Lectures on quantum field theory 2

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ABSTRACT: Notes for lectures that introduce students of physics to advanced quantum field theory with applications to high energy physics, condensed matter and statistical physics. Prepared for a course at Friedrich-Schiller-University Jena in the winter term 2023/2024.

Contents

Literature

There is a large amount of literature on different aspects of quantum field theory. Here is only a fine selection.

Relativistic quantum field theory

- Mark Srednicki, *Quantum field theory*
- Michael Peskin & Daniel Schroeder, *An introduction to quantum field theory*
- Steven Weinberg, *The quantum theory of fields I & II*

Statistical field theory / renormalization group

- Jean Zinn-Justin, *Quantum field theory and critical phenomena*
- Andreas Wipf, *Statistical approach to quantum field theory*
- John Cardy, *Scaling and renormalization in statistical physics*
- Giorgio Parisi, *Statistical field theory*

Non-relativistic quantum field theory / condensed matter

- Alexander Altland & Ben Simons, *Condensed matter field theory*
- Lev Pitaevskii & Sandro Stringari, *Bose-Einstein condensation*
- Crispin Gardiner & Peter Zoller, *The quantum world of ultra-cold atoms and light*

Typos

Please send typos to stefan.floerchinger@uni-jena.de.

1 Generating functionals

Partition function

We start with the partition function for a complex scalar field in the presence of a source field J,

$$Z[J] = \int D\phi \exp \left[iS[\phi] + \int d^d x \left\{ J^*(x)\phi(x) + \phi^*(x)J(x) \right\} \right]$$

We work with a complex field ϕ which we could also express in terms of real and imaginary parts as

$$\phi(x) = \frac{1}{\sqrt{2}} \left[\phi_1(x) + i\phi_2(x) \right],$$

where $\phi_1(x) \in \mathbb{R}$ with $n = 1, 2$ are now real fields. Similarly for the source,

$$J(x) = \frac{1}{\sqrt{2}} \left[J_1(x) + iJ_2(x) \right].$$

With this, the source terms can also be written as

$$J^*(x)\phi(x) + \phi^*(x)J(x) = J_1(x)\phi_1(x) + J_2(x)\phi_2(x).$$

The theory we consider is a special case of an $\mathrm{O}(N)$ model with $N = 2$.

Functional integral

The functional integral is an integral over $\phi_1(x)$ and $\phi_2(x)$ at every spacetime point,

$$\begin{aligned} \int D\phi &= \int D\phi_1 \int D\phi_2 \\ &= \lim_{N\to\infty} \int_{-\infty}^{\infty} d\phi_1(x_1) \int_{-\infty}^{\infty} d\phi_2(x_1) \cdots \int_{-\infty}^{\infty} d\phi_1(x_N) \int_{-\infty}^{\infty} d\phi_2(x_N), \end{aligned}$$

where $x_1, \ldots, x_N$ corresponds to points on some lattice in $d = 1 + 3$ dimensional space-time, and $N \to \infty$ corresponds to the continuum limit.

Transition amplitude

At the initial time $t = t_{\text{in}}$ and at the final time $t = t_{\text{f}}$ we can keep the field fixed, i. e.

$$\begin{aligned} \phi(t_{\text{in}}, \mathbf{x}) &= \phi_{\text{in}}(\mathbf{x}), \\ \phi(t_{\text{f}}, \mathbf{x}) &= \phi_{\text{f}}(\mathbf{x}), \end{aligned}$$

and, up to an overall factor, Z is then the quantum mechanical transition amplitude in the presence of sources,

$$Z = U_{t_{\text{f}} \leftarrow t_{\text{in}}}[\phi_{\text{f}}, \phi_{\text{in}}].$$

For $t_{\text{in}} \to -\infty$, $t_{\text{f}} \to \infty$, and using the $i\epsilon$-prescription, one has for $J = 0$ formally a vacuum-to-vacuum transition amplitude.

One can also use transition amplitudes such as $U_{t_{\text{f}} \leftarrow t_{\text{in}}}$ to construct time evolution of density operators for non-equilibrium dynamics (Schwinger-Keldysh formalism).

Action

We now specify how the action $S[\phi]$ typically looks like. Iy is supposed to be invariant under global U(1) $\cong$ O(2) transformations,

$$\phi(x) \to e^{i\alpha}\phi(x),$$

and it should be local, i. e. an integral over a Lagrange density. In Minkowski space, with mainly plus conventions for the metric, $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$, it has the form

$$S[\phi] = \int_{t_{\text{in}}}^{t_{\text{f}}} dt \int d^3x \left\{ -\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - V(\rho) \right\},$$

where $\rho = \phi^*\phi = \frac{1}{2}[\phi_1^2 + \phi_2^2]$ is invariant under $U(1)$ transformations and the kinetic term can also be written as

$$-\eta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi = (\partial_t\phi^*)(\partial_t\phi) - \boldsymbol{\nabla}\phi^*\boldsymbol{\nabla}\phi.$$

Microscopic potential

The microscopic potential $V(\rho)$ is usually expanded in a Taylor series around its minimum. This can either be at $\rho = 0$, in which case the expansion reads

$$V(\rho) = m^2\rho + \frac{1}{2}\lambda\rho^2 + \dots,$$

where m is now the microscopic (or bare, or un-renormalized) mass of the scalar particles, and λ is similarly a microscopic (or bare or un-renormalized) interaction strength. We have assumed here for simiplicity that $V(\rho)$ vanishes at its minimum, more generally a non-zero value would correspond to a cosmological constant.

The minimum can also be at a positive value $\rho_0 > 0$, and the expansion reads then

$$V(\rho) = \frac{1}{2}(\rho - \rho_0)^2 + \frac{1}{3!}(\rho - \rho_0)^3 + \dots.$$

There is now no linear term. Besides λ, the microscopic potential is then parametrized by the value of ρ_0, and at higher order by γ etc.

Gauge fields

It is interesting and useful to extend the global U(1) symmetry to a local symmetry by introducing an (external) gauge field $A_\mu(x)$. We replace

$$\partial_\mu\phi(x) \to D_\mu\phi(x) = [\partial_\mu - ieA_\mu(x)]\phi(x),$$
$$\partial_\mu\phi^*(x) \to D_\mu\phi^*(x) = [\partial_\mu + ieA_\mu(x)]\phi^*(x),$$

and have now an invariance under the local or gauge transformations

$$\phi(x) \to e^{i\alpha(x)}\phi(x),$$
$$\phi^*(x) \to e^{-i\alpha(x)}\phi^*(x),$$
$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x),$$

such that D_μ transforms as a covariant derivative,

$$D_\mu\phi(x) \to e^{i\alpha(x)}D_\mu\phi(x).$$

One can understand e to be the electromagnetic charge of the scalar particles described by the field $\phi(x)$. The action is now a functional of ϕ and A_μ, and similarly the partition function

$$Z[J, A] = \int D\phi \, e^{iS[\phi,A]+i\int_x \{J^*\phi+\phi^* J\}}.$$

At present $A_\mu(x)$ is just an external gauge field, but we could extend the theory such that it becomes a dynamical field which is also included in the functional integral like in quantum electrodynamics (QED). This is the scalar QED, because the matter fields are scalar particles and not Dirac fermions as in standard or spionor QED.

Electromagnetic current

From the variation of the action with respect to $A_\mu(x)$ we can obtain the microscopic (or bare or un-renormalized) form of the electromagnetic or U(1) current,

$$\delta S = \int d^x \delta A_\mu(x) J^\mu(x).$$

Concretely, we find here

$$J^\mu(x) = ie \left[(D^\mu \phi^*(x))\phi(x) - \phi^*(x) D^\mu \phi(x)\right].$$

As a check, we eveluate this for

$$\phi(x) = \frac{1}{\sqrt{2m}} \varphi(x) e^{-imt}, \qquad A_\mu(x) = 0,$$

and obtain for a constant non-relativistic field $\varphi(x) = \varphi_0$

$$J^0 = e\varphi_0^* \varphi_0,$$

corresponding to the charge times the density, as expected.

Action in general coordinates

It is actually useful to generalize our current formalism in another direction, namely by allowing general (not necessarily cartesian) coordinates. The Minkowski space metric becomes then space- and time-dependent,

$$\eta_{\mu\nu} \to g_{\mu\nu}(x),$$

and we could even go to curved space where we have no cartesian coordinates. This would be needed in the setup of Einsteins theory of gravitation, general relativity. The action becomes

$$S = \int d^d x \sqrt{g} \left\{-g^{\mu\nu}(x) D_\mu \phi^*(x) D_\nu \phi(x) - U(\rho)\right\}.$$

We use here the invariant volume element $d^d x \sqrt{g}$, with $g = -\det(g_{\mu\nu})$.

General coordinate transformations

A nice feature of the action is that it is invariant under general coordinate transformations, or diffeomorphisms,

$$x \to x'(x),$$

with

$$
\begin{aligned}
\frac{\partial}{\partial x^\mu} &\to \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial}{\partial x'^\alpha}, \\
g_{\mu\nu}(x) \to g'_{\mu\nu}(x') &= g_{\alpha\beta}(x(x'))\frac{\partial x^\alpha}{\partial'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}, \\
\sqrt{g} \to \sqrt{g'} &= \sqrt{g}\,\det\left(\frac{\partial x^\alpha}{\partial x'^\mu}\right).
\end{aligned}
$$

Exercise: Show that the action S is indeed invariant under this transformation.

Energy-momentum tensor

Similar to the electromagnetic current we can obtain the energy-momentum tensor from variation of the action with respect to the metric,

$$
\delta S = \frac{1}{2}\int d^d x \sqrt{g}\, T^{\mu\nu}(x)\delta g_{\mu\nu}(x).
$$

From the concrete expression for S we find, using the relations

$$
\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma},
$$

and

$$
\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu},
$$

the microscopic (or bare or un-renormalized) energy-momentum tensor

$$
T^{\mu\nu} = 2D^\mu\phi^* D^\nu\phi - g^{\mu\nu}[g^{\alpha\beta}D_\alpha\phi^* D_\beta\phi + U(\rho)].
$$

Partition function

To summarize, the partition function in the presence of a source field J, an external U(1) gauge field $A_\mu(x)$ and a metric $g_{\mu\nu}(x)$, is given by

$$
Z[J,A,g] = \int D\phi\, e^{iS[\phi,A,g]+i\int_x\{J^*\phi+\phi^* J\}}
$$

with the action

$$
S[\phi,A,g] = \int d^d x\sqrt{g}\left\{-g^{\mu\nu}[\partial_\mu + ieA_\mu]\phi^*[\partial_\nu - ieA_\nu]\phi - U(\phi^*\phi)\right\}.
$$

Field expectation value and correlation function

From the partition function one can obtain correlation functions and expectation values of the form

$$
\langle\phi(x)\rangle = \langle\Omega_\text{f}|\phi(x)|\Omega_\text{in}\rangle = \frac{1}{Z[\phi]}\left[-i\frac{1}{\sqrt{g}(x)}\frac{\delta}{\delta J^*(x)}\right] Z[J],
$$

or correlation functions

$$
\begin{aligned}
\langle\phi(x)\phi(y)\rangle =& \langle\Omega_\text{f}|\phi(x)\phi(y)|\Omega_\text{in}\rangle \\
=& \frac{1}{Z[\phi]}\left[-i\frac{1}{\sqrt{g}(x)}\frac{\delta}{\delta J^*(x)}\right]\left[-i\frac{1}{\sqrt{g}(y)}\frac{\delta}{\delta J^*(y)}\right] Z[J].
\end{aligned}
$$

Typically this will be evaluated at vanishing source, i. e. by setting $J = 0$ after the functional derivatives have been taken. The gauge field A_μ and metric $g_{\mu\nu}$ might be non-trivial, however. For non-equilibrium dynamics one can use a similar formalism based on the Schwinger-Keldysh contour.

Schwinger functional

The generating functional for connected correlation functions can be introduced by setting

$$W[J,A,g] = -i \ln Z[J,A,g]$$

or

$$e^{iW[J,A,g]} = Z[J,A,g].$$

This is the generating functional for conneced correlation functions, e. g. for $g_{\mu\nu} = \eta_{\mu\nu}$ and $A_\mu = 0$,

$$\begin{aligned}\langle\phi(x)\phi(x)\rangle_c = & \left[-i\frac{\delta}{\delta J^*(x)}\right]\left[-i\frac{\delta}{\delta J^*(y)}\right] iW[J] \\ = & \langle\phi(x)\phi(x)\rangle - \langle\phi(x)\rangle\langle\phi(x)\rangle.\end{aligned}$$

Field equation for the Schwinger functional

The Schwinger functional satisfies in particular

$$\frac{1}{\sqrt{g}(x)}\frac{\delta}{\delta J^*(x)}W[J,A,g] = \langle\phi(x)\rangle,$$

which can be seen as a field equation. Because we need the expectation value more often in the following we introduce a separate notation,

$$\Phi(x) = \langle\phi(x)\rangle.$$

Propagator

The connected two-point function or propagator follows from the Schwinger functional as

$$\Delta(x,y) = i\langle\phi^*(x)\phi(y)\rangle_c = \left[\frac{\delta}{\delta J^*(x)}\right]\left[\frac{\delta}{\delta J^*(y)}\right] W[J,A,g].$$

Up to factors of $\sqrt{g}$ this is just the second functional derivative of the Schwinger functional.

The connected correlation function has the property that it decays for large space-like separation,

$$\lim_{|\mathbf{x}-\mathbf{y}|\to\infty} \Delta(x,y) = 0.$$

For the full correlation function this does not hold, but it factorizes into a propduct of expectation values in the large separation limit.

Quantum effective action

We introduce another generating functional, the quantum effective action or one-particle irreducible effective action. It is defined as a Legendre transform of the Schwinger functional,

$$\Gamma[\Phi] = \sup_J \left(\int_x \{J^*(x)\Phi(x) + \Phi^*(x)J(x)\} - W[J] \right).$$

The field $\Phi(x)$ is here the expectation value of $\Phi(x)$, as follows from the condition that the variation with respect to $J^*(x)$ must vanish as the supremum,

$$\Phi(x) = \frac{1}{\sqrt{g}(x)}\frac{\delta}{\delta J^*(x)}W[J].$$

Because J is always evaluated at the supremum, one should understand it to be a functional of the expectation value on the right hand side of the definition for $\Gamma[\phi]$. Because W depends also on A_μ and $g_{\mu\nu}$, this is also the case for the effective action $\Gamma[\Phi, A, g]$.

Quantum field equation

The quantum effective action satisfies a particularly interesting field equation,

$$\frac{\delta}{\delta\Phi(x)}\Gamma[\Phi] = \sqrt{g}(x)J^*(x).$$

In particular, for $J = 0$ it is stationary, $\delta\Gamma/\delta\Phi(x) = 0$. This is very similar to a classical action, which is stationary on the equations of motion (principle of stationary action). In fact, one should see $\Gamma[\Phi]$ as playing the role of the classical action.

Let us prove the field equation. We first use the chain rule,

$$\begin{aligned}\frac{\delta\Gamma}{\delta\Phi(x)} =& - \frac{\delta}{\delta\Phi(x)}W[J] + \int d^dy\sqrt{g}\left\{\frac{\delta J^*(y)}{\delta\Phi(x)}\Phi(y) + \Phi^*(y)\frac{\delta J(y)}{\delta\Phi(x)}\right\} \\ &+ \sqrt{g}J^*(x),\end{aligned}$$

and can use then

$$\begin{aligned}\frac{\delta}{\delta\Phi(x)}W[J] &= \int d^dy\left\{\frac{\delta W}{\delta J(y)}\frac{\delta J(y)}{\delta\Phi(x)} + \frac{\delta W}{\delta J^*(y)}\frac{\delta J^*(y)}{\delta\Phi(x)}\right\} \\ &= \int d^dy\sqrt{g}\left\{\Phi^*(y)\frac{\delta J(y)}{\delta\Phi(x)} + \Phi(y)\frac{\delta J^*(y)}{\delta\Phi(x)}\right\}.\end{aligned}$$

This implies a cancelation of terms and directly yields the field equation as claimed.

Inverse propagator

From the second functional derivative of the quantum effective action one obtains the inverse propagator,

$$\Gamma^{(2)}_{jk}(x,y)[\Phi] = \left[\frac{1}{\sqrt{g}(x)}\frac{\delta}{\delta\Phi_j(x)}\right]\left[\frac{1}{\sqrt{g}(x)}\frac{\delta}{\delta\Phi_k(y)}\right]\Gamma[\Phi] = P_{jk}(x,y).$$

We work here for convenience with a basis of real fields $\Phi_j(x)$. As an (infinite dimensional) matrix, or operator, this is in fact inverse to the second functional derivative of the Schwinger functional

$$W^{(2)}_{jk}(x,y)[J] = \left[\frac{1}{\sqrt{g}(x)}\frac{\delta}{\delta J_j(x)}\right]\left[\frac{1}{\sqrt{g}(x)}\frac{\delta}{\delta J_k(y)}\right]W[J] = \Delta_{jk}(x,y),$$

in the sense that $P^{-1} = \Delta$ or, more explicitely,

$$\int d^d\sqrt{g}\left\{P_{jk}(x,y)\Delta_{kl}(y,z)\right\} = \frac{1}{\sqrt{g}(x)}\delta^{(d)}(x-z)\delta_{jl}.$$

Typically $P = \Gamma^{(2)}$ is actually a generalized derivative operator, and $\Delta(x,y)$ is its Greens function. To prove the above we note that

$$\begin{aligned}P_{jk}(x,y) =& \frac{1}{\sqrt{g}(x)}\frac{\delta}{\delta\Phi_j(x)}J_k(y), \\ \Delta_{kl}(y,z) =& \frac{1}{\sqrt{g}(y)}\frac{\delta}{\delta J_k(y)}\Phi_l(z),\end{aligned}$$

and therefore

$$\begin{aligned}&\int d^dy\sqrt{g}(y)P_{jk}(x,y)\Delta_{kl}(y,z) = \frac{1}{\sqrt{g}(x)}\int d^dy\frac{\delta J_k(y)}{\delta\Phi_j(x)}\frac{\delta\Phi_l(z)}{\delta J_k(y)} \\ &= \frac{1}{\sqrt{g}(x)}\frac{\delta\Phi_l(z)}{\delta\Phi_j(x)} = \frac{1}{\sqrt{g}(x)}\delta^{(d)}(x-z)\delta_{jl},\end{aligned}$$

as claimed.

From the inverse of the second functional derivative of the microscopic action $S[\phi]$ one would obtain a microscopic (or bare or un-renormalized) propagator, while the inverse of $\Gamma^{(2)}$ gives the full propagator, including all quantum corrections!

Abstract index notation

The following is this convenient to use an abstract index notation where a greek index combines the continuous position index and the discrete field index,

$$\alpha = (x, j)$$

and sums over α are abbreviations for integrals over x and sums over j with the appropriate factors, e. g.

$$\sum_\alpha = \sum_j \int d^d x \sqrt{g},$$

and we write

$$\Phi^\alpha = \Phi_j(x), \qquad J_\alpha = J_j(x).$$

In abstract index notation one has also the unit element

$$\delta^\alpha{}_\beta = \frac{1}{\sqrt{g}(x)} \delta^{(d)}(x-y)\delta_{jk},$$

where $\alpha = (x, j)$ and $\beta = (y, k)$.

When working in Minkowski space one can expand fields in momentum modes,

$$\Phi_j(x) = \int \frac{d^d p}{(2\pi)^d} e^{ipx} \Phi_j(p),$$

and α can then also represent the combination (p, j) with

$$\sum_\alpha = \sum_j \int \frac{d^d p}{(2\pi)^d}, \qquad \delta^\alpha{}_\beta = (2\pi)^d \delta^{(d)}(p-q)\delta_{jk}.$$

This works similarly for other expansion schemes. With a bit of experience one can easily translate expressions involving abstract indices to the concrete functional or integral expressions.

One should nevertheless keep in mind that infinite functional vector spaces have partly more involved mathematical properties than finite vector spaces, and the associated mathematical subtleties and complications should not be forgotten through the use of abstract indices.

Legendre transform

We have in abstract index notation

$$\Gamma[\Phi] = \sup_J \left(J_\alpha \Phi^\alpha - W[J] \right).$$

This is a Legendre transform and can be inverted where it is well defined,

$$W[J] = \sup_\Phi \left(J_\alpha \Phi^\beta - \Gamma[\Phi] \right).$$

We now take a few functional derivatives to see what the physical role of $\Gamma[\Phi]$ is. We have the first derivatives

$$\begin{aligned} W^\alpha &= \frac{\delta}{\delta J_\alpha} W[J] = \Phi^\alpha, \\ \Gamma_\alpha &= \frac{\delta}{\delta \Phi^\alpha} \Gamma[\phi] = J_\alpha, \end{aligned}$$

and the second functional derivatives

$$W^{\alpha\beta} = \frac{\delta^2}{\delta J_\alpha \delta J_\beta} W[J] = \frac{\delta}{\delta J_\alpha} \Phi^\beta,$$

as well as its inverse

$$\Gamma_{\alpha\beta} = \frac{\delta^2}{\delta \Phi^\alpha \delta \Phi^\beta} \Gamma[\Phi] = \frac{\delta}{\delta \Phi^\alpha} J_\beta.$$

These are just relation we have derived previously, now written in terms of abstract indices.

Correlation functions and graphical representation

Taking further functional derivatives of W we obtain

$$\begin{aligned} W^{\alpha\beta\gamma} =& \frac{\delta^3}{\delta J_\alpha \delta J_\beta \delta J_\gamma} W[J] = \frac{\delta}{\delta J_\alpha}\left(\Gamma^{(2)}[\Phi]^{-1}\right)^{\beta\gamma} \\ =& \frac{\delta\Phi^{\tilde\alpha}}{\delta J_\alpha}\frac{\delta}{\delta\Phi^\alpha}\left(\Gamma^{(2)}[\Phi]^{-1}\right)^{\beta\gamma} \\ =& - W^{\alpha\tilde\alpha}W^{\beta\tilde\beta}W^{\gamma\tilde\gamma}\Gamma^{(3)}_{\tilde\alpha\tilde\beta\tilde\gamma}. \end{aligned}$$

We used here that $\delta(M^{-1}) = M^{-1}\delta M M^{-1}$ for any matrix M.

Similarly, the connected four-point correlation function is given by

$$\begin{aligned} W^{\alpha\beta\gamma\delta} =& - W^{\alpha\tilde\alpha}W^{\beta\tilde\beta}W^{\gamma\tilde\gamma}W^{\delta\tilde\delta}\Gamma_{\tilde\alpha\tilde\beta\tilde\gamma\tilde\delta} \\ &+ W^{\alpha\tilde\alpha}W^{\delta\tilde\delta}\Gamma_{\tilde\alpha\tilde\delta\lambda}W^{\lambda\kappa}\Gamma_{\kappa\tilde\beta\tilde\gamma}W^{\beta\tilde\beta}W^{\gamma\tilde\gamma} \\ &+ 2 \text{ permutations.} \end{aligned}$$

These are in fact tree-level diagrams where propagators are full propagators,

$$\Delta = W^{(2)} = (\Gamma^{(2)})^{-1},$$

and vertices are given by $\Gamma^{(3)}$ and $\Gamma^{(4)}$!

One-particle irreducible

What is then actually the meaning of one-particle irreducible? This has a diagramatic meaning in terms of perturbation theory. For diagrams that are one-particle-irreducible it is not possible to cut them into pieces – to reduce them – by opening just a single internal line.

One observes that all diagrams can be composed out of sets of one=particle irreducible diagrams and it corresponds to the construction above, with full propagators and vertices from the quantum effective action $\Gamma[\Phi]$. For this reason the latter is also known as one-particle irreducible or 1-P. I. effective action.

Quantum effective action and S-matrix

Recall that the S-matrix elements can be obtained from connceted correlation functions with external propagators removed, or "amputated". But such a removing of external propagators also happens in the transition from connected correlation functions to one-particle-irreducible effective action!

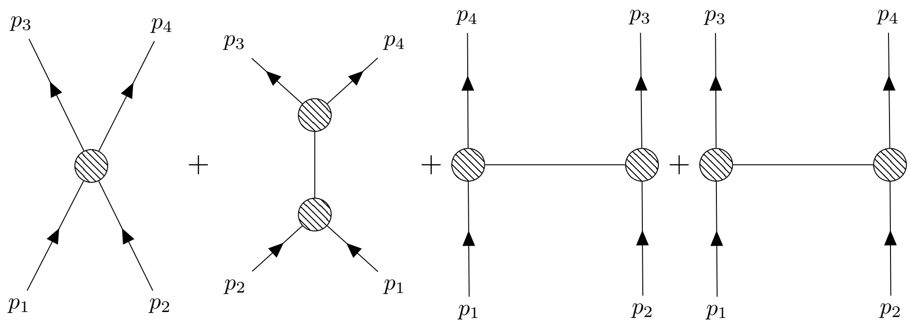
In fact, one can write the transition matrix amplitude for $n \to m$ particle scattering problems in the from

$$\mathcal{T}(2\pi)^d\delta^{(d)}(p^{\text{out}} - p^{\text{in}}) = -\left\{\Gamma^{(n+m)}[\Phi_{\text{eq}}] + \text{tree terms}\right\}.$$

The functional derivatives on the right hand side must be evaluated with Φ_{eq} a solution of the field equation (usually a homogeneous or even vanishing field configuration), and the momenta of the incoing and outgoing particles must be on-shell.

The quantum effective action we use here is in the real time formalism and fundamentally based on the vacuum-to-vacuum transition amplitude. Functional derivatives of this effective action describe particle excitations. The additional tree terms involve lower order derivatives of the quantum effective action, $\Gamma^{(p)}$ with $p < n + m$.

An example is $2 \to 2$ scattering where the transition amplitude $\mathcal{T}(p_1, p_2, p_3, p_4)$ has the contributions



The internal line corresponds to the full propagator. We observe that for known 1-P. I. -vertices only tree diagram appear. There are no more loops, since the fluctuating effects are already incorporated into the computation of the quantum effective action.

Summary

In summary, the quantum effects of fluctuations change the microscopic action $S[\phi]$ to the quantum effective or macroscopic action $\Gamma[\Phi]$. The latter includes all fluctuation effects. Once $\Gamma[\Phi]$ is known, only tree diagrams have to be evaluated for the computation of the transition amplitude $\mathcal{T}$! The full propagator and the full vertices in tree diagrams are given by the propagator $(\Gamma^{(2)}[\Phi])^{-1}$ and the one-particle-irreducible vertices $\Gamma^{(n)}[\Phi]$ with $n \geq 3$. In order to compute the transition amplitude one can follow the receipe:

1. Compute $\Gamma[\Phi]$.
2. Draw all tree diagrams.
3. Insert full propagator for lines and full vertices.

Funnctional integral representation for quantum effective action

We derive now an expresion for the quantum effective action in terms of functional integrals. We start with

$$e^{-i\Gamma[\Phi]} = e^{iW[J]-i\int_x \{J^*\Phi+\Phi^* J\}},$$

and insert there the definition of the Schwinger functional, leading to

$$e^{-i\Gamma[\Phi]} = \int D\phi \, e^{iS[\phi]+i\int_x \{J^*(\phi-\Phi)+(\phi-\Phi)^* J\}}.$$

One may use here the field equation

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = \sqrt{g}(x) J^*(x),$$

and make a change of variables, defining $\phi' = \phi - \Phi$. This leads to the identity

$$\exp\left[-i\Gamma[\Phi]\right] = \int D\phi' \, \exp\left[iS[\Phi+\phi'] + i\int d^dx \left\{\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)}\phi'(x) + \phi'^*(x)\frac{\delta\Gamma[\Phi]}{\delta\Phi^*(x)}\right\}\right].$$

Note that this is an implicit relation because the first derivative of the effective action appears also on the right hand side.

Analytic continuation or Wick rotation

At this point it is useful to recall the idea of Wick rotation or analytic continuation from Minkowski to Euclidean space. We rotatate the time coordinate into an imaginary direction by writing

$$t = -i\tau, \qquad dt = -id\tau,$$

and obtain for the action in Minkowski space

$$\begin{aligned} iS[\phi] =& i\int dt \int d^{d-1}x \left\{ \frac{\partial}{\partial t}\phi^* \frac{\partial}{\partial t}\phi - \boldsymbol{\nabla}\phi^* \boldsymbol{\nabla}\phi - V(\rho) \right\} \\ =& -i^2 \int d\tau \int d^{d-1}x \left\{ -\frac{\partial}{\partial \tau}\phi^* \frac{\partial}{\partial \tau}\phi - \boldsymbol{\nabla}\phi^* \boldsymbol{\nabla}\phi - V(\rho) \right\} \\ =& - S_{\mathrm{E}}[\phi]. \end{aligned}$$

In the last step we employ the Euclidean action

$$S_{\mathrm{E}}[\phi] = \int d\tau \int d^{d-1}x \left\{ \frac{\partial}{\partial \tau}\phi^* \frac{\partial}{\partial \tau}\phi + \boldsymbol{\nabla}\phi^* \boldsymbol{\nabla}\phi + V(\rho) \right\}.$$

This has the nice feature that it is manifestly real and positive, which also motivates the additional sign introduced in the relation between the analytically continued or Wick rotated action and $S_{\mathrm{E}}[\phi]$. Quantum field theory for a scalar field in Euclidean space is in fact a statistical field theory.

Let us also recall here that quantum field theory at non-zero temperature is emplying the Eucliden version of the theory. The time coordinate τ is integrated in the Matsubara formalism from $\tau = 0$ to $\tau = 1/T$ and bosonic fields have periodic boundary conditions, $\phi(0, \mathbf{x}) = \phi(1/T, \mathbf{x})$.

Functional integral representation in Euclidean space

For defineteness we also give the different functional integral expressions in Euclidean space. The Schwinger functional can be written as

$$e^{W_{\mathrm{E}}[J]} = \int d\phi \, e^{-S_{\mathrm{E}}[\phi] + \int_x \{J^*\phi + \phi^* J\}}.$$

The quantum effective action is again defined as the Legendre transform,

$$\Gamma_{\mathrm{E}}[\Phi] = \sup_J \left(\int_x \{J^*\Phi + \Phi^* J\} - W[J] \right),$$

and the same steps as in Minkowski space lead to the background field identity

$$\exp\left(-\Gamma_{\mathrm{E}}[\Phi]\right) = \int D\phi' \exp\left(-S_{\mathrm{E}}[\Phi + \phi'] + \int d^dx \left\{ \frac{\delta\Gamma[\Phi]}{\delta\Phi}\phi' + \phi'^* \frac{\delta\Gamma[\Phi]}{\delta\phi^*} \right\} \right).$$

The concept of the one-particle irreducible effective action can also be used for classical statistical field theories and the action incorporates then the effect of statistical fluctuations.

Classical approximation

In classical field theory, fields are not fluctuating. This corresponds to the functional integral being completely dominated by the expectation value. With Euclidean conventions, the classical field approximation is accordingly

$$\Gamma_{\mathrm{E}}[\Phi] = S_{\mathrm{E}}[\Phi].$$

In Minkowski space there is an additional minus sign, due to the historic convention that the Lagrangian in mechanics is kinetic energy minus potential energy, $L = T - U$, while the potential enters with a positive sign in the quantum effective potential. Accordingly, the classical approximation is there

$$\Gamma[\Phi] = -S[\Phi].$$

In reallity, quantum and statstical fluctuations can have a significant influence on the quantum effective action and lead to substantial deviations from the classical form.

An exception from this general rule occurs for theories that are quadratic in the fields. For an action of the from (emplying abstract index notation)

$$S[\Phi + \phi'] = \frac{1}{2} P_{\alpha\beta}(\Phi^\alpha + \phi'^\alpha)(\Phi^\beta + \phi'^\beta)$$

one finds

$$\Gamma[\Phi] = S[\Phi] + \text{const.}$$

Indeed, using

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi^\alpha} = P_{\alpha\beta}\Phi^\beta,$$

the functional integral representation becomes

$$\begin{aligned}\exp\left(-\frac{1}{2}P_{\alpha\beta}\Phi^\alpha\Phi^\beta - \text{const}\right) &= \int D\phi' \exp\left(-S[\Phi+\phi] + \frac{\delta\Gamma}{\delta\Phi^\alpha}\phi'^\alpha\right) \\ &= \int D\phi' \exp\left(-\frac{1}{2}P_{\alpha\beta}\Phi^\alpha\Phi^\beta - \frac{1}{2}P_{\alpha\beta}\phi'^\alpha\phi'^\beta\right).\end{aligned}$$

This is solved for

$$e^{-\text{const}} = \int D\phi' \exp\left(-\frac{1}{2}P_{\alpha\beta}\phi'^\alpha\phi'^\beta\right).$$

Usually, this constant or field-independent part can be simply dropped. But sometimes, for example in the context of thermodynamics, it depends on external parameters such as temperature, chemical potential, or external gauge field, and must then be taken into account, as well.

Perturbation theory

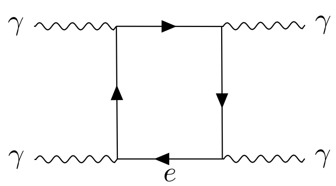
Interactions correspond to terms in $S[\phi]$ that are not quadratic in ϕ, such as cubic or quartic terms. If the couplings characterizing the interaction are small, one expects some kind of perturbation expansion in the small couplings,

$$\Gamma[\Phi] = S[\Phi] + \text{perturbative corrections.} \tag{1.1}$$

We will explore this approach in more detail below.

Quantum vertices

The effective action Γ contains new vertices that are not present in the classical or microscopic action S. For example photon-photon interactions. Classical Maxwell theory has no photon-photon interactions; Maxwell equations are linear. But the quantum one-particle irreducible four point function for photons contains terms like



This implies a one-loop contribution to photon-photon scattering, of order

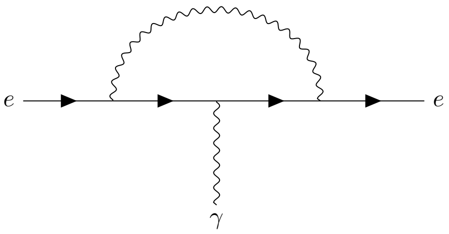
$$\Gamma^{(4)} \sim \alpha^2 \sim e^4.$$

For very small very small momenta below m_e one finds

$$\Gamma^{(4)} \sim \frac{q^4}{m_e^4}.$$

The correction is small, but observable by precisions measurements. Recently light-by-light scattering has been observed by ATLAS experiment in heavy ion collisions at the Large Hadron Collider.

Another example is $g-2$, as generated by one-particle irreducible diagrams of the type



The corresponding piece of the effective action is of the form

$$\Gamma \sim \int_x \bar{\psi}(x)\,[\gamma^\mu, \gamma^\nu]\, F_{\mu\nu}(x)\psi(x),$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength.

An interesting fact about quantum vertices, i. e. terms that are not present on the level of the microscopic action, but that arise purely from quantum fluctuations, is that the corresponding coeffients can actually be determined from quantum field theory! Ultimately this is related to how renormalization works.

Effective potential

The part if the effective action for scalar fields that involves no derivative is the effective potential $U(\rho)$,

$$\Gamma[\Phi] = \int_x U(\rho) + \ldots.$$

The ellipses are here for terms with derivatives which vanish for homogeneous fields. For the O(N)-symmetric scalar model, the effective potential can only depend on the invariant combination

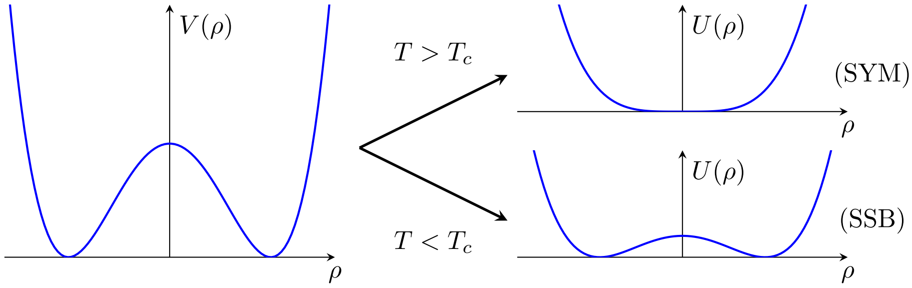
$$\rho = \frac{1}{2}\phi_n\phi_n.$$

The exact quantum field equation for homogeneous fields $\phi_n(x) = \phi_n$ becomes

$$\frac{\partial}{\partial\phi_n}U(\rho) = \phi_n\frac{\partial}{\partial\rho}U(\rho) = 0.$$

There is always a solution $\phi_n = 0$, but this may not be the absolute minimum of U. For reasons of stability the solution should be at least a local minimum. For a minimum at $\rho_0 \neq 0$ one has spontaneous symmetry breaking, see next section.

