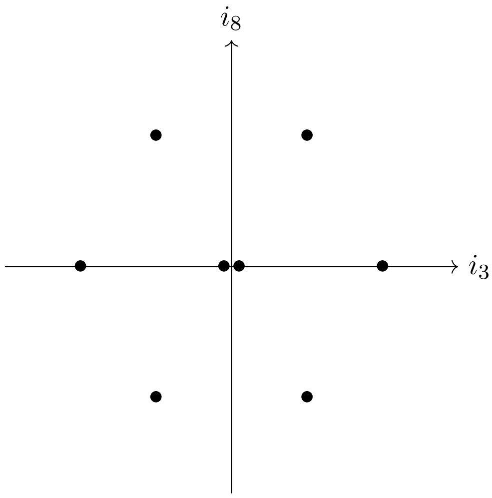
and the associated weight diagram



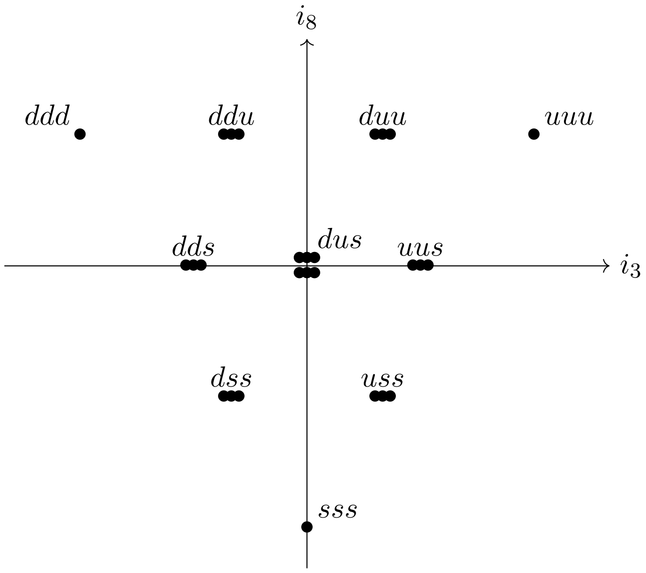
Applying lowering operators to the highest weight state $u \otimes \bar{s}$ give the states

$$u\otimes\bar{s}\,, \quad u\otimes\bar{d}\,, \quad d\otimes\bar{s}\,, \quad d\otimes\bar{u}\,, \quad s\otimes\bar{u}\,, \quad s\otimes\bar{d}\,, \quad \frac{1}{\sqrt{2}}(d\otimes\bar{d}-u\otimes\bar{u})\,, \quad \frac{1}{2}(d\otimes\bar{d}+u\otimes\bar{u}-2s\otimes\bar{s})$$

which furnish the **8** representation with the following weight diagram



There is a singlet representation left with the state $\frac{1}{\sqrt{3}}(u \otimes \bar{u} + d \otimes \bar{d} + s \otimes \bar{s})$. It is clear that this must be a singlet because it is annihilated by all raising or lowering operators $E^m_\pm$. Thus we have decomposed $\mathbf{3} \otimes \mathbf{3}^* = \mathbf{8} \oplus \mathbf{1}$. Finally considering $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ the weight table can be derived analogously and the weight diagram is



In the exact same way as before applying lowering operators to the highest weight state (uuu) we can decompose $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$. The first irreducible representation are the symmetric combinations

$$u \otimes u \otimes u \ , \quad d \otimes d \otimes d \ , \quad s \otimes s \otimes s \ , \quad \frac{1}{\sqrt{3}}(u \otimes u \otimes s + u \otimes s \otimes u + s \otimes u \otimes u) \ , \quad \ldots$$

forming $\mathbf{10}$ whereas the singlet $\mathbf{1}$ is the totally anti-symmetric combination

$$\frac{1}{\sqrt{6}}(s \otimes d \otimes u - s \otimes u \otimes d + d \otimes u \otimes s - d \otimes s \otimes u + u \otimes s \otimes d - u \otimes d \otimes s).$$

The octet states have mixed symmetry.

7.3 Meson and baryon states

One can now assign particle states to the derived decomposed representations, namely mesons to the decomposition

$$\mathbf{3} \otimes \mathbf{3}^* = \mathbf{8} \oplus \mathbf{1},$$

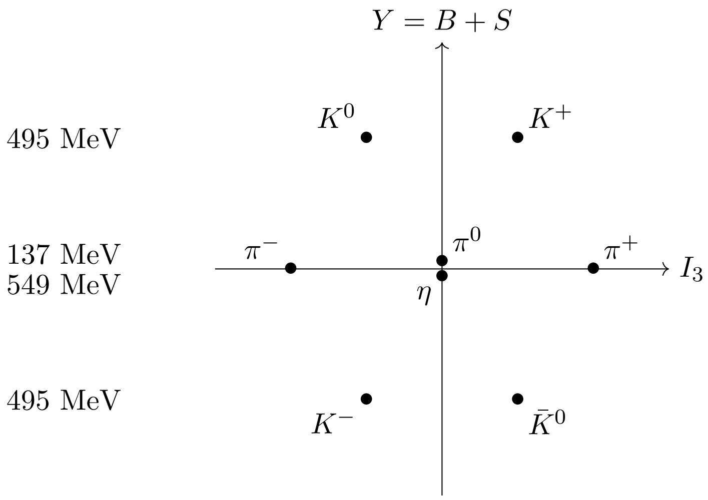
and baryons to

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}.$$

We use the following quantum numbers

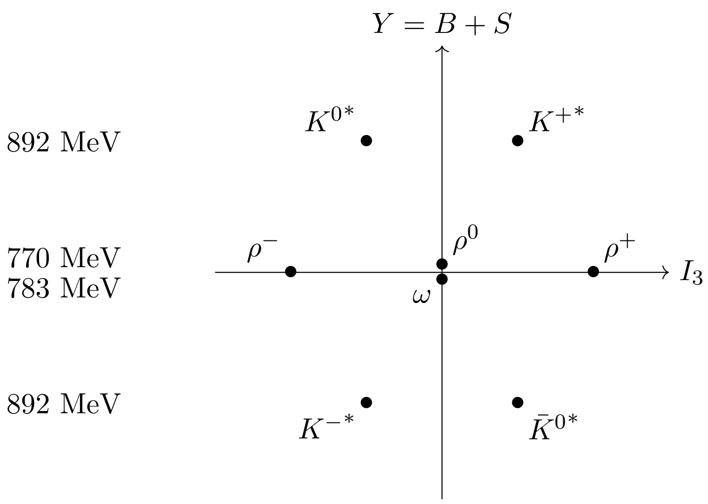
- B: baryon number
- S: strangeness (number of $\bar{s}$ quarks $-$ number of s quarks)
- J: spin
- I_3: isospin (actually the third component of it)
- $Y = B + S$: hypercharge.

We then identify the pseudoscalar mesons ($B = 0, J = 0$) with the weight diagram **8**



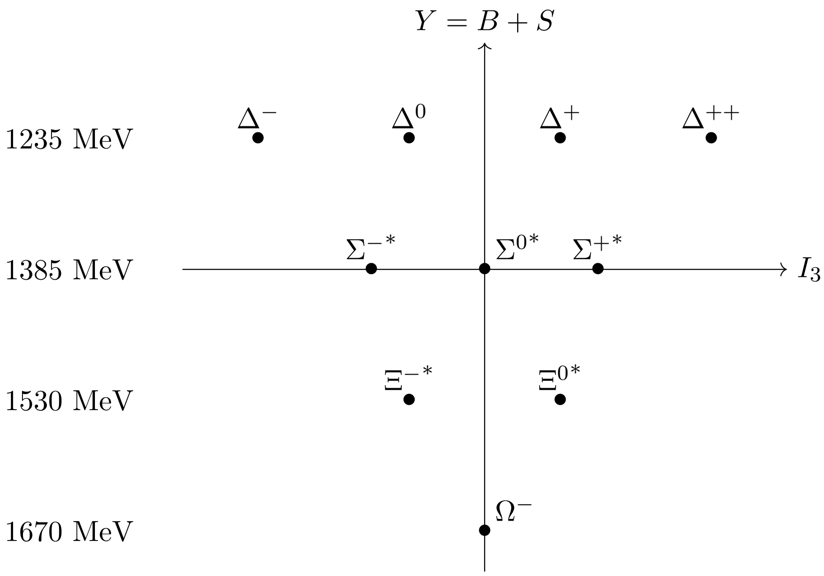
and the additional singlet with the η'-particle. Note that the pion triplet and the η meson appear here again, now as part of a larger structure. One observes that SU(3) symmetry is actually broken to some extend, otherwise all the states in **8** would have the same masses. This breaking is mainly due to the somewhat larger mass of the strange quark compared to the lighter up and down quarks.

Analogous for the vector mesons ($B = 0, J = 1$) with weight diagram

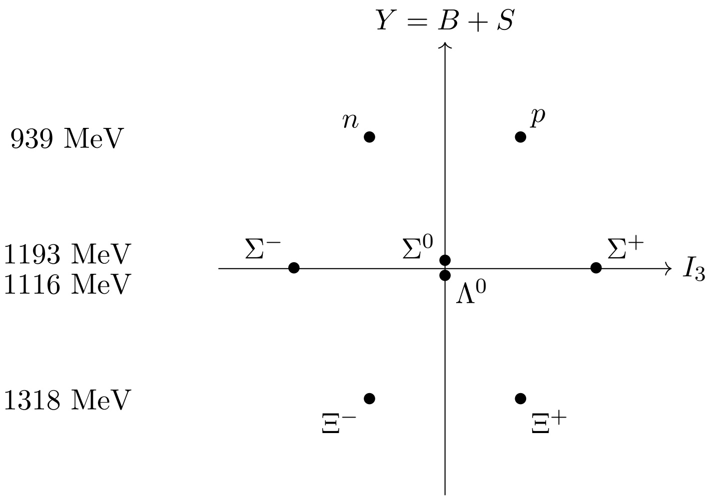


and the additional singlet is the ϕ-particle.

The baryons ($B = 1, J = \frac{3}{2}$) are in the **10** representation with the weight diagram



In contrast the baryons ($B = 1, J = \frac{1}{2}$) form an **8** with weight diagram



Interestingly, the neutron and proton now appear as part of a larger structure which includes other, somewhat heavier baryons.

Now the question might arise why there are no $\mathbf{3} \otimes \mathbf{3}$ or $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}^*$ states. The reason is related to another SU(3) symmetry of quarks that we did not discuss so far: *color*. One assumes that all observable / asymptotic states are singlets with respect to an additional $\mathrm{SU}(3)_C$ colour symmetry. Because we have

$$\mathbf{3}_C \otimes \mathbf{3}^*{}_C = \mathbf{8}_C \oplus \mathbf{1}_C,$$

as well as

$$\mathbf{3}_C \otimes \mathbf{3}_C \otimes \mathbf{3}_C = \mathbf{10}_C \oplus \mathbf{8}_C \oplus \mathbf{8}_C \oplus \mathbf{1}_C,$$

there are singlets under $\mathrm{SU}(3)_C$ in both cases. In contrast,

$$\mathbf{3}_C \otimes \mathbf{3}_C = \mathbf{6}_C \oplus \mathbf{3}^*{}_C,$$

does not contain any color singlets. This is the reason why no states from $\mathbf{3}_C \otimes \mathbf{3}_C$ can be observed.

7.4 Nucleon-meson models

Let us turn again to the isospin model of the Lie algebra of SU(2) in the fundamental representation we already encountered and denote the nucleons and antinucleons as

$$N^i = \begin{pmatrix} p \\ n \end{pmatrix}, \qquad \bar{N}_i = \begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix}.$$

Their transformation in the notation we have established for general SU(N) then is

$$N^i \to U^i{}_j N^j, \qquad \bar{N}_i \to \bar{N}_j (U^\dagger)^j{}_i.$$

Turning to the adjoint representation we can write

$$\begin{aligned}\phi &= \boldsymbol{\pi} \cdot \boldsymbol{\sigma} = \pi_1\sigma_1 + \pi_2\sigma_2 + \pi_3\sigma_3 \\ &= \begin{pmatrix} \pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & -\pi_3 \end{pmatrix} = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix},\end{aligned}$$

which transforms under SU(2) like

$$\phi^i{}_j \to \phi'^i{}_j = U^i{}_l\, \phi^l{}_m\, (U^\dagger)^m_j.$$

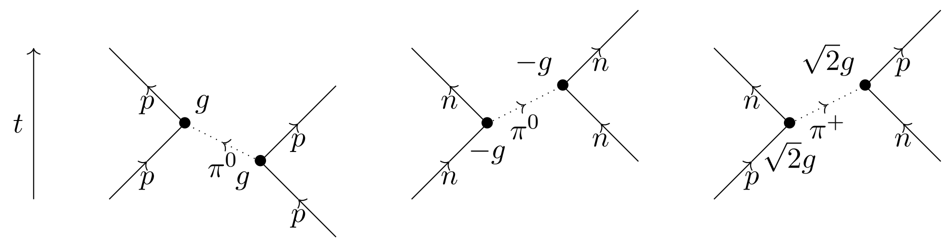
In other words, the pions are now forming a matrix valued field! We now use these fields to construct an invariant Lagrangian. For example, an interaction term of the form

$$\mathcal{L} = \ldots + g\bar{N}_i \phi^i{}_j N^j,$$

is SU(2) invariant. Explicitly, we can expand

$$\begin{pmatrix} \bar{p}, & \bar{n} \end{pmatrix} \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \begin{pmatrix} p \\ n \end{pmatrix} = \bar{p}\pi^0 p - \bar{n}\pi^0 n + \sqrt{2}\bar{p}\pi^+ n + \sqrt{2}\bar{n}\pi^- p.$$

The appearing terms can be expressed in terms of Feynman diagrams.



In a similar manner one may also introduce a meson field as an adjoint representation of SU(3),

$$\phi = \frac{1}{\sqrt{2}} \sum_{a=1}^{8} \phi_a \lambda_a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K^0} & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}.$$

Besides the pions this includes now also the Kaons and the η meson. We may start by assuming a Lagrangian with full SU(3) symmetry

$$\mathcal{L}_0 = \frac{1}{2} g^{\mu\nu} \operatorname{Tr}\{(\partial_\mu \phi)(\partial_\nu \phi)\} - \frac{1}{2} m_0^2 \operatorname{Tr}\{\phi\phi\}.$$

Here all mesons masses would be equal, which clearly is not the case and therefore we have to introduce a term that break the symmetry. Suppose we have a symmetric Hamiltonian H_0 and introduce a perturbation H_1 which breaks the symmetry,

$$H = H_0 + \alpha H_1.$$

As long as α stays small we are in the perturbative regime such that H_1 just shifts the energy eigenvalue by a small amount

$$\begin{aligned} \langle s|H_0|s\rangle &= E_s, \\ \langle s|\alpha H_1|s\rangle &= \Delta E. \end{aligned}$$

We can therefore write

$$E = E_s + \Delta E + \mathcal{O}(\alpha^2).$$

Let us now try to find a suitable symmetry breaking term in the Lagrangian. It is natural to search for a term quadratic in the fields, so we need to use the decomposition

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{10} \oplus \mathbf{10}^* \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1},$$

As usual, these terms can be split into symmetric and anti-symmetric terms,

$$\begin{aligned} (\mathbf{8} \otimes \mathbf{8})_s &= \mathbf{27} \oplus \mathbf{8} \oplus \mathbf{1}, \\ (\mathbf{8} \otimes \mathbf{8})_a &= \mathbf{10} \oplus \mathbf{10}^* \oplus \mathbf{8}. \end{aligned}$$

We need to concentrate here on the symmetric terms because mesons are bosonic and ϕ^2 is symmetric in this sense. The singlet $\mathbf{1}$ from the first line is actually the mass term in $\mathcal{L}_0$. For the breaking term we can choose between $\mathbf{27}$ and $\mathbf{8}$. It is probably natural to start with the simpler case so we pick $\mathbf{8}$ and add a term $\sim \operatorname{Tr}\{\phi\phi\lambda_8\}$ as the breaking term.

Note that we need to preserve the isospin SU(2) in the upper part of ϕ to keep the pion structure. Now we decompose the Gell-Mann matrix λ_8 into two parts,

$$\lambda_8 = \frac{1}{\sqrt{3}} \mathbb{1}_3 - \frac{2}{\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

such that the first term adds to the mass term in the Lagrangian and the second is new. It breaks the SU(3) symmetry in the right way, namely

$$\operatorname{Tr}\left\{ \phi\phi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = K^- K^+ + K^0 \overline{K^0} + \frac{2}{3}\eta^2,$$

and therefore we can write the perturbed Lagragian as

$$\mathcal{L} = \mathcal{L}_0 - \frac{1}{2}\kappa \left[K^- K^+ + K^0 \overline{K^0} + \frac{2}{3}\eta^2 \right].$$

The new term corrects the mass term of the kaons and the η-meson but not of the pion. This needs to be compared to the mass term in the unperturbed Lagrangian,

$$\mathrm{Tr}\{\phi\phi\} = (\pi^0)^2 + \eta^2 + 2K^- K^+ + \ldots .$$

To first order in perturbation theory around the symmetric situation one finds

$$m_\pi^2 = m_0^2, \qquad m_\eta^2 = m_0^2 + \frac{2}{3}\kappa, \qquad m_K^2 = m_0^2 + \frac{1}{2}\kappa,$$

which yields the Gell-Mann Okubo mass formula

$$4m_K^2 = 3m_\eta^2 + m_\pi^2.$$

Indeed this is fulfilled to reasonably good accuracy.

[Exercise: Di-quarks are hypothetical bound states of two quarks (but no anti-quarks). Because they necessarily carry color, they cannot be asymptotic states but might arise as resonances. Moreover, it may be possible to understand baryons as bound states of di-quarks with an additional quark. Using your freshly gained insights into SU(3) representation theory, classify the di-quarks states composed out of the up, down, and strange quarks $\mathbf{3} \otimes \mathbf{3}$ with respect to their transformation behaviour under SU(3) and the color symmetry $\mathrm{SU}(3)_C$.]

8 Classification of compact simple Lie algebras

In the following we will briefly go through a classification of Lie algebras worked out by *Wilhelm Killing* and *Élie Cartan*. This topic is treated in much more depth in mathematics lectures and books; we can only cover the very basics here. More specifically, in this section we will discuss a general classification of a special class of *real* Lie algebras, namely those that are *simple* and for which the corresponding Lie group is *compact* as a manifold. However, this is the basis also for other cases. Let us first clarify these notions.

Simple Lie group and algebra. Recall that we defined *finite* simple groups as those that do not have any non-trivial normal subgroups and that other finite groups could be reduced to those. Recall also that a subgroup H of G was called *normal* when $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$. For Lie groups one can define similarly: a *simple Lie group* is a connected, non-abelian Lie group which does not have non-trivial connected normal subgroups. (Note as an aside that it may have *discrete* normal subgroups.) One can also express this through the algebra: a *simple Lie group* is a connected Lie group for which the corresponding *Lie algebra* is *simple.* A *simple Lie algebra* in turn is a Lie algebra that is non-abelian and that does not contain any non-trivial *ideal.* To understand this, we also need the following notions.

Lie subalgebra and ideal. A *Lie subalgebra* is a sub vector-space of the Lie algebra that is also closed with respect to the Lie bracket. In other words, the Lie bracket of two elements of the subalgebra is again an element of the subalgebra. An *ideal* in this context is a Lie subalgebra that fulfills the stronger condition that the Lie bracket of any element of the full Lie algebra with an element of the ideal is again part of the ideal.

Semi-simple Lie algebra. A *semi-simple* Lie algebra is a Lie algebra that can be written as a *direct sum* of simple Lie algebras. In other words, semi-simple Lie algebras can be composed out of simple Lie algebras.

By the so-called *Levi decomposition* one can actually decompose all finite dimensional real or complex Lie algebras in terms of a *semi-simple* Lie subalgebra and a *solvable* Lie algebra for which an abelian Lie algebra is an example. We see here that simple Lie algebras are an important building block to understand the general case.

Compact Lie group and algebra. A Lie *group* is called *compact* if it is compact as a manifold. (For example the rotation group in four dimensions SO(4) is compact, but the group of special Lorentz transformations SO(1, 3) is not.) Accordingly, one calls a Lie *algebra compact* if it is the Lie algebra of a compact group. This is of course not the general case. However, it turns out that both compact and non-compact simple *real* Lie algebras can be seen as restrictions, so-called *real forms*, of the same simple Lie algebras over the *complex numbers* and for every such complex simple Lie group there is a particular real form that is compact. In this sense, studying the compact real Lie algebras gives insights into the general case. As a physicist one would say that compact and non-compact Lie algebras (and their groups) are related through *analytic continuation.* (For example *Wick rotation* from SO(1, 3) to SO(4).)

8.1 The adjoint representation

Previously we have already introduced the adjoint representation for the Lie algebra. Now we also discuss the adjoint representation of the group. The interesting thing about the adjoint representation is that we need no additional structure; everything is provided by the Lie group already. Consider a group element $h \in G$ on which other group elements $g \in G$ act as follows

$$h \to \mathrm{Ad}_g(h) = ghg^{-1}.$$

This is obviously a representation of the group, in the sense that $\mathrm{Ad}_{g_1}(\mathrm{Ad}_{g_2}(h)) = \mathrm{Ad}_{g_1 g_2}(h)$. Note that this representation acts in fact in the group space itself, because h is simply a group element. However, one may also take h to be infinitesimally close to the unit element, $h = \mathbb{1} + i\zeta^j T_j$, and finds

$$h = \mathbb{1} + i\zeta^k T_k \to ghg^{-1} = \mathbb{1} + ig(\zeta^k T_k)g^{-1}.$$

This shows that one may also consider the adjoint representation to act on the space of the Lie *algebra*,

$$\zeta^k T_k \to g(\zeta^k T_k)g^{-1}.$$

Finally, we may take also g to be infinitesimal, $g = \mathbb{1} + i\xi^j T_j$, and find to linear order in ξ^j,

$$\zeta^k T_k \to g(\zeta^k T_k)g^{-1} = \zeta^k T_k + i\left[\xi^j T_j, \zeta^k T_k\right] = \zeta'^k T_k. \tag{8.1}$$

We can identify the representation of the Lie algebra in the adjoint representation as acting in terms of the *commutator*. Now we can express the Lie bracket in term of the structure constants as

$$[T_j, T_k] = i f_{jk}{}^l T_l. \tag{8.2}$$

It is clear from (8.2) that the structure constants are anti-symmetric in the first two indices, $f_{jk}{}^l = -f_{kj}{}^l$. Let us also recall that they are *real*, $f_{jk}{}^l \in \mathbb{R}$, for Lie algebras with *some* hermitian representation $T_j^\dagger = T_j$. This is the case for the Lie algebras of compact Lie groups.

Eq. (8.2) allows to write the infinitesimal transformation in the adjoint representation (8.1) as

$$\zeta^k \to \zeta'^k = \left[\delta^k{}_l + i\xi^j (T_j^{(A)})^k{}_l\right]\zeta^l, \tag{8.3}$$

with the Lie algebra generators in the adjoint representation introduced already in (4.13)

$$(T_j^{(A)})^k{}_l = i\, f_{jl}{}^k. \tag{8.4}$$

It follows from the Jacobi identity in eq. (4.11) that these matrices satisfy also the commutation relation (8.2). Note that we are carefully distinguishing upper and lower indices here. As it is defined in (8.3), the adjoint representation acts on the Lie algebra or the tangent space of the Lie group at the position of the identity element.