Omitting fluctuations effects one has $U(\rho) = V(\rho)$. In this limit the effective potential equals the microscopic potential. Quantum fluctuation induce a map $V(\rho) \to U(\rho)$. This can also depend on external parameters, such as temperature or chemical potentials.



2 Higgs mechanism and superconductivity

Once the effective action is computed, or a given form is assumed, many properties of the system follow from the field equations and the correlation functions. Often one knows only the symmetries of $\Gamma[\Phi]$ or the generic form of the effective potential, and uses an expansion in the number of derivatives $\partial_\mu \Phi_j(x)$. The derivative expansion is motivated by the interest in macroscopic wave lengths, for which smooth fields often (not always!) play a dominant role. The validity of the derivative expansion may depend on the appropriate choice of macroscopic fields. In general the macroscopic fields can be more complicated than simply $\Phi_j(x) = \langle \phi_j(x) \rangle$. An example are antiferromagnets.

Symmetry of the quantum effective action

Consider again a complex scalar field $\phi(x)$, with $S[\phi]$ invariant under global U(1) transformations, $\phi(x) \to e^{i\alpha}\phi(x)$. Let us take a microscopic action which is U(1) invariant,

$$S[\phi] = -\int_x \{g^{\mu\nu}\partial_\mu \phi^* \partial_\nu \phi + V(\phi^* \phi)\}\,.$$

If the functional measure $\int D\phi$ is invariant under U(1) transformations, it follows that the quantum effective action $\Gamma[\Phi]$ is also invariant under U(1) transformations, where $\Phi(x) \to e^{i\alpha}\Phi(x)$.

To see this, note that the Schwinger functional $W[J]$, defined through

$$e^{iW[J]} = \int D\phi\, e^{iS[\phi]+i\int_x \{J^*\phi + \phi^* J\}},$$

is invariant under the global U(1) transformations $J(x) \to e^{i\alpha}J(x)$. Accordingly, the field expectation value

$$\Phi(x) = \frac{1}{\sqrt{g}(x)}\frac{\delta W[J]}{\delta J^*(x)}$$

transforms as $\Phi(x) \to e^{i\alpha}\Phi(x)$, as one expects for $\Phi(x) = \langle \phi(x) \rangle$. As a consequence, one finds that $\int_x \{J^*\Phi + \Phi^* J\}$ is invariant. This establishes the invariance of $\Gamma[\Phi] = \int_x \{J^*\Phi + \Phi^* J\} - W[J]$.

This generalizes to all symmetry transformations: If $S[\phi]$ is invariant under some symmetry transformation $\phi \to \mathsf{g}\phi$, and the functional measure $\int D\phi$ is invariant under the transformation as well, $\int D\phi = \int D\mathsf{g}\phi$, it follows that $\Gamma[\Phi]$ is invariant under $\Phi \to \mathsf{g}\Phi$. The effective action has the same symmetries as the classical action. This holds unless there is an “anomaly” in the functional measure.

Derivative expansion and Landau theories

In condensed matter physics, the precise microscopic physics is often not known, and the transition from microphysics to macrophysics (the computation of the quantum effective action) is very difficult. In addition, very different microphysical systems often give similar macroscopic phenomena.

This is called universality and plays a crucial role for classical statistical field theories that describe condensed matter systems in the vicinity of second order phase transitions.

A useful approach is a guess for the quantum effective action $\Gamma[\Phi]$. From comparison with experiment and general considerations one makes an assumption on what are the relevant macroscopic degrees of freedom $\phi_j(x)$, without necessarily knowing the microscopic origin. Examples are spin waves for antiferomagnetism, or a complex scalar field $\varphi(x)$ for superconductivity. The miscroscopic degrees of freedom are electrons, and the macroscopic field may represent Cooper pairs or similar composite objects. A second central ingredient is an assumption about the symmetries of the quantum effective action. Third, one employs a derivative expansion, typically up to two derivatives $\partial_\mu \phi(x)$. This restricts the effective action already severely. For the example of the scalar $\mathrm{O}(N)$ model one remains with three functions of the invariant combination

$$\rho = \frac{1}{2} \sum_{n=1}^{N} \Phi_n \Phi_n,$$

appearing in the action

$$\Gamma[\Phi] = \int_x \left\{ U(\rho) + \frac{1}{2} Z(\rho) g^{\mu\nu} \partial_\mu \Phi_n \partial_\nu \Phi_n + \frac{1}{4} Y(\rho) g^{\mu\nu} \partial_\mu \rho \partial_\nu \rho \right\}.$$

This approach can be used in very similar form for relativistic fields in Minkoski space, for condensed matter systems in real time, where the velocity of light is replaced by a smaller velocity for some collective excitations, or for statistical field theories in Euclidean spaces. Even an extension to non-relativistic space-times is possible.

Making further assumptions, as a polynomial expansion of $U(\rho)$ around its minimum and constant Z and Y, one ends with a few parameters. These parameters may be fixed by comparison with experiment. For thermodynamics, they can depend on T and μ. This approach is very successful to gain physical insight without knowledge of the microphycis. The aim of quantum or statistical field theory is to do better by computing the free couplings or relations between them.

For many questions, the most important quantity is the effective potential. For the scalar $O(N)$ model one may write

$$U(\rho) = m^2 \rho + \frac{\lambda}{2} \rho^2 + \ldots$$

In lowest order one has two couplings m^2 and λ. One further assumes constant Z, and the fields can be normalized such that $Z(\rho) = 1$. The coupling Y can often be dropped, $Y(\rho) = 0$. We will concentrate mainly on $N = 2$. The symmetry $\mathrm{U}(1) = \mathrm{O}(2)$ is an abelian symmetry. We employ again the complex field $\Phi(x)$, with $\rho = \Phi^* \Phi$. Our "Landau theory" is

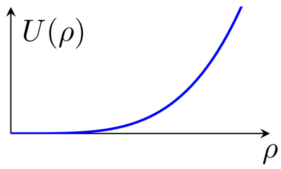
$$\Gamma[\Phi] = \int_x \left\{ g^{\mu\nu} \partial_\mu \Phi^* \partial_\nu \Phi + m^2 \Phi^* \Phi + \frac{\lambda}{2} (\Phi^* \Phi)^2 \right\},$$

where the quartic coupling λ determines the strength of the interaction.

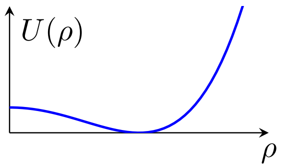
Spontaneous symmetry breaking

Spontaneous symmetry breaking is a key concept in condensed matter and particle physics. It extends to other branches of science as well. The basic ingredient is an effective action that has a given symmetry, while the solution of the field equation breaks this symmetry. The most important example is an effective potential with a minimum at $\Phi \neq 0$. In a Euclidean setting the stable solution of the field equations is the "ground state". It typically corresponds to a minimum of $\Gamma[\Phi]$. We include here the possibility of a local minimum, which would corresponds to a metastable state. A positive kinetic term is minimized by a homogeneous field $\Phi(x) = \Phi_0$, where $\partial_\mu \Phi(x) = 0$. The minimum of $\Gamma[\Phi]$ corresponds then to a minimum of the effective potential $U(\rho)$.

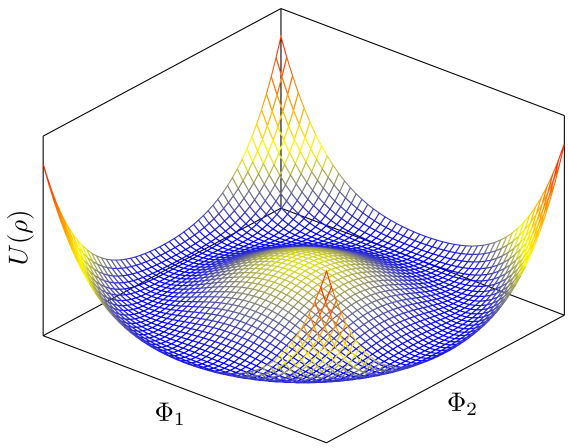
There are two general possibilities for $U(\rho)$. The first is that the minimum is at $\rho_0 = 0$ which implies that $\Phi_0 = 0$ is invariant under the symmetry U(1). This is called the "symmetric phase".



The second possibility is that the minimum of $U(\rho)$ is at $\rho_0 > 0$, such that the field expectation value $\Phi_0 \neq 0$ is not invariant under the symmetry group U(1). This is a "phase with spontaneous symmetry breaking".



A potential with this shape is often called "mexican hat potential", since it is rotation symmetric around the point $\Phi = 0$. The phase of Φ_0 is not determined! Every phase of Φ_0 is equivalent, but the ground state must pick up a fixed direction! Another example for spontaneous symmetry breaking is a rotationally symmetric stick under the influence of gravity. The rotation symmetric state of a vertical stick is unstable, and the ground state of a horizontal stick lying on the floor breaks rotation symmetry spontaneously. Other examples are magnets for which the expectation value of the spin in a Weiss domain singles out some direction.



Inverse propagators as derivative operators

In the following we need an understanding of the physical meaning of two-point functions and their inverses, so it is a good point to recall this here.

Consider a quantum effective action of the form

$$\Gamma[\Phi] = \int d^d z \sqrt{g}(z) \left\{ \frac{1}{2} g^{\mu\nu}(z) \partial_\mu \Phi_n(z) \partial_\nu \Phi_n(z) + U(\Phi(z)) \right\}$$

A summation over field components with $n = 1, \ldots, N$ is implied here. Let us determine the inverse propagator first in position space, through the definition

$$P_{jk}(x, y) = \left[\frac{1}{\sqrt{g}(x)} \frac{\delta}{\delta \Phi_j(x)} \right] \left[\frac{1}{\sqrt{g}(y)} \frac{\delta}{\delta \Phi_k(y)} \right] \Gamma[\Phi].$$

We find from $\delta\Phi_n(z)/\delta\Phi_j(x) = \delta^{(d)}(x-z)\delta_{nj}$

$$
\begin{aligned}
P_{jk}(x,y) = & \frac{1}{\sqrt{g}(x)}\frac{1}{\sqrt{g}(y)}\int d^dz\sqrt{g}(z) \\
\times & \left\{ g^{\mu\nu}(z)\frac{\partial}{\partial z^\mu}\delta^{(d)}(x-z)\delta_{nj}\frac{\partial}{\partial z^\nu}\delta^{(d)}(y-z)\delta_{nk} + \frac{\partial^2 U}{\partial\Phi_j\partial\Phi_k}\delta^{(d)}(x-z)\delta^{(d)}(y-z)\right\}.
\end{aligned}
$$

Derivatives of Dirac distributions are again distributions and one can work with them well, albeit with some care. The derivatives with respect to z can be replaced by derivatives with respect to x and y, and the integral over z and sum over n can then be done, leading to

$$
\begin{aligned}
P_{jk}(x,y) =& \frac{1}{\sqrt{g}(x)}\frac{1}{\sqrt{g}(y)}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\int d^dz\sqrt{g}(z)\left\{g^{\mu\nu}(z)\delta^{(d)}(x-z)\delta_{nj}\delta^{(d)}(y-z)\delta_{nk}\right\} \\
&+ \frac{\partial^2 U}{\partial\Phi_j\partial\Phi_k}\frac{1}{\sqrt{g}(x)}\delta^{(d)}(x-y) \\
=& \delta_{jk}\frac{1}{\sqrt{g}(x)}\frac{1}{\sqrt{g}(y)}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\sqrt{g}(x)g^{\mu\nu}(x)\delta^{(d)}(x-y) + \frac{\partial^2 U}{\partial\Phi_j\partial\Phi_k}\frac{1}{\sqrt{g}(x)}\delta^{(d)}(x-y) \\
=& \frac{1}{\sqrt{g}(y)}\left\{\delta_{jk}\left[\frac{1}{\sqrt{g}(x)}\frac{\partial}{\partial x^\mu}\sqrt{g}(x)g^{\mu\nu}(x)\frac{\partial}{\partial x^\nu}\right] + \frac{\partial^2 U}{\partial\Phi_j\partial\Phi_k}\right\}\delta^{(d)}(x-y).
\end{aligned}
$$

The result is written as a distribution, with the expression in square brackets in the last line being the covariant Laplace-Beltrami operator,

$$
\Box_x = \frac{1}{\sqrt{g}(x)}\frac{\partial}{\partial x^\mu}\sqrt{g}(x)g^{\mu\nu}(x)\frac{\partial}{\partial x^\nu}.
$$

Propagators as Greens functions

The propagator is now a Greens function to this derivative operator such that

$$
\begin{aligned}
\int d^dz\sqrt{g}(z)P_{jn}(x,z)\Delta_{nk}(z,y) =& \left\{\delta_{jn}\Box_x + \frac{\partial^2 U}{\partial\Phi_j\partial\Phi_n}\right\}\Delta_{nk}(x,y) \\
=& \frac{1}{\sqrt{g}(x)}\delta^{(d)}(x-y)\delta_{jk}.
\end{aligned}
$$

As usual, the precise form of a Greens function is also determined by boundary conditions. We will discuss this in more detail later.

Dispersion relations

Speciallizing now to cartesian coordinates in Minkoski space, we can write both inverse propagator and propagator in a Fourier expansion

$$
\begin{aligned}
P_{jk}(x,y) &= \int\frac{d^dp}{(2\pi)^d}e^{ip(x-y)}P_{jk}(p), \\
\Delta_{jk}(x,y) &= \int\frac{d^dp}{(2\pi)^d}e^{ip(x-y)}\Delta_{jk}(p),
\end{aligned}
$$

and find the algebraic relation

$$
P_{jn}(p)\Delta_{nk}(p) = \delta_{jk}.
$$

Propagating particles correspond to poles of the Greens function in momentum space $\Delta_{jk}(p)$. These in turn correspond to vanishing eigenvalues of the inverse propagator $P_{jn}(p)$. They can be detected through the characteristic equation

$$
\det(P_{jk}(p)) = 0,
$$

which encodes the dispersion relations.

Mass squared matrix

Concretely we find here

$$P_{jk}(p) = \delta_{jk} p^2 + \frac{\partial^2 U}{\partial \Phi_j \partial \Phi_k},$$

and the vanishing eigenvalues, or poles of the relativistic propagator, occur when equations of the form

$$p^2 + m^2 = -(p^0)^2 + \mathbf{p} + m^2 = 0,$$

are fulfilled, where m^2 is an eigenvalue of the mass square matrix

$$M^2_{jk} = \frac{\partial^2 U}{\partial \Phi_j \partial \Phi_k}.$$

We will investigate this matrix in more detail below.

Correlation lengths in classical statistical field theory

For static classical statistical field theories one has in a similar way an inverse propagator or correlation function of the form

$$P_{jk}(p) = \delta_{jk} \mathbf{p}^2 + M^2_{jk},$$

and an eigenvalue m^2 of the matrix M^2_{jk} encodes now the information of a correlation length for a specific field, $\xi = 1/m$. A Eucliden correlation function or propagator of the form $(\mathbf{p}^2 + m^2)^{-1}$ becomes large for $\mathbf{p}^2 \to 0$ if $m^2 = 0$ and describes long range correlations. By analogy this is also called a massless field then. In contrast, for $m > 0$ the correlation length $\xi = 1/m$ is finite and correlations decay exponentially for distances much larger than the correlation length.

Energy gap in non-relativistic quantum field theories

For complex non-relativistic quantum fields, the inverse propagator is typically of the form

$$\Delta_{jk}(p) = \delta_{jk} \left[-p^0 + \frac{\mathbf{p}^2}{2m} \right] + M^2_{jk}.$$

Now an eigenvalue ν of M^2_{jk} has the significance of an offset in the energy of particle – or quasi-particle – dispersion relation,

$$p^0 = \frac{\mathbf{p}^2}{2m} + \nu.$$

One should keep in mind that the overall energy scale in non-relativistic quantum mechanics has no physical significance, but relative energies do, and oftentimes ν has the significance of an energy gap that needs to be overcome to create a specific excitation.

Goldstone bosons

Massless or almost massless fields play an important role in high-energy physics because they can be created with little energy and cannot decay for kinematic reasons. In classical statistical field theory they describe long range correlations. Finally, in condensed matter physics or non-relativistic quantum field theory the "massless modes" correspond to gapless excitations, i. e. those that can be excited without much energy cost. Accordingly they dominate thermodynamics and many material properties at small temperatures.

Spontaneous breaking of a continuous global symmetry always introduces massless particles or gapless modes. If the ground state leads to spontaneous breaking of a continuous global symmetry, massless scalar excitations have to be present. They are called "Goldstone bosons".

More specifically, this is a property of the matrix

$$M^2_{jk} = \frac{\partial^2 U}{\partial \Phi_j \partial \Phi_k},$$

and follows from symmetry arguments. Intuitively the reason is clear: A flat direction in the potential (valley in the Mexican hat potential) is dictated by invariance of U with respect to a continuous symmetry. This direction corresponds to a vanishing eigenvalue m^2 of the matrix M^2_{jk}.

For an example with $U(1)$ symmetry we have

$$U = \frac{1}{2}\lambda \left(\rho - \rho_0\right)^2,$$

with

$$\rho = \Phi^*\Phi = \frac{1}{2}\left(\Phi_1^2 + \Phi_2^2\right),$$

where the real fields Φ_1, Φ_2, are related to the complex field $\Phi = (\Phi_1 + i\Phi_2)/\sqrt{2}$. We can choose the expansion point $\Phi_1 = \sqrt{2}\Phi_0$, $\Phi_2 = 0$, $\rho = \Phi_0^2$ since the direction of the expectation value is arbitrary. The mass matrix is now easily computed. With the first derivatives

$$\begin{aligned} \frac{\partial U}{\partial \Phi_1} &= \frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial \Phi_1} = \lambda(\rho - \rho_0)\Phi_1, \\ \frac{\partial U}{\partial \Phi_2} &= \frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial \Phi_2} = \lambda(\rho - \rho_0)\Phi_2, \end{aligned}$$

one obtains

$$\begin{aligned} M^2_{11} &= \frac{\partial^2 U}{\partial \Phi_1^2} = \lambda(\rho - \rho_0) + \lambda\Phi_1^2, \\ M^2_{22} &= \frac{\partial^2 U}{\partial \Phi_2^2} = \lambda(\rho - \rho_0) + \lambda\Phi_2^2, \\ M^2_{12} &= M^2_{21} = \frac{\partial^2 U}{\partial \Phi_1 \partial \Phi_2} = \lambda\Phi_1\Phi_2. \end{aligned}$$

The mass matrix for $\rho = \rho_0$,

$$M^2 = \lambda \begin{pmatrix} \Phi_1^2 & \Phi_1\Phi_2 \\ \Phi_1\Phi_2 & \Phi_2^2 \end{pmatrix},$$

always has one vanishing eigenvalue. In particular, the evaluation at $\Phi_1 = \sqrt{2}\Phi_0$, $\Phi_2 = 0$ yields a diagonal matrix

$$M^2_{ab} = \begin{pmatrix} 2\lambda\rho_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The radial mode Φ_1 has mass squared $m^2 = 2\lambda\rho_0$, while the Goldstone mode Φ_2 is massless, $m^2 = 0$.

The massless field is called "Goldstone boson". This Goldstone boson emerges for example in connection with superfluid Helium-4 or ultracold Bose gases with repulsive interactions. For a non-relativistic spin zero complex field $\varphi(x)$ the $U(1)$ symmetry $\varphi(x) \to e^{i\alpha}\varphi(x)$ is related to particle number conservation. The field equation for $\varphi(x)$ is the Gross-Pitaevskii equation. For the relativistic case it is a Klein-Gordon equation with interaction,

$$\Box_x \Phi(x) - \lambda \left[\Phi^*(x)\Phi(x) - \rho_0\right]\Phi(x) = 0.$$

More generally, the number of massless Goldstone modes that emerges in connection with spontaneous symmetry breaking is given by the number of Lie algebra generators of the symmetry group before spontaneous symmetry breaking minus the number of generators afterwards. Intuitively this is the number of flat directions of the effective potential, and the number of vanishing eigenvalues of the matrix M^2_{jk}.

Gauged scalar field

Let us now generalize our considerations to a field with electromagnetic charge such that the action $\Gamma[\Phi, A]$ is invariant under local U(1) transformations,

$$\Phi(x) \to e^{i\alpha(x)}\Phi(x),$$
$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x).$$

This means that derivatives of the complex fields appear in terms of covariant derivatives

$$D_\mu \Phi(x) = [\partial - ieA_\mu(x)]\Phi(x).$$

What happens now with spontaneous symmetry breaking?

Let us consider the following effective action for scalar QED, playing here the role of a Landau theory,

$$\Gamma[\Phi, A] = \int_x \left\{ D^\mu \Phi^* D_\mu \Phi + \frac{\lambda}{2}(\Phi^*\Phi - \rho_0)^2 + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \right\}.$$

Now assume first a constant background scalar field $\Phi(x) = \Phi_0$, and we can take Φ_0 to be real without loss of generality. For this configuration of the scalar field we find $D_\mu\Phi = -ieA_\mu\Phi_0$, and $D^\mu\Phi^* D_\mu\Phi = e^2\Phi_0^2 A^\mu A_\mu$. But this looks like a mass term for the photons! Indeed, we can evaluate $F^{\mu\nu}F_{\mu\nu} = 2\partial^\mu A^\nu \partial_\mu A_\nu - 2\partial^\nu A^\mu \partial_\mu A_\nu$. With two partial integrations the second term becomes $(\partial_\mu A^\mu)^2$ and when we work in Landau gauge where $\partial_\mu A^\mu = 0$ we can drop it. The part of the action depeding on the gauge field becomes then with another partial integration

$$\Gamma[A] = \int_x \left\{ \frac{1}{2} A^\nu(x) \left[-\partial^\mu \partial_\mu + 2e^2\Phi_0^2 \right] A_\nu(x) \right\}.$$

This is the action of a massive vector field!

Higgs mechanism: photon mass

The field equation for the photon field replaces Maxwells equations in vacuum. In momentum space it reads

$$\left[p^2 + 2e^2\Phi_0^2 \right] A_\nu(p) = 0.$$

The solutions are plane waves with p^2 obeying

$$p^2 + m^2 = -(p^0)^2 + \mathbf{p}^2 + m^2 = 0,$$

where $m^2 = 2e^2\Phi_0^2$ is an effective photon mass squared.

This is an example for the Higgs mechanism: the photon acquires a mass term through the spontaneous breaking of a gauge symmetry!

Screened potential

In non-relativistic physics, the propagation of photons is so fast that it is effectively instantaneous. The photon becomes then a non-dynamical field. If $\mathbf{p}$ is a wave-vector measured in units of inverse length, and p^0 is a frequency, the inverse photon propagtor becomes in the limit where the velocity of light is large

$$-\frac{1}{c^2}(p^0)^2 + \mathbf{p}^2 + m^2 \to \mathbf{p}^2 + m^2.$$

Note that in this counting m has units of inverse length. Accordingly the potential between two unit changes in position space becomes

$$
\begin{aligned}
V(\mathbf{x}-\mathbf{y}) =& \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \frac{1}{\mathbf{p}^2+m^2} \\
=& \frac{1}{4\pi^2} \int_0^\infty dp \int_{-1}^1 d(\cos(\vartheta)) \frac{e^{i\cos(\vartheta)p|\mathbf{x}-\mathbf{y}|}p^2}{p^2+m^2} \\
=& \frac{1}{4\pi^2} \int_0^\infty dp \frac{pe^{ip|\mathbf{x}-\mathbf{y}|} - pe^{-ip|\mathbf{x}-\mathbf{y}|}}{(p^2+m^2)i|\mathbf{x}-\mathbf{y}|} \\
=& \frac{1}{4\pi^2} \int_{-\infty}^\infty dp \frac{pe^{ip|\mathbf{x}-\mathbf{y}|}}{(p^2+m^2)i|\mathbf{x}-\mathbf{y}|} \\
=& \frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}.
\end{aligned}
$$

(The last integral has been done by closing the integral in the upper half of the complex plane and using the residue theorem.) What would be the Coulomb potential for $m = 0$ has now become a Yukawa potential with screening length $1/m$. The electromagnetic interaction becomes a short range interaction! This has a direct consequence for superconductors, where magnetic fields are quickly decaying inside the superconducting material (Meissner effect). In this sense, the Meissner effect can be seen as a variant of the Higgs mechanism for electromagnetism!

In a similar way to what we have described here, the gauge bosons $W^\pm$ and Z in the electroweak standard model gain their mass and the weak nuclear forces becomes short-range.

Photon "eats" Goldstone boson

Massless photons have two internal degrees of freedoms, namely two polarizations or helicities. Massive spin-one particles on the other hand have three internal degrees of freedom, namely their three spin states. Where does this come from? Spontaneous symmetry breaking of a global continuous symmetry produces a Goldstone boson, while the breaking of a local continuous symmetry results in a massive photon, but no Goldstone boson.

Sometimes it is convenient to reparameterize the complex scalar field $\Phi(x)$ in a non-linear way. One such non-linear parameterization is

$$\Phi(x) = \sigma(x)e^{i\pi(x)},$$

where $\sigma(x)$ and $\pi(x)$ are real scalar fields. In the low energy effective action of QCD, $\sigma(x)$ corresponds to the sigma resonance (which is very broad and difficult to observe) and $\pi(x)$ is an analog of the pion. In this parameterization, the gauge transform shifts the pion field $\pi(x) \to \pi(x) + \alpha(x)$, while $\sigma(x)$ is invariant under electromagnetic gauge transformations.

We will now express the effective action in terms of the new fields. First, the effective potential does not depend on $\pi(x)$ at all, $U(\Phi^*\Phi) = U(\sigma^2)$. For a global U(1) symmetry, $\pi(x)$ is the Goldstone boson – it is a massless excitation with no potential term and only the kinetic term in the action. In contrast, $\sigma(x)$ is the radial mode and has a non-vanishing mass

$$m^2 = \frac{\partial^2 U(\sigma^2)}{\partial \sigma^2} = 2\lambda\rho_0.$$

Rewrite action in terms of $\sigma(x)$ and $\pi(x)$

Now we rewrite covariant derivaties of the complex field,

$$
\begin{aligned}
D_\mu\Phi(x) =& [\partial_\mu - ieA_\mu(x)]\sigma(x)e^{i\pi(x)} \\
=& \left[\partial_\mu\sigma(x) + i\sigma(x)\left[\partial_\mu\pi(x) - eA_\mu(x)\right]\right] e^{i\pi(x)}.
\end{aligned}
$$

The kinetic term in the action becomes accordingly

$$D^\mu\Phi^*(x)D_\mu\Phi(x) = \partial^\mu\sigma(x)\partial_\mu\sigma(x) \\ + \sigma(x)^2\left[\partial^\mu\pi(x) - eA^\mu(x)\right]\left[\partial_\mu\pi(x) - eA_\mu(x)\right].$$

We can check that gauge invariance still holds. If $\pi(x) \to \pi(x) + \alpha(x)$ then $\partial_\mu\pi(x) \to \partial_\mu\pi(x) + \partial_\mu\alpha(x)$, while $eA_\mu(x) \to eA_\mu(x) + \partial_\mu\alpha(x)$. Indeed the gauge symmetry is conserved since the pion field only appears in the combination $\partial_\mu\pi(x) - eA_\mu(x)$! Actually we can use the gauge symmetry to make the pion field constant $\partial_\mu\pi(x) = 0$, and even vanishing, $\pi(x) = 0$. This is called unitary gauge. Then $\pi(x)$ disappears from the quantum effective action and the resulting field equations. One says that the photon "eats" the Goldstone boson and becomes massive and we are left with the action

$$\Gamma[\sigma, A] = \int_x \left\{\partial^\mu\sigma\partial_\mu\sigma + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + e^2\sigma^2A^\mu A_\mu + U(\sigma^2)\right\}.$$

It describes an effectively massive gauge field and a massive scalar.

Gauge redundancy

Local gauge theories are "redundant" descriptions. For every generator of the gauge group, there is one degree of freedom on which nothing depends. It can be eliminated by gauge fixing. Different gauge fixings eliminate different fields. For electromagnetism, one may choose Landau gauge $\partial_\mu A^\mu = 0$ which eliminates longitudinal photons or, in the presence of a broken U(1) symmetry unitary gauge $\pi(x) = 0$, which eliminates the Goldstone boson. Of course one cannot apply both conditions simultaneously. Gauge fixings are physically equivalent, even though the gauge fixed actions might look different.

A good reason to accept a redundant description is the locality of the gauge covariant action, which would get lost if one would attempt to elliminate all gauge degrees of freedom.

Electroweak symmetry breaking

A very similar phenomenon occurs for the spontaneous breaking of the electroweak gauge symmetry group SU(2) × U(1) to the electromagnetic gauge symmetry group U(1). The standard model invovles a complex scalar doublet

$$\Phi(x) = \begin{pmatrix} \varphi_1(x) + i\varphi_2(x) \\ \varphi_3(x) + i\varphi_4(x) \end{pmatrix},$$

and four gauge bosons: a triplet $\mathbf{W}_\mu(x)$ for the three generators of SU(2) and a singlet $Y_\mu(x)$ for U(1). The electroweak fields $W^\pm_\mu(x)$ and $Z_\mu(x)$ as well as the electromagnetism gauge field $A_\mu(x)$ are linear combinations of $\mathbf{W}_\mu(x)$ and $Y_\mu(x)$. The symmetry SU(2) × U(1) is broken down to U(1) by an expectation value, which can be brought to the form

$$\langle\Phi(x)\rangle = \begin{pmatrix} \varphi_0 \\ 0 \end{pmatrix}.$$

The $W^\pm$ and Z bosons acquire mass through the Higgs mechanism but the photon remains massless. The real scalars $\varphi_2(x)$, $\varphi_3(x)$ and $\varphi_4(x)$ disappear from the spectrum, much like $\pi(x)$ in the Abelian model. In contrast, $\varphi_1(x)$ plays the role $\sigma(x)$ played before. It descibes a massive scalar parrticle in the low energy description. This is the Higgs boson which has been found at the Large Hadron Collider at CERN.

3 Saddle point approximation and perturbation theory

In this chapter we will start a computation of the quantum effective action $\Gamma_\mathrm{E}[\Phi]$ in the Euclidean domain. We assume that interactions are small, and that some type of perturbation expansion in the small couplings should be possible. We recall that in the absence of interactions the microscopic action $S_\mathrm{E}[\phi]$ is quadratic in ϕ, and we have shown that then $\Gamma_\mathrm{E}[\Phi] = S_\mathrm{E}[\Phi] + \text{const}$, where the constant part can actually depend on external parameters like temperature or chemical potential, or external fields like the metric $g_{\mu\nu}(x)$ or an extneral gauge field $A_\mu(x)$.

Background field identity

Our starting point is the functional integral representation of $\Gamma[\Phi]$, or background field identity, in the Euclidean domain, using abstract index notation,

$$\Gamma_\mathrm{E}[\Phi] = -\ln \int D\phi' \exp\left(-S_\mathrm{E}[\Phi+\phi'] + \frac{\delta\Gamma_\mathrm{E}[\Phi]}{\delta\Phi^\alpha}\phi'^\alpha\right),$$

where Φ^α is the background field or expectation value, and ϕ'^α is the fluctuation field. We separate the classical contribution,

$$\Gamma_\mathrm{E}[\Phi] = \underbrace{S_\mathrm{E}[\Phi]}_{\text{classical contribution}} \underbrace{-\ln \int D\phi' \exp\left(-S_\mathrm{E}[\Phi+\phi'] + S_\mathrm{E}[\Phi] + \frac{\delta\Gamma_\mathrm{E}[\Phi]}{\delta\Phi^\alpha}\phi'^\alpha\right)}_{\text{fluctuation contribution}}.$$

This is similar to the free energy: there are energy and entropy contributions. Every term beyond the classical action is called “fluctuation contribution” or “loop contribution”,

$$\Gamma_\mathrm{E}[\Phi] = S_\mathrm{E}[\Phi] + \Gamma_{\mathrm{E,\ loops}}[\Phi].$$

Here $\Gamma_{\mathrm{E,\ loops}}[\Phi]$ accounts for all loops in perturbation theory. Note that the expression for $\Gamma_{\mathrm{E,\ loops}}[\Phi]$ is implicit because $\delta\Gamma_\mathrm{E}[\Phi]/\delta\Phi^\alpha$ appears on the right hand side.

Saddle point expansion

We expand $S_\mathrm{E}[\Phi+\phi']$ around $\phi'=0$,

$$S_\mathrm{E}[\Phi+\phi'] = S_\mathrm{E}[\Phi] + \frac{\delta S_\mathrm{E}[\Phi]}{\delta\Phi^\alpha}\phi'^\alpha + \frac{1}{2}\frac{\delta^2 S_\mathrm{E}[\Phi]}{\delta\Phi^\alpha\delta\Phi^\beta}\phi'^\alpha\phi'^\beta + \ldots$$

The first derivative term $\delta S_\mathrm{E}[\Phi]/\delta\Phi^\alpha$ cancels against the classical term in $\delta\Gamma_\mathrm{E}[\Phi]/\delta\Phi^\alpha$,

$$\Gamma_{\mathrm{E,\ loops}}[\Phi] = -\ln \int D\phi' \exp\left(-\frac{1}{2}\frac{\delta^2 S_\mathrm{E}[\Phi]}{\delta\Phi^\alpha\delta\Phi^\beta}\phi'^\alpha\phi'^\beta + \ldots + \frac{\delta\Gamma_{\mathrm{E,\ loops}}[\Phi]}{\delta\Phi^\alpha}\phi'^\alpha\right).$$

We can now proceed to an iterative solution. The lowest order is the one-loop approximation. Here one neglects $\delta\Gamma_{\mathrm{E,\ loops}}[\Phi]/\delta\Phi^\alpha$ and higher order terms in the expansion, like $S_\mathrm{E}^{(3)}[\Phi]$ etc. What remains is a Gaussian integral,

$$\begin{aligned}\Gamma_{\mathrm{E,\ 1\text{-}loop}}[\Phi] &= -\ln \int D\chi' \exp\left(-\frac{1}{2}\frac{\delta^2 S_\mathrm{E}[\Phi]}{\delta\Phi^\alpha\delta\Phi^\beta}\phi'^\alpha\phi'^\beta\right)\\ &= -\ln\left(\mathrm{Det}\left(S_\mathrm{E}^{(2)}[\Phi]\right)^{-1/2}\right) + \text{const}\\ &= \frac{1}{2}\mathrm{Tr}\left\{\ln(S_\mathrm{E}^{(2)}[\Phi])\right\} + \text{const}.\end{aligned}$$

Here we used the identity

$$\ln(\mathrm{Det}(M)) = \mathrm{Tr}\{\ln(M)\},$$

The logarithm can only be taken in a basis where M is diagonal. The trace Tr is in the sense of operators and includes here contiuous and discrete indices.

Remarks

(i) The saddle point approximation requires $S_{\rm E}^{(2)}[\Phi]$ to be positive semidefinite (to have positive or at most vanishing eigenvalues). Then the Gaussian integral for a Euclidean functional integral is well defined.

(ii) The second functional derivative of the classical action $S_{\rm E}^{(2)}[\Phi]$ is the inverse of the classical Euclidean propagator in the presence of a background field $\Phi_j(x)$.

(iii) The functional trace is sometimes difficult to evaluate, specifically for inhoogeneous background fields. Special mathematical methods such as the heat kernel expansion have been depeloped for this purpose.

Evaluation in momentum space

In momentum space the 1-loop effective action can be evaluated as

$$\Gamma_{\text{E, 1-loop}}[\Phi] = \frac{1}{2} \underbrace{\int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \, (2\pi)^d \delta^{(d)}(p-q)}_{\rm Tr} (\ln \det S_{\rm E}^{(2)}[\Phi])(p,q).$$

Here det stands for the determinant in the space of discrete field components labed by the indices j and k. For example take $\Phi_n(x) = \Phi_n$ to be constant in space, then the inverse microscopic propagator becomes diagonal in momentum space,

$$(S_{\rm E}^{(2)})_{jk}(p,q)[\Phi] = P_{jk}(p,\Phi)(2\pi)^d \delta^{(d)}(p-q).$$

We need to set here $p = q$ and integrate over it. Using now also that

$$(2\pi)^d \delta^{(d)}(0) = \int_x,$$

which is just the volume of space-time, one finds

$$\Gamma_{\text{E, 1-loop}}[\Phi] = \int_x U_{\text{1-loop}}(\Phi)$$

with the one-loop contribution to the effective potential

$$U_{\text{1-loop}}(\Phi) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \ln \det P(p,\Phi).$$

We will investigate this expression in more detail below.

One loop effective potential

We employ the generic form of the quantum effective action $\Gamma[\Phi]$,

$$\Gamma_{\rm E}[\Phi] = \int_x U(\Phi) + \text{derivative terms}.$$

The effective potential $U(\Phi)$ depends on scalar fields and involves no derivatives. For the computation of $U(\Phi)$ one evaluates $\Gamma[\Phi]$ for homogeneous scalar fields. For $\partial_\mu \Phi = 0$ the derivate terms do not contribute.

We investigate first a real scalar field with classical action

$$S_{\rm E}[\phi] = \int_x \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right\}.$$

In Euclidean space, and when using cartesian coordinates, the metric is simply $g_{\mu\nu} = g^{\mu\nu} = \delta_{\mu\nu}$. We take the microscopic potential to be

$$V(\phi) = \frac{\bar{m}^2}{2}\phi^2 + \frac{\bar{\lambda}}{8}\phi^4.$$

With the derivatives

$$\frac{\partial V}{\partial \phi} = \bar{m}^2\phi + \frac{\bar{\lambda}}{2}\phi^3, \qquad \frac{\partial^2 V}{\partial \phi^2} = \bar{m}^2 + \frac{3}{2}\bar{\lambda}\phi^2,$$

one finds in momentum space for $S_{\mathrm{E}}^{(2)}[\Phi]$, evaluated at $\phi = \Phi = \text{const}$,

$$S_{\mathrm{E}}^{(2)}(p,q)[\Phi] = (p^2 + \bar{m}^2 + \frac{3}{2}\bar{\lambda}\Phi^2)(2\pi)^d\delta^{(d)}(p-q),$$

and therefore the inverse propagator function is

$$P(p,\Phi) = p^2 + \bar{m}^2 + \frac{3}{2}\bar{\lambda}\Phi^2.$$

The propagator depends now on the field expectation value Φ! This yields the one loop contribution to the effective potential

$$U_{\text{1-loop}}(\rho) = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\, \ln(p^2 + \bar{m}^2 + 3\bar{\lambda}\rho)$$

with $\rho = \frac{1}{2}\Phi^2$. Introducing an explicit ultraviolet regulator scale Λ, one finds for the effective potential at this order

$$U(\rho) = \bar{U}_\Lambda + \bar{m}_\Lambda^2\rho + \frac{1}{2}\bar{\lambda}_\Lambda\rho^2 + \frac{1}{2}\int_{p^2<\Lambda^2} \ln(p^2 + \bar{m}_\Lambda^2 + 3\bar{\lambda}_\Lambda\rho).$$

We have introduced also a constant part $\bar{U}_\Lambda$ which could be chosen conveniently.

Full coupling constants and one-loop contributions

We consider a polynomial expansion of $U(\rho)$, with $U'(\rho) = \partial U(\rho)/\partial\rho$, and so on, and define the full coupling constants based on the effective potential,

$$\begin{aligned} m^2 =& U'(0) = \bar{m}_\Lambda^2 + U'_{\text{1-loop}}(0),\\ \lambda =& U''(0) = \bar{\lambda}_\Lambda + U''_{\text{1-loop}}(0),\\ \gamma =& U^{(3)}(0) = U^{(3)}_{\text{1-loop}}(0). \end{aligned}$$

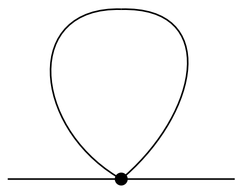
The first two decompose into a microscopic part and a one-loop contribution. Note that γ is a six-point vertex not present in the classical actions. This is a “quantum vertex”.

Feynman graphs

We want to evaluate

$$\begin{aligned} \Delta m^2 =& \frac{\partial}{\partial\rho}U_{\text{1-loop}}(\rho)\\ =& \frac{3\bar{\lambda}_\Lambda}{2}\int_p \frac{1}{p^2 + \bar{m}_\Lambda^2 + 3\bar{\lambda}_\Lambda\rho}. \end{aligned}$$

This corresponds to a one-loop Feynman diagram

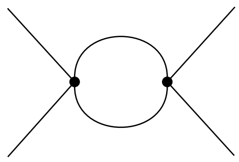


Lines are given by the classical propagator $G = (P^2 + \bar{m}_\Lambda^2 + 3\bar{\lambda}_\Lambda \rho)^{-1}$, and the point denotes the classical vertex $\partial^4 V/\partial\phi^4 = 3\bar{\lambda}_\Lambda$. As usual, a closed line involves a trace, i.e. a momentum integral and a sum over contracted indices.

In a similar way we obtain higher order derivatives,

$$\frac{\partial^2}{\partial\rho^2} U_{\text{1-loop}}(\rho) = -\frac{9\bar{\lambda}_\Lambda^2}{2} \int_p \frac{1}{(p^2 + \bar{m}_\Lambda^2 + 3\bar{\lambda}_\Lambda \rho)^2},$$

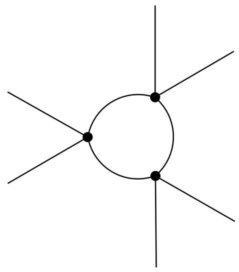
with Feynman diagram



and

$$\frac{\partial^3}{\partial\rho^3} U_{\text{1-loop}}(\rho) = 27\bar{\lambda}_\Lambda^3 \int_p \frac{1}{(p^2 + \bar{m}_\Lambda^2 + 3\bar{\lambda}_\Lambda \rho)^3},$$

with Feynman diagram



Evaluating these expressions at $\rho = 0$, we find the one-loop corrections

$$\Delta m^2 = \frac{3\bar{\lambda}_\Lambda}{2} \int_p \frac{1}{p^2 + \bar{m}_\Lambda^2},$$
$$\Delta\lambda = -\frac{9\bar{\lambda}_\Lambda^2}{2} \int_p \frac{1}{(p^2 + \bar{m}_\Lambda^2)^2},$$
$$\Delta\gamma = 27\bar{\lambda}_\Lambda^3 \int_p \frac{1}{(p^2 + \bar{m}_\Lambda^2)^3}.$$

We recall here the abbreviation

$$\int_p = \int \frac{d^d p}{(2\pi)^d}.$$

Regularization

Some of the expressions we encounter here show divergent behaviour in the high momentum or ultraviolet regime of the momentum integrals. These need to be regularized by imposing a suitable cutoff at a scale Λ. Ultimately we would like to take the limit where the ultraviolet regulator becomes largem, $\Lambda \to \infty$. Also, there could be additional infrared divergences from the lower limit of the momentum integral. In Euclidean space this is not the case for $m_\Lambda^2 > 0$.

For the UV-regularization one has different possibilities, and they have different advantages and disadvantages. Let us start by stating a few general properties of the integrals involved. For the effective potential we are concerned with integrals of the form

$$\begin{aligned}\int \frac{d^d p}{(2\pi)^d} f(p^2) =& \frac{\Omega_d}{(2\pi)^d} \int_0^\infty dp\, p^{d-1} f(p^2) \\ =& \frac{\Omega_d}{2(2\pi)^d} \int_0^\infty dx\, x^{(d-2)/2} f(x)\end{aligned}$$

Here we use the surface are of the unit sphere in d dimensions

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

Well known special values are $\Omega_1 = 2$, $\Omega_2 = 2\pi$, $\Omega_3 = 4\pi$ and $\Omega_4 = 2\pi^2$. However, it is often useful to keep d open, not only because one can then specialize to different physics models, but also because a non-integer value of d provides a possible regularization scheme by itself (dimensional regularization).

Sharp UV cutoff

A simple possibility to regularize the one-loop integrals in the UV is to impose a sharp cutoff by restricting the integrals to $p^2 = x < \Lambda^2$. Let us now also specialize to $d = 4$.

One-loop contribution to mass squared

For the loop correction to the mass term, we need the integral

$$\begin{aligned}\int_p \frac{1}{p^2 + \bar{m}_\Lambda^2} &= \frac{1}{16\pi^2} \int_0^{\Lambda^2} dx \frac{x}{x + \bar{m}_\Lambda^2} = \frac{1}{16\pi^2} \left[\Lambda^2 - \int_0^{\Lambda^2} dx \frac{\bar{m}_\Lambda^2}{x + \bar{m}_\Lambda^2} \right] \\ &= \frac{1}{16\pi^2} \left[\Lambda^2 - \bar{m}_\Lambda^2 \ln \left(\frac{\Lambda^2 + \bar{m}_\Lambda^2}{\bar{m}_\Lambda^2} \right) \right],\end{aligned}$$

resulting in

$$\Delta m^2 = \frac{3\bar{\lambda}_\Lambda}{32\pi^2} \left[\Lambda^2 - \bar{m}_\Lambda^2 \ln \left(\frac{\Lambda^2 + \bar{m}_\Lambda^2}{\bar{m}_\Lambda^2} \right) \right].$$

One concludes that bosonic fluctuations increase the mass term! The loop contribution to the mass squared would have a leading quadratic divergence with the regulator scale Λ in the limit $\Lambda \to \infty$.

In the standard model, one has $m \approx 100\,\text{GeV}$, while grand unification or gravity scales are $\Lambda \approx 10^{15}\,\text{GeV}$ and $\Lambda \approx 10^{18}\,\text{GeV}$ respectively. With

$$m^2 = \bar{m}_\Lambda^2 + g^2 \Lambda^2,$$

a small mass does not seem natural.

One-loop contribution to quartic coupling

For the quartic coupling we can use a simple identity

$$\begin{aligned}\int_p \frac{1}{(p^2+\bar{m}_\Lambda^2)^2} &= -\frac{\partial}{\partial \bar{m}_\Lambda^2}\int_p \frac{1}{p^2+\bar{m}_\Lambda^2} \\ &= \frac{1}{16\pi^2}\left[\ln\left(\frac{\Lambda^2+\bar{m}_\Lambda^2}{\bar{m}_\Lambda^2}\right)+\bar{m}_\Lambda^2\frac{\partial}{\partial \bar{m}_\Lambda^2}\ln\left(\frac{\Lambda^2+\bar{m}_\Lambda^2}{\bar{m}_\Lambda^2}\right)\right] \\ &= \frac{1}{16\pi^2}\left[\ln\left(\frac{\Lambda^2+\bar{m}_\Lambda^2}{\bar{m}_\Lambda^2}\right)+\frac{\bar{m}_\Lambda^2}{\Lambda^2+\bar{m}_\Lambda^2}-1\right].\end{aligned}$$

Neglecting terms $\sim 1/\Lambda^n$ with $n > 0$ yields

$$\int_p \frac{1}{(p^2+\bar{m}_\Lambda^2)^2} = \frac{1}{16\pi^2}\left[\ln\left(\frac{\Lambda^2}{\bar{m}_\Lambda^2}\right)-1\right],$$

which implies the one-loop contribution to the quartic coupling

$$\Delta\lambda = -\frac{9\bar{\lambda}_\Lambda^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{\bar{m}_\Lambda^2}\right)-1\right].$$

The interaction strength is driven to smaller values by the loop corrections. Now the dependence on the regulator scale is only logarithmic.

One-loop contribution to sextic coupling

Finally we consider the loop contribution to the quantum vertex γ. Here we can use our previous result with the calculational trick

$$\begin{aligned}\int_p \frac{1}{(p^2+\bar{m}_\Lambda^2)^3} &= -\frac{1}{2}\frac{\partial}{\partial \bar{m}_\Lambda^2}\int_p \frac{1}{(p^2+\bar{m}_\Lambda^2)^2} \\ &= \frac{1}{32\pi^2\bar{m}_\Lambda^2}.\end{aligned}$$

The last line is the leading expression for $\Lambda \to \infty$ and it is actually finite! Accordingly we find the one-loop contribution to the six-point vertex

$$\gamma = \frac{27\lambda_\Lambda^3}{32\pi^2}\frac{1}{\bar{m}_\Lambda^2}.$$

Fluctuation effects

The one-loop corrections increase the mass, reduce the 4-vertex coupling strength and generates new 6-point vertex,

$$m^2 > \bar{m}_\Lambda^2, \qquad \lambda < \bar{\lambda}_\Lambda, \qquad \gamma > \gamma_\Lambda = 0.$$

Note that $\Delta m^2 \sim \Lambda^2$ and $\Delta\lambda \sim -\ln(\Lambda^2/\bar{m}_\Lambda^2)$ are divergent for $\Lambda \to \infty$, while γ becomes independent of the regulator scale. There is an interesting relation to the canonical mass dimension here. Obviously m^2 has dimensions of mass squared, λ is dimensionless in four dimensions, and γ has dimensions of inverse mass squared.

Separation of scales

Typically one has a separation between large momentum scales of order Λ and much smaller scales of order $\bar{m}_\Lambda$ or m. This assumes small coupling $\lambda_\Lambda \ll 1$. One can infer that different quantities are dominated by rather different momentum ranges in the loop integral:

- The mass squared correction Δm^2 is dominated by modes $p^2 \approx \Lambda^2$. It is UV-dominated, i. e. by microscopic physics.
- The correction to the quartic interaction $\Delta\lambda$ is logarithmically divergent for $\Lambda \to \infty$. All momentum modes contribute here.
- The sextic coupling γ is independent of Λ. It is dominated by modes with $p^2 \approx \bar{m}_\Lambda^2$, or IR-dominated, and independent of microscopic physics.

Impact of fluctuation effects on mass term

Since the mass correction is positive, $\Delta m^2 > 0$, it is possible to have a negative bare mass term $\bar{m}_\Lambda^2 < 0$ but a positive renormalized mass term $m^2 > 0$. Then the system is in the symmetric phase, even for $\bar{m}_\Lambda^2 < 0$. This happens for Ising type models. One has local order but not global order. Strong fluctuation effects destroy the order!

In thermal equilibrium for $T \neq 0$ the fluctuation correction Δm^2 depends on T. This can lead to a phase transition as a function of temperature. The fluctuation contribution $\Delta m^2(T)$ is monotonically increasing with T.

The phase transition occurs at T_c. At the critical temperature the mass term vanishes $m^2(T_c) = 0$. For fermions the sign of the fluctuation effects is opposite. The contribution from fermion fluctuations amounts to $\Delta m^2 < 0$. For the standard model this may lead to top-quark induced electroweak symmetry breaking.

Predictivity of QFT

The sextic vertex γ can be predicted! One may add to the microscopic model a ϕ^6 coupling. By dimension counting it is of the form $\bar{\gamma}_\Lambda \sim 1/\Lambda^2$. The macroscopic coupling γ is dominated by the fluctuation contribution, and an additional microscopic coupling $\bar{\gamma}_\Lambda$ plays no role for $\Lambda \to \infty$.

In other words, we may add to the classical potential a term

$$V_6 = \frac{1}{48}\frac{\tilde{\gamma}}{\Lambda^2}\varphi^6,$$

with dimensionless coupling $\tilde{\gamma}$. This yields for the ratio of the fluctuation contribution and the classical contribution

$$\frac{\text{fluctuation contribution}}{\text{classical contribution}} = \frac{27\bar{\lambda}_\Lambda^3}{32\pi^2}\frac{6}{\tilde{\gamma}}\frac{\Lambda^2}{\bar{m}^2}.$$

One infers that the fluctuation contribution dominates for large ratio $\Lambda/\bar{m}$.

One loop effective potential for three spatial dimensions

The computations for the classical statistics in three dimensions are similar. One now has

$$\Delta m^2 = \frac{3\bar{\lambda}_\Lambda}{8\pi^2}\int_0^{\Lambda^2} dx x^{\frac{1}{2}} \frac{1}{x + \bar{m}_\Lambda^2}$$

and

$$\Delta\lambda = -\frac{9\bar{\lambda}_\Lambda^2}{8\pi^2}\int_0^{\Lambda^2} dx x^{\frac{1}{2}} \frac{1}{(x + \bar{m}_\Lambda^2)^2}.$$

Inspecting the momentum integrals, one finds that $\Delta m^2 \sim \Lambda$ is UV-dominated, while $\Delta\lambda$ is now IR-dominated! One finds

$$\Delta\lambda = -\frac{9\bar{\lambda}_\Lambda^2}{16\pi^2}\frac{1}{\sqrt{\bar{m}_\Lambda^2}}$$

For $\bar{m}_\Lambda \to 0$ one observes an IR-divergence for $\Delta\lambda$. This is a major difficulty for perturbative calculations for $d = 3$ near phase transitions!

Perturbative renormalization

So far we have parametrized our theory in terms of the microscopic parameters $\bar{m}_\Lambda^2$ and $\bar{\lambda}_\Lambda$. But these are not quantities that are easily accessible to any experiment. The strategy is therefore to replace these microscopic parameters by macroscopic quantities m^2 and λ. The microscopic or "bare" parameters $\bar{m}_\Lambda^2$ and $\bar{\lambda}_\Lambda$ are not known. In contrast, the renormalized parameters m^2 and λ can be determined by measurements; for example they enter directly in the computation of propagators, cross sections etc.

Within perturbation theory, one expands in the small renormalized coupling λ. We concentrate here on the leading terms in λ or $\bar{\lambda}_\Lambda$. Moreover, let us concentrate here on $d = 4$ dimensions.

Mass renormalization

The full or renormalized mass is at one-loop order given by

$$m^2 = \bar{m}_\Lambda^2 + \frac{3\bar{\lambda}_\Lambda}{32\pi^2}\Lambda^2 - \frac{3\bar{\lambda}_\Lambda}{32\pi^2}\ln\left(\frac{\Lambda^2 + \bar{m}_\Lambda^2}{\bar{m}_\Lambda^2}\right).$$

This implies that one can express to leading order in the coupling λ the microscopic or bare mass parameter as

$$\bar{m}_\Lambda^2 = m^2 - \frac{3\lambda}{32\pi^2}\Lambda^2 + \frac{3\lambda}{32\pi^2}\ln\left(\frac{\Lambda^2 + m^2}{m^2}\right) + \mathcal{O}(\lambda^2).$$

Coupling renormalization

The full quartic coupling constant is to leading order

$$\lambda = \bar{\lambda}_\Lambda - \frac{9\bar{\lambda}_\Lambda^2}{32\pi^2}\left(\ln\frac{\Lambda^2 + \bar{m}_\Lambda^2}{\bar{m}_\Lambda^2} - 1\right) \approx \bar{\lambda}_\Lambda\left(1 - \frac{9}{32\pi^2}\lambda\ln\frac{\Lambda^2}{m^2}\right)$$

This allows to express the microphysics or bare couling constant as

$$\bar{\lambda}_\Lambda = \frac{\lambda}{1 - \frac{9}{32\pi^2}\lambda\ln\left(\frac{\Lambda^2}{m^2}\right)}.$$

For fixed λ the bare quartic coupling $\bar{\lambda}_\Lambda$ diverges at a "Landau pole" when $\ln(\Lambda^2/m^2) = 32\pi^2/(9\lambda)$. This indicates a limit of validity of the theory. Arbitrary high Λ are not possible for given λ !

In the opposite direction for fixed λ_Λ, Λ and $m^2 \to 0$ one has $\lambda \to 0$. This phenomenon is known as "triviality of Φ^4 theory". For a given finite Λ one finds an upper bound for λ and therefore the mass of the Higgs boson in the standard model. For Λ in the region of a few TeV the resulting bound is $m_{\text{Higgs}} \lesssim 500\,\text{GeV}$.

Sextic coupling

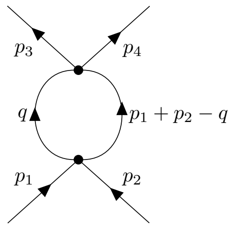
The Φ^6 coupling $\gamma = 27\lambda^3/(32\pi^2 m^2)$ is fixed in terms of λ and m^2. There is no additional free parameter. The theory is specified in terms of only two renormalized parameters, m^2 and λ.

Momentum dependence of Φ^4-vertex

The one-particle irreducible four point vertex depends on the momentum of of the incoming or outgoing particles

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = 3\tilde{\lambda}(p_1, p_2, p_3)(2\pi^4)\delta^{(d)}(p_1 + p_2 - p_3 - p_4)$$

The Feynman graph for the fluctuation contribution is given by



In this notation, the quartic coupling λ computed previously corresponds to $\tilde{\lambda}(0,0,0) = \lambda$. The difference

$$\lambda(p_1, p_2, p_3) - \lambda(0,0,0),$$

is then found to be IR-dominated. It does not depend on Λ. In perturbation theory it is computable and found $\sim \lambda^2$. Cutoff corrections are $\sim 1/\Lambda^2$ and vanish for $\Lambda \to \infty$. The difference is therefore predictable! The whole momentum dependence of $\Gamma^{(4)}$ and the associated cross sections are predicted in terms of the two parameters λ and m^2.

The lesson we learn here is that IR-dominated quantities are predictable! Only a finite number of renormalized couplings needs to be specified! The same strategy works actually for QED, with the renormalized coupling being the electric charge e in addition to the particles masses.

Renormalizable theories miracle

Once all quantities are expressed in terms of renormalized couplings, all momentum integrals become ultraviolet finite for $\Lambda \to \infty$, even for higher loops. Theories with this property are called renormalizable theories. For renormalizable theories, the limit $\Lambda \to \infty$ can be taken! This works very similar in other regularizations schemes, e. g. in the limit $d \to 4$ within dimensional regularizaton.

Typical renormalized parameters are m^2 and λ for ϕ^4 theory, m_e and e for QED and the quark mass m_q and the strong couling constant g_s for QCD. All other quantities are then predictable in terms of these.

Predictivity and irrelevant operators

IR-dominated terms in the quantum effective action $\Gamma[\Phi]$ are known as “irrelevant operators”. Key elements for the predictability of QFT in situations where microphysics is not precisely known are

- symmetries
- fluctuation domination of “irrelevant operators”

Additional predictivity arises in situations where further microphysics is actualy known.

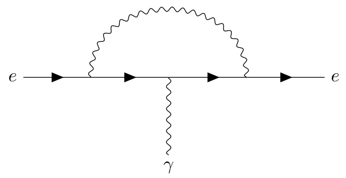
Example in QED, anomalous magnetic momentum of muon

As an example for the principles discussed above, let us take the magnetic moment of the electron, or muon. First, the Dirac theory without radiative corrections gives the magentic moment in Bohr units

$$g = 2.$$

Quantum corrections yield corrections to this result such that $g - 2 \neq 0$.

Specifically, a quantum vertex from the diagram



leading to a term of the form

$$\Delta\Gamma = \int_x \left\{ c\,\bar{\psi}(x)[\gamma^\mu, \gamma^\nu]\psi(x)F_{\mu\nu}(x) \right\}.$$

Such a term is consistent with all all symmetries. The dimension of c is mass^{-1}, and in an unknown misroscopic theory defined at a scale Λ one would expect c to be of order $1/\Lambda$.

Quantum corrections to c are IR-dominated, with leading cutoff dependent terms proportional to $1/\Lambda$. As a consequence, for large scale separation, $\Lambda \to \infty$, the correction $g-2$ becomes predictable! For pure QED (with loops involving electrons, muons, taons and photons) this calculation has been done for the muon anomaleous magnetic moment in perturbation theory up to five loops[1],

$$\left[\frac{g-2}{2}\right]_{\text{QED}} = \frac{\alpha}{2\pi} + 0.765857420(13)\left(\frac{\alpha}{\pi}\right)^2 + 24.05050985(23)\left(\frac{\alpha}{\pi}\right)^3 + 130.8782(60)\left(\frac{\alpha}{\pi}\right)^4 + 751.0(9)\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

The leading term is the classical one-loop result obtained by Schwinger, and will be recalculated as an excercise. With the experimental value for the inverse of the electromagnetic fine structure constant $1/\alpha = 137.035999046(27)$ one finds

$$\left[\frac{g-2}{2}\right]_{\text{QED}} = 0.0011658471893(10).$$

The latest experimental result based on experiments at Brookhaven National Lab and at Fermilab is[2]

$$\left[\frac{g-2}{2}\right]_{\text{experiment}} = 0.00116592061(41).$$

The difference is

$$7342(41) \times 10^{-11},$$

and pure QED (without hadrons) is excluded at more that 100σ level.

Electroweak fluctuation effects ($W^\pm$ bosons, Z bosons and Higgs bosons) also contribute to $g-2$, and have been calculated to two-loop order plus leading three-loop terms,

$$\Delta\left[\frac{g-2}{2}\right]_{\text{electroweak}} = 154(1) \times 10^{-11}.$$

Most difficult are calculations of the hadronic contribution to $g-2$. A recent lattice-QCD simulation gave[3]

$$\Delta\left[\frac{g-2}{2}\right]_{\text{hadronic}} = 7075(55) \times 10^{-11}.$$

With this, there is currently not a big discrepancy between theory and experiment. The hardronic theory calculation is to be confirmed by other groups.

[1] R.L. Workman et al. (Particle Data Group), Prog. Theor. Exp. Phys. 2022, 083C01 (2022).
[2] B. Abi et al. (Muon g-2 Collaboration) Phys. Rev. Lett. 126, 141801 (2021).
[3] Borsanyi, S., Fodor, Z., Guenther, J.N. et al., Nature 593, 51–55 (2021).

4 Gauge theories

Local transformations

Gauge theories are models with a local symmetry. For the example of complex fermionic or scalar fields $\psi(x)$ one has

$$\psi(x) \to U(x)\psi(x).$$

An important example are the strong interactions which are described by the gauge group SU(3). Here the fields $\psi(x)$ are for quarks (up, down, strange, charm, bottom or top) that are each in a color-triplet. In other words, ψ is a complex three component Grassmann field, and U a matrix

$$\psi_j(x), \qquad U_{ij}(x),$$

such that the transformation law becomes

$$\psi_i(x) \to \psi'_i(x) = U_{ij}(x)\psi_j(x),$$

where $i, j = 1, \ldots, 3$ are the color indices.

The color group SU(3)

The transformation matrices $U(x)$ are elements of the group SU(3) of special unitary transformation in three complex dimensions,

$$U^\dagger U = \mathbb{1}, \qquad \det(U) = 1.$$

The group structure is obvious, with the unit matrix being the unit element, inverse $U^{-1} = U^\dagger$, the composition law $U_1 U_2 = U_3$, such that with $U_3^\dagger = (U_1 U_2)^\dagger = U_2^\dagger U_1^\dagger$ one has $U_3^\dagger U_3 = U_2^\dagger U_1^\dagger U_1 U_2 = \mathbb{1}$, and $\det(U_3) = \det(U_1)\det(U_2) = 1$. Also associativity is clear, $U_1(U_2 U_3) = (U_1 U_2)U_3$. The composition is non-Abelian, however, $U_1 U_2 \neq U_2 U_1$.

The weak group SU(2)

Similarly, the weak interactions involve an SU(2)-gauge symmetry, for which left-handed leptons and quarks are doublets, i. e. two-component complex Grassmann fields

$$\begin{pmatrix} \nu(x) \\ e(x) \end{pmatrix}_L, \qquad \begin{pmatrix} u_i(x) \\ d_i(x) \end{pmatrix}_L,$$

and so on. The left-handed part of a Dirac field is obtained with the projection

$$\psi_L(x) = P_L \psi(x) = \frac{1}{2}(\mathbb{1} + \gamma_5)\psi(x).$$

In this case, $U(x)$ is a complex 2×2-matrix. For the standard model, one has an additional Abelian U(1) symmetry under which left- and right-handed fermions transform with different charges.

Lie groups and exponential map

Lie groups are continuous groups which are also differentiable manifolds.[4] They have the interesting property that they can be characterized by the transformations that are infinitesimally close to the unit element, in terms of the Lie algebra.

[4] For an introduction to Lie groups and Lie algebras in physics, see also the course "Symmetries" https://www.tpi.uni-jena.de/~floerchinger/teaching/

Finite group transformations can be composed of many small ones with the exponential map

$$U = \lim_{N\to\infty} \left(\mathbb{1} + \frac{i\alpha^z T_z}{N} \right)^N = \exp\left(i\alpha^z T_z\right).$$

Here $\alpha^z T_z$ (with summation over the index z implied) is an element of the Lie algebra, and the T_z are the generators of the Lie algebra. The coefficients are real, $\alpha^z \in \mathbb{R}$.

One distinguishes between the abstract Lie group and a representation of it. As for any group, a representation has the same composition laws as the abstract group itself. With a representation of the group comes a representation of its algebra and vice versa.

Infinitesimal transformations

Because finite transformations can be composed out of many ininitesimal ones, it is for many applications sufficient to consider infinitesimal transformations. These are very close to the unit element,

$$U = \mathbb{1} + i\alpha^z T_z,$$

in the sense that α^z is infinitesimal. Fields in the fundamental representation transform under an infinitesimal gauge transformation according to

$$\psi(x) \to \psi(x) + \delta\psi(x) = \psi(x) + i\alpha^z(x) T_z \psi(x).$$

Generators for SU(N)

We consider now an SU(N)-symmetry with fermions in the fundamental n-component representation. The generators of the Lie algebra associated with SU(N) are a complete basis for the vector space (over the real numbers) of hermitian, traceless $N \times N$ matrices,

$$T_z^\dagger = T_z, \qquad \mathrm{tr}\{T_z\} = 0.$$

This can be easily checked for infinitesimal transformations, and implies also $U^\dagger = \exp\left(-i\alpha^z(x) T_z\right)$.

Generators for SU(2) and SU(3)

For SU(2), one has $z = 1, \ldots, 3$, and the generators can be written in terms of the three Pauli matrices,

$$T_z = \frac{1}{2}\sigma_z, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For SU(3), there are eight generators, $z = 1, \ldots, 8$,

$$T_z = \frac{1}{2}\lambda_z,$$

and λ_z are the eight "Gell-Mann matrices", to be given explicitly later. The normalization is

$$\mathrm{tr}\{T_z T_y\} = \frac{1}{2}\delta_{zy}.$$

This normalization can be chosen for the fundamental representation of general SU(N).

Composing Lie group elements

To combine two transformations, one needs the Baker-Campbell-Hausdorff formula

$$e^u e^v = e^{w(x,y)},$$

with

$$w(u,v) = u + v + \frac{1}{2}[u,v] + \frac{1}{12}[u,[u,v]] - \frac{1}{12}[v,[u,v]] + \cdots$$

This shows that it is enough to know how to calculate commutators between the Lie algebra generators T_z. To the order given here, the Baker-Campbell-Haussdorff formula can easily be derived from series expansions of exponentials and logarithms.

Commutator and structure constants

The commutator between generators for a given Lie group is of the form

$$[T_y, T_z] = i f_{yz}{}^{w} T_w,$$

where the structure constants $f_{yz}{}^{w}$ characterize the Lie algebra and therefore indirectly the Lie group. Obviously, the structure constants are anti-symmetric in the first two indices, $f_{yz}{}^{w} = -f_{zy}{}^{w}$. When the generators are hermitian (which is the case for compact Lie groups), the structure constants are real, $f_{yz}{}^{w} \in \mathbb{R}$.

As an example, the generators $T_j = \sigma_j/2$ of SU(2) fulfill

$$[T_j, T_k] = i\epsilon_{jkl} T_l,$$

so that the structure constants for SU(2) are given by the Levi-Civita symbol, $f_{jk}{}^{l} = \epsilon_{jkl}$.

Representations of the Lie algebra

The generators T_z can be realized through different representations. For example, this might be matrices such that the above commutation relation is fulfilled. But representations can also be constructed in different ways, for example as differential operators acting on fields.

For example, rotations in three dimensions are defined for many different objects, e. g. fields, particles or solid bodys, corresponding to many different representations of the Lie algebra of SU(2) (which is equal to the Lie algebra of SO(3) in the sense that the structure constants agree).

Jacobi identity and adjoint representation

The generators also satisfy the Jacobi identity

$$[T_x, [T_y, T_z]] + [T_y, [T_z, T_x]] + [T_z, [T_x, T_y]] = 0.$$

For matrix Lie algebras this idendity is easily checked by writing it out explicitly. More generally it is part of the requirements for a Lie algebra. For the structure constants, this implies

$$f_{xu}{}^{v} f_{yz}{}^{u} + f_{yu}{}^{v} f_{zx}{}^{u} + f_{zu}{}^{v} f_{xy}{}^{u} = 0.$$

From the Jacobi identity, one can see that the structure constants can be used to construct another generic representation, the adjoint representation. Here one sets the generator matrices to

$$(T_z^{(A)})^{v}{}_{u} = i f_{zu}{}^{v}.$$

Indeed, one has now

$$[T_x^{(A)}, T_y^{(A)}] = i f_{xy}{}^{w} T_w^{(A)},$$

as a consequence of the Jacobi idendity.

The adjoint representation for the Lie algebra of SU(2) is given by the three 3×3-matrices

$$(T_j^{(A)})^k{}_l = -i\epsilon_{jkl}.$$

These are also the generators for the Lie algebra of SO(3) in the fundamental representation. This shows again that these two Lie algebras agree, SU(2) $\cong$ SO(3). In contrast, for SU(3) the adjoint representation is given by 8×8 matrices.

The fundamental and the adjoint representation are the most important representations of Lie algebras needed in the following. However, there are many more and they all induce corresponding representations of the Lie group through the exponential map.

Gauge fields and covariant derivatives

Partial derivatives do not transform homogeneously under local gauge transformations,

$$\partial_\mu \psi(x) \to \partial_\mu \left[U(x)\psi(x)\right] = U(x)\left[\partial_\mu \psi(x)\right] + \left[\partial_\mu U(x)\right]\psi(x).$$

For constant $U(x)$ (a global transformation) one would only have the first term on the right hand side, but the second term arises in addition for local transformations. One would like to avoid such an additional term, in order to construct invariant actions etc.

Similarly to the local U(1) symmetry of electromagnetism, one introduces gauge fields and defines a covariant derivative. For the SU(2) gauge symmetry of the weak interaction, these additional gauge fields give three W/Z-bosons, and for quantum chromodynamics (QCD), there are eight gluons. We denote the gauge fields by $A_\mu^z(x)$. There is one field for each generator T_z (e. g. $z = 1, \ldots, 3$ for SU(2) or $z = 1, \ldots, 8$ for SU(3)).

The covariant derivative is defined as

$$D_\mu \psi(x) = \left[\partial_\mu - igA_\mu^z(x)T_z\right]\psi(x),$$

with g the gauge coupling.

Transformation law for the gauge field

We want a transformation of the gauge fields such that the covariant derivative transforms homogeneously, or, in other words, just like the field $\psi(x)$ itself,

$$D_\mu \psi(x) = \left[\partial_\mu - igA_\mu^z(x)T_z\right]\psi(x) \to U(x)D_\mu\psi(x).$$

This requires

$$A_\mu^z(x)T_z \to U(x)\left[A_\mu^z(x)T_z\right]U^{-1}(x) - \frac{i}{g}[\partial_\mu U(x)]U^{-1}(x).$$

With this, a covariant derivative transforms as

$$D_\mu \to U(x)D_\mu U^{-1}(x).$$

For an infinitesimal transformation of the form

$$U(x) = \mathbb{1} + i\alpha^y(x)T_y, \qquad U^{-1}(x) = \mathbb{1} - i\alpha^y(x)T_y,$$

one finds the required transformation law for the gauge field

$$\begin{aligned}A_\mu^z(x)T_z \to & A_\mu^z(x)T_z + \frac{1}{g}\partial_\mu\alpha^z(x)T_z + i\left\{\alpha^y(x)T_y A_\mu^z(x)T_z - A_\mu^z(x)T_z\alpha^y(x)T_y\right\} \\ &= A_\mu^z(x)T_z + \frac{1}{g}\partial_\mu\alpha^z(x)T_z + i[\alpha^y(x)T_y, A_\mu^z(x)T_z] \\ &= A_\mu^z(x)T_z + \frac{1}{g}\partial_\mu\alpha^z(x)T_z + i\alpha^y(x)A_\mu^w(x)[T_y, T_w].\end{aligned}$$

Adjoint representation

The commutator of two generators in the last line may be expressed itself as a linear combination of commutators using the structure constants,

$$[T_y, T_w] = i f_{yw}{}^z T_z,$$

and we find the transformation law for the gauge field components

$$A_\mu^z(x) \to A_\mu^z(x) + \frac{1}{g}\partial_\mu\alpha^z(x) - \alpha^y(x) f_{yw}{}^z A_\mu^w(x).$$

Interestingly, the last term is formally of the same form as the change in a matter field

$$\delta\psi(x) = i\alpha^y(x)T_y\psi(x),$$

except that the generator is here in the adjoint representation,

$$i\alpha^y(x)(T_y^{(A)})^v{}_w A_\mu^w(x) = -\alpha^y(x) f_{yw}{}^v A_\mu^w(x).$$

In this sense, a non-Abelian gauge field is actually itself charged, and transforms in the adjoint representation of the gauge group.

Another useful way to write the transformation law for the gauge field is

$$A_\mu^z(x) \to A_\mu^z(x) + \frac{1}{g}(D_\mu[A])^z{}_y\alpha^y(x),$$

where

$$\begin{aligned}(D_\mu[A])^z{}_y =& \partial_\mu\delta^z{}_y - igA_\mu^w(x)(T_w^{(A)})^z{}_y \\ =& \partial_\mu\delta^z{}_y + gA_\mu^w(x) f_{wy}{}^z \\ =& \partial_\mu\delta^z{}_y - gA_\mu^w(x) f_{yw}{}^z\end{aligned}$$

is the covariant derivative in the adjoint representation.