Note that (8.3) can be understood as a matrix representation of the Lie algebra. As usual, an object with a lower index (actually an element of the co-tangent space or a one-form) transforms with the inverse matrix,

$$\omega_k \to \omega'_k = \left[\delta^l{}_k - i\xi^j (T_j^{(A)})^l{}_k\right] \omega_l, \tag{8.5}$$

and continuing in this way we can construct arbitrary tensor representations with upper and lower indices.

We now show that the structure constants are actually *invariant symbols* in the sense of this matrix representation. It is enough to show this for infinitesimal transformations,

$$\begin{aligned} f_{jk}{}^l \to & f_{jk}{}^l + i\xi^m \left[-(T_m^{(A)})^n{}_j f_{nk}{}^l - (T_m^{(A)})^n{}_k f_{jn}{}^l + (T_m^{(A)})^l{}_n f_{jk}{}^n\right] \\ = & f_{jk}{}^l - \xi^m \left[-f_{mj}{}^n f_{nk}{}^l - f_{mk}{}^n f_{jn}{}^l + f_{mn}{}^l f_{jk}{}^n\right] \\ = & f_{jk}{}^l + \xi^m \left[f_{jm}{}^n f_{kn}{}^l + f_{mk}{}^n f_{jn}{}^l + f_{kj}{}^n f_{mn}{}^l\right] = f_{jk}{}^l. \end{aligned} \tag{8.6}$$

In the last line we have used the Jacobi identity written in terms of (4.11).

8.2 Killing-Cartan metric

From the structure constants we can construct another invariant symbol. Let us consider the following matrix known as the *Killing-Cartan metric*,

$$g_{jk} = \mathrm{Tr}\left\{T_j^{(A)} T_k^{(A)}\right\} = -f_{jn}{}^m f_{km}{}^n. \tag{8.7}$$

The definition immediately implies that this is a symmetric matrix, $g_{jk} = g_{kj}$. In principle this transforms as a tensor with two lower indices. However, because it is constructed purely from the structure constants, it is in fact an *invariant symbol* under group transformations in the adjoint representation, $g_{jk} \to g_{jk}$.

For semi-simple Lie algebras the determinant of g_{jk} is non-vanishing, so that also the inverse g^{km} exists, such that $g_{jk} g^{km} = \delta_j{}^m$. In the differential-geometric description of Lie groups one can understand the Killing-Cartan metric as a Riemannian metric on the group manifold.

Now that we know that it is an invariant symbol, we can use g_{jk} and its inverse to pull indices up and down. For example, a variant of the structure constants with only lower indices would be

$$f_{jkl} = f_{jk}{}^m g_{ml} = -f_{jk}{}^m f_{mp}{}^q f_{lq}{}^p = -i\mathrm{Tr}\left\{T_j^{(A)} T_k^{(A)} T_l^{(A)} - T_k^{(A)} T_j^{(A)} T_l^{(A)}\right\}. \tag{8.8}$$

In the last equation we used the Jacobi identity and (8.4). From equation (8.8) one reads off that f_{jkl} is fully anti-symmetric with respect to any interchange of indices.

8.3 Cartan subalgebra

Among the generators T_j there is a maximal number of *mutually commuting* generators, this number r is the *rank* of the Lie algebra. We will denote these generators by H_a with an index like a from the beginning of the alphabet,

$$[H_a, H_b] = 0, \tag{8.9}$$

with $a, b = 1, \dots, r$. In other words, the corresponding components of the structure constants vanish, ${f_{ab}}^j = 0$. The generators H_a form a sub-algebra known as the *Cartan subalgebra*.

As an example, for SU(3) the rank is $r = 2$ and one can take $H_1 = \lambda_3/2$ and $H_2 = \lambda_8/2$, which are both diagonal. The Cartan subalgebra is here the algebra of diagonal matrices.

8.4 Root vectors

In the following we will consider the generators corresponding to the H_a's in the adjoint representation and acting in the space of the remaining $N - r$ generators. They are given by

$$(T_a^{(A)})^m{}_n = i{f_{an}}^m.$$

While a is from the range $1, \dots, r$, the indices m and n run here over the remaining $N - r$ possible values.

Because the H_a mutually commute, one can diagonalize the corresponding generators $(T_a^{(A)})^m{}_n$ in the adjoint representation simultaneously through a similarity transformation in the space of the remaining generators. Moreover, because the generators are hermitian, the eigenvalues need to be real. We assume now that this diagonalization has been done, and write

$$(T_a^{(A)})^m{}_n = \beta_a(n)\, \delta^m{}_n. \tag{8.10}$$

As a side remark let us note that as a consequence of the diagonalization, the remaining $N - r$ generators that do not span the Cartan subalgebra need not be hermitian any more. The function $\beta_a(n)$ associates to every remaining generator with index n a set of r numbers, or a vector (strictly speaking a covector) $\vec{\beta}(n) = (\beta_1(n), \beta_2(n), \dots, \beta_r(n))$.

Consider now

$$[H_a, T_n] = i{f_{an}}^m T_m = \beta_a(n) T_n. \tag{8.11}$$

In other words, the commutator of an H_a with any of the remaining generators is again proportional to this generator with a prefactor given again by the $\beta_a(n)$.

This is interesting because it says that T_n changes the "quantum number" measured by the operator H_a by an amount $\beta_a(n)$. This is precisely as we have seen it happening before for $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$. In other words, we can understand the H_a as operators corresponding to *labels* (similar to T_3 for $\mathfrak{su}(2)$ and T_3 and T_8 for $\mathfrak{su}(3)$) while the remaining operators T_n are generalized raising and lowering operators (similar to $T_\pm$ for $\mathfrak{su}(2)$ and $I_\pm$, $U_\pm$ and $V_\pm$ for $\mathfrak{su}(3)$). The vectors $\vec{\beta}(n)$ by which the labels can be changed are known as the *root vectors*.

Note that the root vectors are vectors in an r dimensional space. We have already seen this at play for $\mathfrak{su}(2)$ which has rank one and $\mathfrak{su}(3)$ which has rank two. More generally, the Lie algebra of SU(N) has rank $r = N - 1$ such that the roots of SU(4) live in a three-dimensional space etc.

It is now convenient to label the remaining generators by their roots, and to introduce a new letter,

$$E_{\vec{\beta}(n)} = T_n.$$

One can conveniently normalize these operators to

$$\mathrm{Tr}\{E^\dagger_{\vec\beta} E_{\vec\beta}\} = 1.$$

We can then write (8.11) as

$$[H_a, E_{\vec\beta}] = \beta_a E_{\vec\beta}.$$

The hermitian conjugate of this equation is

$$[H_a, E^\dagger_{\vec\beta}] = -\beta_a E^\dagger_{\vec\beta}.$$

This shows that for the $E_{\vec\beta}$ hermitian conjugation reverses the sign of the root,

$$E^\dagger_{\vec\beta} = E_{-\vec\beta}.$$

We can also infer that, because the negative of a root is also a root, $(N-r)$ is always even.

In summary, the generators of an N-dimensional Lie algebra with rank r can be divided quite general in into $N-r$ generators $E_{\vec\beta(n)}$, labeled by root vectors $\vec\beta(n)$. Half of them might be called positive, the other half negative roots. Intuitively, they change the commuting "quantum numbers" by an amount $\vec\beta(n)$. In addition there are r hermitian generators H_a and because $[H_a, H_b] = 0$ one may say that they have vanishing roots. They do not change the labels or "commuting quantum numbers", but rather quantify them.

8.5 Scalar product in root space

Consider the commutator $[E_{\vec\beta}, E_{\vec\gamma}]$, where $\vec\beta$ and $\vec\gamma$ are roots. From the Jacobi identity one finds

$$[H_a, [E_{\vec\beta}, E_{\vec\gamma}]] = (\beta_a + \gamma_a)[E_{\vec\beta}, E_{\vec\gamma}]. \tag{8.12}$$

This commutator $[E_{\vec\beta}, E_{\vec\gamma}]$ could vanish, but if it does not, we see that it is a generator belonging to the root $\vec\beta + \vec\gamma$. This implies that one can write

$$[E_{\vec\beta}, E_{\vec\gamma}] = \mathcal{N}_{\vec\beta,\vec\gamma}\, E_{\vec\beta+\vec\gamma}, \tag{8.13}$$

with some coefficients $\mathcal{N}_{\vec\beta,\vec\gamma}$. However, there are only $N-r$ roots, so not all possible sums (or differences) of roots can be roots again. If $\vec\beta + \vec\gamma$ is not a root, the corresponding coefficient vanishes, $\mathcal{N}_{\vec\beta,\vec\gamma} = 0$.

Now set $\vec\gamma = -\vec\beta$. From (8.12) we get now a linear combination of generators with vanishing roots, so that we can write

$$[E_{\vec\beta}, E_{-\vec\beta}] = \beta^a H_a. \tag{8.14}$$

Note that we are now using β^a with an upper index. That this is indeed consistent follows from the following consideration,

$$\mathrm{Tr}\left\{H_b[E_{\vec\beta}, E_{-\vec\beta}]\right\} = \mathrm{Tr}\left\{[H_b, E_{\vec\beta}]E_{-\vec\beta}\right\} = \beta_b \mathrm{Tr}\left\{E_{\vec\beta} E_{-\vec\beta}\right\} = \beta_b = \beta^a\,\mathrm{Tr}\left\{H_a H_b\right\}. \tag{8.15}$$

The last expression is just the Killing-Cartan metric, $\mathrm{Tr}\,\{H_a H_b\} = g_{ab}$, which can be consistently used to pull indices up and down,

$$\beta_a = g_{ab}\beta^b, \qquad \beta^a = g^{ab}\beta_b.$$

We have now also a sensible scalar product in the space of root vectors,

$$(\vec{\alpha},\vec{\beta}) = (\vec{\beta},\vec{\alpha}) = \alpha_a\beta_b\, g^{ab} = \alpha^a\beta^b\, g_{ab}.$$

8.6 Constraints on angles and ratios of root vectors

A detailed consideration based on the fact that the set of root vectors must be finite, and with clever use of the Jacobi identity (for which we refer to the literature) shows now that the scalar product between two non-parallel root vectors is actually not arbitrary but constrained such that

$$2\frac{(\vec{\alpha},\vec{\beta})}{(\vec{\alpha},\vec{\alpha})} = n, \qquad 2\frac{(\vec{\alpha},\vec{\beta})}{(\vec{\beta},\vec{\beta})} = m, \tag{8.16}$$

with two (positive, negative or zero) integers n and m. This implies also the following relation for the angle between the root vectors,

$$\cos^2\theta_{\alpha\beta} = \frac{(\vec{\alpha},\vec{\beta})^2}{(\vec{\alpha},\vec{\alpha})(\vec{\beta},\vec{\beta})} = \frac{mn}{4}. \tag{8.17}$$

Also, the ratio of the length squared of the root vectors must be a rational number (assuming here $n \neq 0$),

$$\frac{(\vec{\alpha},\vec{\alpha})}{(\vec{\beta},\vec{\beta})} = \frac{m}{n}. \tag{8.18}$$

Because $0 \le \cos^2\theta_{\alpha\beta} < 1$ one finds $0 \le mn < 4$. (We exclude $\cos^2\theta_{\alpha\beta} = 1$ because the root vectors would then be parallel.) Because $(\vec{\alpha},\vec{\alpha})$ must be positive, we see that n and m are either both negative, both positive, or both zero. When they are both negative one could just change the sign of one of the roots, $\vec{\alpha} \to -\vec{\alpha}$, so that n and m can be taken both positive or zero. Also, one can take $m \ge n$ without loss of generality.

We find then the following cases

m	n	$\cos^2\theta_{\alpha\beta}$	$\theta_{\alpha\beta}$	$\lvert\vec{\alpha}\rvert/\lvert\vec{\beta}\rvert$
0	0	0	$\pi/2$	indeterminate
1	1	1/4	$\pi/3$	1
2	1	1/2	$\pi/4$	$\sqrt{2}$
3	1	3/4	$\pi/6$	$\sqrt{3}$

We note here that root vectors of at most two different lengths can appear for a given simple Lie algebra. Their ratio can be either $\sqrt{2}$ or $\sqrt{3}$. If both cases would be present simultaneously, we would also have a ratio $\sqrt{2/3}$, in conflict with the statements above. In other words, either the root vectors are all of equal length, or there are long and short roots.

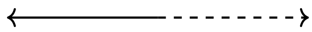
8.7 Positive and simple roots

We have seen that the negative of a root is also a root. It is useful to find some notion of what one calls a positive root and what a negative root, even if this is necessarily dependent on the basis one chooses. In terms of the components β_a one can say that the root is positive/negative when β_1 is positive/negative. When the first component vanishes, $\beta_1 = 0$, we let the second component decide and so on. This allows to classify every non-zero root vector as either positive or negative.

We now still have $(N - r)/2$ positive root vectors in an r-dimensional space and they are not all linearly independent. In particular, some positive roots can typically be written as sums of other positive roots with positive coefficients (see our discussion of $\mathfrak{su}(3)$). It is customary to call those roots that can *not* be written as sums of other positive roots *simple roots*.

8.8 Lie algebras of rank one

At rank one there is in fact only one compact and simple Lie algebra, namely $\mathfrak{su}(2)$. The Cartan subalgebra has a single generator, $H = T_3$. There is one positive and simple root "vector" corresponding to T_+ and its negative corresponds to T_-. The root diagram is as follows (the simple root is shown dashed).



SU(2)

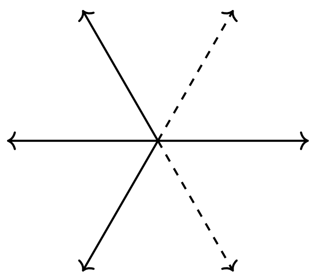
One also uses so-called *Dynkin diagrams* to represent the simple roots. Here we have only a single simple root, so the Dynkin diagram contains just one element.

SU(2)

As we have seen, one can obtain other, non-compact Lie algebras of rank one by analytic continuation, i. e. by multiplying some generators with i.

8.9 Lie algebras of rank two

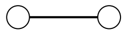
At rank two we have more possibilities. We have already discussed $\mathfrak{su}(3)$ with the following root diagram (again we draw the simple roots as dashed lines).



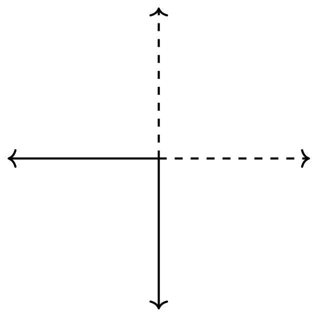
SU(3)

Indeed, the angles are here $\pi/3$ and all root vectors are of equal length. By construction they correspond to the states of the adjoint representation with two additional states corresponding to $H_1 = T_3$ and $H_2 = T_8$ having vanishing root vectors. By dividing this into two halfs we distinguish positive and negative roots and can determine two of the positive roots as simple roots, while the third positive root is the sum of these two simple ones.

Another graphical representation for this Lie algebra $\mathfrak{su}(3)$ is in terms of the *Dynkin diagram.* Because there are only two simple roots it contains two small circles and because they have equal length and intersect at an angle of $\pi/3$, they get linked by a simple line.


SU(3)

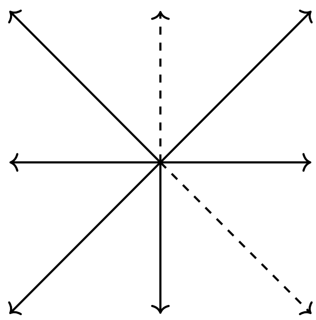
Another possible root diagram is the one of the Lie algebra of SO(4).


SO(4)

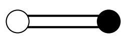
Here we have essentially two copies of the $\mathfrak{su}(2)$ algebra with the corresponding root vectors intersecting at the angle $\pi/2$. Indeed one can establish $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. There are six generators in total, two of them generate the Cartan subalgebra, two simple roots, and their negatives. The Dynkin diagram represents the fact that the algebra breaks up into a direct sum of two $\mathfrak{su}(2)$ algebras. It has two disconnected circles for the two simple roots.


SO(4)

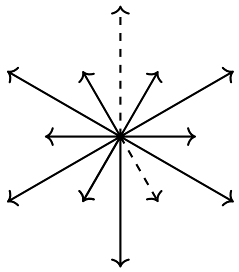
There is also a possibility to have long and short roots when the roots intersect at an angle $\pi/4$, as in the following root diagram of SO(5).


SO(5)

In the Dynkin diagram the short root is now symbolized by a filled circle. Moreover, simple roots that intersect at an angle of $\pi/4$ are connected with two lines.

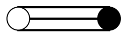

SO(5)

Finally, there is also an *exceptional* Lie group of rank two, G_2.



G_2

Here the two simple roots intersect at an angle $\pi/6$ and have the length ratio $\sqrt{3}$. This is denoted by three lines in the Dynkin diagram.



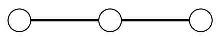
G_2

8.10 Higher rank

Recall that for SU(2) we had one generator for the Cartan subalgebra, $H_1 = T_3 = \text{diag}(1, -1)/2$. Similarly, for SU(3) we worked with $H_1 = T_3 = \text{diag}(1, -1, 0)/2$ and $H_2 = T_8 = \text{diag}(1, 1, -2)/2\sqrt{3}$. More general, for SU($N$) there are $N - 1$ traceless diagonal matrices, so the rank of $\mathfrak{su}(N)$ is $r = N - 1$. For example, for SU(4) one could work with

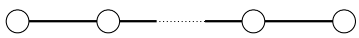
$$
\begin{aligned}
H_1 =& \text{diag}(1, -1, 0, 0)/2, \\
H_2 =& \text{diag}(1, 1, -2, 0)/2\sqrt{3}, \\
H_3 =& \text{diag}(1, 1, 1, -3)/2\sqrt{6}.
\end{aligned}
$$

The root vectors are now in a three-dimensional space and there are three simple roots. They are all of equal length and intersect at angles $\pi/3$ such that the Dynkin diagram is as follows.



SU(4)

Similarly, SU(N) has a Dynkin diagram with $r = N - 1$ connected simple roots.

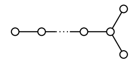


SU(N)

8.11 Classification

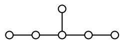
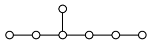
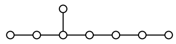
One can now go on and derive rules about how Dynkin diagrams can be constructed. Because they represents the simple roots and the possible angles and length ratios, the Dynkin diagrams can be used to characterize each semi-simple compact real Lie algebra. This leads ultimately to the complete characterization of simple Lie algebras.

There are then four infinite series, labeled by their rank r.

Math. notation	Group	Real compact alg.	Complex form	Dynkin diagram
A_r	SU($r+1$)	$\mathfrak{su}(r+1, \mathbb{R})$	$\mathfrak{sl}(r+1, \mathbb{C})$	
B_r	SO($2r+1$)	$\mathfrak{so}(2r+1, \mathbb{R})$	$\mathfrak{so}(2r+1, \mathbb{C})$	
C_r	USp($2r$)	$\mathfrak{usp}(2r, \mathbb{R})$	$\mathfrak{sp}(2r, \mathbb{C})$	
D_r	SO($2r$)	$\mathfrak{so}(2r, \mathbb{R})$	$\mathfrak{so}(2r, \mathbb{C})$	

Note that one distinguishes here between the orthogonal groups in even and those in odd dimensions. They have different properties and in particular different Dynkin diagrams.

In addition to the four series, there are the exceptional Lie groups G_2, F_4, E_6, E_7 and E_8. For completeness we also give their Dynkin diagrams,

 G_2 , F_4 ,  E_6 ,  E_7,  E_8.

For more details about this classification we refer to the literature.

Note that what we have discussed here is a characterization of the Lie algebras themselves, and not of their possible representations. The root vectors and corresponding operators are useful there, as well; they correspond to generalized ladder operators. The generators of the Cartan subalgebra provide convenient labels for states. We have seen this at play for SU(3) with the meson and baryon states. While the above classification encompasses both compact and non-compact real Lie algebras through their shared complex form, the representation theory is typically simpler for compact real algebras because their Casimir operators are non-negative, while they can be unbounded for non-compact real Lie algebras.

9 Lorentz and Pioncaré groups

We now discuss the symmetry groups of spacetime, both relativistic and non-relativistic. We start with the invariance groups of special relativity, the Lorentz and Pioncaré groups.

9.1 Lorentz group for arbitrary number of time and space dimensions

Real indefinite orthogonal group. We concentrate first on *real* coordinates x^μ and consider d spacetime dimensions divided into r time and $(d-r)$ space dimensions. Technically, we mean by this that there is a real, indefinite but non-degenerate metric, which by a convenient choice of coordinates can be brought to the form

$$\eta_{\mu\nu} = \mathrm{diag}(-1,\ldots,-1,+1,\ldots,+1).$$

The first r entries are -1 for time-like coordinates followed by $(d-r)$ entries $+1$ for space coordinates. One may label the indices μ of coordinates x^μ and the metric such that $\mu = 1-r,\ldots,0$ are time indices and $\mu = 1,\ldots,d-r$ are spatial indices. The most interesting case is $d=4$ and $r=1$ such that the Minkowski space metric has signature $(-,+,+,+)$. (One may alternatively use conventions where time coordinates have positive and space coordinates negative entries.)

The spacetime symmetry group (including rotations, boosts and reflections but without translations) is then the one of the *indefinite orthogonal group* $\mathrm{O}(r, d-r, \mathbb{R})$. The group elements are real $d\times d$ matrices $\Lambda^\mu{}_\nu$, defined through the relation

$$\Lambda^\rho{}_\mu \eta_{\rho\sigma}\Lambda^\sigma{}_\nu = \eta_{\mu\nu}, \qquad \text{or} \qquad \Lambda^T\eta\Lambda = \eta. \tag{9.1}$$

In other words, these transformations are such that the metric is left invariant. The metric and its inverse are *invariant symbols* and they can be used to pull indices up and down.

Complex orthogonal group O$(d,\mathbb{C})$. It is sometimes also convenient to study complex extensions of real spacetime symmetries. Here the coordinates x^μ are *complex* and the transformations consist of complex matrices that satisfy (9.1). However, if the coordinates are anyway complex, one can just multiply the time coordinates by i and effectively map $\eta_{\mu\nu} = \mathrm{diag}(-1,\ldots,-1,+1,\ldots,+1) \to \mathrm{diag}(1,\ldots,1) = \delta_{\mu\nu}$. Of course, one can similarly map to any other metric signature by such a complex change of basis. This shows that the distinction of time and space coordinates looses its meaning, and for given d there is only one complex orthogonal group $\mathrm{O}(d,\mathbb{C})$.

Rotation group O$(d,\mathbb{R})$. We now turn back to the real case and first discuss the simplest and definite case $r=0$ (or, essentially equivalent, $d=r$). In this case there are two disconnected components of the group $\mathrm{O}(d,\mathbb{R})$ with $\det(\Lambda)=\pm 1$. The elements close to the unit element $\Lambda = \mathbb{1}$ have $\det(\Lambda)=1$ and form the group $\mathrm{SO}(d,\mathbb{R})$. They can be combined with reflections to construct other elements of the group $\mathrm{O}(d,\mathbb{R})$. For d *odd* one has a full reflection $\Lambda = -\mathbb{1}$ with $\det(\Lambda) = -1$. This transformation commutes with all other elements. One has therefore the structure $\mathrm{O}(d,\mathbb{R}) = Z_2 \times \mathrm{SO}(d,\mathbb{R})$. For d *even* this is not possible, and reflections with $\det(\Lambda)=-1$ do not commute with all elements of $\mathrm{SO}(d)$. In any case, the topology of $\mathrm{O}(d,\mathbb{R})$ has two disconnected parts with $\det(\Lambda)=\pm1$.

These topological considerations directly extend to the complex group $\mathrm{O}(d,\mathbb{C})$.

Generalized Lorentz group O$(r, d-r, \mathbb{R})$**.** Now we assume $r > 0$ and $(d-r) > 0$. Again there are two disconnected parts with $\det(\Lambda) = \pm 1$. The elements that are continuously connected to $\Lambda = \mathbb{1}$ have $\det(\Lambda) = 1$. *Reflections* along the coordinate axis' can be decomposed into a temporal part P and a spatial part Q,

$$\Lambda = \text{diag}(P, Q).$$

One may have $\det(P) = \pm 1$, $\det(Q) = \pm 1$ with $\det(\Lambda) = \det(P)\det(Q)$. Accordingly, there are now four disconnected components of the group $\text{O}(r, d-r, \mathbb{R})$.

For d odd one can write again $\text{O}(r, d-r, \mathbb{R}) = Z_2 \times \text{SO}(r, d-r, \mathbb{R})$ where $\text{SO}(r, d-r, \mathbb{R})$ has only two disconnected components. Depending on whether the number of time dimensions r or the number of space dimensions $(d-r)$ is odd, these two topologically disconnected components are connected by time reflections or space reflections, respectively. For d even, the group is not of a simple product structure but still has four disconnected components in the real case. We denote the component topologically connected to the unit transformation by $\text{SO}^{\uparrow}(r, d-r, \mathbb{R})$. The following table illustrates the topological structure of $O(r, d-r, \mathbb{R})$ and decomposes it into four sectors I, II, III and IV.

	$\det(Q) = +1$	$\det(Q) = -1$
$\det(P) = +1$	I $\det(\Lambda) = +1$	II $\det(\Lambda) = -1$
$\det(P) = -1$	III $\det(\Lambda) = -1$	IV $\det(\Lambda) = +1$

Note that two subsequent transformations out of a single sector always lead to I. The structure is the one of the finite group $Z_2 \times Z_2$. In other words, $\text{O}(r, d-r, \mathbb{R})/\text{SO}^{\uparrow}(r, d-r, \mathbb{R}) \cong Z_2 \times Z_2$.

In the complexified group $\text{O}(d, \mathbb{C})$ region I and IV are connected, as well as II and III, but the two sectors with $\det(\Lambda) = \pm 1$ remain disconnected from each other. In other words, $\text{O}(d, \mathbb{C})/\text{SO}(d, \mathbb{C}) \cong Z_2$.

Space and time reflections. For $d-r$ even, space reversion (i. e. the reflection along all time-like coordinate axis) $\mathcal{P}$ does not connect different topologically disconnected elements of the group but for $d-r$ odd this is the case. Similarly, when r is even, simple time reversal $\mathcal{T}$ does not connect different components, but for r odd, they do. Combined transformations $\mathcal{PT} = -\mathbb{1}$ connect different components for d odd or for d even with r and $d-r$ odd.

	$(d-r)$ even	$(d-r)$ odd
r even	d even $\mathcal{P} \in \text{I}$, $\mathcal{T} \in \text{I}$, $\mathcal{PT} \in \text{I}$	d odd $\mathcal{P} \in \text{II}$, $\mathcal{T} \in \text{I}$, $\mathcal{PT} \in \text{II}$
r odd	d odd $\mathcal{P} \in \text{I}$, $\mathcal{T} \in \text{III}$, $\mathcal{PT} \in \text{III}$	d even $\mathcal{P} \in \text{II}$, $\mathcal{T} \in \text{III}$, $\mathcal{PT} \in \text{IV}$

Lie algebra. The connected subgroup $SO^{\uparrow}(r, d-r, \mathbb{R})$ may be discussed in terms of the Lie algebra. Infinitesimal transformations are of the form

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}. \tag{9.2}$$

For $\delta\omega^{\mu\nu} = \eta^{\nu\rho}\delta\omega^{\mu}{}_{\rho}$ the condition (9.1) implies anti-symmetry,

$$\delta\omega^{\mu\nu} = -\delta\omega^{\nu\mu}. \tag{9.3}$$

Representations of the Lorentz group with can be written in infinitesimal form as

$$U(\Lambda) = \mathbb{1} + \frac{i}{2}\delta\omega^{\mu\nu}M_{\mu\nu}, \tag{9.4}$$

and finite transformations through the corresponding exponentiation. The generators are anti-symmetric, $M_{\mu\nu} = -M_{\nu\mu}$, and their Lie bracket is

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\right). \tag{9.5}$$

This defines the Lie algebra $\mathfrak{so}(r, d-r, \mathbb{R})$.

The fundamental representation (9.2) has the generators

$$(M^{F}_{\mu\nu})^{\alpha}{}_{\beta} = -i\left(\delta^{\alpha}_{\mu}\eta_{\nu\beta} - \delta^{\alpha}_{\nu}\eta_{\mu\beta}\right). \tag{9.6}$$

It acts on the space of d-dimensional vectors x^{α} and the infinitesimal transformation in (9.2) induces the infinitesimal change

$$\delta x^{\alpha} = \frac{i}{2}\delta\omega^{\mu\nu}(M^{F}_{\mu\nu})^{\alpha}{}_{\beta}\, x^{\beta}.$$

As we have done previously, one can now define tensor representations of various kind, i. e. with an arbitrary number of upper and lower spacetime indices.

9.2 Four dimensional orthogonal group

Let us now specialize to $d = 4$ dimensions and focus first on Euclidean signature $\eta_{\mu\nu} = \delta_{\mu\nu} = \text{diag}(1, 1, 1, 1)$. It is convenient to decompose the generators into the spatial-spatial part (with $j, k, l \in \{1, 2, 3\}$),

$$J_j = \frac{1}{2}\epsilon_{jkl}M_{kl}, \tag{9.7}$$

as well as

$$I_j = M_{j4}. \tag{9.8}$$

Equation (9.5) implies the commutation relations

$$[J_j, J_k] = i\epsilon_{jkl}J_l,$$
$$[J_j, I_k] = i\epsilon_{jkl}I_l,$$
$$[I_j, I_k] = i\epsilon_{jkl}J_l.$$

Interestingly, we have seen this structure before, namely in section 5.4 when we discussed the hidden symmetry of the hydrogen atom. By defining the linear combinations

$$J_j^{\pm} = \frac{1}{2}\left(J_j \pm I_j\right), \tag{9.9}$$

we find two independent representations of the $\mathfrak{su}(2)$ algebra,

$$\left[J_j^{\pm}, J_k^{\pm}\right] = i\epsilon_{jkl} J_l^{\pm}, \qquad \left[J_j^{+}, J_k^{-}\right] = 0. \tag{9.10}$$

In other words, one can write

$$\mathfrak{so}(4, \mathbb{R}) = \mathfrak{su}(2) \oplus \mathfrak{su}(2). \tag{9.11}$$

In fact, we have seen this also in our discussion of Lie algebras of rank two in section 8.9.

The two copies of the SU(2) algebra are linked through the parity transformation $J_j \to J_j$, $I_j \to -I_j$ which implies $J_j^{\pm} \to J_j^{\mp}$.

One can also discuss this correspondence directly in terms of the Lie groups. To that end, let us write an element of SU(2) as

$$M = -ix^4 \mathbb{1} + x^j \sigma_j, \qquad (x^4)^2 + \vec{x}^2 = 1. \tag{9.12}$$

By writing this in components, it is easy to check that iM is the most general unitary 2×2 matrix with unit determinant. On the other side we can also understand x^m as a unit vector in four dimensional Euclidean space. Now consider a transformation

$$M \to M' = U^{\dagger} M V, \tag{9.13}$$

with $U, V \in \mathrm{SU}(2)$. This implies that iM' is also unitary and M' can again be written as in (9.12). In other words, two SU(2) elements U and V induce a SO(4) rotation $x^m \to x'^m = R^m{}_n x^n$. Because $(-U, -V)$ and (U, V) induce the same transformation we have in fact the relation

$$\mathrm{SO}(4, \mathbb{R}) \cong \mathrm{SU}(2) \otimes \mathrm{SU}(2)/Z_2.$$

One says that $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$ is the *double cover* of SO(4). Note that both SO(4) and SU(2) are compact groups so that the correspondence is also consistent from this point of view.

As we have discusses before, representations of SU(2) are characterized by spin j of half integer or integer value. Accordingly, the representations of the "Euclidean Lorentz group" can be classified as $(2j + 1, 2\tilde{j} + 1)$. For example

$$\begin{aligned}
(1, 1) &= \text{scalar or singlet},\\
(2, 1) &= \text{left-handed spinor},\\
(1, 2) &= \text{right-handed spinor},\\
(2, 2) &= \text{vector}.
\end{aligned}$$

Complex Lorentz or orthogonal group. The decomposition of the Lie algebra into a direct sum works also for the complex group $\mathrm{O}(4, \mathbb{C})$. To understand this properly, we need to understand the complexified form of $\mathfrak{su}(N)$. While the Lie algebra of $\mathfrak{su}(N)$ consists of trace-less, hermitian $N \times N$ matrices, the same basis actually spans the vector space of trace-less but otherwise arbitrary complex $N \times N$ matrices when the coefficients are complex. This shows that the complexification of $\mathfrak{su}(N)$ is $\mathfrak{sl}(N, \mathbb{C})$. Applied to (9.11) this implies for the Lie algebra of the *complex orthogonal group*

$$\mathfrak{so}(4, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}).$$

In fact, this follows also from the discussion of Lie algebras of rank two in section 8.9.

Also the direct relation between groups specified by eqs. (9.12) and (9.13) generalizes. In fact when the coefficients x^m are complex, M is simply a complex matrix with unit determinant. For a transformation as in (9.13) this remains the case as long as $U, V \in \mathrm{SL}(2, \mathbb{C})$. With this identification one has

$$\mathrm{SO}(4, \mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C}) \otimes \mathrm{SL}(2, \mathbb{C})/Z_2.$$

9.3 Four dimensional Lorentz group

Let us now come to standard Minkowski space with $r = 1$ and $d = 4$. Again it is convenient to decompose the generators into the spatial-spatial part

$$J_j = \frac{1}{2}\epsilon_{jkl} M_{kl}, \tag{9.14}$$

and a spatial-temporal part,

$$K_j = M_{j0}. \tag{9.15}$$

Equation (9.5) implies now the commutation relations

$$\begin{aligned} [J_j, J_k] &= i\epsilon_{jkl} J_l, \\ [J_j, K_k] &= i\epsilon_{jkl} K_l, \\ [K_j, K_k] &= -i\epsilon_{jkl} J_l. \end{aligned}$$

In the fundamental representation one has

$$(J_i^F)^j{}_k = -i\epsilon_{ijk},$$

where j, k are spatial indices. All other components vanish, $(J_i^F)^0{}_0 = (J_i^F)^0{}_j = (J_i^F)^j{}_0 = 0$. Note that J_i^F is hermitian, $(J_i^F)^\dagger = J_i^F$. The generator K_j has the fundamental representation

$$(K_j^F)^0{}_m = -i\delta_{jm}, \qquad (K_j^F)^m{}_0 = -i\delta_{jm},$$

and all other components vanish, $(K_j^F)^0{}_0 = (K_j^F)^m{}_n = 0$. As a matrix, K_j^F is anti-hermitian, $(K_j^F)^\dagger = -K_j^F$.

One can define the linear combinations of generators

$$N_j = \frac{1}{2}(J_j - iK_j), \qquad \tilde{N}_j = \frac{1}{2}(J_j + iK_j), \tag{9.16}$$

for which the commutation relations become

$$[N_i, N_j] = i\epsilon_{ijk}N_k,$$
$$[\tilde{N}_i, \tilde{N}_j] = i\epsilon_{ijk}\tilde{N}_k,$$
$$[N_i, \tilde{N}_j] = 0.$$

Note that in contrast to (9.9), equation (9.16) consists of *complex* linear combinations. In particular, this implies for an infinitesimal transformation

$$\begin{aligned}\mathbb{1} + \frac{i}{2}\delta\omega^{\mu\nu}M_{\mu\nu} &= \mathbb{1} + \frac{i}{2}\delta\omega^{mn}\epsilon^{mnj}J_j + i\delta\omega^{j0}K_j \\ &= \mathbb{1} + \frac{i}{2}\delta\omega^{mn}\epsilon^{mnj}[N_j + \tilde{N}_j] + i\delta\omega^{j0}[iN_j - i\tilde{N}_j] \\ &= \mathbb{1} + \left[\frac{i}{2}\delta\omega^{mn}\epsilon^{mnj} - \delta\omega^{j0}\right]N_j + \left[\frac{i}{2}\delta\omega^{mn}\epsilon^{mnj} + \delta\omega^{j0}\right]\tilde{N}_j.\end{aligned} \tag{9.17}$$

The coefficients in front of the hermitian generators $N_j = N_j^\dagger$ and $\tilde{N}_j = \tilde{N}_j^\dagger$ are now complex and not purely imaginary as one would have it for a compact group.

In fact, the last line of (9.17) can be understood as a representations of the complexification of $\mathfrak{su}(2)$, namely $\mathfrak{sl}(2, \mathbb{C})$. In other words,

$$\mathfrak{so}(3, 1, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{C}).$$

Representations can still be classified as we would do for two representations of $\mathfrak{su}(2)$ namely in terms of two spins of integer or half-integer value $(2j+1, 2\tilde{j}+1)$. They specify the representation of the hermitian generators N_j and $\tilde{N}_j$, respectively, e. g.

$$\begin{aligned}(1, 1) &= \text{scalar or singlet},\\ (2, 1) &= \text{left-handed spinor},\\ (1, 2) &= \text{right-handed spinor},\\ (2, 2) &= \text{vector}.\end{aligned}$$

Note that the second bracket in the last line of (9.17) is obtained from the first bracket as the negative hermitian conjugate. From a representation $L \in \mathrm{SL}(2, \mathbb{C})$ acting on a left-handed spinor $(2, 1)$ one can get the transformation matrix in the right-handed spinor representation $(1, 2)$ by taking the inverse and hermitian conjugate, $(L^\dagger)^{-1}$. If we take the inverse of the hermitian conjugate of (9.17) we find

$$\left(\mathbb{1} + \frac{i}{2}\delta\omega^{\mu\nu}M_{\mu\nu}\right)^{\dagger -1} = \mathbb{1} + \left[\frac{i}{2}\delta\omega^{mn}\epsilon^{mnj} - \delta\omega^{j0}\right]\tilde{N}_j + \left[\frac{i}{2}\delta\omega^{mn}\epsilon^{mnj} + \delta\omega^{j0}\right]N_j. \tag{9.18}$$

The two generators N_j and $\tilde{N}_j$ now have changed their role.

Let us also work out what happens to the relation between groups induced through eqs. (9.12) and (9.13). Analytic continuation $x^0 = -ix^4$ leads to

$$M = x^0\mathbb{1} + x^j\sigma_j. \tag{9.19}$$

For real coefficients x^μ, the matrix M is now the most general hermitian 2×2 matrix with determinant $\det(M) = (x^0)^2 - \vec{x}^2$. A transformation

$$M \to M' = LML^\dagger, \tag{9.20}$$

with $L \in \mathrm{SL}(2,\mathbb{C})$ is such that M' also of the form (9.19) and it is also preserving the determinant. The sign of x^0 can not be changed through this transformation and the sign of L drops out. This establishes the correspondence

$$\mathrm{SO}^\uparrow(1,3) \cong \mathrm{SL}(2,\mathbb{C})/Z_2.$$

We will later use this to obtain the transformation behaviour of relativistic fermions.

[Exercise: Investigate the group $\mathrm{SO}(2,2)$ as another interesting real form of $\mathrm{SO}(4,\mathbb{C})$. What isomorphism between Lie algebras can you establish? What is the corresponding relation between the Lie groups?]

9.4 Pauli spinors

In the non-relativistic description of spin-1/2 particles due to Pauli they are described by a complex two-component spinor

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

and the generators of rotation are given by the Pauli matrices as given in eq. (5.6),

$$J_i = \frac{1}{2}\sigma_i.$$

We recognize of course the fundamental representation of $\mathfrak{su}(2)$. Concretely, an infinitesimal rotation

$$\Lambda^i{}_j = \delta^i{}_j + \delta\omega^i{}_j,$$

corresponds to

$$L(\Lambda) = \mathbb{1} + \frac{i}{4}\delta\omega^{ij}\,\epsilon_{ijk}\,\sigma_k.$$

By exponentiating this one obtains the rotation map acting on the spinors. Note, however, that the group $\mathrm{SU}(2)$ covers $\mathrm{SO}(3)$ twice in the sense that a rotation by 2π corresponds to $L(\Lambda) = -\mathbb{1}$.

Discrete transformations such as space parity or time reflection need to be established in addition to the rotations.

9.5 Relativistic spinors

Left and right handed spinor representation. We will construct the left and right handed spinor representations of the Lorentz group by using that they agree with the Pauli representation for normal (spatial) rotations. When acting on the left-handed representation (2,1), the generator $\tilde{N}_j$ vanishes. Since $J_j = N_j + \tilde{N}_j$ and $K_j = i(N_j - \tilde{N}_j)$ one has

$$N_j = J_j = -iK_j = \frac{1}{2}\sigma_j, \qquad \tilde{N}_j = 0.$$

Using (9.14) and (9.15) this yields for the left handed spinor representation

$$
\begin{aligned}
(M^L_{jk}) &= \epsilon_{jkl} N_l = \frac{1}{2}\epsilon_{jkl}\,\sigma_l, \\
(M^L_{j0}) &= iN_j = i\frac{1}{2}\sigma_j.
\end{aligned}
\tag{9.21}
$$

As the name suggests, this representation acts in the space of left-handed spinors which are two-components entities, for example

$$
\psi_L = \begin{pmatrix} (\psi_L)_1 \\ (\psi_L)_2 \end{pmatrix}.
$$

We also use a notation with explicit indices $(\psi_L)_a$ with $a = 1, 2$. The infinitesimal transformation in (9.2) reads with the matrices (9.21)

$$
\delta(\psi_L)_a = \frac{i}{2}\delta\omega^{\mu\nu}(M^L_{\mu\nu})_a{}^b\,(\psi_L)_b. \tag{9.22}
$$

Similarly one finds for the right-handed spinor representation (1,2) using

$$
N_j = 0, \qquad \tilde{N}_j = J_j = iK_j = \frac{1}{2}\sigma_j,
$$

the relations

$$
\begin{aligned}
(M^R_{jk}) &= \epsilon_{jkl}\tilde{N}_l = \frac{1}{2}\epsilon_{jkl}\,\sigma_l, \\
(M^R_{j0}) &= -i\tilde{N}_j = -i\frac{1}{2}\sigma_j.
\end{aligned}
\tag{9.23}
$$

The representation (9.23) acts in the space of right handed spinors, for example

$$
\psi_R = \begin{pmatrix} (\psi_R)^1 \\ (\psi_R)^2 \end{pmatrix}.
$$

For right handed spinors we will also use a notation with an explicit index that has a dot in order to distinguish it from a left-handed index, $(\psi_R)^{\dot a}$ with $\dot a = 1, 2$. The infinitesimal transformation in (9.2) reads with the matrices in (9.23)

$$
\delta(\psi_R)^{\dot a} = \frac{i}{2}\delta\omega^{\mu\nu}(M^R_{\mu\nu})^{\dot a}{}_{\dot b}\,(\psi_R)^{\dot b}.
$$

Invariant symbols. From the discussion in section 6.3 or directly from the tensor product decomposition

$$
(2,1)\otimes(2,1) = (1,1)_A \oplus (3,1)_S,
$$

it follows that there must be a Lorentz-singlet with two left-handed spinor indices and that it has to be anti-symmetric. The corresponding invariant symbol can be taken as ε_{ab} with components $\varepsilon_{21} = 1$, $\varepsilon_{12} = -1$ and $\varepsilon_{11} = \varepsilon_{22} = 0$. Indeed one finds that

$$
(M_L^{\mu\nu})_a{}^c\varepsilon_{cb} + (M_L^{\mu\nu})_b{}^c\varepsilon_{ac} = 0. \tag{9.24}
$$

(This is essentially due to $\sigma_j\sigma_2 + \sigma_2\sigma_j^T = 0$ for $j = 1, 2, 3$.) It is natural to use ε_{ab} and its inverse ε^{ab} to pull the indices a, b, c up and down. For clarity the non-vanishing components are

$$\varepsilon^{12} = -\varepsilon^{21} = \varepsilon_{21} = -\varepsilon_{12} = 1. \tag{9.25}$$

The symbol δ^a_b is also invariant when spinors with upper left-handed indices have the Lorentz-transformation behavior

$$\delta(\psi_L)^a = -\frac{i}{2}\delta\omega_{\mu\nu}(\psi_L)^b(M_L^{\mu\nu})_b{}^a.$$

From eq. (9.24) it follows also that

$$(M_L^{\mu\nu})_{ab} = (M_L^{\mu\nu})_{ba},$$

so that

$$\varepsilon^{ab}(M_L^{\mu\nu})_{ab} = (M_L^{\mu\nu})_a{}^a = 0.$$

In a completely analogous way the relation

$$(1, 2) \otimes (1, 2) = (1, 1)_A \oplus (1, 3)_S$$

implies that there is a Lorentz singlet with two right-handed spinor indices. The corresponding symbol can be taken as $\varepsilon^{\dot{a}\dot{b}}$, with inverse $\varepsilon_{\dot{a}\dot{b}}$, with components as in (9.25). This symbol is used to lower and raise right-handed indices. Spinors with lower right handed index transform under Lorentz-transformations as

$$\delta(\psi_R)_{\dot{a}} = -\frac{i}{2}\delta\omega_{\mu\nu}(\psi_R)_{\dot{b}}(M_R^{\mu\nu})^{\dot{b}}{}_{\dot{a}}. \tag{9.26}$$

Consider now an object with a left-handed and a right-handed index. It is in the representation $(2, 2)$ which should also contain the vector. There is therefore an invariant symbol which can be chosen as

$$(\sigma^\mu)_{a\dot{a}} = (\mathbb{1}, \vec{\sigma}),$$

and similarly

$$(\bar{\sigma}^\mu)^{\dot{a}a} = (\mathbb{1}, -\vec{\sigma}).$$

It turns out that the matrices for infinitesimal Lorentz transformations can be written as

$$(M_L^{\mu\nu})_a{}^b = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_a{}^b,$$

$$(M_R^{\mu\nu})^{\dot{a}}{}_{\dot{b}} = \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{a}}{}_{\dot{b}}.$$

Some useful identities are

$$
\begin{aligned}
(\sigma^\mu)_{a\dot a}(\sigma_\mu)_{b\dot b} &= -2\,\varepsilon_{ab}\varepsilon_{\dot a\dot b}\,,\\
(\bar\sigma^\mu)^{\dot a a}(\sigma_\mu)_{b\dot b} &= -2\varepsilon^{ab}\varepsilon^{\dot a\dot b},\\
\varepsilon^{ab}\varepsilon^{\dot a\dot b}(\sigma^\mu)_{a\dot a}(\sigma^\nu)_{b\dot b} &= -2\,\eta^{\mu\nu},\\
(\bar\sigma^\mu)^{\dot a a} &= \varepsilon^{ab}\varepsilon^{\dot a\dot b}(\sigma^\mu)_{b\dot b},\\
(\sigma^\mu\bar\sigma^\nu+\sigma^\nu\bar\sigma^\mu)_a{}^b &= -2\,\eta^{\mu\nu}\delta_a^b,\\
\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) &= \mathrm{Tr}(\bar\sigma^\mu\sigma^\nu) = -2\,\eta^{\mu\nu},\\
\bar\sigma^\mu\sigma^\nu\bar\sigma_\mu &= 2\,\bar\sigma^\nu,\\
\sigma^\mu\bar\sigma^\nu\sigma_\mu &= 2\,\sigma^\nu.
\end{aligned}
$$

Complex conjugation. Note that the matrices (9.21) and (9.23) are hermitian conjugate of each other, i. e.

$$(M^L_{\mu\nu})^\dagger = M^R_{\mu\nu}, \qquad (M^R_{\mu\nu})^\dagger = M^L_{\mu\nu}.$$

The hermitian conjugate of the Lorentz transformation (9.22) is given by

$$[\delta(\psi_L)_a]^\dagger = -\frac{i}{2}\delta\omega^{\mu\nu *}\,[(\psi_L)_b]^\dagger\,\underbrace{\left[(M^L_{\mu\nu})_a{}^b\right]^\dagger}_{=(M^R_{\mu\nu})^{\dot b}{}_{\dot a}}. \tag{9.27}$$

For $\delta\omega^{\mu\nu}\in\mathbb{R}$ this is of the same form as eq. (9.26). In Minkowski space it is therefore consistent to take $\psi_L^\dagger$ to be a right-handed spinor with lower dotted index. We write

$$[(\psi_L)_a]^\dagger = (\psi_L^\dagger)_{\dot a},$$

and in an analogous way one finds that it is consistent to write

$$\left[(\psi_R)^{\dot a}\right]^\dagger = (\psi_R^\dagger)^a.$$

More generally we define new fields

$$(\bar\psi_L)_{\dot a}, \qquad (\bar\psi_R)^a,$$

with transformation laws

$$
\begin{aligned}
\delta(\bar\psi_L)_{\dot a} &= -\frac{i}{2}\delta\omega_{\mu\nu}(\bar\psi_L)_{\dot b}(M_R^{\mu\nu})^{\dot b}{}_{\dot a},\\
\delta(\bar\psi_R)^a &= -\frac{i}{2}\delta\omega_{\mu\nu}(\bar\psi_R)^b(M_L^{\mu\nu})_b{}^a.
\end{aligned}
$$

Only in Minkowski space one may identify $(\bar\psi_L)_{\dot a} = (\psi_L^\dagger)_{\dot a}$ and $(\bar\psi_R)^a = (\psi_R^\dagger)^a$.