Different representations of covariant derivatives

We have seen that the generators of the Lie algebra exist in different representations and so do the covariant derivatives that can be constructed out of them,

$$D_\mu = \partial_\mu - igA_\mu^z(x)T_z.$$

In fact, the appropriate generator for a covariant derivative depends on what object the derivative is acting on. For a field in some representation R, one must use

$$D_\mu^{(R)}\psi^{(R)}(x) = \left[\partial_\mu - igA_\mu^z(x)T_z^{(R)}\right]\psi^{(R)}(x).$$

For example, if the field is in the fundamental representation as for quarks, we ned to use $T_z^{(F)} = T_z$ and for neutral fields, one has the trivial representation, $T_z^{(0)} = 0$, so that a covariant derivative becomes an ordinary derivative.

Covariant derivatives fulfills the Leibniz rule

$$D_\mu[AB] = [D_\mu A]B + A[D_\mu B],$$

even though A and B may be in different representations. This is in particular useful for partial integration.

Commutator of covariant derivatives and field strength

Consider the covariant derivative in some representation of the Lie algebra

$$D_\mu = \partial_\mu - igA_\mu^z(x)T_z = \partial_\mu - i\mathbf{A}_\mu(x).$$

In the last equation we introduced a Lie-algebra-valued gauge field $\mathbf{A}_\mu(x) = gA_\mu^z(x)T_z$. The rescaling by the coupling constant is also convenient. Let us calculate the commutator of the two covariant derivatives,

$$\begin{aligned}[D_\mu, D_\nu] &= [\partial_\mu - i\mathbf{A}_\mu(x), \partial_\nu - i\mathbf{A}_\nu(x)] \\ &= -i\left(\partial_\mu \mathbf{A}_\nu(x) - \partial_\nu \mathbf{A}_\mu(x) - i\left[\mathbf{A}_\mu(x), \mathbf{A}_\nu(x)\right]\right)\end{aligned}$$

The resulting expression can be seen as a non-Abelian, and Lie-algebra-valued generalization of the field strength tensor,

$$\mathbf{F}_{\mu\nu}(x) = \partial_\mu \mathbf{A}_\nu(x) - \partial_\nu \mathbf{A}_\mu(x) - i\left[\mathbf{A}_\mu(x), \mathbf{A}_\nu(x)\right] = gF_{\mu\nu}^z(x)T_z.$$

The field strength tensor in components is

$$F_{\mu\nu}^z(x) = \partial_\mu A_\nu^z(x) - \partial_\nu A_\mu^z(x) + gf_{yw}{}^z A_\mu^y(x) A_\nu^w(x).$$

Note in particular that the field strength contains terms linear and quadratic in the gauge field.

From the definition as a commutator of covariant derivatives one finds that gauge transformations transform the fields strength in a covariant way,

$$\mathbf{F}_{\mu\nu}(x) \to U(x)\mathbf{F}_{\mu\nu}(x)U^{-1}(x).$$

A combination like

$$\frac{1}{2g^2}\text{tr}\left\{\mathbf{F}^{\mu\nu}(x)\mathbf{F}_{\mu\nu}(x)\right\} = \frac{1}{4}F_z^{\mu\nu}(x)F_{\mu\nu}^z(x)$$

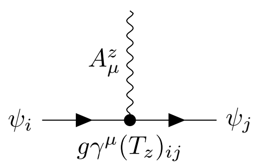
is accordingly invariant and can appear in the Lagrangian. We used here $\text{tr}\{T_z T_w\} = \delta_{zw}/2$.

Gauge invariant action

We now construct gauge invarant actions and start with a gauge invariant kinetic term for the fermions,

$$\begin{aligned}S_{\text{Dirac}} &= \int_x \left\{-\bar{\psi}(x)\gamma^\mu D_\mu \psi(x)\right\} \\ &= \int_x \left\{-\bar{\psi}(x)\gamma^\mu \partial_\mu \psi(x) + igA_\mu^z(x)\bar{\psi}(x)\gamma^\mu T_z \psi(x)\right\}.\end{aligned}$$

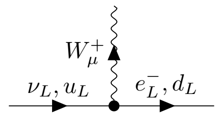
The last terms induces a vertex



similar to the vertex $e\gamma^\mu$ for photons.

Neutron decay

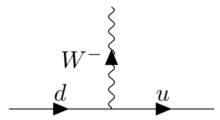
An example in the electroweak SU(2) theory is



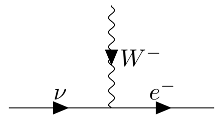
The fermion species can be changed in the vertex due to $(T_z)_{ij}$! This leads to interesting effects, such as the decay of the neutron,

$$n \to p^+ + e^- + \bar{\nu}.$$

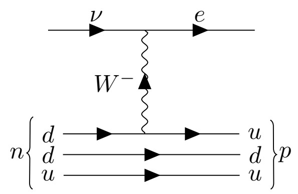
The quark decomposition of the neutron n is udd and of the proton p it is uud. The electric charge of u is $2/3$ and of d is $-1/3$. Out of the vertex



and its relative



one can construct a neutron decay process



For small momenta, the W-propagator can be approximated by $\eta^{\mu\nu}/m_W^2$. This leads to the pointlike four-fermion interaction of the form

$$\frac{g^2}{m_W^2} \left[\bar{u}_L(x)\gamma^\mu d_L(x)\right] \left[\bar{e}_L(x)\gamma_\mu \nu_L(x)\right].$$

Such effective vertices are known from Fermis theory of weak interactions. The mass of the W boson $m_W^2 \sim g^2\phi_0^2$ is generated by the Higgs mechanism, and accordingly the strength of the four-fermion vertex in Fermis theory is proportional to $1/\phi_0^2$. This allows a rough estimate of the Higgs field expectation value ϕ_0.

Self interactions

There is a crucial difference between non-Abelian and Abelian gauge theories. For an Abelian gauge theory, the photon has no self-interaction since it is neutral. As a consequence, Maxwell's equations are linear.

In contrast, for non-Abelian gauge theories such as Yang-Mills theory, there is a self-interaction between gluons. The gluons carry color charge, not only the quarks. The field equations thus become non-linear. The necessity for interaction with gauge bosons is also clear for the $W^\pm$-bosons. In particular they are charged and interact with the photon.

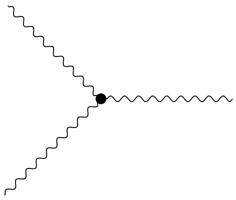
We next need a gauge covariant kinetic term which generalizes the Maxwell action for the photon. It is given in Minkowski space conventions by

$$S_{\text{Yang-Mills}} = \int_x \left\{ -\frac{1}{4} F_z^{\mu\nu}(x) F^z_{\mu\nu}(x) \right\} = \int_x \left\{ -\frac{1}{2g^2} \text{tr} \left\{ \mathbf{F}^{\mu\nu}(x) \mathbf{F}_{\mu\nu}(x) \right\} \right\}.$$

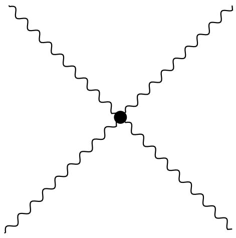
Here we recall that the field strength is given by

$$F^z_{\mu\nu}(x) = \partial_\mu A^z_\nu(x) - \partial_\nu A^z_\mu(x) + g f_{yw}{}^{z} A^y_\mu(x) A^w_\nu(x).$$

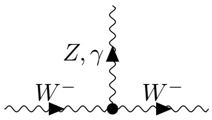
The action $S_{\text{Yang-Mills}}$ contains terms with three gluons leading to a vertex



and with four gluon fields, leading to a vertex



Similarly, in the electroweak theory one has interaction vertices of the form



Let us note here that the Yang-Mills action is actually by itself (without any fermions) a non-trivial theory for interacting gauge fields. We will see later that it becomes strongly interacting in the infrared, and it is not easily solved.

Action for spinor gauge theory

We now have all the components to construct an action for a gauge theory with fermions and gauge bosons. We just need to combine the part for Dirac fermions with the kinetic part for gauge fields,

$$\begin{aligned} S =& S_{\text{Dirac}} + S_{\text{Yang-Mills}} \\ =& \int_x \left\{ -\bar{\psi}(x)\gamma^\mu \left[\partial_\mu - i\mathbf{A}_\mu(x)\right]\psi(x) - \frac{1}{2g^2}\text{tr}\left\{\mathbf{F}^{\mu\nu}(x)\mathbf{F}_{\mu\nu}(x)\right\} \right\} \\ =& \int_x \left\{ -\bar{\psi}(x)\gamma^\mu \left[\partial_\mu - igA^z_\mu(x)T_z\right]\psi(x) - \frac{1}{4}F^{z\mu\nu}(x)F^z_{\mu\nu}(x) \right\}. \end{aligned}$$

This is the basic structure of quantum chromodynamics (QCD), but also of the fermionic and gauge field sector of the electroweak theory.

Problems with gauge freedom

When one attempts to derive a propagator for the Yang-Mills gauge fields, one encounters the same problems as for the photon. In fact, at quadratic level in the gauge field the self-interaction drops out, and Yang-Mills fields are like a collection of several copies of the photon. Recall that an inverse photon propagator is proportional to $p^2\delta^\mu{}_\nu - p^\mu p_\nu = p^2\mathscr{P}^\mu{}_\nu$, and it is not invertible. In fact, this matrix has a vanishing eigenvalue corresponding to the eigenvector p^μ. This problem is a direct consequence of the gauge freedom.

To understand this in more detail, let us decompose the (Abelian) gauge field into a longitudinal or pure gauge part $A^L_\mu(x) = \partial_\mu\beta(x)$, and a remainder term $A^T_\mu(x)$,

$$A_\mu(x) = \partial_\mu\beta(x) + A^T_\mu(x).$$

This decomposition becomes basically unique when $A^T_\mu(x)$ is fixed to obey a specific gauge condition, such as the Landau gauge $\partial^\mu A^T_\mu(x) = 0$. In Landau gauge, there is a residual ambiguity with respect to transformations $A^T_\mu(x) \to \partial_\mu\kappa(x)$, $\beta(x) \to \beta(x) - \kappa(x)$, where $\kappa(x)$ is a harmonic function, $\Box_x\kappa(x) = 0$.

The QED action is invariant under $A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x)$ or $\beta(x) \to \beta(x) + \alpha(x)$. This means that it does not depend on the longitudinal part $A^L_\mu(x) = \partial_\mu\beta(x)$. Accordingy, fluctuations of $A^L_\mu(x) = \partial_\mu\beta(x)$ are not suppressed in the functional integral and contribute an infinite factor. At the same time, physical observables should be gauge invariant and not depend on $A^L_\mu(x)$.

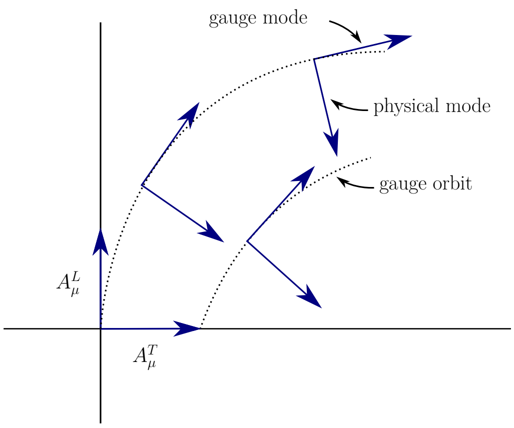
Functional integral for gauge theories

The following discussion will be given in the Euclidean domain formulation of the theory, but works similarly in Minkwoski space. In other words, we work with the Euclidean action S_E, but for simplicity we drop the index E.

Let us try to proceed with the standard definition

$$Z[j] = \int DA\ \exp\left(-S[A] + \int_x \left\{A^z_\mu(x) j^\mu_z(x)\right\}\right).$$

The functional integral contains an integral over the gauge modes that are the extension of the longitudinal gauge bosons to the whole space of non-Abelian gauge fields $A^z_\mu(x)$.



The action does not depend on the gauge modes. As a consequence, the functional integral diverges. Furthermore, perturbation theory cannot be used. In other words, $S^{(2)}$ is not invertible, and we cannot proceed by a saddle point expansion.

Gauge fixing in Abelian gauge theory

Gauge fixing term

For an Abelian theory, we can define a global physical field $A^T_\mu(x)$ and a global gauge degree of freedom $A^L_\mu(x)$. Since the action does not depend on $A^L_\mu(x) = \partial_\mu \beta(x)$, we may simply "take out" this part from the functional integral by invoking a functional Dirac distribution,

$$Z[j] = \int DA\ \delta[A^L_\mu] \exp\left(-S[A] + \int_x j^\mu(x) A_\mu(x)\right).$$

We can replace the functional Dirac distribution by a Gaussian,

$$\delta[A^L_\mu] \to \lim_{\xi\to 0} \exp\left(-\frac{1}{2\xi}\int_x [\partial^\mu A_\mu(x)]^2\right).$$

Note that when $A^T_\mu(x)$ obeys the Landau gauge condition, the functional above vanishes except when $\partial^\mu A_\mu(x) = \partial^\mu \partial_\mu \beta(x) = 0$. This is sufficient gauge fixing, at least for perturbation theory.

One may consider the "gauge fixing term" as an addition to the gauge invariant action S,

$$Z[j] = \int DA\ \exp\left(-S[A] - S_{\text{gf}}[A] + \int_x \{j^\mu_c(x) A^T_\mu(x)\}\right)$$

where

$$S_{\text{gf}}[A] = \frac{1}{2\xi}\int_x \left\{[\partial^\mu A_\mu(x)]^2\right\}.$$

We take $\xi \to 0$ at the end. This is called "Landau gauge fixing".

Propagator with gauge fixing

The gauge fixing term provides an inverse propagator for the longitudinal photon. From

$$S_{\text{gf}}[A] = \frac{1}{2\xi}\int_p A_\nu(-p) p^\nu p^\mu A_\mu(p),$$

we infer

$$(S^{(2)}_{\text{gf}})^{\mu\nu}(p,q) = \frac{1}{\xi} q^\mu q^\nu = \frac{1}{\xi} p^2 \left[\delta^{\mu\nu} - \mathscr{P}^{\mu\nu}\right] (2\pi)^d \delta^{(d)}(p-q).$$

The microscopic or classical inverse propagator with gauge fixing, proportional to

$$\eta^{\mu\nu} p^2 + \left[\frac{1}{\xi} - 1\right] p^\mu p^\nu,$$

is now invertible, with the classical propagator being proportional to

$$\frac{1}{p^2}\left[\eta_{\mu\nu} + (\xi - 1)\frac{p_\mu p_\nu}{p^2}\right].$$

Now perturbation theory can be developed.

Gauge fixing in non-Abelian gauge theories

Background fluctuation splitting

Let us decompose the non-Abelian gauge field we use as integration variable in the functional integral as

$$a^z_\mu(x) = A^z_\mu(x) + a'^z_\mu(x).$$

The field $A^z_\mu(x)$ is a background field, while $a'^z_\mu(x)$ is a fluctuating field. We will eventually do a change of variables and integrate over the field $a'^z_\mu(x)$.

Under an inifinitesimal gauge transformation the full field transforms as

$$
\begin{aligned}
a^z_\mu(x) &\to a^z_\mu(x) + \frac{1}{g}\partial_\mu \alpha^z(x) - \alpha^y(x) f_{yw}{}^z a^w_\mu(x) \\
&= a^z_\mu(x) + \frac{1}{g}(D_\mu[a])^z{}_y \alpha^y(x).
\end{aligned}
$$

One could distribute this transformation in different ways to the background field $A^z_\mu(x)$ and fluctuating field $a'^z_\mu(x)$. We discuss now two possibilities.

Fluctuation field gauge transform

We first consider a "fluctuation field gauge transform", where the background part $A^z_\mu(x)$ is unchanged, and $a'^z_\mu(x)$ gets transformed,

$$
\begin{aligned}
A^z_\mu(x) &\to A^z_\mu(x), \\
a'^z_\mu(x) &\to a'^z_\mu(x) + \frac{1}{g}\partial_\mu \alpha^z(x) - \alpha^y(x) f_{yw}{}^z \left[A^w_\mu(x) + a'^w_\mu(x)\right] \\
&= a'^z_\mu(x) + \frac{1}{g}(D_\mu[A+a'])^z{}_y \alpha^y(x).
\end{aligned}
$$

Here the fluctuation field is transforming almost like the full gauge field, just with an additional background being present.

Background field gauge transform

It is also sometimes useful to consider a "background field gauge transform", where the background field is transformed like a gauge field,

$$
\begin{aligned}
A^z_\mu(x) &\to A^z_\mu(x) + \frac{1}{g}(D_\mu[A])^z{}_y \alpha^y(x), \\
a'^z_\mu(x) &\to a'^z_\mu(x) + i\alpha^y(x)(T^{(A)}_y)z\ {}_w a'^w_\mu(x), \\
&= a'^z_\mu(x) - \alpha^y f_{yw}{}^z a'^w_\mu(x).
\end{aligned}
$$

Here the fluctuation field $a'^z_\mu(x)$ transforms as a matter field in the adjoint representation of the gauge group! The background field transforms as a proper gauge field, instead.

Rewriting unity

In the following it is useful to have a factor unity written in a particularly convenient way. We start from the familiar identity

$$
\int dz\, \delta(f(z)) \frac{df(z)}{dz} = 1,
$$

where we assume that $f(z)$ has a single zero crossing in the relevant range of z and that $df(z)/dz$ is positive there. One can generalize this to several variables,

$$
\left[\prod_{j=1}^{N} \int dz_j\right] \delta^{(N)}(\mathbf{f}(\mathbf{z})) \det\left(\frac{\partial f_k(\mathbf{z})}{\partial z_j}\right) = 1.
$$

The determinant of the Jacobi matrix arises from a change of variables in the vicinity of the zero crossing of the vector valued function, $\mathbf{f}(\mathbf{z}) = 0$.

In the limit $N \to \infty$ this idendity generalizes to a functional idendity,

$$\int D\alpha \; \delta[G[\alpha]] \; \text{Det}\left[\tfrac{\delta}{\delta\alpha} G[\alpha]\right] = 1.$$

For our purpose we use the functional integral over the gauge orbit (the gauge group manifold) at every space-time point,

$$\int D\alpha = \prod_x \prod_z \int d\alpha^z(x).$$

In the theory of Lie groups, a convenient integral measure is known as the Haar measure.

The functional $G[\alpha]$ encodes a gauge condition, in the sense that it should vanish when the condition is fulfilled. We write it as

$$G^z(x)[A, a'[\alpha]] = 0,$$

where $\alpha^z(x)$ is the parameter field of a fluctuation gauge transform and $a'[\alpha]$ is the gauge-transformed fluctuation field. It is enough to know $a'[\alpha]$ for infinitesimal $\alpha^z(x)$ where it is for the fluctuation field gauge transform

$$a'^z_\mu(x)[\alpha] = a'^z_\mu(x) + \frac{1}{g}(D_\mu[A + a'])^z{}_y \alpha^y(x).$$

The Jacobi matrix is now a functional derivative,

$$N^z{}_w(x, y)[A, a'[\alpha]] = \frac{\delta}{\delta\alpha^w(y)} G^z(x)[A, a'[\alpha]].$$

Functional interal and inserting unity

Consider now the partition function for Yang-Mills theory,

$$\begin{aligned} Z[j] = e^{W[j]} &= \int Da \exp\left(-S[a] + \int_x \{j^\mu_z(x) a^z_\mu(x)\}\right) \\ &= \int Da' \exp\left(-S[A + a'] + \int_x \{j^\mu_z(x)(A^z_\mu(x) + a'^z_\mu(x))\}\right) \end{aligned}$$

In the last equation we use the splitting $a^z_\mu(x) = A^z_\mu(x) + a'^z_\mu(x)$.

We now insert a factor of unity under the functional integral,

$$\begin{aligned} e^{W[j]} = &\int Da' \int D\alpha \, \delta\left[G[A, a'[\alpha]]\right] \text{Det}\left[\tfrac{\delta}{\delta\alpha} G[A, a'[\alpha]]\right] \\ &\times \exp\left(-S[A + a'] + \int_x \{j^\mu_z(x)(A^z_\mu(x) + a'^z_\mu(x))\}\right). \end{aligned}$$

The gauge condition $G[A, a'[\alpha]]$ depends on the background field $A^z_\mu(x)$ and the gauge transformed fluctuation field $a'^z_\mu(x)[\alpha]$, with gauge transformation parameter field $\alpha^z(x)$.

Changing the order of integration and transforming the gauge

In a next step we change the order of the two functional integrals,

$$\int Da' \int D\alpha = \int D\alpha \int Da'.$$

Moreover, we can use then that the action is actually gauge invariant, such that

$$S[A + a'] = S[A + a'[\alpha]].$$

The same should hold for the functional integral measure,

$$\int Da' = \int Da'[\alpha].$$

We assume in addition that we can replace the source term

$$\int_x \{j_z^\mu(x)(A_\mu^z(x) + a_\mu^{\prime z}(x))\} \to \int_x \{j_z^\mu(x)(A_\mu^z(x) + a_\mu^{\prime z}(x)[\alpha])\}$$

After all, it is a bit arbitrary whether to introduce a source before or after doing a gauge transformation. This leads us to

$$\begin{aligned} e^{W[j]} = \int D\alpha \int Da'[\alpha]\, \delta\left[G[A, a'[\alpha]]\right] \text{ Det}\left[\tfrac{\delta}{\delta\alpha} G[A, a'[\alpha]]\right] \\ \times \exp\left(-S[A + a'[\alpha]] + \int_x \{j_z^\mu(x)(A_\mu^z(x) + a_\mu^{\prime z}(x)[\alpha])\}\right). \end{aligned}$$

Relabeling integration variables and dropping gauge orbit integral

Now one observes that $a'[\alpha]$ appears everywhere, and it is actually an intgration variable. So one may simply relabel it back to a'. This has the interesting consequence that all dependence on $\alpha^z(x)$ drops out, and the functional integral $\int D\alpha$ is just a large factor, which can be dropped. One arrives at the expression

$$\begin{aligned} e^{W[j]} = \int Da'\, \delta\left[G[A, a']\right] \text{ Det}\left[\tfrac{\delta}{\delta\alpha} G[A, a']\right] \\ \times \exp\left(-S[A + a'] + \int_x \{j_z^\mu(x)(A_\mu^z(x) + a_\mu^{\prime z}(x))\}\right). \end{aligned}$$

Gauge fixing in generalized Landau gauge

A convenient gauge fixing for $a_\mu^{\prime z}(x)$ for a given macroscopic field $A_\mu^z(x)$ is the generalized Landau gauge condition

$$\begin{aligned} G^z(x)[A, a'] =& (D^\mu[A])^z{}_w a_\mu^{\prime w}(x) \\ =& \partial^\mu a_\mu^{\prime z}(x) + g A^{y\mu} f_{yw}{}^z\, a_\mu^{\prime w}(x) = 0. \end{aligned}$$

Here $(D^\mu[A])^z{}_w$ is the covariant derivative in the adjoint representation.

With this choice we can write the functional Dirac distribution as an exponential,

$$\delta[G[A, a']] = \lim_{\xi\to\infty} \exp\left(-\frac{1}{2\xi}\int_x \left\{\delta_{uv}(D^\mu[A])^u{}_w a_\mu^{\prime w}(x)(D^\nu[A])^v{}_z a_\nu^{\prime z}(x)\right\}\right).$$

This adds a gauge fixing term to the action that depends on the background field as well as on the fluctuation field,

$$S_{\text{gauge fixing}}[A, a'] = \int_x \left\{\frac{\delta_{uv}}{2\xi}(D^\mu[A])^u{}_w a_\mu^{\prime w}(x)(D^\nu[A])^v{}_z a_\nu^{\prime z}(x)\right\}.$$

Faddeev-Popov determinant and ghosts

We also need to find a way to deal with the functional Jacobi determinant $\delta\left[G[A, a']\right]$. For the generalize Landau gauge the Jacobi matrix is

$$\begin{aligned} N^z{}_w(x, y)[A, a'[\alpha]] =& \frac{\delta}{\delta\alpha^w(y)} G^z(x)[A, a'[\alpha]] \\ =& (D^\mu[A])^z{}_u \frac{\delta}{\delta\alpha^w(y)}\left[a_\mu^{\prime z}(x) + \frac{1}{g}\partial_\mu\alpha^z(x) - \alpha^y(x) f_{yw}{}^z\left[A_\mu^w(x) + a_\mu^{\prime w}(x)\right]\right] \\ =& \frac{1}{g}(D^\mu[A])^z{}_u (D_\mu[A + a'])^u{}_w \delta^{(d)}(x - y) \end{aligned}$$

In order to take the determinant of this matrix into account one introduces auxilliary Grassmann values scalar fields $\bar{c}_z(x)$ and $c^w(x)$ and writes

$$\text{Det}\left[\tfrac{\delta}{\delta\alpha}G[A,a']\right] = \int D\bar{c}Dc \exp\left(\int_x \left\{\bar{c}_z(x)(D^\mu[A])^z{}_u(D_\mu[A+a'])^u{}_w c^w(x)\right\}\right)$$

The factor $1/g$ could be absorbed into a rescaling of the fields $\bar{c}_z(x)$ and $c^w(x)$, which are known as Fadeev-Popov ghost fields. In this way, the Jacobi determinant adds a so-called ghost term to the action,

$$S_{\text{ghost}}[A,a',\bar{c},c] = \int_x \left\{-\bar{c}_z(x)(D^\mu[A])^z{}_u(D_\mu[A+a'])^u{}_w c^w(x)\right\}.$$

Ghosts are Grassmann variables that belong to the adjoint representation of the gauge group. They cannot be observed as particles.

Action for QCD with gauge fixing

Let us now combine terms for the action of QCD with gauge fixing à la Faddeev-Popov. Without quarks we get

$$S[A,a',\bar{c},c] = S_{\text{Yang-Mills}}[A+a'] + S_{\text{gauge fixing}}[A,a'] + S_{\text{ghost}}[A,a',\bar{c},c].$$

There we use the Yang-Mills action with background-fluctuation spliting

$$S_{\text{Yang-Mills}}[A+a'] = \int_x \left\{\frac{1}{2g^2}\text{tr}\left\{\mathbf{F}^{\mu\nu}(x)\mathbf{F}_{\mu\nu}(x)\right\}\right\},$$

where

$$\begin{aligned}\mathbf{F}_{\mu\nu}(x) =& \partial_\mu\mathbf{A}_\nu(x) + \partial_\mu\mathbf{a}'_\nu(x) - \partial_\nu\mathbf{A}_\mu(x) - \partial_\nu\mathbf{a}'_\mu(x)\\ &- i[\mathbf{A}_\mu(x)+\mathbf{a}'_\mu(x), \mathbf{A}_\nu(x)+\mathbf{a}'_\nu(x)].\end{aligned}$$

In the presence of quarks we also need to add

$$S_{\text{quarks}}[A,a',\bar{\psi},\psi] = \int_x \left\{-\bar{\psi}(x)\gamma^\mu D_\mu[A+a']\psi(x)\right\}.$$

Invariance of gauge fixed action under background field gauge transforms

Although we have fixed the gauge by the Faddeev-Popov method, there actually still is a gauge symmetry remaining, namely with respect to the background field gauge symmetry.

For a finite transformation of this type, the background gauge field expectation value transforms as

$$\mathbf{A}_\mu(x) \to U(x)\mathbf{A}_\mu(x)U^\dagger(x) - iU(x)\partial_\mu U^\dagger(x),$$

such that $D_\mu[A] \to U(x)D_\mu[A]U^\dagger(x)$. In contrast, the fluctuating part of the gauge field transforms simply as

$$\mathbf{a}'_\mu(x) \to U(x)\mathbf{a}'_\mu(x)U^\dagger(x),$$

and similarly, the ghost fields transform as

$$\begin{aligned}\mathbf{c}(x) &= c^z(x)T_z \to U(x)\mathbf{c}(x)U^\dagger(x),\\ \bar{\mathbf{c}}(x) &= \bar{c}^z(x)T_z \to U(x)\bar{\mathbf{c}}(x)U^\dagger(x).\end{aligned}$$

These are the transformation laws for matter fields in the adjoint representation of the gauge group.

Fermions in the fundamental representation transform simply in the familiar way, $\psi(x) \to U(x)\psi(x)$.

It is straight forward to see that the combined action is invariant undert this transformation. For the Yang-Mills term and the quark term this is clear because they are anyway gauge invariant. For the gauge fixing term it is clear when it is written as

$$S_{\text{gauge fixing}}[A, a'] = \int_x \left\{ \frac{1}{\xi} \text{tr} \left\{ D^\mu[A]\mathbf{a}'_\mu(x) D^\nu[A]\mathbf{a}'_\nu(x) \right\} \right\}.$$

It is crucial here that $D^\mu[A]\mathbf{a}'_\mu(x)$, with $D_\mu[A]$ the covariant derivative in the adjoint representation, transforms again like a matter field in the adjoint representation. Similarly, the ghost term can be written as

$$S_{\text{ghost}}[A, a', \bar{c}, c] = \int_x \text{tr} \left\{ -2\bar{\mathbf{c}}(x) D^\mu[A] D_\mu[A + a']\mathbf{c}(x) \right\},$$

with both covariant derivatives in the adjoint representation.

Effective action with quantum corrections

Let us now attempt to calculate the quantum effective action $\Gamma[A]$ to one-loop order. We concentrate on pure Yang-Mills theory, the extension to theories with fermions is straight forward.

Because of the invariance under gauge transformations on $\mathbf{A}_\mu(x)$, only gauge invariant terms can appear in $\Gamma[A]$. The leading term is expected to be of the form

$$\Gamma_{\text{1-loop}}[A] = \int_x \frac{1}{2g^2} \text{tr} \left\{ \mathbf{F}^{\mu\nu}(x) \mathbf{F}_{\mu\nu}(x) \right\},$$

This is the form of the microscopic action but the coupling g may differ from the microscopic coupling by renormalization group running.

We will now perform a one-loop calculation, based on

$$\Gamma[A] = S[A] + \frac{1}{2} \text{STr} \left\{ \ln \left(S^{(2)}[A] \right) \right\} + \ldots.$$

The operation STr is to a trace over all indices of the fields, including momentum and frequency. The S stands for "super" and should be a reminder to add a minus sign for fermionic degrees of freedom, including ghosts.

In order to calculate the quantum correction to $1/g^2$, we need to determine the propagator for the fluctuating fields, $\mathbf{a}'_\mu(x)$, $\bar{\mathbf{c}}(x)$ and $\mathbf{c}(x)$ in the presence of a background field $\mathbf{A}_\mu(x)$. Eventually, we will expand the left hand side and the right hand side of the above relation to at least quadratic order in $\mathbf{A}_\mu(x)$ in order to identify the coefficient of the term $\text{tr}\left\{ \mathbf{F}_{\mu\nu}(x) \mathbf{F}^{\mu\nu}(x) \right\}$.

Yang-Mills term

Our next goal is to derive the second functional derivative $S^{(2)}[A]$. We start by rewriting the Yang-Mills term which involves the field strength tensor for background plus fluctuation fields,

$$\begin{aligned} \mathbf{F}_{\mu\nu}(x) =& \partial_\mu \mathbf{A}_\nu(x) - \partial_\nu \mathbf{A}_\mu(x) - i[\mathbf{A}_\mu(x), \mathbf{A}_\nu(x)] \\ &+ \partial_\mu \mathbf{a}'_\nu(x) - \partial_\nu \mathbf{a}'_\mu(x) - i[\mathbf{A}_\mu(x), \mathbf{a}'_\nu(x)] - i[\mathbf{a}'_\mu(x), \mathbf{A}_\nu(x)] - i[\mathbf{a}'_\mu(x), \mathbf{a}'_\nu(x)] \\ =& \bar{\mathbf{F}}_{\mu\nu}(x) + D_\mu[A]\mathbf{a}'_\nu(x) - D_\nu[A]\mathbf{a}'_\mu(x) - i[\mathbf{a}'_\mu(x), \mathbf{a}'_\nu(x)], \end{aligned}$$

where $D_\mu[A]$ is the covariant derivative in the adjoint representation,

$$D_\mu[A] = \partial_\mu - i[\mathbf{A}, \ldots],$$

and $\bar{\mathbf{F}}^{\text{fnd}}_{\mu\nu}(x) = g\bar{F}^z_{\mu\nu}(x)T_z$ is the background field strength in the fundamental representation.

Quadratic part in fluctuation gauge fields

To calculate $S^{(2)}[A]$, we specifically need the term quadratic in $\mathbf{a}'_\mu(x)$. Collecting terms from the Yang-Mills action we find

$$
\begin{aligned}
S_2 = & \int_x \frac{1}{2g^2} \text{tr} \left\{ (D^\mu[A]\mathbf{a}'^\nu(x) - D^\nu[A]\mathbf{a}'^\mu(x))(D_\mu[A]\mathbf{a}'_\nu(x) - D_\nu[A]\mathbf{a}'_\mu(x)) \right. \\
& \left. \qquad -2i\bar{\mathbf{F}}^{\mu\nu}_{\text{fnd}}(x)[\mathbf{a}'_\mu(x), \mathbf{a}'_\nu(x)] \right\} \\
= & \int_x \frac{1}{g^2} \text{tr} \left\{ \mathbf{a}'_\mu(x) \left[-\eta^{\mu\nu} D^\rho[A] D_\rho[A] + D^\nu[A] D^\mu[A] + i\bar{\mathbf{F}}^{\mu\nu}_{\text{adj}}(x) \right] \mathbf{a}'_\nu(x) \right\}.
\end{aligned}
$$

In the second equation we performed partial integrations and used an identify following from the cyclic property of the trace,

$$
\begin{aligned}
\frac{-i}{g^2} \text{tr}\{\bar{\mathbf{F}}^{\mu\nu}_{\text{fnd}}(x)[\mathbf{a}'_\mu(x), \mathbf{a}'_\nu(x)]\} = & \frac{i}{g^2} \text{tr} \left\{ \mathbf{a}'_\mu(x) \left[\bar{\mathbf{F}}^{\mu\nu}_{\text{fnd}}(x), \mathbf{a}'_\nu(x) \right] \right\} \\
= & \frac{i}{g^2} \text{tr} \left\{ \mathbf{a}'_\mu(x) \bar{\mathbf{F}}^{\mu\nu}_{\text{adj}}(x) \mathbf{a}'_\nu(x) \right\}.
\end{aligned}
$$

One may also use

$$
D^\nu[A] D^\mu[A] = D^\mu[A] D^\nu[A] + [D^\nu[A], D^\mu[A]] = D^\mu[A] D^\nu[A] + i\bar{\mathbf{F}}^{\mu\nu}
$$

where all covariant derivatives and the field strength tensor are in the adjoint representation of the Lie algebra.

Combining terms, and adding now also the gauge fixing term leads to

$$
\begin{aligned}
S_2 = & \int_x \frac{1}{g^2} \text{tr} \left\{ \mathbf{a}'_\mu(x) \left[-\eta^{\mu\nu} D^\rho[A] D_\rho[A] + (1 - 1/\xi) D^\mu[A] D^\nu[A] + 2i\bar{\mathbf{F}}^{\mu\nu} \right] \mathbf{a}'_\nu(x) \right\} \\
= & \int_x \frac{1}{2} \left\{ a'_{z\mu}(x)[-\eta^{\mu\nu} D^\rho[A] D_\rho[A] + (1 - 1/\xi) D^\mu[A] D^\nu[A] + 2i\bar{\mathbf{F}}^{\mu\nu}(x)]^z{}_w a'^w_\nu(x) \right\}
\end{aligned}
$$

In the second equation we used $\text{tr}(T_z T_w) = \delta_{zw}/2$ as well as $\mathbf{a}'_\mu(x) = g a'^w_\mu(x) T_w$ This is finally the form we use in the following.