Weyl fermions. We now have everything to understand massless relativistic fermions. The Lagrange density for Weyl fermions is

$$\mathscr{L} = i(\bar{\psi}_L)_{\dot{a}}(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu(\psi_L)_b.$$

[Exercise: Check invariance of this Lagrangian under rotations and Lorentz boosts.]

To obtain the equation of motion we vary with respect to $\bar{\psi}_L$,

$$\frac{\delta}{\delta(\bar{\psi}_L)_{\dot{a}}(x)}S = \frac{\delta}{\delta(\bar{\psi}_L)_{\dot{a}}(x)}\int d^4x\, \mathscr{L} = i(\bar{\sigma}^\mu)^{\dot{a}b}\partial_\mu(\psi_L)_b(x) = 0.$$

This is the Weyl equation for a left-handed fermion. In Fourier representation, $\psi_L(x) \sim e^{-iEt+i\vec{p}\vec{x}}$, the Weyl equation reads

$$p_\mu\bar{\sigma}^\mu\psi_L = (-E\mathbb{1} - \vec{p}\vec{\sigma})\psi_L = 0.$$

Multiplying with $E\mathbb{1} - \vec{p}\vec{\sigma}$ from the left gives

$$\Big(-E^2\mathbb{1} + p_ip_j\{\sigma_i,\sigma_j\}\Big)\psi_L = 0.$$

With $\{\sigma_i,\sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = \mathbb{1}\delta_{ij}$ the dispersion relation is therefore

$$-E^2 + \vec{p}^2 = 0,$$

which describes massless particles, indeed.

Helicity. The spin of a massive particle is defined in the rest frame where one can choose one axis, say the z-axis for the label of states. For massless particles this does not work, they have no rest frame. One defines spin with respect to the momentum axis and defines helicity to be determined by the operator

$$h = \frac{\vec{\sigma}\vec{p}}{2E} = \frac{\vec{\sigma}\vec{p}}{2|\vec{p}|}.$$

For left-handed Weyl fermions this may be evaluated

$$h\psi_L = \frac{\vec{\sigma}\vec{p}}{2|\vec{p}|}\psi_L = (-1/2)\,\psi_L,$$

so they have helicity $-1/2$. Analogously, there are massless right-handed Weyl fermions with helicity $1/2$. [Exercise: Consider the Lagrangian

$$\mathscr{L} = i(\bar{\psi}_R)^a(\sigma^\mu)_{a\dot{b}}\partial_\mu(\psi_R)^{\dot{b}}$$

and show it describes massless particles with helicity $h = 1/2$.]

Transformations of fields. So far we have discussed how the "internal" indices of a field transform under Lorentz transformations. However, a field depends on a space-time position x^μ which also transforms. This is already the case for a scalar field,

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x).$$

(A maximum at x^μ is moved to a maximum at $\Lambda^\mu{}_\nu x^\nu$.) In infinitesimal form

$$(\Lambda^{-1})^\mu{}_\nu = \delta^\mu{}_\nu - \delta\omega^\mu{}_\nu,$$

and thus

$$\phi(x) \to \phi'(x) = \phi(x) - x^\nu \delta\omega^\mu{}_\nu \partial_\mu \phi(x).$$

This can also be written as

$$\phi'(x) = \left(1 + \frac{i}{2}\delta\omega^{\mu\nu}\mathcal{M}_{\mu\nu}\right)\phi(x),$$

with generator

$$\mathcal{M}_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu).$$

Indeed, these generators form a representation of the Lie algebra (9.5), i. e.

$$[\mathcal{M}_{\mu\nu}, \mathcal{M}_{\rho\sigma}] = i(\eta_{\mu\rho}\mathcal{M}_{\nu\sigma} - \eta_{\mu\sigma}\mathcal{M}_{\nu\rho} - \eta_{\nu\rho}\mathcal{M}_{\mu\sigma} + \eta_{\nu\sigma}\mathcal{M}_{\mu\rho}). \tag{9.28}$$

For fields with non-vanishing spin, the complete generator contains $\mathcal{M}_{\mu\nu}$ and the generator of "internal" transformations, for example

$$(\psi_L)_a(x) \to (\psi'_L)_a(x) = \left(\delta_a{}^b + \frac{i}{2}\delta\omega^{\mu\nu}(M_{\mu\nu})_a{}^b\right)(\psi_L)_b(x),$$

with

$$(M_{\mu\nu})_a{}^b = (M^L_{\mu\nu})_a{}^b + \mathcal{M}_{\mu\nu}\,\delta_a{}^b.$$

This can now be extended to fields in arbitrary representations of the Lorentz group.

9.6 Pioncaré group

Poincaré transformations consist of Lorentz transformations plus translations,

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu - b^\mu.$$

Translations only (without Lorentz transformations) form themselves an abelian Lie group, the additive group $\mathbb{R}^4$. It is clear that Poincaré transformations form a group. The composition law is

$$(\Lambda_2, b_2) \circ (\Lambda_1, b_1) = (\Lambda_2\Lambda_1, b_2 + \Lambda_2 b_1).$$

[Exercise: Show this.] The composition law is an example for a *semi-direct product*, namely of the Lorentz group $\mathrm{O}(1,3)$ and the additive group $\mathbb{R}^4$ of space and time translations,

$$\text{Poincaré group} \cong \mathrm{O}(1,3) \ltimes \mathbb{R}^4.$$

Lorentz transformations can be parametrized by six parameters, which are supplemented by four parameters for translations. The entire symmetry group of Minkowski space has therefore ten parameters.

Let us now find the Lie algebra associated with the Poincaré group. As transformations of fields, translations are generated by the momentum operator

$$P_\mu = -i\partial_\mu.$$

For example, as an infinitesimal transformation,

$$\begin{aligned}\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}(x+b))& \\ &= \phi(x^\mu - \delta\omega^\mu{}_\nu x^\nu + b^\mu) \\ &= \left(1 + \frac{i}{2}\delta\omega^{\mu\nu}\mathcal{M}_{\mu\nu} + ib^\mu P_\mu\right)\phi(x).\end{aligned}$$

One finds easily

$$[P_\mu, P_\nu] = 0, \tag{9.29}$$

and

$$[\mathcal{M}_{\mu\nu}, P_\rho] = i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right), \tag{9.30}$$

which together with (9.28) forms the Lie bracket relations of the Poincaré algebra. The commutator (9.29) tells that the different components of the energy-momentum operator can be diagonalized simultaneously, while (9.30) says that P_ρ transforms as a covector under Lorentz transformations.

9.7 Representations of the Poincaré group

Let us now discuss representations of the Poincaré algebra (and corresponding representations of the Poincaré group). We concentrate here on the part of the group that is connected to the identity transformations, i. e. $\mathrm{SO}^\uparrow(1,3) \ltimes \mathbb{R}^4$. It turns out that single-particle states can be understood as examples for such representations.

As we have done before, we will use a maximal number of commuting generators to label states. In particular, the different components of the momentum operator $P_\mu = -i\partial_\mu$ commute and we can work with corresponding eigenstates, namely plane waves $e^{ip_\mu x^\mu}$. The eigenvalues are then energy and momentum, $p_\mu = (-E, \vec{p})$.

Casimir operators. To classify representations, we first search for Casimir operators, i. e. operators that commute with all generators. One Casimir operator is

$$P^2 = P_\mu P^\mu,$$

which obviously commutes with $\mathcal{M}_{\mu\nu}$ and P_μ. For single particle states of massive particles we have $p_\mu p^\mu + M^2 = 0$ so that $-P^2 = M^2$ gives the particle mass. The other Casimir operator follows from the *Pauli-Lubanski vector*

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{M}_{\nu\rho}P_\sigma.$$

It is orthogonal to the momentum, $W^\mu P_\mu = 0$ and has the commutation relations

$$[W^\mu, P_\nu] = 0, \qquad [\mathcal{M}_{\rho\sigma}, W^\mu] = i\left(\delta^\mu_\rho W_\sigma - \delta^\mu_\sigma W_\rho\right),$$

as well as

$$[W^\mu, W^\nu] = -i\epsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma.$$

The second Casimir of the Poincaré algebra is then given by

$$W^2 = W_\mu W^\mu.$$

The little group. When discussing representations of the Poincaré group it is convenient to first make a case separation in terms of the quadratic Casimir $P^2 = P_\mu P^\mu$. In each of the cases one can then fix a reference choice for p^μ_* and discuss remaining transformations that leave this reference invariant,

$$(\delta^\mu{}_\nu + \delta\omega^\mu{}_\nu) p^\nu_* = p^\mu_*. \tag{9.31}$$

This remaining symmetry group is then known as the *little group*. We will see this working in practise below.

Representations with vanishing momentum. The eigenvalue of the momentum operator P_μ may actually simply vanish, $p^\mu_* = (0,0,0,0)$. In that case the little group corresponds to the entire Lorentz group SO(1, 3). An example for such a state is the vacuum.

Representations with positive mass squared. Let us now first consider situations with $-P^2 = M^2 > 0$. Examples for such representations are single particle states with positive mass.

We can fix a reference momentum $p^\mu_* = (M, 0, 0, 0)$ which corresponds to a particle momentum in its rest frame. The *little group* then consists of transformations that leave p^μ_* invariant. These are just rotations so the little group is here SO(3). More explicitly, this follows from searching solutions to (9.31) which is here equivalent to $\delta\omega^{\mu 0} = 0$. Lorentz boosts are excluded; what is left are rotations.

In the particles rest frame, the Pauli-Lubanski vector evaluates to

$$W_0 = 0, \qquad W_j = \frac{M}{2}\epsilon_{jkl}\mathcal{M}_{kl} = M J_j,$$

with angular momentum or spin operator J_j. The second Casimir of the Poincaré algebra is accordingly $W^2/M^2 = \vec{J}^2$. Single particle states $|p, j, m\rangle$ can be labeled by momentum p^μ, total spin $\vec{J}^2 = j(j+1)$ and eigenvalue m of the spin operator in z-direction J_3.

Representations with negative mass squared. Here we have a situation with $-P^2 = M^2 < 0$. This corresponds to so-called tachyonic modes and if they appear they are usually associated to an instability.

We can fix a reference momentum as $p^\mu_* = (0, 0, 0, M)$. The *little group* consists now of transformations that leave p^μ_* invariant and these are Lorentz transformations in the remaining $1 + 2$ dimensional space, SO(1, 2). We will not discuss these representations in more detail.

Representations with vanishing mass. Let us now consider representations with $P^2 = 0$. This is again a rather interesting case. Examples are here single particle states with vanishing mass $M = 0$.

Massless particles do not have a restframe, so to discuss the little group one must pick another reference momentum, for example $p_*^\mu = (p, 0, 0, p) = p(\delta_0^\mu + \delta_3^\mu)$. The little group consists of transformations that leave this invariant. Specifically, eq. (9.31) implies here $\delta\omega^{\mu 0} = \delta\omega^{\mu 3}$. One can write this as

$$\omega^{\mu\nu} = \begin{pmatrix} 0 & \alpha & \beta & 0 \\ -\alpha & 0 & \theta & -\alpha \\ -\beta & -\theta & 0 & -\beta \\ 0 & \alpha & \beta & 0 \end{pmatrix}.$$

Here, if only θ was non-vanishing, it would be the angle of a rotation in the 1-2-plane, i. e. around the propagation direction of the massless particle. Instead non-vanishing α would parametrize a combination of a boost in 1-direction together with a rotation in the 1-3 plane. Finally, β parametrizes a combination of a boost in the 2-direction with a rotation in 2-3-plane. An infinitesimal group transformation out of the litte group can be written as

$$\mathbb{1} + i\delta\theta J_3 + i\delta\alpha A + i\delta\beta B, \tag{9.32}$$

with

$$A = K_1 + J_2 = M_{10} + M_{31}, \qquad B = K_2 - J_1 = M_{20} + M_{32}.$$

The Lie algebra of the little group is

$$[J_3, A] = iB, \qquad [J_3, B] = -iA, \qquad [A, B] = 0. \tag{9.33}$$

This is in fact the Lie algebra of the so-called *special Euclidean group* $\mathrm{E}^+(2)$ consisting of translations and rotations in the two-dimensional Euclidean plane. It contains an SO(2) subgroup of rotations, as well as a subgroup of translations $\mathbb{R}^2$. The abelian subgroup of translations is in fact a *normal subgroup*. Similar to the Poincaré group itself, the Euclidean group $\mathrm{E}^+(2)$ has the structure of a direct product, $\mathrm{E}^+(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$. [Exercise: Check all this!]

In the fundamental representation of the Lorentz algebra, the operators A and B are actually *nilpotent*. In fact, one has $A^3 = B^3 = AB = BA = 0$. However, there are also representations of (9.33) where A and B are hermitian such that the group has a unitary representation. However, as for any non-compact group, such unitary representations are necessarily infinite dimensional.

Physically, A and B can be related to gauge transformations. To see this consider polarization vectors for photons with momentum p_*^μ,

$$\epsilon_\pm^\mu = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0).$$

These are eigenstates of J_3, namely in the fundamental or vector representation of the Lorentz group,

$$(J_3)^\mu{}_\nu \epsilon_\pm^\nu = \pm\epsilon_\pm^\mu.$$

The two polarizations $\epsilon^\mu_\pm$ describe therefore states with helicity ± 1, respectively. Now consider the action of (9.32) with $\delta\theta = 0$,

$$\epsilon^\mu_\pm \to \epsilon^\mu_\pm + \frac{(\delta\alpha \pm i\delta\beta)}{\sqrt{2}p} p^\mu_*.$$

This is in fact a gauge transformation! For massless particles in the spin-one or vector representation, invariance under the little group transformations implies gauge invariance! This works similarly for massless particles of spin two, where the gauge symmetry is then the one of general relativity.

In a gauge fixed description, physical states of single massless particles can be characterized as having vanishing eigenvalues with respect to the operators A and B. We are then left with J_3 which generates rotations around the direction of propagation. This is in fact helicity, $J_3 = h$.

Fermionic massless particle states can change by a factor -1 under rotations of 2π around the propagation direction. This implies half-integer helicity h. In contrast, bosonic massless particle states should be invariant under 2π rotations, so that helicity h must be integer valued. These quantization conditions arise here from topological properties of the group, and not from properties of the Lie algebra.

[Exercise: Consider massless fermions in the left-handed $(2,1)$ representation. Show that solutions of the corresponding equation of motion (the Weyl equation) are automatically invariant under transformations (9.32) for $\delta\theta = 0$ when the appropriate representations for A and B are chosen.]

9.8 Euclidean group and complex form

Sometimes it is useful to study also the analytic continuation of Minkowski space to Euclidean signature. The extension of $\mathrm{SO}(4,\mathbb{R})$ by Euclidean translations acts like $x^\mu \to R^\mu{}_\nu x^\nu - b^\mu$ and corresponds to the so-called special Euclidean group

$$\mathrm{E}^+(4) = \mathrm{SO}(4,\mathbb{R}) \ltimes \mathbb{R}^4.$$

The discussion of the Lie algebra is very similar as for the Poincaré group itself. Concerning representations, the Casimir operator $P^2 = P_\mu P^\mu$ is now positive semi-definite and we need to distinguish only the vacuum where $p^\mu_* = (0,0,0,0)$ such that the little group is SO(4), and states with $P^2 > 0$ where one may choose $p^\mu_* = (p,0,0,0)$. In the latter case the little group is the rotation group SO(3).

One may more generally also consider a complexified version, which contains complex rotations and translations in $\mathbb{C}^4$, and it has the direct product form

$$\mathrm{SO}(4,\mathbb{C}) \ltimes \mathbb{C}^4.$$

This is a rather large group that contains both the Euclidean group $E^+(4)$ and the Poincaré group as subgroups or real forms.

The Casimir P^2 is now a complex number. The vacuum state with $p^\mu_* = (0,0,0,0)$ has obviously the little group $\mathrm{SO}(4,\mathbb{C})$. There are also states with $P_\mu \neq 0$ but $P^2 = 0$

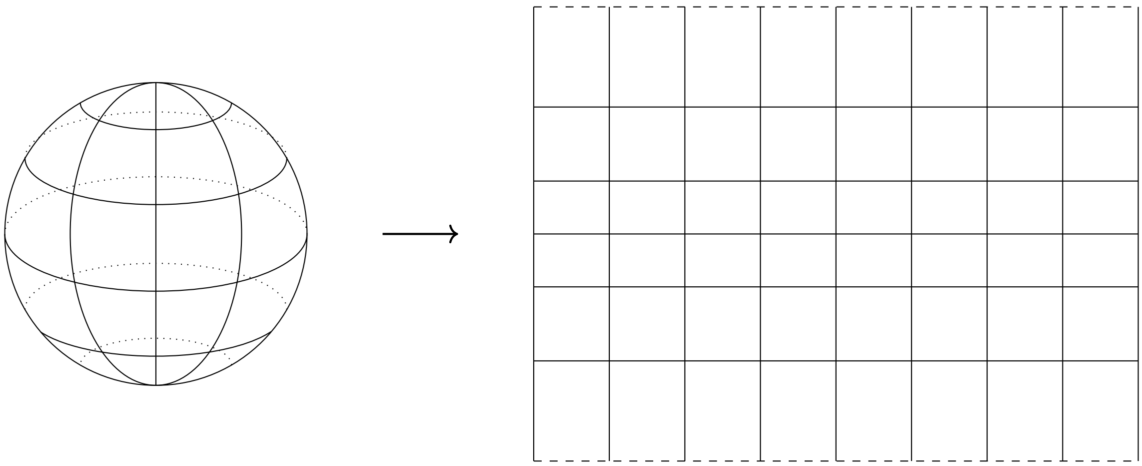
(massless states). Here one may take $p_*^\mu = (ip, 0, 0, p)$. The little group corresponds now to the complexified version of $\mathrm{E}^+(2)$ including complex rotations and translations in a two-dimensional space. This has the important subgroup $\mathrm{SO}(2, \mathbb{C})$ that is isomorphic to the multiplicative group of non-vanishing complex numbers $\mathbb{C}^*$.

Finally, there are states where $P^2 = p^2$ is some non-vanishing complex number. Here one may choose $p_*^\mu = (p, 0, 0, 0)$ and the little group is given by the complex rotation group in three dimensions, $\mathrm{SO}(3, \mathbb{C})$. The latter contains the real group $\mathrm{SO}(3, \mathbb{R})$ but also $\mathrm{SO}(1, 2)$ as subgroups.

10 Conformal group

We will now investigate transformations which include a (local) change of scale but preserve the angles between line segments. These will lead to the conformal group as an extension of the Poincaré group. Not all physical theories are invariant under this larger symmetry group, but a very interesting set of theories is. One example is the free Maxwell theory (without charges), another are theories at a renormalization group fixed point as it describes the critical point at a phase transition of second order.

A well known example for a conformal map is the Mercator projection.



Consider an infinitesimal transformations of the form

$$x^\mu \to x'^\mu(x) = x^\mu + \xi^\mu(x).$$

The differentials are related through

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\rho} dx'^\rho = \left(\delta^\mu_\rho - \frac{\partial}{\partial x^\rho} \xi^\mu(x) \right) dx'^\rho = \left(\delta^\mu_\rho - \partial_\rho \xi^\mu(x) \right) dx'^\rho.$$

The distance between nearby points as measured through the Riemann metric is unchanged by the transformation,

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = g'_{\mu\nu}(x') dx'^\mu dx'^\nu.$$

The transformation of the metric is for a general coordinate change given by

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = g_{\rho\sigma}(x) \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu}.$$

Now, any transformations such that

$$g'_{\mu\nu}(x') = \Omega^2(x') g_{\mu\nu}(x'), \tag{10.1}$$

is called a *conformal transformation.* Indeed, for a given point x this just changes the units with which lengths and times are measured, but does not affect angles.

Transformations of this type constitute a Lie group, the *conformal group*. As usual we will discuss this in terms of the Lie algebra that governs transformations close to the identity.

For an infinitesimal transformation we expand to linear order,

$$\Omega^2(x) = 1 + \kappa(x), \tag{10.2}$$

and we use

$$\frac{\partial x^\rho}{\partial x'^\mu} = \delta^\rho_\mu - \partial_\mu \xi^\rho(x).$$

Inserting this in (10.1) yields the *conformal Killing equation*

$$g_{\mu\sigma}\partial_\rho\xi^\mu + g_{\rho\nu}\partial_\sigma\xi^\nu + \xi^\lambda\partial_\lambda g_{\rho\sigma} + \kappa g_{\rho\sigma} = 0. \tag{10.3}$$

A solution $\xi^\mu(x)$ is called conformal Killing vector field. It parametrizes a change of coordinates for which the metric is invariant up to a conformal factor. Solutions with $\kappa = 0$ is called a Killing vector field and it parametrises a change of coordinates for which the metric does not change at all. This is obviously a stronger condition. [Exercise: Show that the expansion (10.2) together with (10.1) implies the conformal Killing equation (10.3).]

While the above considerations apply in principle to different spacetime geometries, it is not clear that there are always solutions to the (conformal) Killing equation. Not every space-time may have useful symmetries.

For many applications, the special case of the Minkowski metric $g_{\mu\nu}(x) = \eta_{\mu\nu}$ is most interesting. In that case, the conformal Killing equation (10.3) reduces to

$$\partial_\rho\xi_\sigma + \partial_\sigma\xi_\rho + \kappa\,\eta_{\rho\sigma} = 0.$$

For the case $\kappa = 0$ the solutions are at most linear in x,

$$\xi^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu,$$

with antisymmetric $\omega^{\mu\nu} = -\omega^{\nu\mu}$. These are precisely the Poincaré transformations we discussed in the previous section!

The Poincaré group is accordingly a subgroup of the larger conformal group and corresponds to $\Omega^2(x) = 1$. As a first example for the more interesting case of $\kappa \neq 0$ we consider the scaling transformation

$$x^\mu \to \lambda x^\mu. \tag{10.4}$$

This is an infinitesimal transformation for $\lambda = 1 + c$ with infinitesimal $c \in \mathbb{R}$. For the transformation (10.4) the conformal Killing vector and κ are given by

$$\xi^\mu = c x^\mu, \qquad \kappa = -2c.$$

[Exercise: Show this!]

Another example for a conformal transformation is the inversion (we introduce some arbitrary length l so that units match)

$$x^\mu \to x'^\mu(x) = \frac{x^\mu}{x^2} l^2. \tag{10.5}$$

We have

$$dx'^{\mu} = l^2 \frac{\delta^{\mu}_{\lambda} x^2 - 2x^{\mu} x_{\lambda}}{x^4} dx^{\lambda},$$

and thus

$$\eta_{\mu\nu} dx'^{\mu} dx'^{\nu} = \frac{l^4}{x^4} \eta_{\mu\nu} dx^{\mu} dx^{\nu}.$$

This is indeed a conformal transformation! However, such an inversion cannot be continuously deformed to the identity transformation, which means that we need to do more steps to find the Lie algebra of the conformal group.

Let us therefore consider the chain

$$x^{\mu} \overset{\text{inversion}}{\rightarrow} \frac{x^{\mu}}{x^2} l^2 \overset{\text{translation}}{\rightarrow} \frac{x^{\mu}}{x^2} l^2 + b^{\mu} l^2 \overset{\text{inversion}}{\rightarrow} \frac{\frac{x^{\mu}}{x^2} + b^{\mu}}{\eta_{\rho\sigma} \left(\frac{x^{\rho}}{x^2} + b^{\rho} \right) \left(\frac{x^{\sigma}}{x^2} + b^{\sigma} \right)}$$

and expand the final result to linear order in b,

$$x^{\mu} \to x'^{\mu} = x^{\mu} + b_{\lambda} (\eta^{\mu\lambda} x^2 - 2x^{\mu} x^{\lambda}).$$

This implies that

$$\xi^{\mu}(x) = b_{\lambda} (\eta^{\mu\lambda} x^2 - 2x^{\mu} x^{\lambda}),$$

must be a conformal Killing vector. (One may of course check this directly.) The transformations

$$x^{\mu} \to x^{\mu} + b_{\lambda} (\eta^{\mu\lambda} x^2 - 2x^{\mu} x^{\lambda}), \tag{10.6}$$

are also known as infinitesimal versions of *special conformal transformations*. In addition to the dilations (10.4), we have in $d = 4$ dimensions with the transformations (10.6) a $1 + 4 = 5$ parameter class of transformations in addition to the already existing $6 + 4 = 10$ parameters of the Poincaré group. The conformal group is rather large, which makes it quite powerful in practise.

Let us go back to the conformal Killing equation (10.3). By contracting it with $g^{\rho\sigma}$ one finds in Minkowski space with d the number of spacetime dimensions,

$$\kappa = -\frac{2}{d} \partial_{\mu} \xi^{\mu}.$$

Inserting this into the conformal Killing equation and taking another derivative yields

$$d \partial_{\mu} \partial^{\mu} \xi_{\sigma} = (2 - d) \partial_{\sigma} \partial_{\mu} \xi^{\mu}.$$

For $d = 2$ the equation $\partial_{\mu} \partial^{\mu} \xi_{\sigma} = 0$ has infinitely many solutions so that conformal symmetry is even more powerful. In fact, every holomorphic function induces then a conformal map if the two coordinates are identified with real and imaginary parts of a complex variable. In contrast, for $d > 2$, the vector field ξ_{σ} can be at most quadratic in x^{μ},

$$\xi^{\mu} = a^{\mu} + \omega^{\mu}{}_{\nu} x^{\nu} + c x^{\mu} + b_{\lambda} (\eta^{\mu\lambda} x^2 - 2x^{\mu} x^{\lambda}).$$

The analysis given above is complete in this sense.

Let us now find the Lie algebra. This is most easily done by writing the generators in a specific representation. When acting on fields, the generators of the conformal algebra are

$$\begin{aligned} D &= -ix^\mu \partial_\mu, \\ P_\mu &= -i\partial_\mu, \\ C_\mu &= -i(\delta_\mu^\nu x^2 - 2x_\mu x^\nu)\partial_\nu, \\ \mathcal{M}_{\mu\nu} &= -i(x_\mu \partial_\nu - x_\nu \partial_\mu). \end{aligned}$$

As we have seen for Lorentz transformations, it may be necessary to add an internal representation part when generators act on a specific field. For example, the dilation operator also contains a part that goes with the engineering dimension of a field. We already know the commutation relations of P_μ and $\mathcal{M}^{\mu\nu}$. We notice that

$$\begin{aligned} [iD, (x^{\alpha_1} \cdots x^{\alpha_n})] &= n(x^{\alpha_1} \cdots x^{\alpha_n}), \\ [iD, (\partial_{\alpha_1} \cdots \partial_{\alpha_n})] &= -n(\partial_{\alpha_1} \cdots \partial_{\alpha_n}). \end{aligned}$$

This means that iD "measures" in this sense the engineering dimension of an expression. The remaining non-trivial commutation relations can now be easily worked out and read

$$\begin{aligned} [D, P_\mu] &= iP_\mu, \\ [D, C_\mu] &= -iC_\mu, \\ [D, \mathcal{M}_{\mu\nu}] &= 0, \\ [C_\mu, C_\nu] &= 0, \\ [\mathcal{M}_{\mu\nu}, C_\rho] &= i(\eta_{\mu\rho} C_\nu - \eta_{\nu\rho} C_\mu), \\ [C_\mu, P_\nu] &= -2i(\mathcal{M}_{\mu\nu} + \eta_{\mu\nu} D). \end{aligned} \tag{10.7}$$

The first two lines tells that P_μ has dimension of momentum and C_μ has dimension of length, while the third line tells that $\mathcal{M}_{\mu\nu}$ is dimensionless and D transforms as a scalar under Lorentz transformations. The second to last line tells that C_ρ transforms appropriately under Lorentz transformations. [Exercise: Derive the commutation relations (10.7).]

One can actually show that the commutation relations of the conformal group in $d = 1 + 3$ dimensions correspond to the one of the indefinite orthogonal group in $d = 2 + 4$ dimensions. In other words, the conformal Lie algebra is actually $\mathfrak{so}(2, 4)$. [Exercise: Find this correspondence!]

11 Non-relativistic space-time symmetries

Quantum field theories can also be used in the non-relativistic regime, i. e. at energies that are much smaller than mc^2. As an example, consider the action for a scalar field

$$S = \int dt d^{d-1}x \left\{ \varphi^*(t,\vec{x}) \left[i\partial_t + \frac{\vec{\nabla}^2}{2m} - V(t,\vec{x}) \right] \varphi(t,\vec{x}) - \frac{\lambda}{2} \varphi^{*2}(t,\vec{x})\varphi^2(t,\vec{x}) \right\}. \tag{11.1}$$

We include here an external potential $V(t,\vec{x})$ that couples to the density $\varphi^*\varphi$ and a two-particle contact interaction strength λ. The action is written in one time and $d-1$ space dimensions. On a classical level, variation of (11.1) leads to the *Gross-Pitaevskii equation.*

What are the symmetries? Formally one could study the non-relativistic limit of the Poincaré algebra, but it is more transparent to establish the symmetries directly in the non-relativistic regime.

11.1 Symmetries of non-relativistic quantum field theories

Galilei group. As a low energy limit of the Poincaré group, we have in the non-relativistic regime the following space-time transformations

$$(t, x^j) \to (t + a, R^j{}_k x^k + v^j t + b^j). \tag{11.2}$$

This includes translations in time by an amount a, translations in space by $\vec{b}$, rotations in space through the matrix R and Galilei boosts with the velocity $\vec{v}$. A Galilei transformation consists of several elements $(R, a, \vec{v}, \vec{b})$, with the composition law

$$(R_2, a_2, \vec{v}_2, \vec{b}_2) \circ (R_1, a_1, \vec{v}_1, \vec{b}_1) = (R_2R_1, a_2 + a_1, \vec{v}_2 + R_2\vec{v}_1, \vec{b}_2 + R_2\vec{b}_1 + \vec{v}_2 a_1).$$

While the composition law looks a bit involved, it is clear that such Galilei transformations form a group, the Galilei group. In the following we study how the different elements are implemented as transformations of non-relativistic quantum fields.

Rotations. Rotations are realized as in the relativistic case with hermitian generators $J_j = \frac{1}{2}\epsilon_{jkl}M_{kl}$. When acting on a scalar field we have $M_{kl} = \mathcal{M}_{kl}$ where

$$\mathcal{M}_{kl} = -i(x_k\partial_l - x_l\partial_k).$$

The action on a scalar field is simply

$$\varphi(t,\vec{x}) \to \varphi'(t,\vec{x}) = \varphi(t, R^{-1}\vec{x}).$$

For non-relativistic spinor or tensor fields, this is supplemented by the appropriate generator acting on the "internal" representation of the field.

Translations in space and time. Translations in space and time are also implemented as in the relativistic case. They are generated by

$$P_0 = -H = -i\frac{\partial}{\partial t}, \qquad P_j = -i\frac{\partial}{\partial x^j}.$$

The action of a finite group element on the scalar field is

$$\varphi(t,\vec{x}) \to \varphi'(t,\vec{x}) = \varphi(t-a, \vec{x}-\vec{b}).$$

For an infinitesimal transformation this becomes

$$\varphi(t,\vec{x}) \to \varphi'(t,\vec{x}) = \left(1 - ida P_0 - idb^j P_j\right)\varphi(t,\vec{x}).$$

The last equation is for an infinitesimal transformation. Eigenfunctions are plane waves, $e^{-i\omega t + i\vec{p}\vec{x}}$, as usual.

Galilei boosts. Galilei boosts are realized in a somewhat non-trivial way. The scalar field transforms as

$$\varphi(t,\vec{x}) \to \varphi'(t,\vec{x}) = e^{im\vec{v}\vec{x} - i\frac{m\vec{v}^2 t}{2}}\varphi(t, \vec{x}-\vec{v}t). \tag{11.3}$$

The non-trivial point is here the appearance of an additional phase factor which involves the boost velocity $\vec{v}$. Expanding this to linear order in dv^j, we find that an infinitesimal transformation can be written as

$$\varphi(t,\vec{x}) \to \varphi'(t,\vec{x}) = \left(1 - idv^j K_j\right)\varphi(t,\vec{x}),$$

where the boost generator is

$$K_j = -mx_j - it\frac{\partial}{\partial x^j}.$$

In particular, we see that this depends through the first term on the particle mass m. We also make the observation that in contract to the generator of a Lorentz transformation, the generator of a Galilei boost is hermitian, $K_j = K_j^\dagger$. Indeed, the Galilei group can be made compact when time and space are compactified, such as with periodic boundary conditions.

Lie algebra. We can now state the commutation relations. We have

$$\begin{aligned}
&[P_j, P_k] = [P_0, P_k] = [K_j, K_k] = 0,\\
&[J_j, J_k] = i\epsilon_{jkl}J_l,\\
&[J_j, P_0] = 0\\
&[J_j, P_k] = i\epsilon_{jkl}P_l, \qquad [J_j, K_k] = i\epsilon_{jkl}K_l,\\
&[K_j, P_0] = -iP_j,\\
&[K_j, P_k] = -im\delta_{jk}.
\end{aligned} \tag{11.4}$$

Particularly interesting is the last line. It tells that the Lie algebra has been extended by a so-called *central charge*, given here by the particle mass m. Note that such a mass was not present when we first specified Galilei transformations in (11.2). As they stand, the commutation relations (11.4) form a *central extension* of the Galilei algebra. There are many different such central extensions of the algebra; in particular the mass m is not constrained, except for being real and positive. There are also representations of the Galilei algebra on fields that do not need such a central extension, for example in non-relativistic fluid dynamics.

Related to the central charge is the appearance of the U(1) phase factor in (11.3). One says that this is a *projective representation*; in some sense a representation of the Galilei group without central extension up to a phase. This comes with a *superselection rule*: linear superpositions of states with different mass are not allowed. Alternatively one may introduce one more element of the Lie algebra, a *mass operator* M which has eigenvalue m for a specific representation of a field for a particle with mass m. In that case, the last line in (11.4) is replaced by

$$\begin{aligned}&[K_j, P_k] = -iM\delta_{jk},\\ &[M, P_0] = [M, P_j] = [M, J_j] = [M, K_j] = 0.\end{aligned}$$

With this prescription we have again an ordinary representation (instead of a projective one), but now in a larger algebra. The mass operator commutes with all other operators, that is why one says that it is in the *center* of the algebra.

Galilei covariant derivative. It is instructive to work out how different elements in the action (11.1) transform under Galilei boost transformations. First of all, the interaction term is covariant,

$$\frac{\lambda}{2}\varphi^{*2}(t,\vec{x})\varphi^2(t,\vec{x}) \to \frac{\lambda}{2}\varphi^{*2}(t,\vec{x}-\vec{v}t)\varphi^2(t,\vec{x}-\vec{v}t).$$

The phase factors in (11.3) cancel out and the net effect is just the one of a translation in space by and amount $\vec{v}t$. Now consider the time derivative term,

$$\varphi^*(t,\vec{x})\,[i\partial_t]\,\varphi(t,\vec{x}) \to \varphi^*(t,\vec{x}-\vec{v}t)\left[i\partial_t + \frac{m\vec{v}^2}{2} - i\vec{v}\,\vec{\nabla}\right]\varphi(t,\vec{x}-\vec{v}t).$$

This is not covariant! Let us also consider the spatial derivative term

$$\varphi^*(t,\vec{x})\left[\frac{\vec{\nabla}^2}{2m}\right]\varphi(t,\vec{x}) \to \varphi^*(t,\vec{x}-\vec{v}t)\left[\frac{\vec{\nabla}^2}{2m} - \frac{m\vec{v}^2}{2} + i\vec{v}\,\vec{\nabla}\right]\varphi(t,\vec{x}-\vec{v}t).$$

We observe that neither the time derivative term nor the spatial derivative terms are Galilei covariant by themselves. However, their combination is. One may state this by saying that the combination

$$\mathscr{D} = i\partial_t + \frac{\vec{\nabla}^2}{2m}, \tag{11.5}$$

is acting as a *covariant derivative* with respect to Galilei boost transformations. In fact, for this combined derivative operator one has the Galilei boost transformation

$$\mathscr{D}\varphi(t,\vec{x}) \to \mathscr{D}e^{im\vec{v}\vec{x}-i\frac{m\vec{v}^2 t}{2}}\varphi(t,\vec{x}-\vec{v}t) = e^{im\vec{v}\vec{x}-i\frac{m\vec{v}^2 t}{2}}\mathscr{D}\varphi(t,\vec{x}-\vec{v}t).$$

The covariant derivative of a scalar field $\mathscr{D}\varphi$ transforms just as the field φ itself. Any power of $\mathscr{D}$ also has this property, this is important to construct invariant terms that can appear in a *quantum effective action*.

Global U(1) transformations. The action (11.1) has another symmetry, namely under global U(1) transformations,

$$\varphi(t,\vec{x}) \to e^{i\alpha}\varphi(t,\vec{x}), \qquad \varphi^*(t,\vec{x}) \to e^{-i\alpha}\varphi^*(t,\vec{x}).$$

This is rather easy to check. The physical consequence of this symmetry is the conservation law for particle number which can be stated in local form as

$$\partial_t n + \vec{\nabla}\vec{n} = 0.$$

Here $n = \varphi^*\varphi$ is the particle number density and

$$\vec{n} = -\frac{i}{m}\left(\varphi^*\vec{\nabla}\varphi - \varphi\vec{\nabla}\varphi^*\right),$$

is the corresponding current.

Time-dependent U(1) transformations. There is also a way to make the U(1) transformations time dependent,

$$\varphi(t,\vec{x}) \to e^{i\alpha(t)}\varphi(t,\vec{x}), \qquad \varphi^*(t,\vec{x}) \to e^{-i\alpha(t)}\varphi^*(t,\vec{x}).$$

One would say that this is in conflict with the time derivative term which transforms as

$$[i\partial_t]\,\varphi \to [i\partial_t]\,e^{i\alpha(t)}\varphi = e^{i\alpha(t)}\,[-\partial_t\alpha(t) + i\partial_t]\,\varphi.$$

However, this change can be compensated by a change in the external potential,

$$V(t,\vec{x}) \to V(t,\vec{x}) - \partial_t\alpha(t).$$

In other words, the combination

$$i\partial_t - V(t,\vec{x}),$$

acts as a *covariant derivative* with respect to time-dependent U(1) transformations! The extended combination

$$\mathscr{D} = i\partial_t + \frac{\vec{\nabla}^2}{2m} - V(t,\vec{x}),$$

is now a covariant derivative for both Galilei boost and time-dependent U(1) transformations.

As a special case of a time-dependent U(1) transformation consider

$$\varphi(t,\vec{x}) \to e^{-i\Delta V t}\varphi(t,\vec{x}), \qquad V(t,\vec{x}) \to V(t,\vec{x}) + \Delta V.$$

This changes the field φ by an oscillating phase and the external potential by an additive constant. For a massive *relativistic* scalar field, the solutions to the equations of motion have oscillations with frequency proportional to the rest mass, actually mc^2. In the non-relativistic limit we have taken $mc^2 \to \infty$ and the above symmetry is a direct consequence of this limit. In the non-relativistic quantum mechanical formalism this corresponds to the observation that the absolute scale of potential energy can be chosen at will.

11.2 Scaling transformations in the non-relativistic regime

The question comes up what happens to conformal transformations in the non-relativistic regime. There is in fact an algebra known as the *conformal Galilean algebra* that contains analogously to the relativistic conformal group also 15 generators. However, this is *not* the invariance group of the free Schrödinger equation. In fact, for the non-relativistic limit of a relativistic quantum field theory, the particle mass m is crucial. It is intuitively clear that this leads to a partial breaking of relativistic conformal invariance, because this is only a symmetry for theories with massless particles (or no quasi-particles at all).

Despite this, there is also a non-relativistic version of scaling symmetry and the symmetry group of the free Schrödinger equation, the so-called *Schrödinger group* has a Lie algebra that contains in addition to Galilei algebra (with central extension) in (11.4) two more generators.

Dilatations. To find the non-relativistic version of a dilatation let us first note that the operator in (11.5) contains a quadratic space and a linear time derivative. It is therefore not consistent to rescale time and space coordinates by the same factor but rather one should do the rescaling according to

$$(t, \vec{x}) \to \left(\lambda^2 t, \lambda \vec{x}\right). \tag{11.6}$$

In this sense, time counts in the non-relativistic domain as having dimension of length squared! (More precisely this works in units where mass m is effectively dimensionless.)

In the following we also need to know the scaling dimension of a scalar field as it appears in the action (11.1). The action S itself must be dimensionless. The combination $dt d^{d-1}x$ has with the counting we just established dimension of length to the power $d+1$. The time derivative term has length dimension (-2) and accordingly the field φ must have dimension of length to the power $(1-d)/2$, where $d-1$ is the number of spatial dimensions. We can now state the action of a dilation transformation acting on the field,

$$\varphi(t, \vec{x}) \to \varphi'(t, \vec{x}) = \lambda^{(1-d)/2} \varphi(\lambda^{-2} t, \lambda^{-1} \vec{x}).$$

For an infinitesimal transformation $\lambda = 1 + c$ this can be written as

$$\varphi(t, \vec{x}) \to \varphi'(t, \vec{x}) = (1 - icD)\, \varphi(t, \vec{x}),$$

with the dilatation operator acting acting on the scalar fields given by

$$D = -i \left(x^j \frac{\partial}{\partial x^j} + 2t \frac{\partial}{\partial t} \right) - i \frac{d-1}{2}.$$

This is such that iD measures the non-relativistic scaling dimensions of an expression, for example $[iD, x^j] = x^j$, $[iD, \partial_j] = -\partial_j$, $[iD, t] = 2t$ and $[iD, \partial_t] = -2\partial_t$.

We can immediately state the commutation relation with the generators of the Galilei algebra. They are given by

$$\begin{aligned} &[D, J_j] = 0, \qquad [D, P_j] = iP_j, \qquad [D, P_0] = 2iP_0, \\ &[D, K_j] = -iK_j, \qquad [D, M] = 0. \end{aligned} \tag{11.7}$$

Note again that mass is effectively dimensionless in the units measured by D. For vanishing external potential $V(t,\vec{x}) = 0$, and vanishing interaction strength $\lambda = 0$, the action (11.1) contains no dimensionfull external parameter and is then invariant under the dilatations generated by D.

Scaling symmetry in two dimensions. One may also work out the scaling dimension of the interaction λ and finds that it has dimension of length to the power $d-3$. In particular for $d-1=2$ spatial dimensions one finds that λ is dimensionless. On a classical level there is then accordingly a scaling symmetry also for the interacting system. However, this scaling symmetry is broken by quantum fluctuations. More specifically, the interaction parameter λ becomes scale dependent as a consequence of the renormalization group flow.