Expanding the inverse gluon propagator

It is useful to decompose the inverse gluon propagator in the presence of background field into different terms

$$
\begin{aligned}
& [-\eta^{\mu\nu} D^\rho[A] D_\rho[A] + (1 - 1/\xi) D^\mu[A] D^\nu[A] + 2i\bar{\mathbf{F}}^{\mu\nu}(x)]^z{}_w \\
& = [P^{\mu\nu} + V_1^{\mu\nu} + V_2^{\mu\nu} + V_J^{\mu\nu}]^z{}_w.
\end{aligned}
$$

Here the leading term is the free inverse propagator

$$
P^{\mu\nu} = -\eta^{\mu\nu} \partial^\rho \partial_\rho + (1 - 1/\xi) \partial^\mu \partial^\nu,
$$

which is easily inverted in momentum space, leading to the free gluon propagator

$$
\frac{\eta_{\mu\nu} - (1 - \xi) p_\mu p_\nu / p^2}{p^2}.
$$

We also have a term linear in the background gauge field

$$
\begin{aligned}
V_1^{\mu\nu} = & i[\partial^\rho \mathbf{A}_\rho(x)]\eta^{\mu\nu} + 2i\mathbf{A}^\rho(x)\partial_\rho \eta^{\mu\nu} \\
& - i(1 - 1/\xi)[\partial^\mu \mathbf{A}^\nu(x)] - i(1 - 1/\xi)[\mathbf{A}^\mu(x)\partial^\nu + \mathbf{A}^\nu(x)\partial^\mu]
\end{aligned}
$$

The gauge fields $\mathbf{A}_\rho(x)$ are here in the adjoint representation. This corresponds to a vertex couling the background field to a gluon line. A similar term is quadratic in the background field,

$$V_2^{\mu\nu} = \mathbf{A}^\rho(x)\mathbf{A}_\rho(x)\eta^{\mu\nu} - (1-1/\xi)\mathbf{A}^\mu(x)\mathbf{A}^\nu(x).$$

Finally, there is a term linear in the background field strength, which is best written as

$$V_J^{\mu\nu} = 2\mathbf{F}^{\rho\sigma}(x)(J_{\rho\sigma})^{\mu\nu},$$

with

$$(J_{\rho\sigma})^{\mu\nu} = i(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\nu_\rho\delta^\mu_\sigma),$$

being the generator of Lorentz transformations in the vector representation. This corresponds to a vertex directly coupling to the background field strength.

Quadratic part in ghost fields and inverse propagator

We also need the quadratic part of the ghost action,

$$S_2 = \int_x \bar{c}_z(x)[-D^\rho[A]D_\rho[A]]^z{}_w c^w(x).$$

The inverse propagator can be expanded similar as for the gluons,

$$[-D^\rho[A]D_\rho[A]]^z{}_w = [P + V_1 + V_2]^z{}_w,$$

with the inverse free ghost propagator

$$P = -\partial^\rho\partial_\rho,$$

which can be inverted in momentum space to $1/p^2$. The vertices are

$$V_1 = i[\partial^\rho \mathbf{A}_\rho(x)] + 2i\mathbf{A}^\rho(x)\partial_\rho,$$

and

$$V_2 = \mathbf{A}^\rho(x)\mathbf{A}_\rho(x),$$

with all background gauge fields in the adjoint representation.

Field expansion of one-loop action

We now discuss a general method to expand a one-loop expressions in terms of fields. Generically, the inverse propagator is of the form

$$S^{(2)} = \mathscr{P} + \mathscr{F}_1 + \mathscr{F}_2 + \ldots.$$

Here $\mathscr{P}$ is independent of the fields, $\mathscr{F}_1$ is linear in field, $\mathscr{F}_2$ is quadratic and so on. Usually one can invert $\mathscr{P}$ by going to momentum space, and we denote here

$$\mathscr{G} = \mathscr{P}^{-1}.$$

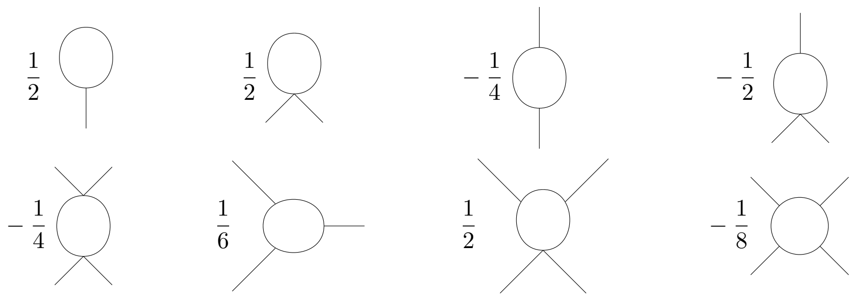
Consider now a one-loop expression

$$\begin{aligned}\frac{1}{2}\mathrm{STr}\{\ln S^{(2)}\} =& \frac{1}{2}\mathrm{STr}\left\{\ln\left(\mathscr{P} + \mathscr{F}_1 + \mathscr{F}_2 + \ldots\right)\right\} \\ =& \frac{1}{2}\mathrm{STr}\left\{\ln(\mathscr{P})\right\} + \frac{1}{2}\mathrm{STr}\left\{\ln\left(1 + \mathscr{G}\mathscr{F}_1 + \mathscr{G}\mathscr{F}_2 + \ldots\right)\right\}.\end{aligned}$$

Here one can expand the logarithm, which yields

$$\begin{aligned}\frac{1}{2}\text{STr}\{\ln S^{(2)}\} = &\frac{1}{2}\text{STr}\{\ln(\mathscr{P})\} + \frac{1}{2}\text{STr}\left\{\mathscr{G}\mathscr{F}_1 + \mathscr{G}\mathscr{F}_2 + \ldots\right\}\\ &- \frac{1}{4}\text{STr}\left\{\mathscr{G}\mathscr{F}_1\mathscr{G}\mathscr{F}_1 + 2\mathscr{G}\mathscr{F}_1\mathscr{G}\mathscr{F}_2 + \mathscr{G}\mathscr{F}_2\mathscr{G}\mathscr{F}_2 + \ldots\right\}\\ &+ \frac{1}{8}\text{STr}\{\mathscr{G}\mathscr{F}_1\mathscr{G}\mathscr{F}_1\mathscr{G}\mathscr{F}_1 + 3\mathscr{G}\mathscr{F}_1\mathscr{G}\mathscr{F}_1\mathscr{G}\mathscr{F}_2 + \ldots\}\\ &- \frac{1}{8}\text{STr}\{\mathscr{G}\mathscr{F}_1\mathscr{G}\mathscr{F}_1\mathscr{G}\mathscr{F}_1\mathscr{G}\mathscr{F}_1 + \ldots\}\end{aligned}$$

The terms on the left hand side have a diagrammatic interpretation. The first term is independent of the background field and contributes only to the constant part of the effective action. The subsequent terms are loop expressions with external background field insertions.



Specifically, we kept here all terms with up to four external fields while higher orders have been suppressed.

Projection to field strength squared term

The effective action $\Gamma[A]$ contains not only the term proportional to $\text{tr}\{\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}\}$, but also many other structures allowed by the symmetries. In order to project to the coefficient of this term, we could determine the coefficient of the term quadratic, of the term cubic, or of the term quartic in the background gauge field in $\text{tr}\{\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}\}$. We follow here the last strategy, but gauge invariance implies that all these coefficients must actually be equal.

Note that

$$\int_x \frac{1}{2g^2}\text{tr}\left\{\mathbf{F}^{\mu\nu}(x)\mathbf{F}_{\mu\nu}(x)\right\} = \ldots + \int_x \frac{1}{2g^2}\text{tr}\{-[\mathbf{A}^\mu(x), \mathbf{A}^\nu(x)][\mathbf{A}_\mu(x), \mathbf{A}_\nu(x)]\}.$$

Because the right hand side contains no derivatives, we can evaluate it in momentum space at vanishing momentum, or, in other words, for homogeneous backgound fields $\mathbf{A}_\mu(x) = \mathbf{A}_\mu$.

Simplified vertices for homogeneous fields

For homogeneous background fields, and working from now on in Feynman gauge where $\xi = 1$, the gluon vertex terms simplify substantially to

$$\begin{aligned}V_1^{\mu\nu} &= -2\mathbf{A}_\rho p^\rho \eta^{\mu\nu},\\ V_2^{\mu\nu} &= \mathbf{A}^\rho \mathbf{A}_\rho \eta^{\mu\nu},\\ V_J^{\mu\nu} &= 2\mathbf{F}^{\rho\sigma}(J_{\rho\sigma})^{\mu\nu}.\end{aligned}$$

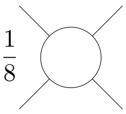
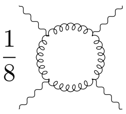
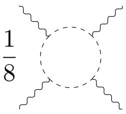
Here p^μ is the gluon momentum running through the loop. Similarly, the ghost vertices become

$$\begin{aligned}V_1 &= -2\mathbf{A}_\rho p^\rho,\\ V_2 &= \mathbf{A}^\rho \mathbf{A}_\rho.\end{aligned}$$

With this we may proceed with the loop calculation.

Loops with four single legs

We start by considering loop diagrams with four single legs. There is a gluon loop and a ghost loop, where the latter contributes with a negative sign because it is fermionic.

$-\frac{1}{8}$  $=$ $-\frac{1}{8}$  $+$ $\frac{1}{8}$ 

We find in Feynman gauge

$$\begin{aligned}&-\frac{1}{8}\text{tr}\left\{\mathbf{A}_{\mu_1}\cdots\mathbf{A}_{\mu_4}\right\}\int_p 2p^{\mu_1}2p^{\mu_2}2p^{\mu_3}2p^{\mu_4}\frac{\delta^\rho{}_\rho}{(p^2)^4}\\&+\frac{2}{8}\text{tr}\left\{\mathbf{A}_{\mu_1}\cdots\mathbf{A}_{\mu_4}\right\}\int_p 2p^{\mu_1}2p^{\mu_2}2p^{\mu_3}2p^{\mu_4}\frac{1}{(p^2)^4}\end{aligned}$$

The second loop is from ghosts and has an additional minus sign for Grassmann fields and a factor 2 for involving complex instead of real fields. The trace over Lorentz indices in the first diagram gives a factor $d = 4$.

Combining the diagrams gives

$$-\frac{(4-2)}{8}\text{tr}\left\{\mathbf{A}_{\mu_1}\cdots\mathbf{A}_{\mu_4}\right\}\int_p\left\{\frac{2p^{\mu_1}}{p^2}\cdots\frac{2p^{\mu_4}}{p^2}\right\}$$

The momentum integral is of a structure that can be simplified based on rotation invariance,

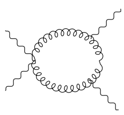
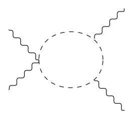
$$\begin{aligned}&\int\frac{d^dp}{(2\pi)^d}\;p^{\mu_1}p^{\mu_2}p^{\mu_3}p^{\mu_4}f(p^2)\\&=\frac{1}{d(d+2)}\left[\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4}+\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}+\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}\right]\int\frac{d^dp}{(2\pi)^d}\;p^4f(p^2).\end{aligned}$$

This leads to a relatively simple result for these two diagrams in $d = 4$ dimensions,

$$-\frac{1}{6}\text{tr}\{\mathbf{A}^\mu\mathbf{A}^\nu\mathbf{A}_\mu\mathbf{A}_\nu+2\mathbf{A}^\mu\mathbf{A}_\mu\mathbf{A}^\nu\mathbf{A}_\nu\}\int_p\frac{1}{p^4}.$$

Loops with two single legs and one double leg

Let us now adress the diagrams with two single and one double field vertex. We count also V_J as a double field vertex.

$\frac{1}{2}$  $-$ $\frac{1}{2}$ 

We find here after performing the trace over Lorentz indices in the gluon diagram and subtracting the ghost diagram

$$=\frac{1}{2}\text{tr}\left\{\mathbf{A}^\rho\mathbf{A}_\rho\mathbf{A}_\mu\mathbf{A}_\nu\right\}(4-2)\int_p\left\{2p^\mu2p^\nu\frac{1}{(p^2)^3}\right\}.$$

There is no contribution from the vertex $V_J^{\mu\nu}$ because it is anti-symmetric in the Lorentz indices. Using

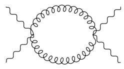
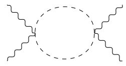
$$\int\frac{d^4p}{(2\pi)^4}p^\mu p^\nu f(p^2)=\frac{1}{d}\eta^{\mu\nu}\int\frac{d^4p}{(2\pi)^4}p^2f(p^2)$$

leads for $d = 4$ to

$$\text{tr}\{\mathbf{A}^\mu\mathbf{A}_\mu\mathbf{A}^\nu\mathbf{A}_\nu\}\int_p\frac{1}{p^4}.$$

Loops with two double legs

Finally we also have diagrams with two double legs.

$-\frac{1}{4}$  $+$ $\frac{1}{4}$ 

Here we obtain

$$-\frac{1}{4}\mathrm{tr}\left\{\mathbf{A}^\mu\mathbf{A}_\mu\mathbf{A}^\nu\mathbf{A}_\nu\right\}(4-2)\int_p\frac{1}{p^4}$$
$$-\frac{1}{4}\mathrm{tr}\left\{\mathbf{F}^{\mu\nu}\mathbf{F}^{\rho\sigma}\right\}(J_{\mu\nu})^{\alpha\beta}(J_{\rho\sigma})_{\beta\alpha}\int_p\frac{1}{p^4}.$$

where the last line is a contribution from the $V_J^{\mu\nu}$ vertices. We can calculate explicitly

$$(J_{\mu\nu})^{\alpha\beta}(J_{\rho\sigma})_{\beta\alpha}=2\left[\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\mu\sigma}\eta_{\nu\rho}\right].$$

This allows to combine the two terms to

$$-\frac{1}{2}\mathrm{tr}\{\mathbf{A}^\mu\mathbf{A}_\mu\mathbf{A}^\nu\mathbf{A}_\nu\}\int_p\frac{1}{p^4}-\mathrm{tr}\{\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}\}\int_p\frac{1}{p^4}.$$

Combination of all diagrams

We can now combine all gluon and ghost diagrams

$$\left[\frac{1}{6}\mathrm{tr}\left\{\mathbf{A}^\mu\mathbf{A}_\mu\mathbf{A}^\nu\mathbf{A}_\nu\right\}-\frac{1}{6}\mathrm{tr}\left\{\mathbf{A}^\mu\mathbf{A}^\nu\mathbf{A}_\mu\mathbf{A}_\nu\right\}-\mathrm{tr}\left\{\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}\right\}\right]\int_p\frac{1}{p^4}.$$

Here it is useful to recall that for homogeneous fields one has

$$\mathrm{tr}\{\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}\}=\mathrm{tr}\{-2\mathbf{A}^\mu\mathbf{A}^\nu\mathbf{A}_\mu\mathbf{A}_\nu+2\mathbf{A}^\mu\mathbf{A}_\mu\mathbf{A}^\nu\mathbf{A}_\nu\}.$$

This allows to combine everything into

$$-\frac{11}{12}\mathrm{tr}\{\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}\}\int_p\frac{1}{p^4}.$$

This is almost what we were looking for. We need to remember here, however, that the field strengths are in the adjoint representation. We can remedy this, using an identify for adjoint and fundamental SU(N) Lie algebra generators,

$$\mathrm{tr}\{T_u^{(A)}T_v^{(A)}\}=N\delta_{uv}=2N\mathrm{tr}\{T_uT_v\}.$$

One-loop quantum effective action

We thus find for the quantum effective action including the one-loop term

$$\Gamma[A]=\int_x\left\{\frac{1}{2\bar{g}^2}-\frac{11N}{6}\int_p\frac{1}{p^4}\right\}\mathrm{tr}\{\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}\}.$$

Here the field strength is in the fundamental representation as usual, and $\bar{g}$ is the microscopic or bare coupling constant.

Finally, we note that the momentum integral is logarithmically divergent both in the UV and in the IR in $d=4$ dimensions, but can be regularized easily

$$\int_p\frac{1}{p^4}=\frac{1}{(2\pi)^4}2\pi^2\int_0^\infty\frac{dp}{p}\to\frac{1}{(4\pi)^2}2\int_\mu^\Lambda\frac{dp}{p}=\frac{2}{(4\pi)^2}\ln\left(\frac{\Lambda}{\mu}\right)$$

We have introduced a UV cutoff Λ and and IR cutoff μ.

Running coupling constant

We find the effective coupling constant g with quantum corrections (one loop) to obey

$$\frac{1}{g^2} = \frac{1}{\bar{g}^2} - \frac{11N}{3}\frac{1}{(4\pi)^2}2\ln\left(\frac{\Lambda}{\mu}\right).$$

In particular, the effective coupling constant depdends on the infrared regulator scale μ!

If we would have done the calculation at non-vanishing external momenta, a corresponding scale would have appeared instead of μ naturally. Renormalized coupling constants are scale-dependent as a consequence of quantum fluctuatuions! For Yang-Mills theory, such quantum fluctuations are particularly important because the gluons are massless.

The logarithmic derivative of the coupling constant with respect to the infrared scale is called the beta function,

$$\mu\frac{\partial}{\partial\mu}g = \beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}N\right].$$

This is in fact the renormalization group equation for the coupling constant of $SU(N)$ gauge theory at one loop, and so far without fermions in the adjoint representation.

More generally, for SU(N) gauge theory, with n_f massless Dirac fermions in the fundamental representation, the beta function is

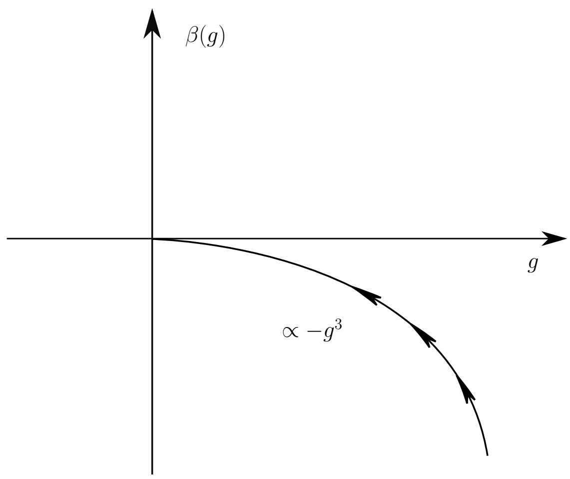
$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left[\frac{11}{3}N - \frac{2}{3}n_f\right].$$

Specifically for QCD, one has $N = 3$, and at high energies where all quarks can be counted massless, $n_f = 6$. The beta function is then

$$\beta(g) = -\frac{g^3}{(4\pi)^2}\left[11 - 4\right].$$

Asymptotic freedom

A very interesting property of non-Abelian gauge theories is that for small enough n_f, the beta function is negative, $\beta(g) < 0$. This implies that the coupling becomes weaker at higher momentum scale!



The coupling constant flows into a fixed point at $g = 0$. This property is called asymptotic freedom. At asymptotically large momenta, the theory becomes free.

While QCD as a quantum field theory becomes non-interacting or free at very high energies, the renormalization group beta function also tells that it becomes very strongly interacting at small energies, or in the infrared regime.

Solution to one-loop flow equation*

In terms of the combination

$$\alpha_s = \frac{g^2}{4\pi}$$

we obtain

$$\mu^2 \frac{\partial}{\partial \mu^2} \alpha_s(\mu) = \frac{\alpha_s(\mu)^2}{4\pi} \left[\frac{11}{3} N - \frac{2}{3} n_f \right],$$

or

$$\mu^2 \frac{\partial}{\partial \mu^2} \left(\frac{4\pi}{\alpha_2} \right) = - \left[\frac{11}{3} N - \frac{2}{3} n_f \right].$$

This can be easily solved,

$$\frac{4\pi}{\alpha_s(\mu)} - \frac{4\pi}{\alpha_s(\Lambda)} = \left[\frac{11}{3} N - \frac{2}{3} n_f \right] \ln \left(\mu^2 / \Lambda^2 \right)$$

Imagine we start with some coupling $\alpha_s(\mu)$ at a scale μ and take Λ to be smaller and increase the ratio μ^2/Λ^2. The coupling $\alpha_s(\Lambda)$ becomes larger and larger until a divergence $\alpha_s(\Lambda) \to \infty$ some scale, which is by convention called Λ_{QCD}. We find the solution to the one-loop flow equation

$$\alpha_s(\mu) = \frac{4\pi}{\left[\frac{11}{3} N - \frac{2}{3} n_f \right] \ln \left(\mu^2 / \Lambda_{\text{QCD}}^2 \right)}.$$

One should keep in mind that perturbation theory breaks down when α_s becomes too large. It is only applicable at large energies $\mu \gg \Lambda_{\text{QCD}}$.

5 Wegner-Wilson loops and confinement

Wilson links

Let us consider pure Yang-Mills theory (without quarks) with the Euclidean action

$$S_{\text{Yang-Mills}}[A] = \int_x \left\{ \frac{1}{2g^2} \text{tr} \left\{ \mathbf{F}^{\mu\nu}(x) \mathbf{F}_{\mu\nu}(x) \right\} \right\}.$$

Let us take two spacetime points x^μ and $x^\mu + \varepsilon^\mu$ where ε^μ is infinitesimal. We define the Wilson link as

$$\begin{aligned} \mathbf{W}(x+\varepsilon, x) &= \exp \left(i \varepsilon^\mu \mathbf{A}_\mu(x) \right) \\ &= \mathbb{1} + i \varepsilon^\mu \mathbf{A}_\mu(x) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Because $\mathbf{A}_\mu(x)$ is an $N \times N$-matrix, this is also the case for $\mathbf{W}(x+\varepsilon, x)$. We now determine how the Wilson link transforms under gauge transformations. Recall the finite gauge transformations for the fermion and gauge fields,

$$\begin{aligned} \psi(x) &\to U(x) \psi(x), \\ \mathbf{A}_\mu(x) &\to U(x) \mathbf{A}_\mu(x) U^\dagger(x) - i [\partial_\mu U(x)] U^\dagger(x). \end{aligned}$$

The Wilson link (to order ε) transforms as

$$\mathbf{W}(x+\varepsilon, x) \to \mathbb{1} + i \varepsilon^\mu U(x) \mathbf{A}_\mu(x) U^\dagger(x) + \varepsilon^\mu [\partial_\mu U(x)] U^\dagger(x).$$

Here one can use that to order ε,

$$U(x+\varepsilon) = U(x) + \varepsilon^\mu \partial_\mu U(x).$$

Accordingly we obtain a simple transformation law,

$$\mathbf{W}(x+\varepsilon,x) \to U(x+\varepsilon)\mathbf{W}(x+\varepsilon,x)U^\dagger(x).$$

This indicates the geometric significance of the Wilson link as connecting the gauge groups at different points in space-time.

Wilson line

We can now consider a Wilson line as a chain of infinitesimal Wilson links. It goes along some path ξ, connecting two spacetime points x and $y = x + \varepsilon_1 + \cdots + \varepsilon_n$,

$$\mathbf{W}_\xi(y,x) = \mathbf{W}(y, y-\varepsilon_n)\cdots\mathbf{W}(x+\varepsilon_1+\varepsilon_2, x+\varepsilon_1)\mathbf{W}(x+\varepsilon_1, x).$$

The transformation behaviour under gauge transformations is rather simple,

$$\mathbf{W}_\xi(y,x) \to U(y)\mathbf{W}_\xi(y,x)U^\dagger(x).$$

For a Wilson link, one can write

$$\mathbf{W}^\dagger(x+\varepsilon,x) = \mathbf{W}(x,x+\varepsilon).$$

For a finite Wilson line, this extends to

$$\mathbf{W}^\dagger_\xi(y,x) = \mathbf{W}_{\bar\xi}(x,y),$$

where $\bar\xi$ denotes the reverse of the path ξ.

Wegner-Wilson loop

Consider now a closed path or oriented curve $\xi = \mathcal{C}$. The Wegner-Wilson loop is the trace of the Wilson line along the closed curve,

$$W_\mathcal{C} = \mathrm{tr}\{\mathbf{W}_\mathcal{C}(x,x)\}.$$

The trace goes over the $\mathrm{SU}(N)$ matrix indices and the Wilson loop is accordingly not a matrix but a scalar.

Note that the Wegner-Wilson loop depends on the entire path of the loop and is in this sense not a local field of a standard type.

From the transformation law of the Wilson line, it follows that the Wegner-Wilson loop is gauge invariant,

$$W_\mathcal{C} \to W_\mathcal{C}.$$

Furthermore, the complex conjugate is

$$W^*_\mathcal{C} = W_{\bar{\mathcal{C}}},$$

with reversed path $\bar{\mathcal{C}}$.

Interaction potential from Wegner-Wilson loop

One may obtain the interaction potential between two very heavy or static particles from the Wegner-Wilson loop. To this end, consider a rectangular closed path with lengths T in euclidean time and R in space direction, such that $T \gg R$. When $\langle W_\mathcal{C}\rangle$ is computed, we are actually solving the functional integral in the presence of static, opposite charges with separation R. The Euclidean path integral will then be proportional to $\exp(-E(R)T)$, where $E(R)$ is the interaction energy of the two heavy particles.

QED or weak coupling expansion

Let us consider the expectation value of the Wegner-Wilson loop in the Abelian theory (QED). A weak-coupling expansion of Yang-Mills theory leads to a very similar theory.

One can write

$$\langle W_{\mathcal{C}} \rangle = \int DA \exp\left(ig \int_{\mathcal{C}} dx^\mu A_\mu(x) \right) \exp\left(-S[A]\right).$$

The line integral in the exponential has the form of a current term in the partition function,

$$\exp\left(\int_x \{ j^\mu(x) A_\mu(x) \} \right),$$

if we write the current in the form of a line integral,

$$j^\mu(x) = ig \int_{\mathcal{C}} d\tilde{x}^\mu \delta^{(4)}(x - \tilde{x}).$$

For the free Abelian gauge field, the partition function is quadratic and one finds

$$\left\langle \exp\left(\int_x j^\mu(x) A_\mu(x) \right) \right\rangle = \exp\left(\frac{1}{2} \int_{x,y} \{ j^\mu(x) \Delta_{\mu\nu}(x-y) j^\nu(y) \} \right),$$

with Euclidean photon propagator $\Delta_{\mu\nu}(x-y)$. The Wegner-Wilson loop eveluates to

$$\langle W_{\mathcal{C}} \rangle = \exp\left(-\frac{g^2}{2} \int_{\mathcal{C}} dx^\mu \int_{\mathcal{C}} dy^\nu \Delta_{\mu\nu}(x-y) \right).$$

Coulomb potential

Consider now the rectangular configuration with $T \gg R$. For the photon propagator we can take the Feynman gauge expression (in $d = 4$ Euclidean dimensions)

$$\Delta_{\mu\nu}(x-y) = \frac{\delta_{\mu\nu}}{4\pi^2 (x-y)^2}.$$

Doing the integrals (exercise) one finds with $\alpha = g^2/(4\pi)$,

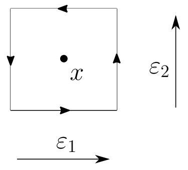
$$\langle W_{\mathcal{C}} \rangle = \text{const} \times \exp\left(-\frac{\alpha}{R} T \right).$$

The constant part is in fact divergent and corresponds to the infinite self energy of a classical charged particle. From the R-dependece, one can read off the static potential $V(R) = -\alpha/R$. This is Coulombs potential.

In a weak coupling expansion of a non-abelian gauge theory like QCD, one also finds the Coulomb potential. However this cannot be the full story. We now describe a strong coupling expansion for the Wegner-Wilson loop.

Wegner-Wilson plaquette

Imagine that we formulate the gauge theory on a discrete spacetime lattice with lattice spacing a. We now consider a Wilson loop of the form



going around a point x. The gauge fields and link variables are in the lattice formulation living between the lattice sites. With two space-time vectors ε_1 and ε_2 of length a, we use the link variables at $x \pm \varepsilon_1/2$ and $x \pm \varepsilon_2/2$ to formulate a discrete version of the Wegner-Wilson loop. We define this loop to be the plaquette. Multiplying the Wilson links around the loop and taking the trace gives the Wegner-Wilson loop

$$W_{\text{plaq}} = \text{Tr}\left\{e^{-ia\mathbf{A}_2(x-\frac{\varepsilon_1}{2})}e^{-ia\mathbf{A}_1(x+\frac{\varepsilon_2}{2})}e^{ia\mathbf{A}_2(x+\frac{\varepsilon_1}{2})}e^{ia\mathbf{A}_1(x-\frac{\varepsilon_2}{2})}\right\}.$$

Assume now that the gauge fields are smooth and expand to quadratic order in the lattice spacing a,

$$\begin{aligned} W_{\text{plaq}} = \text{Tr}\Big\{ & e^{-ia\mathbf{A}_2(x)+ia^2\partial_1\mathbf{A}_2(x)/2}e^{-ia\mathbf{A}_1(x)-ia^2\partial_2\mathbf{A}_1(x)/2} \\ & \times e^{ia\mathbf{A}_2(x)+ia^2\partial_1\mathbf{A}_2(x)/2}\ e^{ia\mathbf{A}_1(x)-ia^2\partial_2\mathbf{A}_1(x)/2}\Big\}. \end{aligned}$$

With help of the Bakeer-Campbell-Hausdorff relation

$$e^A e^B = e^{A+B+[A,B]/2+\dots},$$

one can combine the exponentials in the first line and in the second line and then both lines together. The result is

$$W_{\text{plaq}} = \text{Tr}\left\{e^{ia^2(\partial_1\mathbf{A}_2-\partial_2\mathbf{A}_1-i[\mathbf{A}_1,\mathbf{A}_2])}\right\} = \text{Tr}\left\{e^{ia^2\mathbf{F}_{12}}\right\}.$$

We see the non-Abelian field strength naturally appearing! The Wilson loop of the same plaquette in the opposite sense gives

$$W_{\overline{\text{plaq}}} = \text{Tr}\left\{e^{-ia^2(\partial_1\mathbf{A}_2-\partial_2\mathbf{A}_1-i[\mathbf{A}_1,\mathbf{A}_2])}\right\} = \text{Tr}\left\{e^{-ia^2\mathbf{F}_{12}}\right\}.$$

Action for lattice gauge theory

Adding them and expanding the exponentials gives

$$W_{\text{plaq}} + W_{\overline{\text{plaq}}} = 2N - a^4\text{Tr}\left\{\mathbf{F}_{12}\mathbf{F}^{12}\right\} + \dots$$

Interestingly, this is precisely of the form we need for the action. We can take the lattice action of Yang-Mills theory to be

$$S[U] = -\frac{1}{2g^2}\sum_{\text{plaq}} W_{\text{plaq}},$$

where the sum goes over the plaquettes around each lattice point including both orientations. Each plaquette is expressed as the product of four link matrices U.

Functional integral

The functional integral can be written as an integral over these link matrices,

$$Z = \int \mathcal{D}U\ e^{-S[U]},$$

where

$$\mathcal{D}U = \prod_{\text{links}} dU_{\text{links}},$$

and dU is the Haar measure associated with the group SU(N). It has the properties (for $N \geq 3$)

$$\int dU \; U_{ij} = 0,$$
$$\int dU \; U_{ij} U_{kl} = 0,$$
$$\int dU \; U_{ij} U^*_{kl} = \frac{1}{N} \delta_{ik} \delta_{jl}.$$

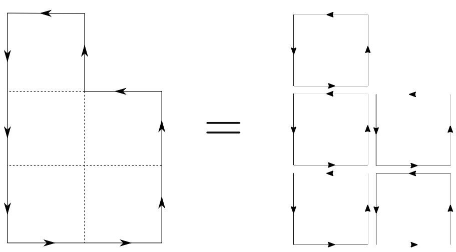
Wegner-Wilson loop at strong coupling

Let us now consider a Wilson loop composed of a sequence of link variables,

$$\langle W_{\mathcal{C}} \rangle = \frac{1}{Z} \int \mathcal{D}U \, W_{\mathcal{C}} \, e^{-S[U]}.$$

We will evaluate this in the strong coupling expansion in powers of $1/g^2$. To lowest order, $e^{-S[U]} \to 1$ and the result vanishes because $\int dU \; U_{ij} = 0$ for each link. Clearly, each link U in $W_{\mathcal{C}}$ must be balanced by a conjugate link U^* from the expansion of e^{-S}.

To get a non-zero result, we must fill the interior of the Wilson loop by opposite plaquettes from the action using a division like the following.



Each plaquette comes with a factor $1/g^2$ and the number of plaquettes needed is the area A/a^2. Accordingly,

$$\langle W_{\mathcal{C}} \rangle = \text{const} \times \left[\frac{1}{g^2} \right]^{A/a^2} = \text{const} \times e^{-\sigma A},$$

where

$$\sigma = \frac{\ln(g^2)}{a^2},$$

is the string tension.

Area law and confinement

This area law is a signal for confinement. The consideration of a static quark-antiquark pair now leads to

$$\langle W_{\mathcal{C}} \rangle = \text{const} \times e^{-\sigma R T},$$

and thus the potential between static charges in linear,

$$V(R) = \sigma R.$$

The energy becomes infinitely large when one tries to separate the quark and anti-quark. In reality, the string breaks when the energy is large enough to produce another quark-antiquark pair.

Implementing lattice QCD numerically, one can go beyond the strong and weak coupling expansions.

6 The standard model

Gauge group of the standard model

The gauge group of the standard model of elementary particle physics is

$$\mathrm{SU}(3) \otimes \mathrm{SU}(2) \otimes \mathrm{U}(1).$$

The fermion fields and the Higgs boson scalar field can be classified into representations of the corresponding Lie algebras. With respect to the strong interaction group $\mathrm{SU}(3)_{\mathrm{colour}}$ we need the representations

$$\begin{array}{ll} \text{singlet} & \mathbf{1}, \\ \text{triplet} & \mathbf{3}, \\ \text{anti-triplet} & \mathbf{3}^*. \end{array}$$

The triplet is the fundamental representation, and the anti-triplet its complex conjugate.

With respect to the weak interaction group SU(2) we need

$$\begin{array}{ll} \text{singlet} & \mathbf{1}, \\ \text{doublet} & \mathbf{2}. \end{array}$$

The group SU(2) is pseudo-real so there is no independent $\mathbf{2}^*$. Finally with respect to the hypercharge group $\mathrm{U}(1)_Y$ we will classify fields by their charge as generalisations of electric charge q. The charges turn out to be

$$0\ , \quad \pm\frac{1}{6}\ , \quad \pm\frac{1}{3}\ , \quad \pm\frac{1}{2}\ , \quad \frac{2}{3}\ , \quad \pm 1\ .$$

Moreover the fermions transform as Weyl spinors under the Lorentz group, either left- or right-handed.

Field content

There are the following fields

$\begin{pmatrix}\nu_L \\ e_L\end{pmatrix}$	neutrino electron	left-handed	$\left(\mathbf{1},\ \mathbf{2},\ -\frac{1}{2}\right)$
$\begin{pmatrix}\bar{\nu}_L & \bar{e}_L\end{pmatrix}$	anti-neutrino anti-electron	right-handed	$\left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right)$
e_R	electron	right-handed	$\left(\mathbf{1},\ \mathbf{1},\ -1\right)$
$\bar{e}_R$	anti-electron	left-handed	$\left(\mathbf{1},\ \mathbf{1},\ 1\right)$
$\begin{pmatrix}u_L \\ d_L\end{pmatrix}$	up-quark down-quark	left-handed	$\left(\mathbf{3},\ \mathbf{2},\ \frac{1}{6}\right)$
$\begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix}$	anti-up-quark anti-down-quark	right-handed	$\left(\mathbf{3}^*,\ \mathbf{2},\ -\frac{1}{6}\right)$
u_R	up-quark	right-handed	$\left(\mathbf{3},\ \mathbf{1},\ \frac{2}{3}\right)$
$\bar{u}_R$	anti-up-quark	left-handed	$\left(\mathbf{3}^*,\ \mathbf{1},\ -\frac{2}{3}\right)$
d_R	down-quark	right-handed	$\left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right)$
$\bar{d}_R$	anti-down-quark	left-handed	$\left(\mathbf{3}^*,\ \mathbf{1},\ \frac{1}{3}\right)$
ϕ	Higgs-doublet	scalar	$\left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right)$

where the last expression determines the representations under the gauge symmetries. The fields have several indices corresponding to the different groups, for example

$$\left(u_R\right)^{\dot{a}m}(x),$$

where $\dot{a} \in \{1,2\}$ is the Lorentz spinor index and $m \in \{1,2,3\}$ is the $\mathrm{SU}(3)_{\text{colour}}$ index.

Three families

The leptons and quarks come in three copies, also known as *families*. For example, in addition to the electrons and anti-electrons there are also muons and tau leptons with their corresponding anti-particles and an associated neutrino. For the quarks we have discussed this already.

Gauge bosons

In addition to these "matter fields", there are corresponding gauge bosons, specifically for $\mathrm{SU}(3)_{\text{color}}$ the eight real gluons, for SU(2) three real gauge bosons and one for the abelian $\mathrm{U}(1)_Y$ subgroup. After spontaneous symmetry breaking, the $\mathrm{SU}(2) \otimes \mathrm{U}(1)_Y$ bosons combine into the two massive complex $W^{\pm}$ bosons, the neutral and massive Z boson and the massless photon. The symmetry breaking itself is due to an expectation value for the scalar Higgs field.

Microscopic action

The microscopic action of the standard model is rather simple. It is a collection of terms we have discussed before. There are kinetic terms for the non-Abelian gauge fields of SU(3), of SU(2) and the Abelian U(1) fields. To do perturbation theory one needs to do gauge fixing, which introduces additional gauge fixing terms and ghost fields.