Time-dependent scaling or "Schrödinger expansion". There is another interesting symmetry of the free Schrödinger equation, namely it is invariant with respect to the time-dependent scaling

$$(t,\vec{x}) \to (t',\vec{x}') = \left(\frac{t}{1+ft}, \frac{\vec{x}}{1+ft}\right). \tag{11.8}$$

Here we are using a real parameter f with dimension of inverse time. As an infinitesimal transformation this reads

$$(t,\vec{x}) \to (t',\vec{x}') = (t - dft^2, \vec{x} - dft\vec{x}).$$

To see that the scaling (11.8) is indeed a symmetry requires a few calculations. We concentrate on the infinitesimal transformation where one has

$$dt = (1+2dft)dt', \qquad dx^j = (1+dft)dx'^j,$$

such that we can replace in the action (11.8)

$$dtd^{d-1}x \to [1+(d+1)dft]\, dt'd^{d-1}x'.$$

To transform also the other terms we need to study how partial derivatives are related. For the spatial derivative this is straight-forward,

$$\frac{\partial}{\partial x^j} = \frac{\partial t'}{\partial x^j}\frac{\partial}{\partial t'} + \frac{\partial x'^k}{\partial x^j}\frac{\partial}{\partial x'^k} = (1-dft)\frac{\partial}{\partial x'^j},$$

so there is just a time-dependent scaling. For the time derivative there is an extra term,

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'^k}{\partial t}\frac{\partial}{\partial x'^k} = (1-2dft)\frac{\partial}{\partial t'} - dfx^j\frac{\partial}{\partial x'^j}.$$

Without this extra term one could establish a symmetry of the action (11.8) by just rescaling also the fields in an appropriate way. Now we need also an additional position dependent phase factor. Accordingly, the infinitesimal transformation of the fields is

$$\varphi(t,\vec{x}) \to \varphi'(t,\vec{x}) = (1 - idfC)\,\varphi(t,\vec{x}),$$

with the generator for a *special conformal transformation* in the representation φ

$$C = -i\left(t^2\frac{\partial}{\partial t} + tx^j\frac{\partial}{\partial x^j}\right) - it\frac{d-1}{2} - \frac{1}{2}m\vec{x}^2. \tag{11.9}$$

Full Schrödinger algebra. We can now state the full Lie algebra of symmetries for the free Schrödinger equation. The Galilei algebra with central extension (11.4) is extended by the operators for non-relativistic dilations D with the commutation relations (11.7). In addition to this, there are special conformal transformations C with the additional commutation relations

$$\begin{aligned} &[C, J_j] = 0, \qquad [C, K_j] = 0, \qquad [C, P_j] = iK_j, \\ &[C, P_0] = iD, \qquad [C, D] = 2iC. \end{aligned} \tag{11.10}$$

This closes the algebra. Not counting the central generator for mass M we have 12 generators. Let us emphasize again that the full Schrödinger group can only be applied in exceptional cases such as the free Schrödinger equation or at a renormalization group fixed point of an interacting system.

The subgroup of Galilei transformations is more useful and applies quite generally to few-body systems in the non-relativistic regime. It can also be broken, however, as in a state of condensed matter. As an example, the fluid rest frame singles out a specific frame and therefore breaks Galilei boost symmetry. Also translation invariance and rotation symmetry can be broken in various ways, for example through the external potential $V(t, \vec{x})$.

[Exercise: Investigate the subgroup of the Schrödinger group formed by translations in time, dilatations, and special conformal transformations. Find the infinitesimal and finite group transformations and establish a correspondence to $\mathrm{SL}(2, \mathbb{R})$.]

12 Consequences of symmetries for effective actions

Now that we have discussed many different symmetries that appear in different physics problems, specifically in the context of quantum theory, let us pause for a moment and discuss what consequences follow from this. We will perform this discussion in the context of quantum field theory after analytic continuation from a space with Minkowski signature of the metric to a Euclidean signature.

12.1 Functional integral, partition function and effective action

We consider a theory for quantum fields $\chi(x)$ which we do not specify in further detail here. In practise $\chi(x)$ stands typically for a collection of different fields and may encompass components transforming as scalars, vectors, tensors, or spinors. The theory is described by a microscopic action $S[\chi]$. The latter enters the *partition function*

$$Z[J] = \int D\chi \, e^{-S[\chi]+\int_x J(x)\chi(x)}. \tag{12.1}$$

We use here the *functional integral* $\int D\chi$. We cannot go here into the details of its construction and refer to a lecture course on quantum field theory. From the partition function one defines the *Schwinger functional* $W[J] = \ln Z[J]$ and from there the *quantum effective action* or *one-particle irreducible effective action* Γ as a Legendre transform,

$$\Gamma[\phi] = \sup_J \left(\int_x J(x)\phi(x) - W[J] \right). \tag{12.2}$$

The effective action $\Gamma[\phi]$ depends on ϕ, which is the *expectation value* of the field χ. To see this one evaluates the supremum by varying the source field $J(x)$ leading to

$$\phi(x) = \frac{\delta}{\delta J(x)} W[J] = \frac{1}{Z[J]} \frac{\delta}{\delta J(x)} Z[J] = \frac{\int D\chi \, \chi(x) \, e^{-S[\chi]+\int J\chi}}{\int D\chi \, e^{-S[\chi]+\int J\chi}}.$$

One also writes this as

$$\phi(x) = \langle \chi(x) \rangle,$$

with the obvious definition of the expectation value $\langle \cdot \rangle$ in the presence of sources J. An interesting property of $\Gamma[\phi]$ is its equation of motion. It follows from the variation of (12.2) as

$$\frac{\delta}{\delta \phi(x)} \Gamma[\phi] = J(x). \tag{12.3}$$

In particular, for vanishing source $J = 0$, one obtains an equation that resembles very much the classical equation of motion, $\delta S/\delta\chi = 0$. However, in contrast to the latter, (12.3) contains all corrections from quantum fluctuations! Another interesting property is that tree-level Feynman diagrams become formally exact when propagators and vertices are taken from the effective action $\Gamma[\phi]$ instead of the microscopic action $S[\chi]$.

12.2 Symmetry transformations of microscopic action

Both the microscopic action $S[\chi]$ and the quantum effective action $\Gamma[\phi]$ are *functionals* of fields. When one speaks of a symmetry transformation of the action one means in practise a symmetry transformation of the fields on which the action depends. A symmetry of the microscopic action means an identity of the form

$$S[g\chi] = S[\chi].$$

Here the group element $g \in G$ is acting on the fields (not necessarily linearly) and the symmetry implies that the action is unmodified by this transformation. So far, G could be either a finite group, an infinite discrete group or a continuous Lie group. In the latter case one can decompose finite group transformations into infinitesimal transformations. One can write

$$S[g\chi] = S\left[\left(\mathbb{1} + id\xi^j T_j\right)\chi\right] = S[\chi] + \int_x \frac{\delta S[\chi]}{\delta\chi(x)}\, id\xi^j (T_j\chi)(x).$$

One also abbreviates this as $S[\chi] + dS[\chi]$ and a continuous symmetry corresponds then to the statement $dS[\chi] = 0$. This needs in particular the infinitesimal change in the field

$$\chi(x) \to \chi'(x) = \chi(x) + d\chi(x) = \chi(x) + id\xi^j (T_j\chi)(x), \tag{12.4}$$

where T_j is an appropriate representation of the Lie algebra acting on the fields (not necessarily linearly).

12.3 Symmetry transformation of integral measure

The functional integral measure can also be transformed. One says that it is invariant if

$$D\chi = D(g\chi).$$

An example for this would be a generalization of a change of integration variable $x^j \to f^j(\vec{x})$, where one would in general expect a Jacobian determinant, $d^N x = |\det(\partial x^j/\partial f^k)| d^N f(x)$. The above equation tells that this determinant is unity. (More generally one may also allow a field independent constant that can be dropped for many purposes.) Again this goes for elements $g \in G$ of finite, discrete or continuous groups G.

It can happen that one finds a symmetry of a microscopic action $S[\chi]$ but that the functional integral measure is *not* invariant. In that case one speaks of a *quantum anomaly*.

12.4 Continuous symmetries of effective actions

We now specialize to continuous transformations which we can study in infinitesimal form (12.4). After a change of integration variable $\chi \to g\chi$ we write the Schwinger functional (12.1) as

$$Z[J] = \int D\left(\chi + id\xi^j T_j\chi\right)\; e^{-S[\chi + id\xi^j T_j\chi] + \int_x J(x)\left(\chi(x) + id\xi^j T_j \chi(x)\right)}.$$

We now assume the invariance of the measure $D\left(\chi + id\xi^j T_j \chi\right) = D\chi$, or, in other words, the absence of an anomaly. This leads for small $d\xi^j$ to

$$\begin{aligned} Z[J] &= \int D\chi \; e^{-S[\chi + id\xi^j T_j\chi] + \int_x J(x)\left(\chi(x) + id\xi^j T_j \chi(x)\right)} \\ &= \int D\chi \; \left[1 + \int_x \left\{\left(-\frac{\delta}{\delta\chi(x)} S[\chi] + J(x)\right) id\xi^j T_j \chi(x)\right\}\right] \; e^{-S[\chi] + \int_x J(x)\chi(x)}. \end{aligned}$$

The leading term on the right hand side is just $Z[J]$ itself. Subtracting it we find using (12.3) the *Slavnov-Taylor identity*

$$\langle dS[\chi]\rangle = \left\langle \int_x \left(\frac{\delta}{\delta\chi(x)} S[\chi]\right) id\xi^j T_j \chi(x) \right\rangle = \int_x \left(\frac{\delta}{\delta\phi(x)}\Gamma[\phi]\right) id\xi^j \left\langle T_j \chi(x)\right\rangle. \tag{12.5}$$

An important class of transformations is such that the Lie algebra generators T_j act on the fields χ in a linear way. In that case one can write

$$\langle T_j \chi(x)\rangle = T_j \langle \chi(x)\rangle = T_j \phi.$$

In that case the right hand side of (12.5) can be written as $d\Gamma[\phi]$ and one has

$$\langle dS[\chi]\rangle = d\Gamma[\phi]. \tag{12.6}$$

In particular, the most important case is here that the microscopic action is invariant, $dS[\chi] = 0$, from which it follows that also the effective action is invariant, $d\Gamma[\phi] = 0$, or

$$\Gamma[g\phi] = \Gamma[\phi].$$

In summary, in the absence of a quantum anomaly, and for a linear representation of the Lie algebra on the fields, we conclude that the effective action $\Gamma[\phi]$ shares the symmetries of the microscopic action $S[\chi]$. This is very useful in practice because it constrains very much the form the effective action can have. This is important for example for proofs of renormalizability or also for solving renormalization group equations in practice.xx

Extended symmetries. Interestingly, eq. (12.6) is also useful when the microscopic action $S[\chi]$ is not fully invariant, i. e. $dS[\chi] \neq 0$. Specifically, if $dS[\chi]$ is *linear* in the field χ, one can infer that the effective action $\Gamma[\phi]$ must change in the same way such that (12.6) remains fulfilled. This can also constrain the form of $\Gamma[\phi]$ substantially. All this assumes invariance of the functional integral measure, though.

One can also often make use out of anomalous symmetries, i. e. symmetry transformations where the functional measure is not invariant, if the corresponding Jacobian determinant can be absorbed into a change of the action.

12.5 Conservation laws for field theories

We have already briefly discussed the close relation between continuous symmetries and conservation laws in terms of Noethers theorem earlier. Here we will briefly discuss how this extends to *classical* field theories as described by the action $S[\chi]$. An extension to quantum field theories and a description in terms of the effective action $\Gamma[\phi]$ will be discussed later.

Assume that we have an action $S[\chi]$ that is invariant under a *global* transformation of the fields

$$\chi(x) \to \chi(x) + id\xi^j T_j(x).$$

Here the adjective *global* refers to $d\xi^j$ being independent of position x. By the statement that $S[\chi]$ is invariant we mean $dS = 0$. More generally, one may allow the change of $S[\chi]$ by a boundary term, such that one can write dS for a global transformation as an integral over a total divergence,

$$dS = i\xi^j \int_x \partial_\mu \mathcal{F}_j^\mu(x).$$

Because such boundary terms do not contribute to the equations of motion one can drop them for many purposes and we will do so below.

As a calculational trick we allow now the transformation become local, $d\xi^j \to d\xi^j(x)$. We can then write up to boundary terms

$$dS = i\int_x \left\{ \mathcal{J}_j^\mu(x)\partial_\mu d\xi^j(x) + \mathcal{K}_j^{\mu\nu}(x)\partial_\mu\partial_\nu d\xi^j(x) + \ldots \right\}$$

Because we have assumed a global symmetry, the derivative expansion on the right hand side starts with the terms proportional to the first derivative $\partial_\mu d\xi^j$. The objects $\mathcal{J}_j^\mu(x)$ and $\mathcal{K}_j^{\mu\nu}(x)$ depend on the field χ which is evaluated at the solution to the classical field equation $\delta S/\delta\chi = 0$. In principle, higher orders in derivatives may appear, but typically, for microscopic actions $S[\chi]$ this is not the case and already $K^{\mu\nu}(x)$ often vanishes.

Let us now perform a partial integration. Dropping again boundary terms we find

$$dS = i\int_x \left\{ \left[-\partial_\mu \mathcal{J}_j^\mu(x) + \partial_\mu\partial_\nu \mathcal{K}_j^{\mu\nu}(x) - \ldots \right] d\xi^j(x) \right\}.$$

Because $d\xi^j(x)$ is arbitrary and for stationary action $dS = 0$, we find one local conservation law for each Lie algebra generator T_j,

$$\partial_\mu J_j^\mu(x) = 0,$$

where

$$J_j^\mu(x) = \mathcal{J}_j^\mu(x) - \partial_\nu \mathcal{K}_j^{\mu\nu}(x) + \ldots.$$

In principle this derivation would extend also to the quantum effective action $\Gamma[\phi]$ and should then give a conservation law for currents where quantum effects have been taken into account. However, in that case it is less clear that the expansion in powers of derivatives terminates, and the construction is therefore not as useful. A possibility to solve this problem is to extend the global symmetry to a local symmetry in a more sophisticated way, namely by introducing (external) gauge fields. This will be discussed in a later section.

13 Non-Abelian gauge theories

A gauge theory has a local symmetry as opposed to a global symmetry. Let us discuss this immediately for a non-abelian group, such as SU(N) (recall our discussion of matrix Lie groups in section 6.3). For some matter field $\psi^a(x)$ in the fundamental representation the local transformation reads

$$\psi^a(x) \to U^a{}_b(x)\psi^b(x) = \exp[i\xi^j(x)T_j]^a{}_b\,\psi^b(x) \tag{13.1}$$

where $U^a{}_b(x)$ depends now through the parameter fields $\xi^j(x)$ on space and time. In order to construct theories involving such local symmetries one needs a *covariant derivative* such that with the field transforming like (13.1) its covariant derivative transforms similarly,

$$(D_\mu(x))^a{}_b\psi^b(x) \to U^a{}_b(x)(D_\mu(x))^b{}_c\psi^c(x). \tag{13.2}$$

The covariant derivative is written as

$$(D_\mu(x))^a{}_b = \partial_\mu\delta^a{}_b - i(A_\mu(x))^a{}_b = \partial_\mu\delta^a{}_b - iA^j_\mu(x)(T_j)^a{}_b.$$

We use in the first equation a notation where the *gauge field* $(A_\mu(x))^a{}_b$ is a field which for every space-time position x and index μ is an element of the Lie algebra, i. e. it is matrix valued. One may decompose this further in terms of a set of generators of the Lie algebra T_j in the fundamental representation,

$$(A_\mu(x))^a{}_b = A^j_\mu(x)(T_j)^a{}_b, \tag{13.3}$$

so that $A^j_\mu(x)$ is the corresponding coefficient field.

Under the local transformation (13.1), the gauge field transforms as

$$(A_\mu(x))^a{}_b \to U^a{}_c(x)(A_\mu(x))^c{}_d(U^{-1})^d{}_b(x) - iU^a{}_c(x)\partial_\mu(U^{-1})^c{}_b(x). \tag{13.4}$$

Indeed this makes sure that (schematically) $D_\mu(x) \to U(x)D_\mu(x)U^{-1}(x)$, such that (13.2) is indeed fulfilled. Note that the first term in the transformation (13.4) is such that it is corresponds to the standard transformation law for a field with one upper and one lower index. (This is essentially how matter fields transform.) More precisely, we recognize here the *adjoint representation* of the gauge group. However, the second term spoils this and is characteristic for a *gauge field*. [Exercise: Consider two fields $(A_\mu(x))^a{}_b$ and $(B_\mu(x))^a{}_b$ that transform both as gauge fields. Show that the difference $(A_\mu(x))^a{}_b - (B_\mu(x))^a{}_b$ transforms as an ordinary matter field.]

One may also define the *field strength tensor* (also a Lie algebra valued field; we drop the indices $a, b, \dots$ for simplicity)

$$F_{\mu\nu}(x) = i[D_\mu(x), D_\nu(x)] = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i[A_\mu(x), A_\nu(x)].$$

Under gauge transformations this transforms covariantly (as a matter field in the adjoint representation of the gauge group),

$$F_{\mu\nu}(x) \to U(x)F_{\mu\nu}(x)U^{-1}(x).$$

Similar to (13.3) one can also write the field strength tensor as

$$(F_{\mu\nu}(x))^a{}_b = F^j_{\mu\nu}(x)(T_j)^a{}_b,$$

and using the structure constants

$$[T_j, T_k] = i f_{jk}{}^l T_l,$$

one finds then

$$F^j_{\mu\nu}(x) = \partial_\mu A^j_\nu(x) - \partial_\nu A^j_\mu(x) + f_{mn}{}^j A^m_\mu(x) A^n_\nu(x).$$

Finally, let us write down a kinetic term for gauge fields, as it appears in the Lagrangian. The construction principle is the action should be invariant. An allowed term involving only the gauge field is

$$S = \int d^dx \left\{ -\frac{1}{2g^2} \text{tr}\{F_{\mu\nu}(x)F^{\mu\nu}(x)\} \right\} = \int d^dx \left\{ -\frac{1}{2g^2} (F_{\mu\nu}(x))^a{}_b (F^{\mu\nu}(x))^b{}_a \right\}.$$

It is immediately clear that this is gauge invariant. One may also write this in terms of generators T_j. In that case the action becomes

$$S = \int d^dx \left\{ -\frac{c_\mathcal{R} \delta_{jk}}{2g^2} F^j_{\mu\nu}(x) F^{k\mu\nu}(x) \right\}.$$

We assumed here that the generators are normalized according to

$$\text{Tr}\{T_j T_k\} = c_\mathcal{R} \delta_{jk},$$

with a constant $c_\mathcal{R}$ that depends on the representation $\mathcal{R}$. For $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ we had chosen generators such that in the fundamental representation $c_\mathcal{R} = 1/2$ and that can be done more generally for $\mathfrak{su}(N)$. In fact one can relate $c_\mathcal{R}$ to the dimension $d_\mathcal{R}$ of the representation $\mathcal{R}$ (such that the generators T_j are in this representation $d_\mathcal{R} \times d_\mathcal{R}$ matrices), the dimension d_A of the adjoint representation (equal to the number of generators), and the eigenvalue of the quadratic Casimir operator in the representation $\mathcal{R}$ defined by

$$C_2 = T_j T_j = c_2^\mathcal{R} \mathbb{1}.$$

The relation is [Exercise: Show this!]

$$c_\mathcal{R} = \frac{d_\mathcal{R} c_2^\mathcal{R}}{d_A}.$$

13.1 Gauge group of the standard model

The gauge group of the standard model of elementary particle physics is

$$\text{SU}(3) \otimes \text{SU}(2) \otimes \text{U}(1).$$

The fermion fields and the Higgs boson scalar field can be classified into representations of the corresponding Lie algebras. With respect to the strong interaction group $SU(3)_{\text{colour}}$ we need the representations

$$\begin{array}{ll} \text{singlet} & \mathbf{1}, \\ \text{triplet} & \mathbf{3}, \\ \text{anti-triplet} & \mathbf{3}^*. \end{array}$$

With respect to the weak interaction group SU(2) we need

$$\begin{array}{ll} \text{singlets} & \mathbf{1}, \\ \text{doublets} & \mathbf{2}. \end{array}$$

Recall that SU(2) is pseudo-real so there is no independent $\mathbf{2}^*$. Finally with respect to the hypercharge group $U(1)_Y$ we will classify fields by their charge as generalisations of electric charge q. The charges turn out to be

$$0\,, \quad \pm\frac{1}{6}\,, \quad \pm\frac{1}{3}\,, \quad \pm\frac{1}{2}\,, \quad \frac{2}{3}\,, \quad \pm 1\,.$$

Moreover the fermions transform as Weyl spinors under the Lorentz group, either left- or right-handed. There are the following fields

$\begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	neutrino electron	left-handed	$\left(\mathbf{1},\ \mathbf{2},\ -\frac{1}{2}\right)$
$\left(\bar\nu_L\ \bar e_L\right)$	anti-neutrino anti-electron	right-handed	$\left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right)$
e_R	electron	right-handed	$\left(\mathbf{1},\ \mathbf{1},\ -1\right)$
$\bar e_R$	anti-electron	left-handed	$\left(\mathbf{1},\ \mathbf{1},\ 1\right)$
$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	up-quark down-quark	left-handed	$\left(\mathbf{3},\ \mathbf{2},\ \frac{1}{6}\right)$
$\left(\bar u_L\ \bar d_L\right)$	anti-up-quark anti-down-quark	right-handed	$\left(\mathbf{3}^*,\ \mathbf{2},\ -\frac{1}{6}\right)$
u_R	up-quark	right-handed	$\left(\mathbf{3},\ \mathbf{1},\ \frac{2}{3}\right)$
$\bar u_R$	anti-up-quark	left-handed	$\left(\mathbf{3}^*,\ \mathbf{1},\ -\frac{2}{3}\right)$
d_R	down-quark	right-handed	$\left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right)$
$\bar d_R$	anti-down-quark	left-handed	$\left(\mathbf{3}^*,\ \mathbf{1},\ \frac{1}{3}\right)$
ϕ	Higgs-doublet	scalar	$\left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right)$

where the last expression determines the representations under the gauge symmetries. The fields have several indices corresponding to the different groups, for example

$$\left(u_R\right)^{\dot{a}m}$$

where $\dot{a} \in \{1,2\}$ is the Lorentz spinor index and $m \in \{1,2,3\}$ is the $\mathrm{SU}(3)_{\mathrm{colour}}$ index.

The leptons and quarks come in three copies, also known as *families*. For example, in addition to the electrons and anti-electrons there are also muons and tau leptons with their corresponding anti-particles and an associated neutrino. For the quarks we have discussed this already.

In addition to these "matter fields", there are corresponding gauge bosons, specifically for $\mathrm{SU}(3)_{\mathrm{color}}$ the eight real gluons, for SU(2) three real gauge bosons and one for the abelian $\mathrm{U}(1)_Y$ subgroup. After spontaneous symmetry breaking, the $\mathrm{SU}(2) \otimes \mathrm{U}(1)_Y$ bosons combine into the two massive complex $W^{\pm}$ bosons, the neutral and massive Z boson and the massless photon. The symmetry breaking itself is due to an expectation value for the scalar Higgs field.

14 Grand unification

14.1 SU(5) unification

We now discuss a proposed extension of the Standard Model which leads to a unification of the gauge groups into SU(5). This has been proposed by *Howard Georgi* and *Sheldon Glashow* in 1974.

Note that the SU(3) and SU(2) generators naturally fit into SU(5) generators and similar for the spinors

$$\begin{pmatrix} \begin{pmatrix} 3\times 3 \end{pmatrix}_{\mathrm{SU}(3)} & \\ & \begin{pmatrix} 2\times 2 \end{pmatrix}_{\mathrm{SU}(2)} \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \\ \psi^5 \end{pmatrix}.$$

There are $5^2 - 1 = 24$ generators of SU(5) corresponding to the hermitian traceless 5×5 matrices. Out of them, eight generate SU(3), while three generate SU(2).