In addition we have Dirac or actually Weyl kinetic terms for the chiral fermions featureing gauge covariant derivatives for the appropriate representation of SU(3), SU(2), and U(1). This part does not feature any Dirac or Majorana masses, and they are actually not allowed by the symmetries of the model.

For the scalar field in the doublet representation of SU(2) there is an appropriate kinetic terms with covariant derivatives, and there is an effective potential with a non-vanishing minimum value featureing spontaneous symmetry breaking.

Finally, there is a set of Yukawa interaction terms coupling fermion and anti-fermion fields, as well as the Higgs-doublet scalar field $\phi(x)$. When the latter obtains a vacuum expectation value by spontaneous symmetry breaking this also induces Dirac masses for the quarks and leptons, with the exception of the neutrions, which remain massless in the standard model.

7 Grand unification

SU(5) unification

We now discuss a proposed extension of the Standard Model which leads to a unification of the gauge groups into SU(5). This has been proposed by *Howard Georgi* and *Sheldon Glashow* in 1974.

Note that the SU(3) and SU(2) generators naturally fit into SU(5) generators and similar for the spinors

$$\begin{pmatrix} \begin{pmatrix} 3\times 3 \end{pmatrix}_{\text{SU}(3)} & \\ & \begin{pmatrix} 2\times 2 \end{pmatrix}_{\text{SU}(2)} \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \\ \psi^5 \end{pmatrix}.$$

There are $5^2 - 1 = 24$ generators of SU(5) corresponding to the hermitian traceless 5×5 matrices. Out of them, eight generate SU(3), while three generate SU(2).

Moreover, within SU(5) there is one hermitian traceless matrix

$$\frac{1}{2}Y = \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{pmatrix}.$$

That generates a U(1) subgroup which actually gives $\text{U}(1)_Y$. The remaining generators correspond to additional gauge bosons not present in the Standard Model so they are supposedly very heavy or confined. We find the embedding

$$\text{SU}(5) \to \text{SU}(3) \otimes \text{SU}(2) \otimes \text{U}(1).$$

Fundamental representation 5

Now let us consider representations. Take the fundamental representation of SU(5) the spinor ψ^m. From the above illustration one sees that it decomposes like

$$\mathbf{5} = \left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) \oplus \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right),$$

in a natural way. The conjugate decomposes

$$\mathbf{5}^* = \left(\mathbf{3}^*,\ \mathbf{1},\ \frac{1}{3}\right) \oplus \left(\mathbf{1},\ \mathbf{2},\ -\frac{1}{2}\right).$$

Indeed these could be the representations for the right-handed down quark and the anti-lepton doublet,

$$d_R, \qquad \left(\bar{\nu}_L\ \bar{e}_L\right),$$

and their anti-particles

$$\bar{d}_R, \qquad \begin{pmatrix} \nu_L \\ e_L \end{pmatrix},$$

respectively.

Note that the hypercharges indeed conspire to be such that the generator $\frac{1}{2}Y$ is indeed traceless. It can therefore be one of the generators of SU(5).

Antisymmetric tensor representation 10

So what about the other representations? The next smallest representation is the anti-symmetric tensor ψ^{mn} with dimension ten. We still need

$$\left(\mathbf{3},\ \mathbf{2},\ \frac{1}{6}\right), \qquad \left(\mathbf{3}^*,\ \mathbf{1},\ -\frac{2}{3}\right), \qquad \left(\mathbf{1},\ \mathbf{1},\ 1\right),$$

and the corresponding anti-fields. These are ten fields indeed. Now ψ^{mn} decomposes into irreducible representations according to

$$\begin{aligned}
\left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) \otimes_A \left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) &= \left(\mathbf{3}^*,\ \mathbf{1},\ -\frac{2}{3}\right), \\
\left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) \otimes_A \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right) &= \left(\mathbf{3},\ \mathbf{2},\ \frac{1}{6}\right), \\
\left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right) \otimes_A \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right) &= \left(\mathbf{1},\ \mathbf{1},\ 1\right).
\end{aligned}$$

This matches indeed to $\bar{u}_R$, the left-handed quark doublet (u_L, d_L) and $\bar{e}_R$, respectively.

Note that we have used here tensor product decomposition relations discussed before such as for SU(3)

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}^*_A \oplus \mathbf{6}_S,$$

or for SU(2)

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_A \oplus \mathbf{3}_S.$$

The U(1) charges are simply added. Indeed things work out! Also in this sector one finds that the hypercharges add up to zero, $3 \times 1 \times (-\frac{2}{3}) + 3 \times 2 \times \frac{1}{6} + 1 \times 1 \times 1 = 0$.

All fermions

The fermion fields of a single generation in the Standard Model can be organised into the $SU(5)$ representations

$$\mathbf{5}^* : \quad \bar{d}_R,\ \begin{pmatrix} \nu_L \\ e_L \end{pmatrix},$$

and

$$\mathbf{10} : \quad \bar{u}_R,\ \bar{e}_R,\ \begin{pmatrix} u_L \\ d_L \end{pmatrix},$$

as well as the corresponding anti-fields. There is no space here for a right handed neutrino, it would have to be a singlet **1** under SU(5).

The scalar Higgs field could be part of a **5** scalar representation but the corresponding field with quantum numbers

$$\left(\mathbf{3},\ \mathbf{1},\ -\frac{2}{3}\right)$$

is not present in the Standard Model and must be very heavy or otherwise suppressed.

Gauge bosons

The gauge bosons of SU(5) can be found from decomposing $\mathbf{5}\otimes\mathbf{5}^* = \mathbf{24}+\mathbf{1}$. In terms of SU(3) $\otimes$ SU(2) $\otimes$ U(1) the **24** decomposes into

$$\mathbf{24} = \left(\mathbf{1},\ \mathbf{3},\ 0\right)\oplus\left(\mathbf{8},\ \mathbf{1},\ 0\right)\oplus\left(\mathbf{1},\ \mathbf{1},\ 0\right)\oplus\left(\mathbf{3},\ \mathbf{2},\ \frac{2}{3}\right)\oplus\left(\mathbf{3}^*,\ \mathbf{2},\ -\frac{2}{3}\right).$$

We recognize the W boson triplet, the gluons, the hypercharge photon and two more gauge bosons that transform under both $\text{SU}(3)_{\text{color}}$ and the electroweak group SU(2) $\times$ U(1).

Proton decay?

The latter type of gauge bosons could in principle induce transitions of the type

$$d \to e^+,$$
$$u \to \bar{u},$$

and thus $u + d \to \bar{u} + e^+$ causing

$$uud \to u\bar{u} + e^+,$$
$$p \to \pi^0 + e^+.$$

The proton could therefore decay! This is actually one of the main experimental signatures for such grand unified theories.

Proton decay has not been observed so the transition rate must be very small. This also implies that the unification scale where the three forces $\text{SU}(3)_{\text{color}}$, SU(2) and $\text{U}(1)_Y$ unite, must be very high. The latest experimental constraint is that the proton half-life time must be at least 1.6×10^{34} years [Super-Kamiokande, PRD 95, 012004 (2017)]. If the decay rate goes like

$$\Gamma \approx \frac{m_p^5}{M_{\text{GUT}}^4},$$

one can estimate for the unification scale $M_{\text{GUT}} > 10^{16}$ GeV.

The Georgi-Glashow model we discussed so far is not very realistic, in some sense it is already ruled out. For example it predicts massless neutrinos, which is in conflict with the observation of neutrino oscillations. Also the unification of renormalization group trajectories to a single SU(5) coupling constant at the scale M_{GUT} does not seem to work as it should.

Charge quantization

Besides the nice matching of the representations, there is another theoretical reason that speaks for a unified gauge theory. In the standard model it is not explained by electric charge to be a multiple of the electron charge (with fractional charges 1/3 for the quarks). In the SU(5) model the $\text{U}(1)_Y$ generator is part of SU(5) and it is naturally explained why the charges have the values they have.

$\mathbf{SO}(10)$ unification

There are further possibilities to construct unified theories. The *Pati-Salam* model for example has the gauge group $\mathrm{SU}(4) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$.

Both the Georgi-Glashow and the Pati-Salam model can be further unified and embedded into the group SO(10). The unified gauge theory based on SO(10) is particularly elegant, but we will not discuss it in detail here. Let us just mention that there is a *spinorial* representation (similar to the left-handed or right-handed spinor representations of $\mathrm{SO}(1,3)$ for chiral fermions) **16**, that decomposes in terms of SU(5) representations as

$$\mathbf{16} = \mathbf{10} \oplus \mathbf{5}^* \oplus \mathbf{1}.$$

This contains all the representations we need for the Georgi-Glashow model and therefore the standard model fermions of one generation as well as one additional fermion that has the quantum numbers of the right-handed neutrino! The latter is anyway needed for the *seesaw mechanism* to give mass to the observable neutrinos. For this mechanism to work, the right-handed neutrino is supposed to be very heavy. On the other side, in the SO(10) model, it is part of the **16** representation together with all the other fermions, so it is supposed to be massless. There must be some mechanism that breaks SO(10) at some high energy or mass scale and this mechanism needs to give the right-handed neutrino its mass. One can infer that the scale of SO(10) breaking shows up (albeit somewhat indirectly) through the seesaw mechanism in the observable neutrino masses and more directly in the right-handed neutrino mass, which is also a candidate for dark matter.

8 Functional renormalization

We have seen that loop integrals often contain ultraviolet divergencies if the UV cutoff is moved to infinity, or infrared divergencies if a particle mass scale is sent to zero. Quantities dominated by infrared fluctuations become predictable in terms of a few "renormalised couplings". This idea is the central point why quantum field theory has predictive power. Rather than dealing with this idea in a technical fashion, we will develop the concepts that explain why "technical miracles" (as the cancellation of the divergencies) occur in perturbation theory. This is done by introducing functional renormalization as developed by Wilson, Wegner, Symanzik, Kadanoff and others.

The main idea is to relate the microphysical laws encoded by the classical action $S[\phi]$ to the macrophysical laws that can be extracted from the quantum effective action $\Gamma[\Phi]$. This is done in a continuous way by the effective average action $\Gamma_k[\Phi]$, which intuitively describes the laws at a length scale $1/k$. This effective average action interpolates smoothly between the classical action (at $k = \Lambda$ or $k \to \infty$) and the quantum effective action (at $k = 0$). In this way, the effect of fluctuations is included incrementally. The way in which $\Gamma_k[\Phi]$ depends on k is described by a so-called "flow-equation" or "renormalization group equation".

Intuitively, the effective average action includes the effects of all fluctuations with Ecuclidean momenta p larger than k, $p^2 > k^2$, but does not include those with smaller momenta, $p^2 < k^2$. The small momentum fluctuations are "cut off" by an infrared regulator function.

We discuss the principle ideas of functional renormalization using abstract index notation for bosonic theories. Extensions to theories with fermions, gauge fields and so on are also available.

Regulator

In order to implement these ideas, we add to the action a regulator piece

$$\Delta S_k[\phi] = \frac{1}{2} R_{k\alpha\beta} \phi^\alpha \phi^\beta.$$

Typically, the role of $R_{k\alpha\beta}$ is to regularize fluctuations in the infrared, for example it can in the form of a mass term

$$\Delta S_k[\phi] = \int_x \left\{ \frac{1}{2} k^2 \phi_n(x)\phi_n(x) \right\}.$$

However, we keep the form of $R_{k\alpha\beta}$ open for the moment. In most applications it vanishes in one limit, and becomes large in another.

Generating functionals in presence of regulator

We can now repeat all definitions for generating functionals in the presence of a regulator for $k > 0$. The Schwinger functional becomes k-dependent,

$$e^{W_k[J]} = \int D\phi \, \exp\left(-S[\phi] - \Delta S_k[\phi] + J_\alpha \phi^\alpha\right).$$

Only the action, not the construction, is modified. The same holds for the Legendre transform

$$\tilde{\Gamma}_k[\Phi] = \sup_J \left(J_\alpha \Phi^\alpha - W_k[J]\right).$$

The relation between Φ and J depends now on k,

$$\frac{\delta W_k[J]}{\delta J(x)} = \Phi(x), \qquad \frac{\delta \tilde{\Gamma}_k[\Phi]}{\delta \Phi(x)} = J(x),$$

since $W_k[J]$ and $\tilde{\Gamma}_k[\Phi]$ depend on k.

Flowing action

For the definition of the flowing action or effective average action $\Gamma_k[\Phi]$, we subtract from $\tilde{\Gamma}_k[\Phi]$ the regulator piece, now in terms of $\Phi(x)$,

$$\Gamma_k[\Phi] = \tilde{\Gamma}_k[\Phi] - \Delta S_k[\Phi] = \tilde{\Gamma}_k[\Phi] - \frac{1}{2} R_{k\alpha\beta} \Phi^\alpha \Phi^\beta.$$

In this notation one can write

$$\frac{\delta \Gamma_k[\Phi]}{\delta \Phi^\alpha} = J_\alpha - R_{k\alpha\beta} \Phi^\beta.$$

Background field identity

Inserting these definitions, one obtains

$$\exp\left(-\Gamma_k[\Phi]\right) = \int D\phi' \; \exp\left(-S[\Phi + \phi'] + \frac{\delta \Gamma_k[\Phi]}{\delta \Phi^\alpha} \phi'^\alpha - \frac{1}{2} R_{k\alpha\beta} \phi'^\alpha \phi'^\beta \right)$$

Interestingly, the infrared regulator only acts on the fluctuations $\phi'^\alpha = \phi^\alpha - \Phi^\alpha$.

Limit of large regulator scale

From the background field identity one can derive a few important results concerning limits of the effective average action. Consider first the limit of very large k and assume

$$R_{k\alpha\beta} \to k^2 \delta_{\alpha\beta} \qquad (k \to \infty).$$

This is indeed the case for the constant or mass-type regulator k^2. In this case the regulator part suppresses all fluctuations in the functional integral, and forms a representation of a functional Dirac delta,

$$\delta[\phi] = \lim_{k \to \infty} \exp\left(-\frac{1}{2} R_{k\alpha\beta} \phi^\alpha \phi^\beta - c_k \right).$$

The constant c_k is not important for most purposes. With the functional Dirac delta and the background field identity one obtains

$$\lim_{k\to\infty} \Gamma_k[\Phi] + c_k = S[\Phi].$$

For large regulator scale, the effective average action approaches the microscopic action!

If one wants to start at some large scale $k = \Lambda$, one can often work with $\Gamma_\Lambda[\Phi]$ being equal to $S[\Phi]$ plus one-loop terms, where the latter are calculated in the presence of the regulator.

Limit or small regulator scale

The limit of small regulator scale is very easily taken, at least on the formal level. Here the regulator is supposed to vanish,

$$R_{k\alpha\beta} \to 0 \qquad\qquad (k \to 0).$$

This ensures that the effective average action approaches the quantum effective action,

$$\lim_{k\to 0} \Gamma_k[\Phi] = \Gamma[\Phi].$$

In this lmimit, all fluctuations get included in the quantum effective action.

The flowing action $\Gamma_k[\Phi]$ interpolates between the microscopic action $S[\Phi]$ and the macroscopic quantum effective action $\Gamma[\Phi]$.

Some remarks

1. In practice, there is often a finite microscopic (UV) scale Λ. Instead of taking the limit $k \to \infty$, one sets $k \to \Lambda$. Thus, $\Gamma_\Lambda[\Phi]$ can be associated with the microscopic action (though in principle, the first step is to compute $\Gamma_\Lambda[\Phi]$ from $S[\phi]$).

2. Symmetries of $\Gamma_k[\Phi]$: All symmetries of $S[\phi] + \Delta S_k[\phi]$ (in absence of anomalies) or of $\Gamma_\Lambda[\phi]$ and $\Delta S_k[\phi]$ are also symmetries of $\Gamma_k[\Phi]$. Sometimes the infrared regulator can violate certain symmetries.

3. Effective laws: $\Gamma_k[\Phi]$ encodes the effective laws at the momentum scale k, i. e. at the length scale $1/k$. Thus, the flow to lower k can intuitively be understood as “zooming out” with a microscope that enables to adjust to variable resolutions. When fluctuations $p^2 < k^2$ are not yet included, $\Gamma_k[\Phi]$ describes a situation analogous to an experiment with a finite probe size $1/k$. Therefore $\Gamma_k[\Phi]$ is called the “flowing action”.

4. To take into account fluctuations only down to a certain momentum k can also be interpreted as averaging of fields, taking into account all interactions within a range $1/k$. Therefore, one talks about the “effective average action”.

Exact flow equation

The flowing action obeys an exact flow equation (C. Wetterich 1993),

$$\partial_k \Gamma_k[\Phi] = \frac{1}{2} \mathrm{Tr} \left\{ (\Gamma^{(2)}[\Phi] + R_k)^{-1} \partial_k R_k \right\}.$$

It is a functional differential equation and both $\Gamma_k[\Phi]$ and $\Gamma_k^{(2)}[\Phi]$ are functionals of the field expectation values $\Phi^\alpha = \langle \phi^\alpha \rangle$.

Beyond purely bosonic theories, the trace is replaced by a supertrace operation which includes an additional sign for fermionic fields.

Derivation of the flow equation

In order to derive the exact flow equation we use several steps.

1. We first show that the definition of $\tilde{\Gamma}_k[\Phi]$ as a Legendre transform of $W_k[J]$,

$$\tilde{\Gamma}_k[\Phi] = \sup_J \left(J_\alpha \Phi^\alpha - W_k[\Phi] \right)$$

implies

$$\partial_k \tilde{\Gamma}_k[\Phi] = -\partial_k W_k[J].$$

This holds as one can see by carefully taking derivatives using the chain rule,

$$\partial_k \tilde{\Gamma}_k[\Phi]\big|_\Phi = -\partial_k W_k[J]\big|_J - \frac{\delta W_k[J]}{\delta J_\alpha} \partial_k J_\alpha\big|_\Phi + \Phi^\alpha \partial_k J_\alpha\big|_\Phi = -\partial_k W_k[J]\big|_J.$$

2. Now we evaluate this further

$$\begin{aligned} -\partial_k W_k[J] &= -\partial_k \ln \int D\phi \, \exp\left(-S[\phi] - \Delta S_k[\phi] + J_\alpha \phi^\alpha\right) \\ &= \frac{1}{Z} \int D\phi \, \exp\left(-S[\phi] - \Delta S_k[\phi] + J_\alpha \phi^\alpha\right) \partial_k \Delta S_k[\phi] \\ &= \partial_k \langle \Delta S_k[\phi] \rangle, \end{aligned}$$

where we used that only $\Delta S_k[\phi]$ depends on k.

3. The formula for the propagator is derived as follows. We start from the definition

$$G^{\alpha\beta} = \langle \phi^\alpha \phi^\beta \rangle - \langle \phi^\alpha \rangle \langle \phi^\beta \rangle,$$

which can be rewritten to

$$\langle \phi^\alpha \phi^\beta \rangle = G^{\alpha\beta} + \Phi^\alpha \Phi^\beta.$$

Note that for bosonic fields $G^{\alpha\beta} = G^{\beta\alpha}$. This results in

$$\partial_k \tilde{\Gamma}_k[\Phi] = \frac{1}{2} (\partial_k R_{k\alpha\beta}) G^{\alpha\beta} + \frac{1}{2} (\partial_k R_{k\alpha\beta}) \Phi^\alpha \Phi^\beta.$$

For $\Gamma_k[\Phi] = \tilde{\Gamma}_k[\Phi] - \Delta S_k[\Phi]$ we find

$$\partial_k \Gamma_k[\Phi] = \frac{1}{2} (\partial_k R_{k\alpha\beta}) G^{\alpha\beta} = \frac{1}{2} \mathrm{Tr}\{G \partial_k R_k\}.$$

Here,

$$G^{\alpha\beta} = \frac{\delta^2 W_k[J]}{\delta J_\alpha \delta J_\beta},$$

is the propagator matrix in the presence of the infrared regulator at scale k. It depends on sources or fields.

4. Let us now rewrite the propagator part further. We employ the general matrix identity for Legendre transforms

$$G^{\alpha\beta} (\tilde{\Gamma}_k^{(2)})_{\beta\gamma} = G^{\alpha\beta} (\Gamma_k^{(2)} + R_k)_{\beta\gamma} = \delta^\alpha{}_\gamma.$$

This is easily shown,

$$\frac{\delta^2}{\delta J_\alpha \delta J_\beta} W_k[J] \frac{\delta^2}{\delta \Phi^\beta \delta \Phi^\gamma} \tilde{\Gamma}_k[\Phi] = \frac{\delta \Phi^\beta}{\delta J_\alpha} \frac{\delta J_\gamma}{\delta \Phi^\beta} = \frac{\delta J_\gamma}{\delta J_\alpha} = \delta^\alpha{}_\gamma.$$

This yields the final form of the closed flow equation and concludes the proof.

Rewriting the flow equation

A dimensionless form of the flow equation is obtained by multiplying with k and by defining $\partial_t = k\partial_k$,

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\left\{(\Gamma^{(2)}[\Phi] + R_k)^{-1}\partial_t R_k\right\}.$$

Here $t = \ln(k/\Lambda)$ is sometimes called RG time, but has nothing to do with physical time.

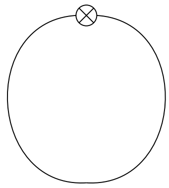
Another rewriting is sometimes useful,

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2}\mathrm{Tr}\left\{\tilde{\partial}_t \ln(\Gamma_k^{(2)}[\Phi] + R_k)\right\},$$

with the formal scale derivative $\tilde{\partial}_t$ which hits only the regulator term R_k. Recalling the one-loop formula

$$\Gamma_{\text{1-loop}}[\Phi] = \frac{1}{2}\mathrm{Tr}\left\{\ln S^{(2)}[\Phi]\right\},$$

one sees the close correspondence to perturbation theory and Feynman graphs. A Feynman diagram for the flow equation can be given as

$\partial_t \Gamma_k =$ 

The line represents an exact and field-dependent propagator.

Properties of the flow equation

1. The flow equation is exact, we have made no approximations. All non-perturbative effects are included, such as topological defects, etc.

2. The particular form of the matrix $R_{k\alpha\beta}$ is not important. It is only important that $\Delta S_k[\phi]$ is a quadratic form in the fields. This allows generalisations to a wide range of situations where R_k is not necessarily interpreted as a momentum cutoff.

3. Finite momentum integrals. For certain choices of the regulator R_k one obtains finite momentum integrals. This is best discussed in examples later on.

4. Flow equations for correlation functions. One may take functional derivatives with respect to the field Φ^α and obtain further flow equations for correlation functions.

Derivative expansion

The flow equation is a functional differential equation. Except for a few particular cases (leading order large N expansion, few non-relativistic particles), it cannot be solved exactly.

Approximate solutions are constructed by truncation. A truncation is an ansatz for the general form of the effective average action in terms of a few free parameters or free functions. One computes the flow of these parameters or functions by inserting the ansatz on the right hand side of the flow equation, specifically for the computation of $\Gamma^{(2)}[\Phi]$.

For the derivative expansion, one expands $\Gamma_k[\Phi]$ in terms of powers of derivatives of the fields. For example, for a theory with $\mathrm{O}(N)$ symmetry, this leads to

$$\Gamma_k[\Phi] = \int_x \left\{ U_k(\rho) + \frac{g^{\mu\nu}}{2} Z_k(\rho)\partial_\mu \Phi_n(x)\partial_\nu \Phi_n(x) + \frac{g^{\mu\nu}}{4} Y_k(\rho)\partial_\mu \rho(x)\partial_\nu \rho(x) + \ldots \right\},$$

where $\rho(x) = \frac{1}{2}\Phi_n(x)\Phi_n(x)$. The derivative expansion to the given order neglects terms with four or more derivatives. In this order, one has three functions, $U_k(\rho)$, $Z_k(\rho)$ and $Y_k(\rho)$ which parametrize the flowing action. If we simplify further: $Y_k = 0$, Z_k independent of ρ, then this is called "leading (order) potential approximation".

Flow of effective potential

We want to compute the flow equation for the effective potential $U_k(\rho)$. For this purpose, we evaluate $\partial_t \Gamma_k[\Phi]$ for homogeneous fields. One needs to evaluate $\Gamma_k^{(2)}$ for constant Φ. In momentum space, it reads

$$(\Gamma_k^{(2)})_{nm}(p,q)[\Phi] = \left[Z_k p^2 \delta_{nm} + \frac{\partial^2 U_k(\rho)}{\partial \Phi_n \partial \Phi_m} \right] (2\pi)^{(d)} \delta^{(d)}(p-q).$$

We also specify the regulator term to be of the form

$$\Delta S_k[\Phi] = \int_x \left\{ \frac{1}{2} \Phi_n(x) R_k(-\Box_x) \Phi_n(x) \right\} = \int_p \left\{ \frac{1}{2} \Phi_n^*(p) R_k(p^2) \Phi_n(p) \right\},$$

with some function $R_k(p^2)$. The functional trace for the flow equation is then easily performed in momentum space, and one obtains the flow equation for the effective potential,

$$\partial_t U_k(\rho) = \frac{1}{2} \int_q \frac{\partial_t R_k(q^2)}{Z_k q^2 + R_k(q^2) + U_k'(\rho) + 2\rho U_k''(\rho)} + \frac{N-1}{2} \int_q \frac{\partial_t R_k(q^2)}{Z_k q^2 + R_k(q^2) + U_k'(\rho)}.$$

This can be compared to the one-loop approximation

$$U_{\text{1-loop}} = \frac{1}{2} \int_q \ln(Zq^2 + V'(\rho) + 2\rho V''(\rho)) + \frac{N-1}{2} \int_q \ln(Zq^2 + V'(\rho)).$$

Replace $V(\rho) \to U(\rho)$, add $R_k(q^2)$ by $Zq^2 \to Z_k q^2 + R_k(q^2)$ and take the $\tilde{\partial}_t$-derivative. This leads from the one-loop approximation to the flow equation.

Ultraviolet finite flow

There are choices of regulator functions such that no ultraviolet divergencies arise in $\partial_t U_k(\rho)$. As an example, consider the function

$$R_k(q^2) = Z_k(k^2 - q^2)\theta(k^2 - q^2).$$

The anomalous dimension is defined by

$$\eta = -\partial_t \ln Z_k.$$

It is typically very small and we can neglect the term proportional to η in $\partial_t R_k$. The result is a simple expression for the flow of the effective potential,

$$\partial_t U_k(\rho) = \frac{1}{2} \int_{q^2<k^2} \frac{2Z_k k^2}{Z_k k^2 + U_k'(\rho) + 2\rho U_k''(\rho)} + \frac{N-1}{2} \int_{q^2<k^2} \frac{2Z_k k^2}{Z_k k^2 + U_k'(\rho)}.$$

We define the renormalized dimensionless mass terms w_1 for the radial mode and w_2 for the Goldstone modes.

$$w_1 = \frac{U_k'(\rho) + 2\rho U_k''(\rho)}{Z_k k^2}, \qquad w_2 = \frac{U_k'(\rho)}{Z_k k^2}.$$

The momentum integrals are trivial,

$$\int_{q^2<k^2} = \alpha_d k^d,$$

Examples for the coefficients that arise here are $\alpha_2 = 1/(4\pi)$, $\alpha_3 = 1/(6\pi^2)$ and $\alpha_4 = 1/(32\pi^2)$. We arrive at a very simple flow equation for the effective potential,

$$\partial_t U_k(\rho) = \alpha_d k^d \left[\frac{1}{1+w_1} + \frac{N-1}{1+w_2} \right].$$

Since w_1 and w_2 involve ρ-derivatives of $U_k(\rho)$, this is a differential equation for a single function U of the two variables k and ρ. For a given η or $\eta = 0$, it is closed.

Scale dependent minimum

We now specialize to $d = 4$ dimensions and investigate the implications of the flow equation in more detail. If the effective average potential $U_k(\rho)$ has a minimum at $\rho_0(k)$, the condition for the minimum is for all k,

$$U_k'(\rho_0(k)) = 0.$$

The flow equation for $U'(\rho)$ at fixed ρ is obtained by taking a ρ-derivative of the flow equation for the potential,

$$\begin{aligned} \partial_t U_k'(\rho) &= \frac{k^4}{32\pi^2} \frac{\partial}{\partial\rho} \frac{1}{1+w_1} + \frac{N-1}{1+w_2} \\ &= -\frac{k^4}{32\pi^2} - \frac{1}{(1+w_1)^2} \frac{\partial w_1}{\partial \rho} + \frac{N-1}{(1+w_2)^2} \frac{\partial w_2}{\partial \rho}. \end{aligned}$$

For simplicity, we take $Z = 1$ (equivlent to $\eta = 0$), and use

$$\begin{aligned} w_1 &= \frac{U_k'(\rho) + 2\rho U_k''(\rho)}{k^2}, & w_2 &= \frac{U_k'(\rho)}{k^2}, \\ \frac{\partial w_1}{\partial \rho} &= \frac{3U_k''(\rho) + 2\rho U_k'''(\rho)}{k^2}, & \frac{\partial w_2}{\partial \rho} &= \frac{U_k''(\rho)}{k^2}. \end{aligned}$$

One infers

$$\partial_t U_k'(\rho) = -\frac{k^2}{32\pi^2} \left[-\frac{3U''(\rho) + 2\rho U'''(\rho)}{(1+w_1)^2} + \frac{(N-1)U''(\rho)}{(1+w_2)^2} \right].$$

For $\rho = \rho_0$, one has $U_k'(\rho_0) = 0$, $w_2 = 0$, and $w_1 = 2\rho_0 U_k''(\rho_0)/k^2$. We define $\lambda = U_k''(\rho_0)$ and $\nu = U_k'''(\rho_0)$ such that

$$\partial_t U_k'(\rho_0) = -\frac{k^2}{32\pi^2} \left[\frac{3\lambda + 2\rho_0\nu}{(1 + 2\rho_0\lambda/k^2)^2} + (N-1)\lambda \right].$$

For a fixed location ρ_0, the derivation $U_k'(\rho_0)$ does not remain zero. The location of the minimum therefore depends on k, according to

$$\partial_t U_k'(\rho_0) + \partial_t U_k''(\rho_0) \frac{\partial \rho}{\partial t} = 0,$$

which implies

$$\frac{\partial \rho_0}{\partial t} = -\frac{1}{\lambda} \partial_t U_k'(\rho_0).$$

The location of the minimum moves according to

$$\frac{\partial \rho_0}{\partial t} = \frac{k^2}{32\pi^2} \left[\frac{3 + 2\rho_0\nu/\lambda}{(1 + 2\rho_0\lambda/k^2)^2} + (N-1) \right].$$

As k is lowered, ρ_0 becomes smaller. Depending on the initial value at $k = \Lambda$, it may reach zero at some $k > 0$ or not.

Leading order at small coupling

For small λ, we will see that $\nu \sim \lambda^3$. To lowest order in λ, the flow equation for ρ_0 simplifies,

$$\frac{\partial \rho_0}{\partial t} = \frac{k^2}{32\pi^2}(N+2).$$

This has the simple solution

$$\rho_0(k) = \frac{k^2}{64\pi^2}(N+2) + c_\Lambda,$$

with integration constant c_Λ, or, with $\rho_\Lambda = \rho_0(k=\Lambda)$,

$$\rho_0(k) = \rho_\Lambda - \frac{\Lambda^2 - k^2}{64\pi^2}(N+2).$$

Different initial values ρ_Λ label different "flow trajectories".

Phase transition

There is a critical value $\rho_{\Lambda,\text{critilcal}}$ for which $\rho_0(k=0) = 0$, namely

$$\rho_{\Lambda,\text{critical}} = \frac{\Lambda^2}{64\pi^2}(N+2).$$

For $\rho_\Lambda > \rho_{\Lambda,\text{critical}}$, one has $\rho_0(k=0) > 0$. This corresponds to the phase with spontaneous symmetry breaking (SSB).

On the other hand, for $\rho_\Lambda < \rho_{\Lambda,\text{critical}}$, one finds $\rho_0(k_t) = 0$ for $k_t > 0$. For $k < k_t$, the minimum is located at $\rho = 0$. The flow of U is then better described by the flow of $m_0^2 = U'(0)$. It increases with decreasing k. At $k = 0$, one finds $m_0^2 > 0$. Then the model is in the symmetric phase (SYM).

Quadratic divergence

For a given macroscopic or "remormalized" $\rho_{0,\text{R}} = \rho_0(k=0)$, one finds for the microscopic or "bare" ρ_Λ,

$$\rho_\Lambda = \rho_{0,\text{R}} + \frac{\Lambda^2}{64\pi^2}(N+2).$$

For $\Lambda \to \infty$, this diverges quadratically. The divergence arises from the relation between bare and renormalized parameters which in turn arises due to the flow that is generated by fluctuations.

Running quartic coupling

For the flow of $\lambda = U_k''(\rho_0)$, one needs

$$\partial_t \lambda = \partial_t U_k''(\rho_0) + U_k'''(\rho_0)\frac{\partial \rho_0}{\partial t},$$

where $\partial_t U_k''(\rho)$ is obtained from $\partial_t U_k'(\rho)$ by taking a further ρ-derivative. Thus,

$$\begin{aligned}\partial_t U_k''(\rho) =& \frac{1}{16\pi^2}\left[\frac{(3U_k'' + 2\rho U_k''')^2}{(1+w_1)^3} + (N-1)\frac{(U'')^2}{(1+w_2)^3}\right] \\ &- \frac{k^2}{32\pi^2}\left[\frac{3U''' + 2\rho U_k''''}{(1+w_1)^2} + (N-1)\frac{U'''}{(1+w_2)^2}\right].\end{aligned}$$

Subsequently, we neglect $U_k'''(\rho_0)$ and $U_k''''(\rho_0)$ because they are of higher order in small λ (see below). This results in

$$\partial_t \lambda = \frac{1}{16\pi^2}\left[\frac{9\lambda^2}{(1+2\lambda\rho_0/k^2)^2} + (N-1)\lambda^2\right].$$

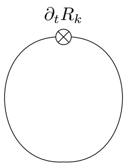
The leading order in a perturbative expansion in λ yields the one-loop β-function,

$$\partial_t \lambda = \beta_\lambda = \frac{\lambda^2}{16\pi^2}(N+8).$$

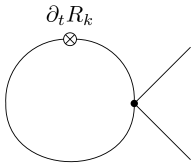
This is a typical "renormalization group equation" for a dimensionless coupling, that can also be found by perturbative renormalization. The β-function involves only λ at a given scale k, not the bare coupling $\lambda_\Lambda = \lambda(k=\Lambda)$. It does not involve k explicitly.

Feynman diagrams

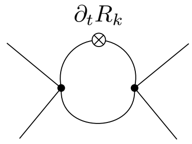
The flow of the effective potential is symbolized by

$\partial_t U_k =$ 

A ρ-derivative inserts external legs attached to a vertex,

$\partial_t U_k' =$ 

and similarly,

$\partial_t U_k'' =$ 

This is the perturbative Feynman diagram with the insertion of $\partial_t R_k$. Renormalized vertices replace the bare vertices. The $\partial_t R_k$ insertion removes divergencies from the momentum integral. The flow equation has the important property that only the couplings at a given scale k appear, not the bare couplings. The computation of a change of $\Gamma_k[\Phi]$ only involves $\Gamma_k[\Phi]$!

Running coupling

The solution of the flow equation is easily found by integration. In a first step one writes

$$\frac{d\lambda}{\lambda^2} = -d(1/\lambda) = \frac{N+8}{16\pi^2} d\ln(k),$$

which integrates to

$$\frac{1}{\lambda(k)} = \frac{1}{\lambda_\Lambda} + \frac{N+8}{16\pi^2}\ln(\Lambda/k).$$

Finally the solution is

$$\lambda(k) = \frac{\lambda_\Lambda}{1 + \frac{(N+8)\lambda_\Lambda}{16\pi^2}\ln(\Lambda/k)}.$$

As k decreases, $\lambda(k)$ decreases.

Triviality

For any fixed Λ and $\lambda_\Lambda > 0$, one finds for $k \to 0$ that $\lambda(k \to 0) = 0$.

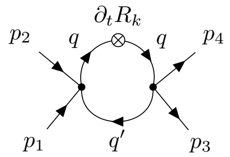
The interaction vanishes in this limit and one ends up with a free theory. This is called "triviality". One can use the flow equation in order to show triviality without the assumption of small λ.

External momenta

Consider the momentum-dependent four-parent vertex

$$\lambda_k(p_1, p_2, p_3, p_4) = \frac{\delta^4 \Gamma_k}{\delta\Phi(p_1)\delta\Phi(p_2)\delta\Phi^*(p_3)\delta\Phi^*(p_4)}$$

(We omit the internal indices, e. g. $N = 1$.) In lowest order, the flow equation is given by a one-loop diagram.

$\partial_t \lambda =$ 

It involves the renormalized vertices $\lambda_k(p_1, p_2, q, q')$ and $\lambda_k(q, q', p_3, p_4)$. Momentum conservation at the vertices implies

$$q - q' = p_1 + p_2 = p_3 + p_4, \qquad (q')^2 = (p_1 + p_2 - q)^2.$$

For $(p_1 + p_2)^2 = \mu^2$ and $k^2 \ll \mu^2$, only momenta $q^2 < k^2$ or momenta $(q')^2 \approx \mu^2$ contribute to the flow. This replaces in one of the propagators $1/(k^2 + m^2)$ by $1/(\mu^2 + m^2)$, leading to a suppression $\propto k^2/\mu^2$. The flow effectively stops for $k^2 < \mu^2$. One can associate

$$\lambda(\mu) \approx \lambda_{k^2=\mu^2}(0),$$

where the left hand side represents the non-vanishing momenta at vanishing regulator scale $k = 0$, and the right hand side the vanishing momenta, at the regulator scale $k^2 = \mu^2$. Taking $\lambda_R(\mu) = \lambda_{k=0}(\mu)$, one has

$$\lambda_R(\mu) = \frac{\lambda_\Lambda}{1 + \frac{(N+8)\lambda_\Lambda}{16\pi^2} \ln(\Lambda/\mu)}.$$

The flow equation describes now the dependence of the vertex on the scale of the external momenta. It is equivalent to the perturbative renormalization group.

Landau pole and "incomplete theories"

For the standard model, the Fermi scale ϕ_0 contributes an effective infrared cutoff. The renormalized coupling at $k = \phi_0$ is measured by the observation of the mass of the Higgs boson

$$m_H^2 = 2\lambda(\phi_0)\phi_0^2.$$

We can use the flow equation in order to compute λ at shorter distance scales $k > \phi_0$. We have

$$\frac{1}{\lambda(k)} - \frac{1}{\lambda(\phi_0)} = -\frac{N+8}{16\pi^2} \ln(k/\phi_0),$$

or

$$\lambda(k) = \frac{\lambda(\phi_0)}{1 - \frac{(N+8)\lambda(\phi_0)}{16\pi^2}\ln(k/\phi_0)}.$$

We observe that $\lambda(k)$ diverges at a Landau pole at a scale k_{Landau}, i. e. $\lambda(k) \to \infty$ for $k \to k_{\text{Landau}}$.

One concludes that the O(N)-model with non-zero renormalized coupling $\lambda_R(\mu)$ can not be continued to infinitely short scales. This is an “incomplete theory”. One finds similarly that the standard model is an incomplete theory. Some new physics is necessary at very short length scales in order to make the standard model a well-defined QFT. The Landau pole appears far beyond the Planck scale for gravity. The completion of the standard model could therefore be provided by quantum gravity, changing the flow of couplings for $k > M_{\text{Planck}}$, where $M_{\text{Planck}} \approx 10^{18}$ GeV.

Predictivity

Let us compute the flow equation for $\nu = U'''(\rho_0)$. We neglect $U^{(4)}(\rho)$ and higher ρ-derivatives.

$$\begin{aligned}\partial_t U_k'''(\rho) = &-\frac{3}{16\pi^2 k^2}\left[\frac{(3U'' + 2\rho U''')^3}{(1+w_1)^4} + (N-1)\frac{(U'')^3}{(1+w_2)^4}\right] \\ &- \frac{1}{16\pi^2}\left[\frac{5U'''(3U'' + 2\rho U'')}{(1+w_1)^3} + (N-1)\frac{U'''U''}{(1+w_2)^3}\right].\end{aligned}$$

With $U'' = \lambda,\ U''' \propto \lambda^3$, the leading term in an expansion in small λ is

$$\partial_t \nu = -\frac{3(N+26)\lambda^3}{16\pi^2 k^2}.$$

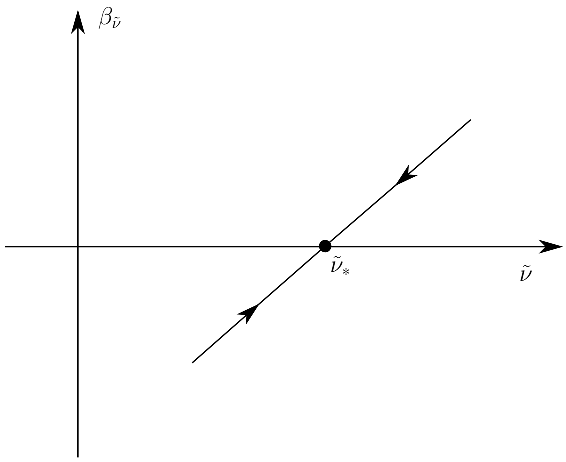
For the dimensionless ratio $\tilde{\nu} = \nu k^2$, one has

$$\partial_t \tilde{\nu} = \beta_{\tilde{\nu}} = 2\tilde{\nu} - \frac{3(N+26)\lambda^3}{16\pi^2}.$$

The function $\beta_{\tilde{\nu}}$ has a zero for

$$\tilde{\nu}_* = \frac{3(N+26)\lambda^3}{32\pi^2 k^2}.$$

Indicating the flow for decreasing k by arrows, one obtains the diagram



The solution of the flow equation attracts $\tilde{\nu}$ to the partial-IR fixed point $\tilde{v}_*$. For a given k, this predicts

$$\nu(k) = \frac{3(N+26)\lambda(k)^3 k^2}{32\pi^2}.$$

More precisely, one has

$$\partial_t \left[\frac{\tilde{\nu}}{\lambda^3}\right] = 2\left[\frac{\tilde{\nu}}{\lambda^3}\right] - \frac{3(N+26)}{16\pi^2} - \frac{\tilde{\nu}}{\lambda^4}\partial_t \lambda$$
$$= \left[2 - \frac{N+8}{16\pi^2}\lambda\right]\frac{\tilde{\nu}}{\lambda^3} - \frac{3(N+26)}{16\pi^2}.$$

The effects of the running of λ can be neglected and one finds for the ratio $z = \tilde{\nu}/\lambda^3$,

$$\partial_t z = 2(z - z_*),$$

with the fixed point value $z_* = \tilde{\nu}_*/\lambda^3$. The solution is a power law behaviour

$$z - z_* = c_0 \frac{k^2}{\Lambda^2}.$$

This implies

$$\tilde{\nu} - \tilde{\nu}_* = c_0 \lambda^3 \frac{k^2}{\Lambda^2},$$

or

$$\nu = \frac{\tilde{\nu}_*}{k^2} + \frac{c_0 \lambda^3}{\Lambda^2}.$$

For $k^2 \ll \Lambda^2$, the initial value (a bare coupling) $\nu(\Lambda)$ that is specified by c_0 plays no role. The flow "looses the memory about its microphysics".

This happens to all couplings except for the two renormalizable couplings λ_R and $\rho_{0,R}$. All other couplings can be produced, and can ultimately be expressed in terms of λ_R and $\rho_{0,R}$!

9 Quantum field theory in thermal equilibrium

Quantum field theory can not only describe vacuum states and its excitations, the particles, but also states with non-zero density and temperature. We discuss now the main steps and concepts for that.

Thermal states

Thermal states are mixed states, described by density matrices. For a relativistic quantum field theory, the grand canonical ensemble is most relevant, where energt and particles number can be exchanged between subsystems. Quantum mechanically, this is described by the density matrix

$$\rho = \frac{1}{Z} e^{-\frac{1}{T}(H-\mu N)} = e^{-\frac{1}{T}(H-\mu N) - \ln Z},$$

where H and N are operators. The entropy is obtained by von-Neumanns formula,

$$S = \mathrm{Tr}\{-\rho \ln \rho\} = \frac{1}{T}(\langle H\rangle - \mu\langle N\rangle) + \ln Z.$$

The expectation values of energy and conserved particle number (usually proportional to a number of particles minus anti-particles) are here simply

$$E = \langle H\rangle = \mathrm{Tr}\{\rho H\},$$
$$N = \langle N\rangle = \mathrm{Tr}\{\rho N\}.$$

Grand canonical partition function

Recall from statistical mechanis that the grand canonical partition function is given by

$$Z(T,\mu,V) = \mathrm{Tr}\left\{e^{-\frac{1}{T}(H-\mu N)}\right\}.$$

Related to the partition function is the grand canonical potential $\Omega(T,\mu,V)$ through the relation

$$Z = e^{-\Omega/T}.$$

The differential of the grand potential is

$$d\Omega = -SdT - Nd\mu - pdV,$$

and it can be expressed as

$$\Omega = E - TS - \mu N = -pV,$$

Thermodynamic quantities can be directly derived from Z or Ω, for example

$$S = -\left.\frac{\partial\Omega}{\partial T}\right|_{\mu,V} \qquad N = -\left.\frac{\partial\Omega}{\partial\mu}\right|_{T,V}.$$

Other observables follow from Legendre transforms, for example

$$E = \Omega + TS + \mu N = \Omega - T\frac{\partial\Omega}{\partial T} - \mu\frac{\partial\Omega}{\partial\mu}.$$

Thermodynamics for fluids

For fluids it is convenient to work with pressure $p(T,\mu)$ as a thermodynamic potential. For constant volume V one has

$$Z(T,\mu) = \exp\left(\frac{Vp(T,\mu)}{T}\right) = \exp\left(\int_0^{\frac{1}{T}} d\tau \int d^3x\, p(T,\mu)\right),$$

and the differential Gibbs-Duhem relation

$$dp = sdT + nd\mu.$$

For a homogeneous state of a fluid one has constant energy density $\epsilon = E/V$, particle density $n = N/V$ and entropy density $s = S/V$. With this, together with the relation

$$\epsilon + p = Ts + \mu n,$$

one can all thermodynamic quantities of interest from the potential $p(T,\mu)$. This also generalizes directly to situations with several conserved quantum numbers such as (net) baryon number, (net) electric charge, strangeness and so on. One can introduce a chemical potential associated to each of them.

Exercise on thermodynamics for fluids

Derive expresions for the heat capacity densities

$$c_v = \frac{C_v}{V} = \frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_{V,N}, \qquad c_p = \frac{C_p}{V} = \frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_{p,N},$$

and the thermal expansion coefficient

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{p,V},$$

in terms of $p(T,\mu)$ and its derivatives.

Partition function for quantum fields

From these considerations, it becomes clear that it would be very useful to have a method to calculate the grand canonical partition function for matter described by a quantum field theory.

Note that $e^{-\frac{1}{T}(H-\mu N)}$ resembles the kernel of transition amplitude. Transition amplitudes in Minkowski space are of the form

$$\langle\phi_\mathrm{f}|e^{-i(t_\mathrm{f}-t_\mathrm{in})H}|\phi_\mathrm{in}\rangle = \int\limits_{\phi_\mathrm{in},\phi_\mathrm{f}} D\phi\, e^{iS_\mathrm{M}[\phi]}.$$

The right hand side involves the Minkowski space action. For a complex scalar field

$$S_\mathrm{M}[\phi] = -\int_{t_\mathrm{in}}^{t_\mathrm{f}} dt \int d^3x\sqrt{g}\left\{g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + m^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2\right\}.$$

The metric $g_{\mu\nu}$ has here the signature $(-,+,+,+)$. The functional integral has the boundary conditions

$$\phi(t_\mathrm{in},\mathbf{x}) = \phi_\mathrm{in}(\mathbf{x}), \qquad \phi(t_\mathrm{f},\mathbf{x}) = \phi_\mathrm{f}(\mathbf{x}),$$

for the initial and final states.

Imaginary or Euclidean time

In order to come from a transition amplitude to a thermodynamic partition function a few steps are needed. The first is to choose the time to be imaginary such that

$$t_\mathrm{f} - t_\mathrm{in} = -i\frac{1}{T}.$$

For example we may choose without loss of generality $t_\mathrm{in} = 0$ and $t_\mathrm{f} = -i\beta$ where $\beta = 1/T$. Propagation will be along imaginary time! It is convenient to introduce a variable τ which is integrated from 0 to $\beta = 1/T$ with $dt = -id\tau$. This is actually a Euclidean time variable, as becomes clear from the invariant length element

$$ds^2 = -dt^2 + d\mathbf{x}^2 = d\tau^2 + d\mathbf{x}^2.$$

Accordingly, the symmetry of Minkowski space $\mathrm{SO}(1,d-1)$ becomes simply an $\mathrm{SO}(d)$ symmetry in Euclidean space.

Periodic boundary conditions

Taking the trace as needed for the partition function means to identify initial and final states and to sum over them. In other words, we need to set

$$\phi_\mathrm{in}(\mathbf{x}) = \phi_\mathrm{f}(\mathbf{x}) = \phi(0,\mathbf{x}) = \phi(t=-i\beta,\mathbf{x}),$$

and include a functional integral over $\phi(0,\mathbf{x})$. This leads to a functional integral without boundaries but with the periodic identification

$$\phi(0,\mathbf{x}) = \phi(t=-i/T,\mathbf{x}).$$

The imaginary time dimension is periodic, the geometry is like the one of a cylinder with times $t=0$ and $t=-i/T$ or $\tau=0$ and $\tau=\beta$ identified. The geometry is thus the one of a torus, with the spatial dimensions unchanged, but time periodic. This geometry is called Matsubara torus. In the limit $T \to 0$ the circumference of the torus becomes infinitely large, while for $T \to \infty$ is becomes very small.

Euclidean action

With the imaginary time element $dt = -id\tau$ comes also a change in the action. We write $iS_{\mathrm{M}}[\phi] = -S_{\mathrm{E}}[\phi]$ with Euclidean action

$$S_E[\phi] = \int_0^{1/T} d\tau \int d^3x \left\{ \delta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + m^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \right\}$$

and

$$\delta^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi = \frac{\partial}{\partial\tau}\phi^*\frac{\partial}{\partial\tau}\phi + \boldsymbol{\nabla}\phi^*\boldsymbol{\nabla}\phi.$$

In this form, the action is real and positive.

Chemical potential

We also need to introduce the chemical potential term. One can see this as a modification of the Hamiltonian or of the action. To introduce it properly, let us first go back to real time and let us couple the theory to an external gauge field $A_\mu(x)$,

$$S_{\mathrm{M}}[\phi] = -\int_{t_{\mathrm{in}}}^{t_{\mathrm{f}}} dt \int d^3x\sqrt{g}\left\{ g^{\mu\nu}(\partial_\mu + iA_\mu)\phi^*(\partial_\nu - iA_\nu)\phi + m^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \right\}.$$

The conserved current on the microscopic or classical level then follows from

$$N^\mu(x) = -\frac{\delta}{\delta A_\mu(x)}S_{\mathrm{M}}[\phi] = g^{\mu\nu}\left[i\phi^*(x)\partial_\nu\phi(x) - i\phi(x)\partial_\nu\phi^*(x)\right].$$

The conserved particle number is

$$N(t) = \int d^3x\; N^0(t, \mathbf{x}).$$

If we take the chemical potential to be the time component of an external gauge field, $A_0 = \mu$, it will automatically couple to the conserved number density $n = N^0$. One may check that signs and factors of i indeed come out correctly. After analytic continuation to Euclidean time, one obtains

$$\frac{\partial}{\partial t} - iA_0 \to \frac{\partial}{\partial(-i\tau)} - iA_0 = i\left(\frac{\partial}{\partial\tau} - A_0\right) \to i\left(\frac{\partial}{\partial\tau} - \mu\right),$$

and the Euclidean action becomes

$$S_{\mathrm{E}}[\phi] = \int_0^{1/T} d\tau \int d^3x \left\{ \left(\frac{\partial}{\partial\tau} + \mu\right)\phi^*\left(\frac{\partial}{\partial\tau} - \mu\right)\phi + \boldsymbol{\nabla}\phi^*\boldsymbol{\nabla}\phi + m^2\phi^*\phi + \frac{\lambda}{2}(\phi^*\phi)^2 \right\}.$$

Matsubara frequencies

Bosonic fields on the Matsubara torus have periodic boundary conditions with respect to Euclidean time τ being changed by an amout $\beta = 1/T$,

$$\phi(0, \mathbf{x}) = \phi(\beta, \mathbf{x}).$$

For fermionic or Grassmann fields, a careful consideration (best done with discretized time) shows that they must be anti-periodic instead,

$$\psi(0, \mathbf{x}) = -\psi(\beta, \mathbf{x}).$$

This has consequences for the Fourier expansion. We expand the fields as

$$\chi(\tau, \mathbf{x}) = T\sum_n \int \frac{d^{d-1}p}{(2\pi)^{d-1}} e^{-i\omega_n\tau + i\mathbf{p}\mathbf{x}}\,\chi(i\omega_n, \mathbf{p}),$$

where $\chi(\tau, \mathbf{x})$ could be either bosonic or fermionic.

As a consequence of the peridicity, or anti-peridicity, the Matsubara frequencies ω_n are discrete. They must be integer multiples of $2\pi T$ for bosons,

$$\omega_n = 2\pi T n,$$

and half-integer multiples for fermions,

$$\omega_n = 2\pi T(n + 1/2),$$

where $n \in \mathbb{Z}$.

In the limit of $T \to 0$, the discrete Matsubara sum becomes again an integral,

$$T \sum_n = \sum_n \frac{\Delta\omega_n}{2\pi} \to \int \frac{d\omega}{2\pi}.$$

In that limit we are back to the standard quantum field theory setup after Wick rotation to Euclidean space.

In the high-temperature limit $T \to \infty$ the distance between nearby Matsubara frequncies becomes very large. We will see that a large Matsubara frequency has in the Euclidean theory a similar effect as a large mass parameter and suppresses fluctuations. Only bosonic fields have a zero mode for which the Matsubara frequency vanishes, $\omega_0 = 0$. It is the only mode that survives in the limit $T \to \infty$ ane leads to a theory of classical fields in termal equilibrium. Fermionic field fluctuations are not contributing in the classical limit.

Propagator on the Matsubara torus

For perturbative calculation and beyond we need the propagator

$$\Delta(\tau - \tau', \mathbf{x} - \mathbf{x}') = \langle \phi(\tau, \mathbf{x}) \phi^*(\tau', \mathbf{x}') \rangle_c$$

As a consequence of translational symmetry in space this connected two-point function depends only on $\mathbf{x} - \mathbf{x}'$. Similarly there is a translational invariance for Euclidean time τ on the torus, keeping in mind that τ and $\tau + n\beta$ describe the same time. This implies also that $\Delta\tau = \tau - \tau'$ can be restriced to the range $-\beta < \Delta\tau < \beta$.

We assume now that the inverse propagator is of the form

$$P(i\omega_n, \mathbf{p}) = (\omega_n - i\mu)^2 + \mathbf{p}^2 + m^2.$$

This yields

$$\Delta(\Delta\tau, \Delta\mathbf{x}) = T \sum_{n=-\infty}^{\infty} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{e^{-i\omega_n \Delta\tau + i\mathbf{p}\Delta\mathbf{x}}}{(\omega_n - i\mu)^2 + \mathbf{p}^2 + m^2}.$$

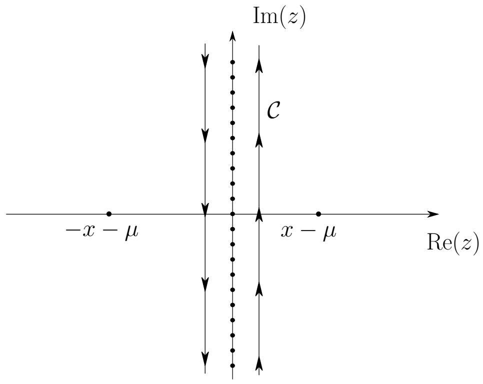
We now face the problem to perform the infinite sum over the Matsubara frequencies.

Contour integral around imaginary axis

In order to find a way to sum over n, we first consider a complex contour integral

$$J = \frac{1}{2\pi i} \int_{\mathcal{C}} dz \left\{ \frac{e^{-z\Delta\tau}}{-(z+\mu)^2 + x^2} \left[\theta(\Delta\tau) + n_B(z)\right] \right\}.$$

Here, μ and $x \geq 0$ as well as $\Delta\tau$ with $-1/T < \Delta\tau < 1/T$ are parameters. The integration contour $\mathcal{C}$ goes downwards slightly left of the imaginary z-axis and up again slightly to the right of it.



We use here the Bose distribution function

$$n_B(z) = \frac{1}{e^{z/T} - 1},$$

which has poles at $z = i2\pi Tn$ for all integer $n \in \mathbb{Z}$, with residue T. It is also useful to keep in mind that for positive argument $z > 0$ one has the zero temperature limit

$$\lim_{T\to 0} n_B(z) = 0.$$

Similarly, in the high-temperature limit one has

$$\lim_{T\to\infty} n_B(z) = \frac{T}{z}.$$

The term $-(z+\mu)^2 + x^2$ has zero-crossings at

$$z = \pm x - \mu.$$

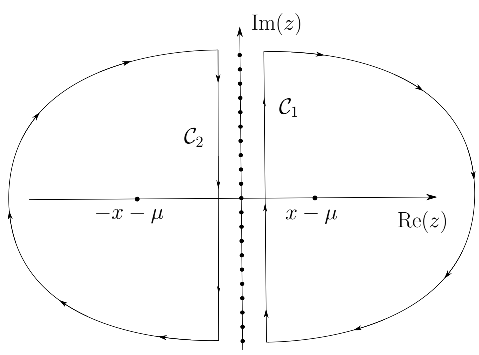
We assume $|\mu| < x$, so that those poles stay away from the imaginary z-axis. The contour can be closed at $z = \pm i\infty$ and we find from the residue theorem

$$J = T \sum_{n=-\infty}^{\infty} \frac{e^{-i2\pi Tn\Delta\tau}}{(2\pi nT - i\mu)^2 + x^2}.$$

This is precisely the infinite sum we need to calculate!

Contour integral around real axes

On the other side, wen can also close the contour somewhat differently without changing the result.



We now get two contributions, $J = J_1 + J_2$, one contour $\mathcal{C}_1$ that closes on the right and one contour $\mathcal{C}_2$ that closes on the left. From

$$\frac{1}{-(z+\mu)^2 + x^2} = \left(-\frac{1}{z - x + \mu} + \frac{1}{z + x + \mu} \right) \frac{1}{2x}$$

one can read off where the poles are. Closing the contour $\mathcal{C}_1$ at large positive real values of z is unproblematic when $0 \leq \Delta\tau < \beta$, because $e^{-z\Delta\tau}$ decays quickly. However, also when $-\beta < \Delta\tau \leq 0$ this is fine because the constant part in $\theta(\Delta\tau) + n_B(z)$ is then absent and $n_B(z)$ decays asymptotically more quickly than $e^{-z\Delta\tau}$ grows. Taking into account that the contour has now a clockwise orientation gives a contribution from $\mathcal{C}_1$

$$J_1 = \frac{1}{2x} \left[\theta(\Delta\tau) + n_B(x - \mu) \right] e^{-(x-\mu)\Delta\tau}.$$

On the other side, closing the contour $\mathcal{C}_2$ at large negative real z is also possible, because one can use the identity

$$1 + n_B(z) + n_B(-z) = 0,$$

to replace $\theta(\Delta\tau) + n_B(z)$ by $-[\theta(-\Delta\tau) + n_B(-z)]$ and then use similar arguments as for J_1. We find thus a contribution from $\mathcal{C}_2$

$$J_2 = \frac{1}{2x} [\theta(-\Delta\tau) + n_B(x + \mu)] e^{(x+\mu)\Delta\tau}.$$

In summary, from the different possibilities to perform the contour integration, $J = J_1 + J_2$, we obtain the formula

$$\begin{aligned} T \sum_{n=-\infty}^{\infty} \frac{e^{-i2\pi T n \Delta\tau}}{(2\pi T n - i\mu)^2 + x^2} = & \frac{1}{2x} \left[\theta(\Delta\tau) + n_B(x - \mu) \right] e^{-(x-\mu)\Delta\tau} \\ & + \frac{1}{2x} [\theta(-\Delta\tau) + n_B(x + \mu)] e^{(x+\mu)\Delta\tau}. \end{aligned}$$

This identity allows to calculate the Matsubara sums for bosons! It also gives the right result for $T \to 0$ as one may check by a direct integration. In that limit the terms proprtional to n_B simply drop out. Similar techniques involving contour integrals can be used also for the Matsubara sums appearing in perturbative diagrams or renormalization group flow equations at finite temperature.

Result for scalar boson propagator

We can now combine terms and find for the boson propagator on the Matsubara torus

$$\begin{aligned} \Delta(\Delta\tau, \Delta\mathbf{x}) = \int_{\mathbf{p}} \frac{e^{i\mathbf{p}\mathbf{x}}}{\sqrt{\mathbf{p}^2 + m^2}} \Big(& \Big[\theta(\Delta\tau) + n_B(\sqrt{\mathbf{p}^2 + m^2} - \mu) \Big] e^{-(\sqrt{\mathbf{p}^2+m^2} - \mu)\Delta\tau} \\ & + \Big[\theta(-\Delta\tau) + n_B(\sqrt{\mathbf{p}^2 + m^2} + \mu) \Big] e^{(\sqrt{\mathbf{p}^2+m^2} + \mu)\Delta\tau} \Big). \end{aligned}$$

Contour integral around imaginary axis for fermions

For fermionic Matsubara sums we need to go through similar steps. Let us start with the contour integral

$$K = \frac{1}{2\pi i} \int_{\mathcal{C}} dz \left\{ \frac{e^{-z\Delta\tau}}{-(z+\mu)^2 + x^2} \left[\theta(\Delta\tau) - n_F(z) \right] \right\}.$$

The integration contour goes again up slightlt to the right of the imaginary z axis, and down slightly to the left of it.

We use the Fermi distribution function

$$n_F(z) = \frac{1}{e^{z/T}+1}.$$

The latter has poles at $z = i2\pi T(n+\frac{1}{2})$ with residue $-T$. Closing the contour at $z = i \pm \infty$ and using the residue theorem gives

$$K = T \sum_{n=-\infty}^{\infty} \frac{e^{-i2\pi T(n+\frac{1}{2})\Delta\tau}}{(2\pi(n+\frac{1}{2})T - i\mu)^2 + x^2}.$$

This is a typical Matsubara sum appearing when working with fermions.

Contour integral around real axes for fermions

Again one can alternatively close the contour integrals around the real z axes. Many arguments are similar as in the bosonic case. A noteable difference is that we should not assume $|\mu| < x$ because this would not allow a Fermi sphere which builds up at small temperature. However, because there is no Matsubara frequency with $\omega_n = 0$, it is possible to deform the integration contours such that it goes through between the two poles in the decomposition

$$\frac{1}{-(z+\mu)^2 + x^2} = \left(-\frac{1}{z - x + \mu} + \frac{1}{z + x + \mu} \right) \frac{1}{2x}.$$

Proceeding carefully gives from $\mathcal{C}_1$ the contribution

$$K_1 = \frac{1}{2x} [\theta(\Delta\tau) - n_F(x-\mu)] e^{-(x-\mu)\Delta\tau}.$$

Similarly, from $\mathcal{C}_2$ we obtain

$$K_2 = \frac{1}{2x} [\theta(-\Delta\tau) - n_F(x+\mu)] e^{(x+\mu)\Delta\tau}.$$

In summary, using $K = K_1 + K_2$ we find a pocket formula for ferminic Matsubara sums

$$\begin{aligned} T \sum_{n=-\infty}^{\infty} \frac{e^{-i2\pi T(n+\frac{1}{2})\Delta\tau}}{(2\pi(n+\frac{1}{2})T - i\mu)^2 + x^2} =& \frac{1}{2x} [\theta(\Delta\tau) - n_F(x-\mu)] e^{-(x-\mu)\Delta\tau} \\ &+ \frac{1}{2x} [\theta(-\Delta\tau) - n_F(x+\mu)] e^{(x+\mu)\Delta\tau}. \end{aligned}$$

Using this formula, as well as convenient derivatives of it, one can perform many of the Matsubara sums needed in perturbative or renormalization group calculations.

Pressure from quantum effective action

From the quantum effective action in a Matsubara torus geometry we can directly obtain the pressure. The partition function at vanishing source, $J = 0$, is

$$Z[0] = \exp\left(\int_0^\beta d\tau \int d^{d-1}x p(T,\mu) + \text{const} \right) = \exp\left(W[0]\right) = \exp\left(-\Gamma[\Phi_{\text{eq}}]\right),$$

where Φ_{eq} is a solution to the field equation $\delta\Gamma[\Phi]/\delta\Phi = 0$. For a homogeneous solution Φ_{eq}, the effective potential is the integral of the effective potential,

$$\Gamma[\Phi_{\text{eq}}] = \int_0^\beta d\tau \int d^{d-1}x U(\Phi_{\text{eq}}, T, \mu).$$

We find that the pressure is directly related to the effective potential at its minimum,

$$p(T,\mu) = -U(T,\mu,\Phi_{\text{eq}}) + \text{const}.$$

The additive constant can be determined such that $p = 0$ at $T = \mu = 0$.

Pressure at one-loop

For a complex scalar field in the symmetric phase, the one-loop contribution to the effective potential is

$$\Delta U_{\text{1-loop}}(\Phi_{\text{eq}}, T, \mu) = T \sum_n \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \ln\left((\omega_n - i\mu)^2 + \mathbf{p}^2 + m^2\right).$$

It is useful to perform partial integration with respect to the momenta, assuming that the boundary term at $\mathbf{p}^2 \to \infty$ vanishes. This gives

$$\begin{aligned}\Delta U_{\text{1-loop}} &= -T \sum_n \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{1}{d-1} \left(\sum_{j=1}^{d-1} p_j \frac{\partial}{\partial p_j} \right) \ln\left((\omega_n - i\mu)^2 + \mathbf{p}^2 + m^2\right) \\ &= -T \sum_n \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{2\mathbf{p}^2}{d-1} \frac{1}{(\omega_n - i\mu)^2 + \mathbf{p}^2 + m^2}.\end{aligned}$$

Here we can use the formula for the Matsubara frequencies we derived previously (setting $\Delta\tau = 0$). This yields

$$\Delta U_{\text{1-loop}} = -\int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{\mathbf{p}^2}{(d-1)\sqrt{\mathbf{p}^2 + m^2}} \left[1 + n_B(\sqrt{\mathbf{p}^2 + m^2} - \mu) + n_B(\sqrt{\mathbf{p}^2 + m^2} + \mu) \right].$$

The terms in square bracket has here a nice physical interpretation. The term $1 = 1/2 + 1/2$ comes from quantum fluctuations of quantum modes for particles and anti-particles, or two real field components of the complex scalar field. The terms propotional to n_B give the contributions of thermal fluctuations or particle and anti-particle modes, repectively. For thermodynamics, the vacuum part that is independent of T and μ must be subtracted, so that the pressure is at this order the one of free particles and anti-particles,

$$p(T, \mu) = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{\mathbf{p}^2}{(d-1)\sqrt{\mathbf{p}^2 + m^2}} \left[n_B(\sqrt{\mathbf{p}^2 + m^2} - \mu) + n_B(\sqrt{\mathbf{p}^2 + m^2} + \mu) \right].$$

A similar formula holds for fermionic contributions to pressure.

Density

It is illustrative to take a derivative with respect to chemical potential to obtain the conserved density,

$$\begin{aligned} n = \frac{\partial}{\partial\mu} p &= \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \frac{\mathbf{p}^2}{(d-1)\sqrt{\mathbf{p}^2 + m^2}} \frac{\partial}{\partial\mu} \left[n_B(E - \mu) + n_B(E + \mu) \right] \\ &= \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \left(\frac{1}{(d-1)} \sum_j p_j \frac{\partial}{\partial p_j} E \right) \left(-\frac{\partial}{\partial E} n_B(E - \mu) + \frac{\partial}{\partial E} n_B(E + \mu) \right) \\ &= \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \left(-\frac{1}{d-1} \sum_j p_j \frac{\partial}{\partial p_j} \right) \left(n_B(E - \mu) - n_B(E + \mu) \right) \\ &= \int \frac{d^{d-1}p}{(2\pi)^{d-1}} \left[n_B(\sqrt{\mathbf{p}^2 + m^2} - \mu) - n_B(\sqrt{\mathbf{p}^2 + m^2} + \mu) \right]. \end{aligned}$$

We have done the partial integration backwards here. We see that the conserved particle number is indeed the difference between integrals over occupation numbers for particles and anti-particles.

Massless limit

Let us discuss the case of massless particles at vanishing chemical potential in $d = 1 + 3$ spacetime dimensions. This is a good approximation for many quantum fields in the early universe but also for quarks and gluons in relativistic heavy ion collisions at the Large Hadron Collider (LHC) at CERN.

For a single real massless relativistic field degree at vanishing chemical potential we find through a simple change of variables $x = p/T$,

$$p(T) = T^d \frac{\Omega_{d-1}}{(d-1)(2\pi)^{d-1}} \int_0^\infty dx\, x^{d-1} \frac{1}{e^x \mp 1}.$$

Here $\Omega_{d-1} = 2\pi^{\frac{d-1}{2}}/\Gamma(\frac{d-1}{2})$ is the area of the unit sphere in $d-1$ dimensions. The upper sign is for bosons, the lower sign for fermions. The remaining integral can be expressed as $\Gamma(d)\zeta(d)$ in the case of bosons, and $(1-2^{1-d})\Gamma(d)\zeta(d)$ in the case of fermions. Together this implies for N_B real massless bosons and N_F fermions in d spacetime dimensions

$$p(T) = T^d \frac{\zeta(d)\Gamma(d)}{2^{d-1}\pi^{\frac{d-1}{2}}\Gamma(\frac{d+1}{2})} \left[N_B + (1-2^{1-d})N_F\right].$$

Specialializing now to $d = 4$ one finds

$$p(T) = T^4 \frac{\pi^2}{90} \left(N_B + \frac{7}{8} N_F\right).$$

Note that a complex scalar field has $N_B = 2$, and similarly the photon field with two polarizations. A Dirac fermion has two spin states and Dirac fermions are complex (in contrast to Majorana fermions), so $N_F = 4$.

It is very easy to derive the entropy density

$$s(T) = \frac{\partial p(T)}{\partial T} = 4\frac{\pi^2}{90} \left(N_B + \frac{7}{8} N_F\right) T^3.$$

The energy density is given by

$$\epsilon = -p + sT = 3\frac{\pi^2}{90} \left(N_B + \frac{7}{8} N_F\right) T^4 = 3p.$$

In particular, the relation $p = \epsilon/3$ is a consequence of a scaling symmetry[5] that arises if the theory has no mass scale besides the temperature, which characterises the state, and shows that the velocity of sound is (in units of c)

$$c_s = \sqrt{\frac{\partial p}{\partial \epsilon}} = \frac{1}{\sqrt{3}}.$$

Re-deriving theromdynamics

Thermodynamics is usually developed in the context of classical or quantum mechanics. It is interesting to re-derive it here directly using relativistic and field theoretic concepts. This will help to clarfy what are the detailed conditions for global thermal equilibrium states and how they can be described in general coordinates.

The starting point for the derivation of thermodynamics are conservation laws, which for a quantum field theory are best stated in local form. The conservation of energy and momentum is

[5] For more on scaling and conformal symmetry, see the Lectures on Symmetries.

a consequence of diffeomorphism symmetry if the theory is formulated in general coordinates with Riemannian metric $g_{\mu\nu}(x)$,

$$\nabla_\mu T^{\mu\nu}(x) = 0.$$

In addition the theory may exhibit a U(1) symmetry leading to a covariantly conserved particle number current,

$$\nabla_\mu N^\mu(x) = 0.$$

One should read here $T^{\mu\nu}(x)$ and $N^\mu(x)$ as beging the expectation values with respect to some quantum state.

Furthermore, one also introduces an entropy current $s^\mu(x)$. In a phenomenological approach it is postulated to be governed by a local form of the second law

$$\nabla_\mu s^\mu(x) \geq 0,$$

where equality is reached in global thermal equilibrium. Unlike the two former equations the local second law does not follow from symmetry considerations and needs a more careful justification. Moreover, it is not clear whether a local entropy current is well-defined in out-of-equilibrium situations.

One should note that the above equations could be supplemented by additional conservation laws or equations for additional order parameters $\Phi(t)$, which we assume here to vanish for simplicity.