Moreover, within SU(5) there is one hermitian traceless matrix

$$\frac{1}{2}Y = \begin{pmatrix} -\frac{1}{3} &&&& \\ & -\frac{1}{3} &&& \\ && -\frac{1}{3} && \\ &&& \frac{1}{2} & \\ &&&& \frac{1}{2} \end{pmatrix}.$$

That generates a U(1) subgroup which actually gives $\mathrm{U}(1)_Y$. The remaining generators correspond to additional gauge bosons not present in the Standard Model so they are supposedly very heavy or confined. We find the embedding

$$\mathrm{SU}(5) \to \mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1).$$

Fundamental representation 5. Now let us consider representations. Take the fundamental representation of SU(5) the spinor ψ^m. From the above illustration one sees that it decomposes like

$$\mathbf{5} = \left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) \oplus \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right),$$

in a natural way. The conjugate decomposes

$$\mathbf{5}^* = \left(\mathbf{3}^*,\ \mathbf{1},\ \frac{1}{3}\right) \oplus \left(\mathbf{1},\ \mathbf{2},\ -\frac{1}{2}\right).$$

Indeed these could be the representations for the right-handed down quark and the anti-lepton doublet,

$$d_R, \qquad \begin{pmatrix} \bar\nu_L & \bar e_L \end{pmatrix},$$

and their anti-particles

$$\bar d_R, \qquad \begin{pmatrix} \nu_L \\ e_L \end{pmatrix},$$

respectively.

Note that the hypercharges indeed conspire to be such that the generator $\frac{1}{2}Y$ is indeed traceless. It can therefore be one of the generators of SU(5).

Antisymmetric tensor representation 10. So what about the other representations? The next smallest representation is the anti-symmetric tensor ψ^{mn} with dimension ten. We still need

$$\left(\mathbf{3},\ \mathbf{2},\ \frac{1}{6}\right), \qquad \left(\mathbf{3}^*,\ \mathbf{1},\ -\frac{2}{3}\right), \qquad \left(\mathbf{1},\ \mathbf{1},\ 1\right),$$

and the corresponding anti-fields. These are ten fields indeed. Now ψ^{mn} decomposes into irreducible representations according to

$$\begin{aligned}\left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) \otimes_A \left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) &= \left(\mathbf{3}^*,\ \mathbf{1},\ -\frac{2}{3}\right),\\ \left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) \otimes_A \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right) &= \left(\mathbf{3},\ \mathbf{2},\ \frac{1}{6}\right),\\ \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right) \otimes_A \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right) &= \left(\mathbf{1},\ \mathbf{1},\ 1\right).\end{aligned}$$

This matches indeed to $\bar{u}_R$, the left-handed quark doublet (u_L, d_L) and $\bar{e}_R$, respectively.

Note that we have used here tensor product decomposition relations discussed before such as for SU(3)

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}^*_A \oplus \mathbf{6}_S,$$

or for SU(2)

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_A \oplus \mathbf{3}_S.$$

The U(1) charges are simply added. Indeed things work out! Also in this sector one finds that the hypercharges add up to zero, $3 \times 1 \times (-\frac{2}{3}) + 3 \times 2 \times \frac{1}{6} + 1 \times 1 \times 1 = 0$.

All fermions. The fermion fields of a single generation in the Standard Model can be organised into the $SU(5)$ representations

$$\mathbf{5}^*: \quad \bar{d}_R, \ \begin{pmatrix} \nu_L \\ e_L \end{pmatrix},$$

and

$$\mathbf{10}: \quad \bar{u}_R,\ \bar{e}_R,\ \begin{pmatrix} u_L \\ d_L \end{pmatrix},$$

as well as the corresponding anti-fields. There is no space here for a right handed neutrino, it would have to be a singlet **1** under SU(5).

The scalar Higgs field could be part of a **5** scalar representation but the corresponding field with quantum numbers

$$\left(\mathbf{3},\ \mathbf{1},\ -\frac{2}{3}\right)$$

is not present in the Standard Model and must be very heavy or otherwise suppressed.

Gauge bosons. The gauge bosons of SU(5) can be found from decomposing $\mathbf{5} \otimes \mathbf{5}^* = \mathbf{24} + \mathbf{1}$. In terms of SU(3) $\otimes$ SU(2) $\otimes$ U(1) the $\mathbf{24}$ decomposes into

$$\mathbf{24} = \left(\mathbf{1},\ \mathbf{3},\ 0\right) \oplus \left(\mathbf{8},\ \mathbf{1},\ 0\right) \oplus \left(\mathbf{1},\ \mathbf{1},\ 0\right) \oplus \left(\mathbf{3},\ \mathbf{2},\ \frac{2}{3}\right) \oplus \left(\mathbf{3}^*,\ \mathbf{2},\ -\frac{2}{3}\right).$$

We recognize the W boson triplet, the gluons, the hypercharge photon and two more gauge bosons that transform under both $\mathrm{SU}(3)_{\mathrm{color}}$ and the electroweak group SU(2) $\times$ U(1). The latter type of gauge bosons could in principle induce transitions of the type

$$\begin{aligned} d &\to e^+, \\ u &\to \bar{u}, \end{aligned}$$

and thus $u + d \to \bar{u} + e^+$ causing

$$\begin{aligned} uud &\to u\bar{u} + e^+, \\ p &\to \pi^0 + e^+. \end{aligned}$$

The proton could therefore decay! This is actually one of the main experimental signatures for such grand unified theories.

Proton decay has not been observed so the transition rate must be very small. This also implies that the unification scale where the three forces $\mathrm{SU}(3)_{\mathrm{color}}$, SU(2) and $\mathrm{U}(1)_Y$ unite, must be very high. The latest experimental constraint is that the proton half-life time must be at least 1.6×10^{34} years [Super-Kamiokande, PRD 95, 012004 (2017)]. If the decay rate goes like

$$\Gamma \approx \frac{m_p^5}{M_{\mathrm{GUT}}^4},$$

one can estimate for the unification scale $M_{\mathrm{GUT}} > 10^{16}$ GeV.

The Georgi-Glashow model we discussed so far is not very realistic, in some sense it is already ruled out. For example it predicts massless neutrinos, which is in conflict with the observation of neutrino oscillations. Also the unification of renormalization group trajectories to a single SU(5) coupling constant at the scale M_{GUT} does not seem to work as it should.

Charge quantization. Besides the nice matching of the representations, there is another theoretical reason that speaks for a unified gauge theory. In the standard model it is not explained by electric charge to be a multiple of the electron charge (with fractional charges 1/3 for the quarks). In the SU(5) model the $\mathrm{U}(1)_Y$ generator is part of SU(5) and it is naturally explained why the charges have the values they have.

14.2 SO(10) unification

There are further possibilities to construct unified theories. The *Pati-Salam* model for example has the gauge group $\mathrm{SU}(4) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$.

Both the Georgi-Glashow and the Pati-Salam model can be further unified and embedded into the group SO(10). The unified gauge theory based on SO(10) is particularly elegant, but we will not discuss it in detail here. Let us just mention that there is a *spinorial* representation (similar to the left-handed or right-handed spinor representations of SO$(1,3)$ for chiral fermions) **16**, that decomposes in terms of SU(5) representations as

$$\mathbf{16} = \mathbf{10} \oplus \mathbf{5}^* \oplus \mathbf{1}.$$

This contains all the representations we need for the Georgi-Glashow model and therefore the standard model fermions of one generation as well as one additional fermion that has the quantum numbers of the right-handed neutrino! The latter is anyway needed for the *seesaw mechanism* to give mass to the observable neutrinos. For this mechanism to work, the right-handed neutrino is supposed to be very heavy. On the other side, in the SO(10) model, it is part of the **16** representation together with all the other fermions, so it is supposed to be massless. There must be some mechanism that breaks SO(10) at some high energy or mass scale and this mechanism needs to give the right-handed neutrino its mass. One can infer that the scale of SO(10) breaking shows up (albeit somewhat indirectly) through the seesaw mechanism in the observable neutrino masses and more directly in the right-handed neutrino mass, which is also a candidate for dark matter.

15 General coordinate and local Lorentz transformations

We started these lectures by stating that symmetries are closely related to conservation laws. In particular, the translational symmetries of the Poincaré group are responsible for energy and momentum conservation. However, when gravity is taken into account, space-time becomes curved, and Minkowski space is only a local approximation to it. One may then ask what happens to the conservation laws and also to concepts like spin or helicity that seem to be bounded to the symmetries of Minkowski space. To answer these questions, we will now extend our considerations to general coordinates on some quite general space-time manifold. The latter may be different from Minkowski space and may also be curved as is the case in general relativity as a consequence of gravitation. We will discuss here the *tetrad formalism* which allows to describe also fermionic in such a situation. The basic idea of the tetrad formalism is that one the one hand we use a general system of (non-cartesian) coordinates x^μ, but that on the other hand even a curved manifold is locally flat. Moreover, one can locally choose a frame such that different directions are orthogonal. The tetrad field will mediate between these two points of view and it will also allow to make Lorentz transformations *local*.

The present section assumes some familiarity with standard concepts of differential geometry as they are taught for example in a course on general relativity.

15.1 Tetrad formalism

The tetrad field can be defined through a local choice of an orthogonal frame, or formally as a Lorentz vector valued one-form $V_\mu{}^A(x)dx^\mu$. The latin index A is here a Lorentz index (in a sense to be made more precise below), while the Greek index μ is a general coordinate index. With Minkowski metric $\eta_{AB} = \text{diag}(-1, +1, +1, +1)$ one can write the Riemannian coordinate metric $g_{\mu\nu}(x)$ as

$$g_{\mu\nu}(x) = \eta_{AB} V_\mu{}^A(x) V_\nu{}^B(x). \tag{15.1}$$

We also introduce the *inverse* tetrad $V^\mu{}_A(x)$ such that

$$V_\mu{}^A(x)V^\nu{}_A(x) = \delta_\mu{}^\nu, \qquad V_\mu{}^A(x)V^\mu{}_B(x) = \delta^A{}_B.$$

Under a *general coordinate transformation* or *diffeomorphism* $x^\mu \to x'^\mu(x)$, the tetrad transforms like a coordinate covector or one-form

$$V_\mu{}^A(x) \to V'^A_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu{}^A(x).$$

Note that no transformation acts here on the Lorentz index A. In this formalism one must carefully distinguish between Lorentz and coordinate indices.

Changing afterwards the coordinate label from x'^μ back to x^μ gives the transformation rule

$$V_\mu{}^A(x) \to V'^A_\mu(x) = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu{}^A(x) - \left[V'^A_\mu(x') - V'^A_\mu(x)\right].$$

For an infinitesimal transformation $x'^\mu = x^\mu - \varepsilon^\mu(x)$ this reads

$$V_\mu{}^A(x) \to V_\mu{}^A(x) + \varepsilon^\nu(x)\partial_\nu V_\mu{}^A(x) + \left(\partial_\mu \varepsilon^\rho(x)\right) V_\rho{}^A(x) = V_\mu{}^A(x) + \mathcal{L}_\varepsilon V_\mu{}^A(x). \tag{15.2}$$

We are using here the *Lie derivative* $\mathcal{L}_\varepsilon$ in the direction $\varepsilon^\mu(x)$. More general, any coordinate tensor transforms under such infinitesimal coordinate transformations with the corresponding Lie derivative $\mathcal{L}_\varepsilon$.

In addition to coordinate transformations one may also consider *local Lorentz transformations* acting on the tetrad according to

$$V_\mu{}^A(x) \to V_\mu'^A(x) = \Lambda^A{}_B(x)\, V_\mu{}^B(x), \tag{15.3}$$

where $\Lambda^A{}_B(x)$ is at every point x a Lorentz transformation matrix such that

$$\Lambda^A{}_B(x)\Lambda^C{}_D(x)\eta_{AC} = \eta_{BD}.$$

Note that these local Lorentz transformations are *internal*, i. e. they do not act on the space-time argument x^μ of a field as a conventional Lorentz transformation would do.

In infinitesimal form, the local Lorentz transformation (15.3) reads

$$V_\mu{}^A(x) \to V_\mu'^A(x) = V_\mu{}^A(x) + \delta\omega^A{}_B(x)V_\mu{}^B(x), \tag{15.4}$$

where $\omega^{AB}(x) = -\omega^{BA}(x)$ is anti-symmetric.

Coordinate vectors and tensors can be transformed using the tetrad and its inverse to become scalars under general coordinate transformations according to

$$A^B(x) = V_\mu{}^B(x)A^\mu(x), \qquad T^{AB}(x) = V_\mu{}^A(x)V_\nu{}^B(x)T^{\mu\nu}(x).$$

The results are then Lorentz vectors and tensors, respectively.

More generally, a field Ψ might transform in some representation $\mathcal{R}$ with respect to the local, internal Lorentz transformations

$$\Psi(x) \to \Psi'(x) = L_\mathcal{R}(\Lambda(x))\Psi(x), \tag{15.5}$$

or infinitesimally

$$\Psi(x) \to \Psi'(x) = \Psi(x) + \frac{i}{2}\omega^{AB}(x)M^\mathcal{R}_{AB}\Psi(x).$$

In addition to various fields, we also need to consider their derivatives. The standard coordinate *covariant derivative* ∇_μ using the *Levi-Civita connection* creates coordinate tensors of higher rank when acting on coordinate scalar, vector or tensor fields. Moreover, one could contract with the inverse tetrad $V^\mu{}_A(x)$ to change the additional index from a coordinate index into a Lorentz index.

It is less clear at this point how to deal with fields that are already in some non-trivial representation with respect to local Lorentz transformations. For Lorentz vectors or tensors one could use the strategy to first transform them to coordinate vectors and tensors with help of the tetrad, use then the standard coordinate covariant derivative, and then transform the result back using again the tetrad. This shows that a real difficulty arises only for spinor fields in a corresponding representation $\mathcal{R}$ with respect to the Lorentz group. The problem is solved by the Lorentz covariant derivative in terms of the *spin connection*.

It is possible to define a covariant derivative $\mathscr{D}_\mu$ such that for the spinor field $\Psi(x)$ transforming under local Lorentz transformations according to (15.5) one has

$$V^\mu{}_A(x)\mathscr{D}_\mu\Psi(x) \to \Lambda_A{}^B(x)V^\mu{}_B(x)L_\mathcal{R}(\Lambda(x))\mathscr{D}_\mu\Psi(x).$$

In other words, the covariant derivative of some field transforms as before, with an additional transformation matrix for the new index, but without any extra non-homogeneous term.

The full covariant derivative is now

$$\mathscr{D}_\mu = \nabla_\mu + \mathbf{\Omega}_\mu(x),$$

where ∇_μ is the standard coordinate covariant derivative and where $\mathbf{\Omega}_\mu$ depends on the Lorentz representation of the field the derivative acts on. The spin connection $\mathbf{\Omega}_\mu(x)$ must transform like a non-abelian gauge field for local Lorentz transformations,

$$\mathbf{\Omega}_\mu(x) \to \mathbf{\Omega}'_\mu(x) = L_\mathcal{R}(\Lambda(x))\mathbf{\Omega}_\mu(x)L^{-1}_\mathcal{R}(\Lambda(x)) - \left[\partial_\mu L_\mathcal{R}(\Lambda(x))\right]L^{-1}_\mathcal{R}(\Lambda(x)).$$

We also write this for an infinitesimal Lorentz transformation $\Lambda^A{}_B(x) = \delta^A{}_B + \delta\omega^A{}_B(x)$ as

$$\mathbf{\Omega}_\mu(x) \to \mathbf{\Omega}'_\mu(x) = \mathbf{\Omega}_\mu(x) + \frac{i}{2}\delta\omega^{AB}(x)\left[M^\mathcal{R}_{AB}, \mathbf{\Omega}_\mu(x)\right] - \frac{i}{2}M^\mathcal{R}_{AB}\partial_\mu\delta\omega^{AB}(x).$$

This is the transformation rule of the adjoint representation plus the additional term required for a gauge field. Quite generally, one may write the spin connection as

$$\mathbf{\Omega}_\mu(x) = \Omega_\mu{}^{AB}(x)\frac{i}{2}M^\mathcal{R}_{AB}, \tag{15.6}$$

where $\Omega_\mu{}^{AB}(x)$ is anti-symmetric in the Lorentz indices A and B and now independent of the representation $\mathcal{R}$. Sometimes it is also called spin connection. Es examples we note here the covariant derivative of a Lorentz vector with upper index,

$$\mathscr{D}_\mu A^B(x) = \partial_\mu A^B(x) + \Omega_\mu{}^B{}_C(x)A^C(x),$$

and similarly for a lower Lorentz index,

$$\mathscr{D}_\mu A_B(x) = \partial_\mu A_B(x) - \Omega_\mu{}^C{}_B(x)A_C(x) = \partial_\mu A_B(x) + \Omega_{\mu B}{}^C(x)A_C(x).$$

We will define the spin connection $\Omega_\mu{}^A{}_B$ such that the fully covariant derivative of the tetrad vanishes,

$$\mathscr{D}_\mu V_\nu{}^A = \partial_\mu V_\nu{}^A + \Omega_\mu{}^A{}_B V_\nu{}^B - \Gamma^\rho_{\mu\nu}V_\rho{}^A = 0.$$

This implies directly the so-called metric compatibility condition $\nabla_\rho g_{\mu\nu} = 0$. One may solve the above relation for the spin connection, leading to

$$\Omega_\mu{}^A{}_B = -\left(\nabla_\mu V_\nu{}^A\right)V^\nu{}_B = -\left(\partial_\mu V_\nu{}^A\right)V^\nu{}_B + \Gamma^\rho_{\mu\nu}V_\rho{}^A V^\nu{}_B. \tag{15.7}$$

Note that the spin connection is here fully determined by the tetrad and its derivatives. As a consequence of (15.1) one can also express the Christoffel symbols through the tetrad. An expression for the spin connection that uses only the tetrad can be given as follows

$$\Omega_\mu{}^{AB} = \frac{1}{2} V^{A\nu} \left(\partial_\mu V^B_\nu - \partial_\nu V^B_\mu \right) - \frac{1}{2} V^{B\nu} \left(\partial_\mu V^A_\nu - \partial_\nu V^A_\mu \right) - \frac{1}{2} V^{A\rho} V^{B\sigma} \left(\partial_\rho V_{C\sigma} - \partial_\sigma V_{C\rho} \right) V^C_\mu. \tag{15.8}$$

Even though the spin connection is directly determined by the tetrad it is sometimes useful to keep it open and to vary it independent of the tetrad, at least at intermediate stages of a calculation.

It is useful to note also a relation known as *Cartans first structure equation for torsion*

$$T_{\mu\nu}{}^A = \partial_\mu V_\nu{}^A - \partial_\nu V_\mu{}^A + \Omega_\mu{}^A{}_B V_\nu{}^B - \Omega_\nu{}^A{}_B V_\mu{}^B.$$

In particular the right hand side vanishes in situations without space-time torsion. We also give the behavior under infinitesimal Lorentz transformations directly for $\Omega_\mu{}^A{}_B(x)$,

$$\Omega_\mu{}^A{}_B(x) \to \Omega'_\mu{}^A{}_B(x) = \Omega_\mu{}^A{}_B(x) + \delta\omega^A{}_C(x)\Omega_\mu{}^C{}_B(x) - \Omega_\mu{}^A{}_C(x)\delta\omega^C{}_B(x) - \partial_\mu \delta\omega^A{}_B(x). \tag{15.9}$$

This relation will be useful below. It is automatically fulfilled by (15.7).

Finally we note a useful identity for the variation of the spin connection that can be easily derived from (15.7),

$$\delta\Omega_\mu{}^A{}_B(x) = -\left(\mathscr{D}_\mu \delta V_\nu{}^A \right) V^\nu{}_B + \delta\Gamma^\rho_{\mu\nu} V_\rho{}^A V^\nu{}_B. \tag{15.10}$$

We use here the fully covariant derivative $\mathscr{D}_\mu$ and the variation of the Christoffel symbol

$$\delta\Gamma_\mu{}^\rho{}_\nu = \frac{1}{2} g^{\rho\lambda} \left(\nabla_\mu \delta g_{\nu\lambda} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu} \right). \tag{15.11}$$

Note that in contrast to the spin connection $\Omega_\mu{}^A{}_B(x)$ itself, which is a gauge field for local Lorentz transformations, its variation $\delta\Omega_\mu{}^A{}_B(x)$ transforms simply as a tensor with one upper and one lower index under local Lorentz transformations. Under coordinate transformations both $\Omega_\mu{}^A{}_B(x)$ and $\delta\Omega_\mu{}^A{}_B(x)$ transform as one-forms.

15.2 Response to coordinate and local Lorentz transformations

Let us discuss here the response of a quantum field theory to both general coordinate transformations and local Lorentz transformations. We will perform this discussion for a quantum effective action which we take to depend on a collection of matter fields $\Psi(x)$ (actually field expectation values), as well as the tetrad field $V_\mu{}^A(x)$ and the spin connection field $\Omega_\mu{}^A{}_B(x)$,

$$\Gamma[\Psi, V, \Omega].$$

Because of relation (15.7) or equivalently (15.8) the spin connection and the tetrad field are actually not independent of each other. Nevertheless, one may at an intermediate stage consider the spin connection as an independent field (for the moment anti-symmetric in A and B) and vary the effective action with respect to it. For stationary matter fields $\delta\Gamma/\delta\Psi = 0$, the variation of the effective action is

$$\delta\Gamma = \int d^dx\sqrt{g}\left\{\mathscr{T}^\mu{}_A(x)\delta V_\mu{}^A(x) - \frac{1}{2}\mathscr{S}^\mu{}_{AB}(x)\delta\Omega_\mu{}^{AB}(x)\right\}. \tag{15.12}$$

The field $\mathscr{T}^\mu{}_A(x)$ is defined through a variation with respect to the tetrad at fixed spin connection. It's physical significance will become more clear below. The variation with respect to the spin connection with otherwise fixed tetrad defines the current $\mathscr{S}^\mu{}_{AB}(x)$ which is in fact the *spin current*. Both $\mathscr{T}^\mu{}_A(x)$ and $\mathscr{S}^\mu{}_{AB}(x)$ transform as mixed coordinate and Lorentz tensors under coordinate transformations and local Lorentz transformations as indicated by their indices. The reason is that $\delta V_\mu{}^A(x)$ and $\delta\Omega_\mu{}^{AB}(x)$ are both transforming as tensors in this sense and the variation of the action itself must be a scalar.

We should also state here that a full variation of the effective action with respect to the tetrad (with the spin connection taken to obey relation (15.7) and therefore not taken as independent) leads to the energy momentum tensor as a mixed coordinate and Lorentz tensor,

$$\delta\Gamma = \int d^dx\sqrt{g}\, T^\mu{}_A(x)\delta V_\mu{}^A(x). \tag{15.13}$$

Using (15.10) we can relate the quantities in (15.12) and (15.13) and find

$$T^\mu{}_A(x) = \mathscr{T}^\mu{}_A(x) - \frac{1}{2}\mathscr{D}_\rho\left[-\mathscr{S}^{\rho\mu}{}_A + \mathscr{S}_A{}^{\rho\mu} + \mathscr{S}^{\mu\rho}{}_A\right]. \tag{15.14}$$

One can recognize this as the *Belinfante-Rosenfeld form* of the energy-momentum tensor with the first term $\mathscr{T}^\mu{}_A(x)$ being the canonical energy-momentum tensor and $T^\mu{}_A(x)$ its symmetric relative, written here as a mixed Lorentz and coordinate tensor. Note that the expression in square brackets in (15.14) is anti-symmetric in ρ and μ. This implies $\mathscr{D}_\mu T^\mu{}_A(x) = \mathscr{D}_\mu\mathscr{T}^\mu{}_A(x)$.

Let us now first discuss general coordinate transformations. The tetrad transforms according to eq. (15.2) and the spin connection in a fully analogous way with the Lie derivative $\mathcal{L}_\varepsilon$. It suffices at this point to consider the variation with spin connection taken as dependent, as in eq. (15.13). One finds after a bit of algebra

$$\begin{aligned}\delta\Gamma &= \int d^dx\sqrt{g}\, T^\mu{}_A(x)\left[\epsilon^\nu(x)\partial_\nu V_\mu{}^A(x) + V_\nu{}^A(x)\partial_\mu\epsilon^\nu(x)\right]\\ &= \int d^dx\sqrt{g}\,\epsilon^\nu(x)\left[-V_\nu{}^A(x)\mathscr{D}_\mu T^\mu{}_A(x) + T_{AB}(x)\Omega_\nu{}^{AB}(x)\right].\end{aligned} \tag{15.15}$$

We will see below that $T_{AB}(x) = T_{BA}(x)$ so that the second term on the right hand side of (15.15) drops out. Accordingly, because the variation $\delta\Gamma$ must vanish for any $\epsilon^\nu(x)$, the energy-momentum tensor $T^\mu{}_A(x)$ is covariantly conserved. One may easily bring this conservation law to the standard form $\nabla_\mu T^{\mu\nu}(x) = \nabla_\mu\mathscr{T}^{\mu\nu}(x) = 0$.

Besides general coordinate transformations, the action is invariant under local Lorentz transformations. We consider now such a transformation in infinitesimal form. The matter fields are still assumed to be stationary, $\delta\Gamma/\delta\Psi = 0$, so that it suffices to consider the variations of the tetrad and spin connection.

We first consider a variation where only the tetrad is being varied, and the spin connection is taken as dependent according to eq. (15.7). One finds

$$\delta\Gamma = \int d^dx\sqrt{g}\, T^{\mu A}(x)V_\mu{}^B(x)\delta\omega_{AB}(x).$$

Because this must vanish for arbitrary $\delta\omega_{AB}(x)$ one finds that the energy-momentum tensor is symmetric,

$$T^{AB}(x) = T^{BA}(x).$$

However, one may also do the calculation in an alternative way where the spin connection is first varied independent of the tetrad and we use then (15.4) and (15.9),

$$\begin{aligned}
\delta\Gamma &= \int d^dx\sqrt{g}\left\{\mathscr{T}^\mu{}_A(x)\delta V_\mu{}^A(x) - \frac{1}{2}\mathscr{S}^\mu{}_{AB}(x)\delta\Omega_\mu{}^{AB}(x)\right\}\\
&= \int d^dx\sqrt{g}\left\{\mathscr{T}^\mu{}_A(x)\delta\omega^A{}_B(x)V_\mu{}^B(x)\right.\\
&\qquad\left. - \frac{1}{2}\mathscr{S}^\mu{}_{AB}(x)\left[\delta\omega^A{}_C(x)\Omega_\mu{}^C{}_B(x) - \Omega_\mu{}^A{}_C(x)\delta\omega^C{}_B(x) - \partial_\mu\delta\omega^A{}_B(x)\right]\right\}.
\end{aligned} \tag{15.16}$$

Using partial integration one can rewrite this as

$$\delta\Gamma = \int d^dx\sqrt{g}\left[\mathscr{T}^{BA}(x) - \frac{1}{2}\mathscr{D}_\mu\mathscr{S}^{\mu AB}(x)\right]\delta\omega_{AB}(x).$$

For this to vanish for arbitrary $\delta\omega_{AB}(x)$ the expression in square brackets must be symmetric. Because $\mathscr{S}^{\mu AB} = -\mathscr{S}^{\mu BA}$ is anti-symmetric, we find for the divergence of the spin current

$$\mathscr{D}_\mu\mathscr{S}^{\mu AB}(x) = \left[\mathscr{T}^{BA}(x) - \mathscr{T}^{AB}(x)\right]. \tag{15.17}$$

We note that the spin current is in general *not* conserved. What needs to be conserved as a consequence of full Lorentz symmetry (also including a coordinate transformation) is the sum of spin current and orbital angular momentum current,

$$\mathscr{M}^{\mu AB}(x) = x^A(x)\mathscr{T}^{\mu B}(x) - x^B(x)\mathscr{T}^{\mu A}(x) + \mathscr{S}^{\mu AB}(x).$$

We assume here $\mathscr{D}_\mu x^A(x) = V_\mu{}^A(x)$ (which essentially defines what is meant by $x^A(x)$ in non-cartesian coordinates) one has indeed $\mathscr{D}_\mu\mathscr{M}^{\mu AB}(x) = 0$ as a consequence of (15.17) and the conservation law $\mathscr{D}_\mu\mathscr{T}^{\mu A}(x) = 0$.

16 Symmetries of the harmonic oscillator

This section considers a rather simple problem, but with enormous importance for physics: the harmonic oscillator. It has so many incarnations, in classical mechanics, quantum mechanics, non-relativistic and relativistic quantum field theory and also statistical physics that it is worth to study it in detail. The symmetry group of the harmonic oscillator is surprisingly large.

16.1 Classical and quantum mechanics

We consider first the mechanical formulation in terms of the action for a trajectory $\vec{x}(t)$,

$$S = \int dt \left\{ \frac{m}{2} \sum_{j=1}^{d-1} \left[\dot{x}_j^2(t) - \omega^2 x_j^2(t) \right] \right\}. \tag{16.1}$$

What are the symmetries that leave this action invariant? The only obvious symmetries are translations in time, $t \to t+\Delta t$, and spatial rotations, $x_j \to R_{jk} x_k$. Note that rotations are here seen as an *internal* transformation, because the "field" is the trajectory $x_j(t)$.

In addition to these obvious symmetries there are a number of somewhat hidden symmetries. The general transformation includes a transformation of the time coordinate,

$$t \to t' = \frac{1}{\omega} \arctan\left(\frac{\alpha \tan(\omega t) + \beta}{\gamma \tan(\omega t) + \delta} \right), \tag{16.2}$$

with $\alpha\delta - \gamma\delta = 1$. Interestingly, there is (at least locally) an isomorphism to $\mathrm{SL}(2,\mathbb{R})$, given by the relation

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \cosh\left(\frac{\sqrt{s_1^2 + s_2^2 + s_3^2}}{2} \right) \mathbb{1} + \sinh\left(\frac{1}{2}(s_1\sigma_1 + i s_2 \sigma_2 + s_3 \sigma_3) \right) \in \mathrm{SL}(2,\mathbb{R}).$$

In infinitesimal form, i. e. when s_j are infinitesimal, one has

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \mathbb{1} + i \left(s_1 \frac{-i\sigma_1}{2} + s_2 \frac{\sigma_2}{2} + s_3 \frac{-i\sigma_3}{2} \right) = \begin{pmatrix} 1 + s_3/2 & (s_1+s_2)/2 \\ (s_1 - s_2)/2 & 1 - s_3/2 \end{pmatrix},$$

and the transformation of the time coordinate reads then

$$t \to t' = t + s_1 \frac{\cos(2\omega t)}{2\omega} + s_2 \frac{1}{2\omega} + s_3 \frac{\sin(2\omega t)}{2\omega}.$$

This contains a translation $\sim s_2$, but also two non-linear periodic transformations $\sim s_1$ and $\sim s_3$. Note that for $\omega \to 0$ the term $\sim s_3$ becomes a dilation of time while the term $\sim s_1$ becomes to leading order a translation in time, and contains at subleading order also a an infinitesimal "Schrödinger expansion", see section 11.2. This explains how the symmetry group of the harmonic oscillator is related to the $\mathrm{SL}(2,\mathbb{R})$ part of the Schrödinger group.

The general symmetry transformation of the trajectory $x_j(t)$ is a linear map of the form

$$\begin{aligned} x_j(t) \to x_j'(t') = & \sqrt{\frac{1+\tan^2(\omega t)}{(\alpha \tan(\omega t) + \beta)^2 + (\gamma \tan(\omega t) + \delta)^2}} \\ & \times \left[R_{jk} x_k(t) + v_j \sin(\omega t) + a_j \cos(\omega t) \right], \end{aligned} \tag{16.3}$$

with rotation matrix $R \in \mathrm{O}(d-1)$ in $d-1$ spatial dimensions, vectors $\vec{v}, \vec{a} \in \mathbb{R}^{d-1}$.

The $\mathrm{SL}(2,\mathbb{R})$ transformation acts here both on the time argument of $x_j(t)$ as well as explicitly through a prefactor. In addition, there are rotations mediated by R_{jk} and two time-periodic translations in space proportional to v_j and a_j respectively. Note that in the limit $\omega \to 0$ these become Galilei boosts and spatial translations, respectively. Note that the full symmetry group is now as large as the one of the free Schrödinger equation we have studied in section 11.2 and in fact there is an isomorphism between the groups.

It is very useful to express (16.3) in infinitesimal form,

$$x_j(t) \to x'_j(t') = \left[1 - \frac{s_1}{2}\sin(2\omega t) + \frac{s_3}{2}\cos(2\omega t)\right] x_j(t) + \omega_{jk} x_k(t) + v_j \sin(\omega t) + a_j \cos(\omega t), \tag{16.4}$$

where again the right hand side must be evaluated at $t(t')$.

Let us establish that these are indeed symmetries of the action (16.1) step by step.

- First, rotations as parametrized in infinitesimal form by ω_{jk} are an obvious symmetry of (16.1). Note that the group $\mathrm{O}(d-1)$ includes also discrete reflections such as the *parity transform* $x_j \to -x_j$.

- The infinitesimal parameter s_2 parametrizes translations in time. Because (16.1) does not contain any explicit time dependence, this is also an obvious symmetry.

- The time dependent translations in space proportional to v_j and a_j are less obvious symmetries but easy to check. Up to a total derivative, the change in the action is

$$\delta S = \int dt \left\{ m \sum_{j=1}^{d-1} x_j(t) \left[-\delta\ddot{x}_j(t) - \omega^2 \delta x_j(t)\right] \right\}.$$

  With $\delta x_j(t) = v_j \sin(\omega t) + a_j \cos(\omega t)$ this vanishes, indeed.

- The transformations proportional to s_1 and s_3 are non-trivial because they act in a non-linear way on time t. Concentrating on the term proportional to s_1 (the one proportional to s_3 goes similarly) we have the leading order relations

$$\begin{aligned} dt &\to dt' = dt\left[1 - s_1 \sin(2\omega t)\right], \\ x_j(t) &\to x'_j(t') = \left[1 - \frac{s_1}{2}\sin(2\omega t)\right] x_j(t), \\ \dot{x}_j(t) &\to \frac{d}{dt'}x'(t') = \frac{dt}{dt'}\frac{d}{dt}x'(t') = \left[1 + \frac{s_1}{2}\sin(2\omega t)\right]\dot{x}_j(t) - s_1\omega\cos(2\omega t)x_j(t). \end{aligned} \tag{16.5}$$

  Inserting this in the action (16.1) one finds again to leading order

$$S \to \int dt \left\{ \frac{m}{2} \sum_{j=1}^{d-1} \left[\dot{x}_j^2(t) - \omega^2 x_j^2(t) - \frac{d}{dt}\left[s_1 x_j^2(t)\omega\cos(2\omega t)\right]\right] \right\}.$$

  Up to boundary terms, the action is indeed invariant!

In summary, we have found that the d-dimensional harmonic oscillator has a substantially larger symmetry group than apparent on first sight. In particular, the full symmetry group of the classical action (16.1) contains a factor $\mathrm{SL}(2,\mathbb{R})$. This group is particularly interesting because it contains time translations. This implies that the Hamiltonian (which generates time translations) is actually part of the corresponding Lie algebra.

16.2 Lie algebra $\mathfrak{sl}(2,\mathbb{R})$

Now that we have seen that $\mathrm{SL}(2,\mathbb{R})$ with its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ plays such an important role for the harmonic oscillator, let us say a few words about it.

Fundamental representation. The fundamental representation follows from writing an infinitesimal $\mathrm{SL}(2,\mathbb{R})$ matrix element as usual as

$$M = \mathbb{1} + i\xi^j T_j,$$

where one can choose the generators of the Lie algebra in terms of Pauli matrices,

$$T_1 = -i\frac{1}{2}\sigma_1, \qquad T_2 = \frac{1}{2}\sigma_2, \qquad T_3 = -i\frac{1}{2}\sigma_3. \tag{16.6}$$

Note that the generators differ from the ones of $\mathrm{SU}(2)$ by factors $-i$ for T_1 and T_3 and they agree with the ones we have introduced for $\mathrm{Sp}(2,\mathbb{R})$ in eq. (5.15). The Lie algebras $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{sp}(2,\mathbb{R})$ and also $\mathfrak{so}(2,1)$ actually agree.

Incidentally, this given another view on the $\mathrm{SL}(2,\mathbb{R})$ part of the symmetry group of the harmonic oscillator. It is in fact the group of symplectic transformations that leave the commutation relation $[x,p] = i$ invariant as we have discussed it in section (5.6). The latter can also be seen as the group of canonical transformations or of Bogoliubov transformations.

Note that the generators T_1 and T_3 in (16.6) are not hermitian (in agreement with the fact that the group is not compact). This implies that the fundamental representation of $\mathrm{SL}(2,\mathbb{R})$ is not unitary.

The commutation relations and structure constants of $\mathfrak{sl}(2,\mathbb{R})$ follow immediately through (16.6) and the commutation relations of the Pauli matrices,

$$[T_1, T_2] = iT_3, \qquad [T_2, T_3] = iT_1, \qquad [T_3, T_1] = -iT_2. \tag{16.7}$$

Because $\mathrm{SL}(2,\mathbb{R})$ is non-compact, the representation theory of the Lie algebra is more involved than for $\mathrm{SU}(2)$. For example, non-trivial finite representations are not unitary. Many of them can be obtained by analytic continuation from a representation of $\mathfrak{su}(2)$.

Singleton representations. As we have mentioned in section (5.6), the quadratic Casimir

$$C_2 = -T_1^2 + T_2^2 - T_3^2.$$

is not positive semi-definite. This implies that there can be infinite representations where C_2 remains finite. Two of them are particularly interesting, the so-called singleton representations. They can be constructed with the choice

$$T_1 = \frac{1}{4}(a^\dagger a^\dagger + aa), \qquad T_2 = \frac{1}{4}(aa^\dagger + a^\dagger a) = \frac{1}{4} + \frac{1}{2}a^\dagger a, \qquad T_3 = \frac{1}{4}(ia^\dagger a^\dagger - iaa), \tag{16.8}$$

where $a^\dagger$ and a are creation and annihilation operators with $[a, a^\dagger] = 1$. Note that the generators in (16.8) are actually hermitian so that the corresponding representation of the group is in fact unitary. However, we will see below that they act in an infinite space.

Note that T_2 is essentially the familiar Hamiltonian of the harmonic oscillator, $H = 2\hbar\omega T_2$. Moreover,

$$T_+ = T_1 - iT_3 = \frac{1}{2}a^\dagger a^\dagger, \qquad T_- = T_1 + iT_3 = \frac{1}{2}aa,$$

act as double creation and annihilation operators, respectively.

Assume now that there is a vacuum state $|0\rangle$ such that $a|0\rangle = 0$. Through application of T_+ we can create the states

$$|0\rangle, \qquad |2\rangle, \qquad |4\rangle, \qquad |6\rangle, \qquad \ldots$$

These form now a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. These states are eigenstates of T_2 and accordingly also energy eigenstates of the harmonic oscillator problem.

In a similar way we can start from the state $|1\rangle$ and create through application of T_+ the set

$$|1\rangle, \qquad |3\rangle, \qquad |5\rangle, \qquad |7\rangle, \qquad \ldots$$

This set is another representation of $\mathfrak{sl}(2,\mathbb{R})$. For the harmonic oscillator, the representation $|2n\rangle$ contains the energy eigenstates that are even under spatial parity $x \to -x$, while the representation $|2n+1\rangle$ contains parity odd energy eigenstates. For both of these representations the quadratic Casimir has the value $C_2 = -3/16$.

In summary, we find that the energy spectrum of the harmonic oscillator follows nicely from the Lie algebra representation theory of its $\mathrm{SL}(2,\mathbb{R})$ symmetry group. The reason is that time translations, as they are generated by the Hamiltonian, are actually a subgroup of $\mathrm{SL}(2,\mathbb{R})$.

16.3 Lie superalgebra

One can extend the Lie algebra generated by T_1, T_2 and T_3 to a so-called *Lie superalgebra*. This is an extension of the Lie algebra which includes a Z_2 grading. In the present context, this Z_2 grading corresponds to spatial parity (which can be either even or odd).

In general, a Lie superalgebra is closed with respect to a generalized bracket $[\![\cdot,\cdot]\!]$ for which antisymmetry is replaced by so-called super skew-symmetry,

$$[\![A, B]\!] = -(-1)^{|A||B|}[\![B, A]\!].$$

Here, $|A|$ is the Z_2 *grade* of the algebra element A. Obviously, for operators with grade $|A| = 0$ this corresponds to the standard anti-symmetry. In other words, for Z_2 even operators the generalized bracket is just the standard anti-commutator $[\![A, B]\!] = [A, B]$. For Z_2 odd operators it is the anti-commutator instead, $[\![A, B]\!] = \{A, B\}$. The Jacobi identity is also generalized and becomes for a Lie superalgebra

$$(-1)^{|A||C|}[\![A, [\![B, C]\!]]\!] + (-1)^{|B||A|}[\![B, [\![C, A]\!]]\!] + (-1)^{|C||B|}[\![C, [\![A, B]\!]]\!] = 0.$$

For the harmonic oscillator we extend the algebra of the parity even operators T_1, T_2 and T_3 in (16.8) by the two parity odd generators a and $a^\dagger$. Note that parity $x \to -x$ or $a \to -a$ is itself a part of the symmetry group of the harmonic oscillator. The graded bracket relations become then in addition to (16.7)

$$\begin{aligned}
\{a, a\} &= 4(T_1 + iT_3), & \{a^\dagger, a^\dagger\} &= 4(T_1 - iT_3), & \{a, a^\dagger\} &= 8T_2, \\
[a, T_1] &= \frac{1}{2}a^\dagger, & [a, T_2] &= \frac{1}{2}a, & [a, T_3] &= \frac{i}{2}a^\dagger, \\
[a^\dagger, T_1] &= -\frac{1}{2}a, & [a^\dagger, T_2] &= -\frac{1}{2}a^\dagger, & [a^\dagger, T_3] &= \frac{i}{2}a.
\end{aligned} \tag{16.9}$$

Graded algebras appear also in other contexts in physics, for example BRST transformations in the context of gauge theories, certain condensed matter systems, or supersymmetric extensions of the standard model.

As a representation of the Lie superalgebra, the two singleton representations of parity even and odd states combine, and form a single series of states,

$$|0\rangle, \qquad |1\rangle, \qquad |2\rangle, \qquad |3\rangle, \qquad \ldots$$

This infinite series contains now all energy eigenstates of the simple quantum harmonic oscillator.

16.4 Non-relativistic quantum field theory

Let us now discuss the symmetries of the harmonic oscillator in the non-relativistic quantum field theoretic formalism. The action is the one of a non-relativistic quantum field theory for scalar particles with harmonic external potential,

$$S = \int dt d^{d-1}x \left\{ \varphi^*(t, \vec{x}) \left[i\partial_t + \frac{\vec{\nabla}^2}{2m} - \frac{m\omega^2}{2}\vec{x}^2 \right] \varphi(t, x) \right\}. \tag{16.10}$$

Our goal is now to find the appropriate transformations of time and space coordinates, as well as of the field φ corresponding to the symmetry group introduced in section (16.1). For simplicity we concentrate on $R = \mathbb{1}$, $\vec{v} = \vec{a} = 0$, and work with infinitesimal transformations. (For a more complete discussion see [U. Niederer, Helvetica Physica Acta 46, 191 (1973)].)

For time we keep the transformations we introduced previously,

$$t \to t' = t + s_1 \frac{\cos(2\omega t)}{2\omega} + s_2 \frac{1}{2\omega} + s_3 \frac{\sin(2\omega t)}{2\omega}.$$

The transformation of spatial coordinates is adapted from (16.4) and reads in infinitesimal form

$$x_j \to x_j' = \left[1 - \frac{s_1}{2}\sin(2\omega t) + \frac{s_3}{2}\cos(2\omega t)\right] x_j.$$

Together, these transformations imply for the integration measure

$$dt d^{d-1}x \to dt' d^{d-1}x' = \left[1 - (1 + \tfrac{d-1}{2})s_1 \sin(2\omega t) + (1 + \tfrac{d-1}{2})s_3 \cos(2\omega t)\right] dt d^{d-1}x.$$

The field theoretic action in (16.10) is in fact invariant under this coordinate transformation together with the time and space dependent linear field transformation

$$\varphi(t,\vec{x}) \to \left[1 + \tfrac{d-1}{4} s_1 \sin(2\omega t) - \tfrac{d-1}{4} s_3 \cos(2\omega t) - i\tfrac{m\omega\vec{x}^2}{2}\left(s_1\cos(2\omega t) - s_3 \sin(2\omega t)\right)\right]\varphi(t,\vec{x}).$$

We leave the proof as an [exercise].

16.5 Relativistic quantum field theory

For a free real scalar field theory we can write the action as

$$S = \int dt \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \left\{ \frac{1}{2}\partial_t\phi^*(t,\vec{p})\partial_t\phi(t,\vec{p}) - \frac{1}{2}(\vec{p}^2 + m^2)\phi^*(t,\vec{p})\phi(t,\vec{p}) \right\}.$$

We are using here a representation in (spatial) Fourier modes such that

$$\phi(t,\vec{x}) = \int \frac{d^{d-1}x}{(2\pi)^{d-1}} e^{i\vec{p}\vec{x}}\phi(t,\vec{p}),$$

and because $\phi(t,\vec{x}) \in \mathbb{R}$ we have $\phi^*(t,\vec{p}) = \phi(t,-\vec{p})$. In $d = 1+0$ dimensions the situation is particularly simple and the action is just

$$S = \int dt \left\{ \frac{1}{2}\partial_t\phi(t)\partial_t\phi(t) - \frac{1}{2}m^2\phi(t)\phi(t) \right\}.$$

Here it is immediately clear that this is just the action of a harmonic oscillator in the quantum mechanical formalism. Similarly, for $d > 1$ we can consider a fixed momentum $\vec{p}$. The action is then formally equivalent to the one of a two-dimensional harmonic oscillator (with real and imaginary part of $\phi(t,\vec{p})$ corresponding to the two directions). The oscillation frequency is of course determined by the dispersion relation $\omega^2 = \vec{p}^2 + m^2$.

We have learned that the symmetry group for the harmonic oscillator contains a factor $\mathrm{SL}(2,\mathbb{R})$ including time translations and its "conformal extension". The Hamiltonian is part of the corresponding Lie algebra. Energy eigenstates form representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{sp}(2,\mathbb{R}) = \mathfrak{so}(2,1)$.

In a quantum field theory, the excitations correspond actually to particles. So we can say that the fact that a quantum field theory contains discrete excitations with quantized amounts of energy can actually be traced back to the representation theory of Lie algebras!