Vanishing entropy production

With the conservation relations as well as the local form of the second law one can discuss relativistic thermodynamics. Thermal equilibrium states are supposed to be fully specified by the energy-momentum tensor and conserved particle number expectation values. Accordingly, one can assume the entropy current to be a function of the conserved energy-momentum tensor and particle current $s^\mu(T^{\lambda\nu}, N^\sigma)$, and write the second law

$$\nabla_\mu s^\mu = \frac{\partial s^\mu}{\partial T^{\lambda\nu}} \nabla_\mu T^{\lambda\nu} + \frac{\partial s^\mu}{\partial N^\sigma} \nabla_\mu N^\sigma \geq 0.$$

Because this should reduce to an equality in thermal equilibrium as a consequence of the two covariant conservation laws for energy-momentum and particle number, one should have

$$\frac{\partial s^\mu}{\partial T^{\lambda\nu}} = -\beta_\nu \, \delta^\mu_\lambda, \qquad \frac{\partial s^\mu}{\partial N^\sigma} = -\alpha \, \delta^\mu_\sigma,$$

or, in other words, the differential shoud be

$$ds^\mu = -\beta_\nu dT^{\mu\nu} - \alpha dN^\mu.$$

Here

$$\beta^\nu(x) = \frac{u^\nu(x)}{T(x)}$$

is a vector field corresponding to the ratio of fluid velocity and temperature. The fluid velocity is a time-like vector field normalized with the metric to $g_{\mu\nu}(x)u^\mu(x)u^\nu(x) = -1$, and points in time-direction, in the fluid rest frame.

Similarly,

$$\alpha(x) = \frac{\mu(x)}{T(x)}$$

is a scalar field corresponding to the ratio of chemical potential and temperature. One should understand these relations as definitions, very similar as the standard definitions of temperature and chemical potential from the differential of entropy $S(E, N)$ in the microcanonical ensemble,

$$dS = \frac{1}{T} dE - \frac{\mu}{T} dN.$$

Stationary entropy production

The divergence of the entropy current $\nabla_\mu s^\mu$ is non-negative. Accordingly, it must not only vanish in thermal equilibrium, but also be stationary. One finds for its differential

$$d(\nabla_\mu s^\mu) = \nabla_\mu ds^\mu = -(\nabla_\mu \beta_\nu) dT^{\mu\nu} - (\partial_\mu \alpha)\, dN^\mu = 0,$$

which leads to the condition that $\beta^\nu(x)$ must be a Killing vector field and α a constant,

$$\nabla_\mu \beta_\nu(x) + \nabla_\nu \beta_\mu(x) = 0, \qquad \partial_\mu \alpha(x) = 0.$$

These are two conditions for global thermal equilibrium states, and they hold in any coordinate system. In Minkoski space, a simple possibility is for example that the fluid velocity u^μ, temperature T and chemical potential μ are all constant.

A time-like Killing vector does not exist for all spacetimes, the condition is equivalent to the spacetime being stationary. Evolving spacetimes as they are needed in cosmology do not have any time-like Killing vectors, for example, and accordingly no equilibrium states.

In terms of the fluid velocity u^μ, and the projector orthogonal to the fluid velocity $\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$, one can decompose the Killing vector equilibrium condition as a set of identities,

$$\begin{aligned} u^\mu \partial_\mu T =& 0, \\ T \Delta^\mu{}_\rho u^\nu \nabla_\nu u^\rho + \Delta^{\mu\rho} \partial_\rho T =& 0, \\ \sigma^{\mu\nu} = \frac{1}{2}\left[\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma} - \frac{1}{d-1}\Delta^{\mu\nu}\Delta^{\rho\sigma}\right] \nabla_\rho u_\sigma =& 0, \\ \nabla_\mu u^\mu =& 0. \end{aligned}$$

We introduced here $\sigma^{\mu\nu}$ as a combination of derivatives of the fluid velocity that is symmetric, trace-less, and orthogonal to the fluid velocity. In deriving these equations we used the identity $u_\rho \nabla_\mu u^\rho = 0$, which follows from the normalization condition $u_\rho u^\rho = -1$.

Global equilibrium states

We can now formulate what are global thermal equilibrium states in a relativistic quantum field theory. Global states can be defined as density matrices or density matrix functionals $\rho[\phi_1, \phi_2]$ on Cauchy hypersurfaces Σ of spacetime. In the grand canonical ensemble they are given by

$$\begin{aligned} \rho =& \frac{1}{Z} \exp\left(\int d\Sigma_\mu \left\{ -\beta_\nu(x) T^{\mu\nu}(x) - \alpha(x) N^\mu(x) \right\} \right) \\ =& \exp\left(\int d\Sigma_\mu \left\{ -\beta_\nu(x) T^{\mu\nu}(x) - \alpha(x) N^\mu(x) + \beta^\mu(x) p(x) \right\} \right) \end{aligned}$$

We use here the surface element $d\Sigma_\mu = \sqrt{g} n_\mu(x) d^{d-1}x = \sqrt{g} \epsilon_{\mu\alpha\beta\gamma} dx^\alpha dx^\beta dx^\gamma / 3!$, with the last equation holding for $d-1 = 3$. The time-like normal vector $n^\mu(x)$ is assumed to be oriented towards the future direction, which together with our choice of metric signature $(-,+,+,+)$ explains the minus signs.

In the second equation we wrote the partition function for the thermal state as

$$Z = \exp\left(- \int d\Sigma_\mu \beta^\mu(x) p(x) \right),$$

which generalizes our previous expression $Z = \exp(pV/T)$ on a constant time hypersurface of Minkowski space.

Similar to what we have discussed before, one can understand the density matrix as an evolution operatore from points x^μ on the hypersurface Σ to $x^\mu - i\beta^\mu(x)$. The chemical potential term can be conveniently rewritten in terms of an external gauge field $A_\rho(x) = \mu(x)u_\rho(x)$. The density matrix functional becomes

$$\rho[\phi_1, \phi_2] = \frac{1}{Z} \int\limits_{\phi_1,\phi_2} D\phi \exp\left(-S[\phi]\right),$$

where $\phi(x) = \phi_1(x)$ and $\phi(x - i\beta) = \phi_2(x)$ is kept fixed at the boundary, and $S[\phi]$ is the action with Euclidean conventions in a space with Euclidean metric $g_{\mu\nu}(x) + 2u_\mu(x)u_\nu(x)$.

Action for equilibrated fluid in general coordinates

It is usful to express the action of a fluid in thermal equilibrium in terms of general coordinates. For vanishing field expectation values we have an action that still depends on the matric $g_{\mu\nu}(x)$ and the external gauge field $A_\mu(x)$,

$$\Gamma[g, A] = -\int d^dx\sqrt{g}\,\{p(T,\mu) + \text{const}\}\,.$$

This follows by simple analytic continuation of the action in Matsubara space to real times. Thermal equilibrium states are fixed by the periodicity condition

$$\chi(x^\mu) = \pm\chi(x^\mu - i\beta^\mu(x)),$$

The temperature can thus be written as

$$T(x) = \frac{1}{\sqrt{-g_{\mu\nu}(x)\beta^\mu(x)\beta^\nu(x)}},$$

and the chemical potential in terms of $A_\rho(x) = \mu(x)u_\rho(x)$ as

$$\mu(x) = \frac{-A_\rho(x)\beta^\rho(x)}{\sqrt{-g_{\mu\nu}(x)\beta^\mu(x)\beta^\nu(x)}}.$$

We have intentionally expressed everything in terms of the metric $g_{\mu\nu}(x)$, the external gauge field $A_\rho(x)$ and the vector field $\beta^\mu(x)$, because we can then easily vary the action with respect to the metric and the gauge field to obtain the energy-momentum tensor and conserved number current. The inverse temperature vector $\beta^\mu(x)$, that defines the periodicity condition of fields, is kept fixed in this variantions.

Energy-momentum tensor and current from action

From the quantum effective one can get the energy-momentum tensor and conserved number current through the variations

$$\delta\Gamma = \int d^dx\sqrt{g}\left\{-\frac{1}{2}T^{\mu\nu}(x)\delta g_{\mu\nu}(x) + N^\mu(x)\delta A_\mu(x)\right\}.$$

where the action is here defined such that the effective potential apears with positive sign, and the pressure accordingly with negative sign. To do the variations, recall $g = -\det(g_{\mu\nu})$ and

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu},$$

such that

$$\begin{aligned} T^{\rho\sigma} =& p\, g^{\rho\sigma} + 2\frac{\partial p}{\partial T}\frac{\partial T}{\partial g_{\rho\sigma}} + 2\frac{\partial p}{\partial \mu}\frac{\partial \mu}{\partial g_{\rho\sigma}} \\ =& p\, g^{\rho\sigma} + sT^3\beta^\rho\beta^\sigma + n\mu T^2\beta^\rho\beta^\sigma \\ =& p\, g^{\rho\sigma} + (\epsilon + p)u^\rho u^\sigma. \end{aligned}$$

We have used here thermodynamic identities such as $dp = sdT + nd\mu$ and $\epsilon + p = sT + \mu n$. The resulting energy-momentum tensor is indeed of the expected form for a thermal equilibrium state.

In a similar way we find the conserved current

$$N^\rho = -\frac{\partial p}{\partial \mu}\frac{\partial \mu}{\partial A_\rho} = nT\beta^\rho = nu^\rho.$$

This is also of the form expected.

Ideal fluid dynamics

In thermal equilibrium, the energy-momentum tensor is fixed by the fluid velocity, or the frame where the fluid is at rest, and two thermodynamic variables which can be T and μ or ϵ and n for example. The idea behind ideal fluid dynamics is to postulate that thermal equilibrium states can be made local, such that the fluid velocity $u^\mu(x)$, the energy density $\epsilon(x)$ and particle number density $n(x)$ can become general functions of space and time but that the energy-momentum tensor and number current still have locally the same form as in global equilibrium. It is not guarantied that this works, because terms proportional to gradients of the fluid fields emerge in general out-of-global equilibrium. In this sense, ideal fluid dynamics can only be the leading order of a derivative expansion.

Assuming now that the global equilibrium expressions also hold at local thermal equilibrium, we find for the conservation of energy and momentum

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu \left((\epsilon + p)u^\mu u^\nu + pg^{\mu\nu}\right) = 0.$$

This equation can be contracted with u_ν, which yields

$$u^\mu \partial_\mu \epsilon + (\epsilon + p)\nabla_\mu u^\mu = 0,$$

and it can be contracted with the projector orthogonal to the fluid velocity $\Delta^\rho{}_\sigma = u^\rho u_\sigma + \delta^\rho{}_\sigma$, which yields

$$(\epsilon + p)u^\mu \nabla_\mu u^\rho + \Delta^{\rho\sigma}\partial_\sigma p = 0.$$

These two equations can be seen as spefiying the time evolution of energy density and fluid velocity, respectively. They get supplemented by the conservation law for particle number,

$$u^\mu \partial_\mu n + n\nabla_\mu u^\mu = 0,$$

which can be seen as a time evolution equation for the density n.

Let us emphasize again that ideal fluid dynamics in only an approximation and higher order corrections in a derivative expansion are needed when one goes further away from global equilibrium.

Observables at equal and un-equal times

With the thermal equilibrium formalism one can directly calculate observables such as expectation values or correlation functions on a single Cauchy hypersurface Σ, for example at some time t_0. Formally this is done by finding the corresponding operator $A_\Sigma[\phi_1, \phi_2]$ and its expectation value

$$\langle A \rangle = \mathrm{Tr}\left\{\rho A_\Sigma\right\} = \int D\phi D\phi' \rho[\phi, \phi'] A_\Sigma[\phi', \phi].$$

For a global equilibrium state, the state is actually time translation invariant in the direction of the Killing vector field $\beta^\mu(x)$, so the hypersurface Σ could be placed at different times.

In addition to this, it is also possible to calculate expectation values and correlation functions at different times. For this, one needed to supplement the operators with pieces for time evolution - forwards and backwards. This leads to a functional integral with braches for forward and backward time evolution, the Schwinger-Keldysh double time path, and it is also the idea underlying the Heisenberg picture of quantum theory, which we shall use in the following.

10 Linear response theory

Two-point function for thermal equilibrium states

We will now discuss two-point correlation functions for a quantum field theory in thermal equilibrium. We work here in Minkowski space with cartesian coordinates where $\beta^\mu(x) = \beta^\mu$ is constant. One can also find a Lorentz frame, the fluid rest frame, where $\beta^\mu = (1/T, 0, 0, 0)$, but we largely keep the frame general.

Let us start from the correlation function of two fields $\chi_a(x)$ and $\chi_b(y)$ which might be elementary fields but could as well be composite operators such as pairing fields or for example energy density, particle density or similar,

$$\Delta^+_{ab}(x-y) = \langle \chi_a(x)\chi_b(y)\rangle = \text{tr}\left\{\rho\, \chi_a(x)\chi_b(y)\right\}.$$

We can allow $\chi_a(x)$ and $\chi_b(x)$ to be both bosonic or fermionic. The mixed case can be excluded because a mixed correlation function needs to vanish. The indices a and b are also used to label representations of the Lorentz group. For example they could label spinor components for the Dirac field of vector components for currents etc.

Note that the fields are not necessarily time-ordered here. When x and y are at different times, we need to insert convenient time evolution operators to evaluate the trace on the Cauchy surface where ρ is defined. We use here the notation of the operator formalism, but one could of course work in a functional representation.

We take the density matrix to correspond to a thermal equilibrium with temperature T and fluid velocity u^μ (a possible chemical potential can be included as an external gauge field),

$$\rho = \frac{1}{Z} \exp\left(P_\mu \beta^\mu\right),$$

where $P^\mu = (H, \mathbf{P})$ is a four-momentum operator, which we could write as a Cauchy surface integral involving the energy-momentum tensor.

The fields $\chi_a(x)$ and $\chi_b(y)$ have the Heisenberg representation

$$\chi(x) = \exp(-iP_\mu x^\mu)\chi(0)\exp(iP_\mu x^\mu),$$

which also involves the four-momentum operator.

Using complete basis of momentum eigenstates

Introduce now a complete set of states which are eigenstates of the four-momentum operator,

$$P^\mu |m\rangle = p^\mu_m |m\rangle.$$

The energies are non-negative, $p^0 \geq 0$. The normalization is taken to be such that

$$\mathbb{1} = \sum_m |m\rangle\langle m|, \qquad \langle m|n\rangle = \delta_{mn}.$$

The thermal density matrix becomes

$$\rho = \frac{1}{Z}\sum_m e^{p_m \beta}|m\rangle\langle m|. \tag{10.1}$$

For simplicity, we use a notation as for discrete states, but in reality there will also be a continuum of states $|m\rangle$. One obtains

$$\begin{aligned}\Delta^+_{ab}(x-y) &= \sum_{m,l}\frac{1}{Z}e^{p_m\beta}\langle m|e^{-iPx}\chi_a(0)e^{iPx}|l\rangle\langle l|e^{-iPy}\chi_b(0)e^{iPy}|m\rangle\\ &= \sum_{m,l}\frac{1}{Z}e^{p_m\beta}e^{i(p_l-p_m)(x-y)}\langle m|\chi_a(0)|l\rangle\langle l|\chi_b(0)|m\rangle.\end{aligned}$$

We introduce also the momentum space representation

$$\Delta^{+}_{ab}(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \; \Delta^{+}_{ab}(p),$$

and find by Fourier transform

$$\Delta^{+}_{ab}(p) = \sum_{m,l} \delta^{(4)}(p - p_l + p_m) \frac{1}{Z} e^{p_m \beta} \langle m|\chi_a(0)|l\rangle\langle l|\chi_b(0)|m\rangle.$$

One can see this as the probability amplitude for the process

$$p_m \xrightarrow{\chi_b} p_m + p \xrightarrow{\chi_a} p_m.$$

Starting from a state with thermal occupation and energy-momentum p_m, $\chi_b(0)$ mediates a transition to a state with energy $p_l = p_m + p$ and $\chi_a(0)$ mediates a transition back to the original state.

Positive and negative energies

The sum over m contains a Boltzmann weight where energies that are large compared to the temperature are exponentially suppressed. For $T \to 0$ only the ground state with $p_m = 0$ survives in the sum over m and one has

$$\Delta^{+}_{ab}(p) = \sum_{l} \delta^{(4)}(p - p_l)\langle 0|\chi_a(0)|l\rangle\langle l|\chi_b(0)|0\rangle \qquad\qquad (\text{for } T \to 0).$$

One observes that there is only a contribution with $p \neq 0$ when $\chi_b(0)$ creates an excitation with momentum p which is subsequently destroyed by $\chi_a(0)$. The energy p_0 must be above the energy of the vacuum state and accordingly $\Delta^{+}_{ab}(p)$ only has support for $p^0 \geq 0$ and it is then a transition amplitude. This is the reason for labeling it with a plus.

For non-vanishing temperature, the part of $\Delta^{+}_{ab}(p)$ at negative frequency $p^0 < 0$ corresponds to a transition from a positive energy state with thermal occupation to one with smaller energy induced by $\chi_b(0)$ before the energy is recovered again through $\chi_a(0)$.

Reversed argument correlation function and detailed balance

Using analogous steps, the correlation function with reversed field arguments,

$$\Delta^{+}_{ba}(y-x) = \langle \chi_b(y)\chi_a(x)\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \Delta^{+}_{ba}(-p),$$

can be written as

$$\Delta^{+}_{ba}(-p) = \sum_{m,l} \delta^{(4)}(p - p_m + p_l) \frac{1}{Z} e^{p_m \beta} \langle m|\chi_b(0)|l\rangle\langle l|\chi_a(0)|m\rangle.$$

One can see this as the probability amplitude for the process

$$p_m \xrightarrow{\chi_a} p_m - p \xrightarrow{\chi_b} p_m.$$

Starting from a state with thermal occupation and energy-momentum p_m, $\chi_a(0)$ mediates a transition to a state with energy $p_l = p_m - p$ and $\chi_b(0)$ mediates a transition back to the original state. This is the reverse process to the oe described by $\Delta^{+}_{ab}(p)$. In thermal equilibrium there must be detailed balance between all the intermediate processes. In other words, the rates, which are transition probabilities or squared transition amplitudes times Boltzmann weights, for forward and backward processes, must be equal. This intuitively explains the identity

$$\Delta^{+}_{ba}(-p) = \exp(p\beta)\Delta^{+}_{ab}(p),$$

which can be derived easily from the explicit expressions.

Spectral and statistical correlation functions

Define now also the spectral and statistical correlation functions by

$$\begin{aligned}\Delta^{\rho}_{ab}(x-y) =& \langle [\chi_a(x), \chi_b(y)]_{\mp} \rangle = \mathrm{tr}\left\{\rho\, [\chi_a(x), \chi_b(y)]_{\mp}\right\} = \int_p e^{ip(x-y)} \Delta^{\rho}_{ab}(p), \\ \Delta^{S}_{ab}(x-y) =& \frac{1}{2} \langle [\chi_a(x), \chi_b(y)]_{\pm} \rangle = \frac{1}{2}\mathrm{tr}\left\{\rho\, [\chi_a(x), \chi_b(y)]_{\pm}\right\} = \int_p e^{ip(x-y)} \Delta^{S}_{ab}(p).\end{aligned}$$

For bosonic fields the spectral function ρ_{ab} involves the commutator (upper sign), for fermionic fields the anti-commutator (lower sign). For the statistical propagator it is the opposite.

Intuitively, the statistical correlation function carries information about fluctuations, for example it could be evaluated at equal time to yield the correlation function in the thermal equilibrium state. In contrast, the spectral function contains information about propgation in time and response to perturbations (see below).

One can express the spectral correlation function in terms of $\Delta^{+}_{ab}(p)$,

$$\Delta^{\rho}_{ab}(p) = \Delta^{+}_{ab}(p) \mp \Delta^{+}_{ba}(-p) = (1 \mp \exp(p\beta))\, \Delta^{+}_{ab}(p),$$

and similarly the statistical correlation function,

$$\Delta^{S}_{ab}(p) = \frac{1}{2}\left[\Delta^{+}_{ab}(p) \pm \Delta^{+}_{ba}(-p)\right] = \frac{1}{2}\,(1 \pm \exp(p\beta))\, \Delta^{+}_{ab}(p).$$

Fluctuation-dissipation relation

Comparing the two expressions for the spectral and statistical functions yields the important fluctuation-dissipation relation (H. B. Callen and T. A. Welton, 1951)

$$\Delta^{S}_{ab}(p) = \left[\frac{1}{2} \pm n_{B/F}(\omega)\right] \Delta^{\rho}_{ab}(p),$$

where $\omega = -u_\mu p^\mu$ is the frequency in the fluid rest frame, and the Bose and Fermi occupation number functions introduced previously

$$n_{B/F}(\omega) = \frac{1}{e^{\omega/T} \mp 1}.$$

Note that the square bracket in the fluctuation dissipation relation is anti-symmetric under $p^\nu \to -p^\nu$, as was shown previously.

The fluctuation-dissipation theorem holds in thermal equilibrium only, and it corresponds to the statement of detailed balance between forward and backward processes. More generally, the statistical and spectral correlation functions are not related in a simple way. In the vacuum state at zero temperature only the quantum fluctuations survive, corresponding to the term $1/2$ in the square bracket. In the high-temperature or classical limit, the square bracket becomes in the bosonic case T/ω.

From the definitions one can also obtain the expression for the spectral function

$$\Delta^{\rho}_{ab}(p) = \sum_{m,l} \delta^{(4)}(p - p_l + p_m) \frac{1}{Z} \left(e^{p_m\beta} \mp e^{p_l\beta}\right) \langle m|\chi_a(0)|l\rangle\langle l|\chi_b(0)|m\rangle.$$

For the statistical correlation function one has a similar expression, but with an additional factor $1/2$ and the opposite sign between the two Boltzmann weights in the round bracket. For bosonic fields this implies that the spectral function must vanish, $\Delta^{\rho}_{ab}(p) \to 0$, in the zero frequency limit $p\beta \to 0$, because the Dirac distribution implies $p_l\beta = p_m\beta$ then. For fermionic fields a similar statement holds for the statistical correlation function $\Delta^{S}_{ab}(p)$.

More Greens functions

It is useful to define also the Feynman, retarded and advanced propagators through the equations

$$
\begin{aligned}
-i\Delta^F_{ab}(x-y) =&\langle T\,\chi_a(x)\chi_b(y)\rangle = \theta(x^0-y^0)\langle\chi_a(x)\chi_b(y)\rangle \pm \theta(y^0-x^0)\langle\chi_b(y)\chi_a(x)\rangle,\\
-i\Delta^R_{ab}(x-y) =&\theta(x^0-y^0)\langle[\chi_a(x),\chi_b(y)]_\mp\rangle,\\
-i\Delta^A_{ab}(x-y) =& -\theta(y^0-x^0)\langle[\chi_a(x),\chi_b(y)]_\mp\rangle,
\end{aligned}
$$

with corresponding momentum space representations $\Delta^F_{ab}(p)$, $\Delta^R_{ab}(p)$ and $\Delta^A_{ab}(p)$. Typically these are Greens functions to some inverse propagator which is a combination of derivative operators, and they differ through their boundary conditions. While the Feynman propagator is time-ordered, the retarded propagator is only non-vanishing for $x^0 > y^0$ and the advanced propagator for $y^0 > x^0$.

From the Feynman propagator one obtains via analytic continuation through Wick rotation of the frequency axis the Matsubara propagator. In momentum space, and for simplicity in the fluid rest frame,

$$\Delta^M(i\omega_n, \mathbf{p}) = \Delta^F(\omega = i\omega_n, \mathbf{p}).$$

Useful relations

The following relations between retarded and advanced functions follow directly from the definitions

$$\Delta^R_{ab}(x-y) = \pm\Delta^A_{ba}(y-x), \qquad \Delta^A_{ab}(x-y) = \pm\Delta^R_{ba}(y-x),$$

or, in momentum space,

$$\Delta^R_{ab}(p) = \pm\Delta^A_{ba}(-p), \qquad \Delta^A_{ab}(p) = \pm\Delta^R_{ba}(-p).$$

If one known the retarded function one also knows the advanced one, and vice versa.

The spectral correlation function can be written in terms of the difference of retarded and advanced Greens functions,

$$\Delta^\rho_{ab}(p) = -i\Delta^R_{ab}(p) + i\Delta^A_{ab}(p),$$

an identity that also follows directly from the definitions. Finally, the statistical correlation function can be obtained from this via the fluctuation-dissipation relation.

Complex argument Greens function and spectral representation

For simplicity we specialize now to the fluid rest frame. For homogeneous equilibrium states (which includes the vacuum as a special case), the spectral function $\Delta^\rho_{ab}(\omega, \mathbf{p})$ plays a special role. It determines all other correlation functions through an integral representation.

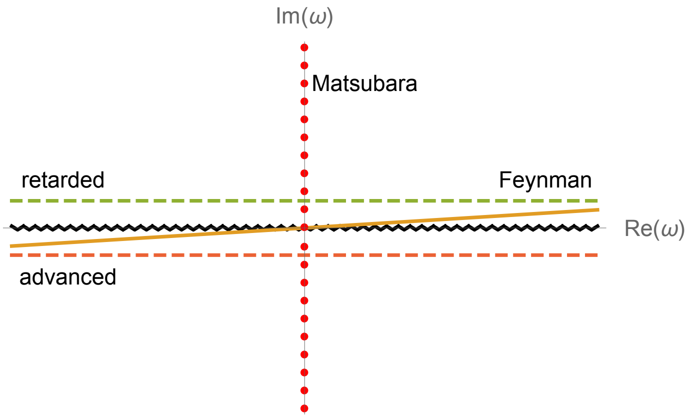
We define the complex argument Greens function by the integral over the spectral function

$$G_{ab}(\omega, \mathbf{p}) = \int_{-\infty}^{\infty} \frac{dz}{2\pi}\, \Delta^\rho_{ab}(z, \mathbf{p}) \frac{1}{z-\omega}.$$

The integral over z is along the real axis.

The function $G_{ab}(\omega, \mathbf{p})$ can be evaluated for complex frequency argument $\omega \in \mathbb{C}$. It has a brach cut or poles along the real ω axis, but, importantly, nowhere else! This follows from the integral relation above, which is known as the spectral representation.

One can show that one obtains the Feynman, retarded, advanced and Matsubara Greens functions by evaluating G on the contours close to the real ω axis, but shifted slightly away it.



Specifically, the retarded Greens function is obtained by evaluating $G_{ab}(\omega, \mathbf{p})$ slightly above the real frequency axis. It has then by construction on poles or brach cuts in the upper half of the complex frequency plane, which leads to the right causality structure when one goes back to real space. Similarly the advanced Greens function is obtained by evaluating $G_{ab}(\omega, \mathbf{p})$ slightly below the real frequency axis, and the Feynman propagator by evaluating $G_{ab}(\omega, \mathbf{p})$ below the real frequency argument when $\text{Re}(\omega)$ is negative, and above it when $\text{Re}(\omega)$ is positive. This is equivalent to the usual $i\epsilon$ prescription.

In terms of formula,

$$\begin{aligned}
\Delta^R_{ab}(p) =& G_{ab}\left(\omega + i\epsilon, \mathbf{p}\right), \\
\Delta^A_{ab}(p) =& G_{ab}\left(\omega - i\epsilon, \mathbf{p}\right), \\
\Delta^F_{ab}(p) =& G_{ab}\left(\omega + i\epsilon\,\text{sign}(\omega), \mathbf{p}\right), \\
\Delta^M_{ab}(p) =& G_{ab}\left(i\omega_n, \mathbf{p}\right).
\end{aligned}$$

Using the identity

$$\frac{1}{x \pm i\epsilon} = \mp i\pi\delta(x) + \text{P.V.}\ \frac{1}{x},$$

where the second term is the Cauchy principal value, one can see that the difference between retarded and advanced Greens functions leads indeed back to the spectral density.

Implications of symmetries

Besides unitarity and translational symmetries one can also make use of the Lorentz group in order to constrain the spectral function $\Delta^\rho_{ab}(z, \mathbf{p})$. For example, when a and b are indices for scalar fields we infer immediately that $\Delta^\rho_{ab}(z, \mathbf{p})$ must be a scalar under Lorentz transformations and can depend on $\mathbf{p}$ trough its magnitude or $\mathbf{p}^2$, only. When a and b correspond to spinor, vector or tensor fields it is best to decompose them into irreducibles with respect to the rotation group to find a convenient form for the spectral function.

For the vacuum state at vanishing temperature and density, $T = \mu = 0$, we are dealing with a Lorentz invariant situation. Instead of using a spectral representation with an integral over a frequency it is then much more elegant to write it as in integral over a mass parameter. The latter is a singlet under the Lorentz group, while z is only a singlet under the part of it (rotations) that remains unbroken in the fluid rest frame.

Källén-Lehmann spectral representation in vacuum

In vacuum Lorentz symmetry implies that $\Delta^\rho_{ab}(\omega, \mathbf{p})$ can, for a and b labeling scalar fields, only depend on $-\omega^2 + \mathbf{p}^2$ and $\text{sign}(\omega)$. For spinor, vector or tensor fields the analysis is slightly more involved but by decomposing the fields into irreducible representations one can eventually arrive at expressions very similar to the scalar case.

Concentrating on vaccum fields, one can write the complex argument Greens function as an integral over a mass squared parameter,

$$\begin{aligned} G_{ab}(p) &= \int_0^\infty d\mu^2\, \rho_{ab}(\mu^2)\, \frac{1}{p^2+\mu^2} \\ &= \int_0^\infty d\mu^2\, \rho_{ab}(\mu^2) \left[\frac{1}{\sqrt{\mathbf{p}^2+\mu^2}-\omega} - \frac{1}{-\sqrt{\mathbf{p}^2+\mu^2}-\omega} \right] \frac{1}{2\sqrt{\mathbf{p^2}+\boldsymbol{\mu}^\mathbf{2}}}, \end{aligned}$$

which is known as the Källén-Lehmann representation.

Performing the variable substitution $z = \sqrt{\mathbf{p}^2+\mu^2}$ one can rewrite this as

$$G_{ab}(p) = \int_0^\infty dz\, \rho_{ab}(z^2-\mathbf{p}^2) \left[\frac{1}{z-\omega} - \frac{1}{-z-\omega} \right].$$

We can write this as in integral over z in the range $(-\infty, \infty)$ in the form discussed previously when we identify

$$\Delta^\rho_{ab}(\omega, \mathbf{p}) = 2\pi\, \mathrm{sign}(\omega)\, \rho_{ab}(\omega^2-\mathbf{p}^2).$$

As before, one can immediatley obtain the retarded, advanced, Feynman or Matsubara propagator by specializing to the approriate frequency domain.

Complex conjugate fields

So far our discussion was very general and we did not assume much about the fields for which we considered the correlation functions. For special cases one can make further going statements. In particular for two complex conjugate fields, $\chi_a(x) = \varphi(x)$ and $\chi_b(y) = \varphi^*(y)$, we find

$$\Delta^\rho_{ab}(p) = \sum_{m,l} \delta^{(4)}(p - p_l + p_m) \frac{1}{Z} \left(e^{p_m \beta} \mp e^{p_l \beta} \right) |\langle m|\varphi(0)|l\rangle|^2,$$

which clearly shows that the spectral density must be real,

$$\Delta^\rho_{ab}(p) \in \mathbb{R}.$$

In a similar way one can also find reality constraints for spinor, vector or tensor fields.

Sum rules

If one field is the canonical conjugate momentum field of the other, e. g. $\chi_a(x) = \varphi(x)$ and $\chi_b(y) = -i\Pi(y)$, they fulfill a canonical commutation relation at equal time,

$$[\varphi(t, \mathbf{x}), \Pi(t, \mathbf{y})]_\mp = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

or

$$[\chi_a(t, \mathbf{x}), \chi_b(t, \mathbf{y})]_\mp = \delta^{(3)}(\mathbf{x}-\mathbf{y}).$$

This impies for the spectral function

$$\Delta^\rho_{ab}(0, \mathbf{x}-\mathbf{y}) = \langle [\chi_a(t, \mathbf{x}), \chi_b(t, \mathbf{y})]_\mp \rangle = \int \frac{d\omega}{2\pi} \int_\mathbf{p} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \Delta^\rho_{ab}(p) = \delta^{(3)}(\mathbf{x}-\mathbf{y}).$$

This only works when

$$\int_{-\infty}^\infty \frac{dz}{2\pi} \Delta^\rho_{ab}(z, \mathbf{p}) = 1,$$

for any value of $\mathbf{p}$, which is called a sum rule. It is a direct consequence of the non-perturbative spectral representation and the canonical commutation relation.

Remarks

1. Our discussion did not rely on perturbation theory and the spectral representation is a non-perturbative statement.

2. We did use unitarity at the fundamental level, to have a Heisenberg representation.

3. We also used translation invariance in time, which is closely related to the assumption of a thermal equilibrium or vacuum state. Translation invariance in space has also been used but could have been avoided.

4. Assuming real spectral function one finds

$$\Delta^{\rho}_{ab}(\omega, \mathbf{p}) = 2\,\mathrm{Im}\left(\Delta^{R}_{ab}(p)\right) = -2\,\mathrm{Im}\left(\Delta^{A}_{ab}(p)\right) = 2\,\mathrm{sign}(\omega)\,\mathrm{Im}\left(\Delta^{F}_{ab}(p)\right).$$

5. For a free scalar field with mass m one has simply

$$\Delta^{\rho}(p) = 2\,\mathrm{sign}(\omega)\,\mathrm{Im}\left(\frac{1}{p^2+m^2-i\epsilon}\right) = 2\pi\,\mathrm{sign}(\omega)\,\delta(p^2+m^2).$$

   More generally, stable particles correspond to Dirac peaks in the spectral function.

6. For unstable resonances the peak in the spectral function is broadened. For example, in the Breit-Wigner parametrization it becomes

$$\Delta^{\rho}(p) = 2\,\mathrm{sign}(\omega)\mathrm{Im}\left(\frac{1}{p^2+m^2-im\Gamma}\right) = 2\,\mathrm{sign}(\omega)\frac{m\Gamma}{(p^2+m^2)^2+m^2\Gamma^2}.$$

   The decay width parameter Γ has the interpretation of an inverse life time. The Breit-Wigner parametrization is not fully realistic, however, because Δ^{ρ} does not vanish for $\omega \to 0$.

7. The spectral representation constrains strongly the analytic structure of two-point functions. This can be very useful for non-perturbative investigations of quantum field theory.

8. The spectral representation is for two-point functions, but can also involve composite operators. In this sense it constrains also particular limits of higher-order correlation functions.

9. Using similar concepts (unitarity, analytic continuation in momenta) one can also investigate the analytic structure higher order correlation functions. This is known as the analytic S-matrix programm.

Analytic structure of inverse propagator

The function $G_{ab}(p)$ is obtained via analytic continuation from the second functional derivative of the Schwinger functional $W[J]$, if the latter is a priori defined in the Euclidean domain. Similarly, its inverse

$$P_{ab}(p) = G^{-1}_{ab}(p),$$

is obtained from the analytic continuation of the second functional derivative of the effective action $\Gamma[\Phi]$. Similar to $G_{ab}(p)$, the function $P_{ab}(p)$ (or more specific its eigenvalues) might have brach cuts and zero-crossings along the axis of real frequency ω but nowhere else.

One can decompose the inverse complex-argument propagator

$$P_{ab}(p) = P_{1,ab}(p) - is_{\mathrm{I}}(\omega)\,P_{2,ab}(p),$$

where $s_{\mathrm{I}}(\omega) = \mathrm{sign}(\mathrm{Im}(\omega))$. Both functions $P_{1,ab}(p)$ and $P_{2,ab}(p)$ are regular when crossing the real frequency axis. However, the sign $s_{\mathrm{I}}(\omega)$ changes, which leads to a brach cut for the function $P_{ab}(p)$.

The term $P_{2,ab}(p)$ parametrizes the strength of the branch cut. Going back to real space one has $\omega \to i\partial_t$ and

$$s_\mathrm{I}(\omega) = \mathrm{sign}(\mathrm{Im}(\omega)) \to s_\mathrm{R}(\partial_t) = \mathrm{sign}(\mathrm{Re}(\partial_t)).$$

It might be surprising that such terms can arise in an effective action, but it is a consequence of the spectral representation and analytic continuation.

Damping terms

It is interesting to analyse the influence of a branch cut behaviour in a simple model for one real degree of freedom $\phi(t)$. We take the effective action to be

$$\begin{aligned}\Gamma[\phi] &= \int dt \left\{ -\frac{1}{2}\dot{\phi}(t)^2 + \frac{1}{2}m\phi(t)^2 + \zeta m\, \phi(t)\, s_\mathrm{R}(\partial_t)\dot{\phi}(t) \right\} \\ &= \int \frac{d\omega}{2\pi} \left\{ \frac{1}{2}\phi^*(\omega) \left[-\omega^2 + m^2 - 2i\, s_\mathrm{I}(\omega)\zeta m\omega \right] \phi(\omega) \right\}.\end{aligned}$$

From the frequency representation in the second line we can read off the inverse propagator,

$$P(\omega) = 1/G(\omega) = P_1(\omega) - is_\mathrm{I}(\omega)P_2(\omega) = -\omega^2 + m^2 - is_\mathrm{I}(\omega)2\zeta m\omega.$$

For $\zeta = 0$ there are two zero crossings of the inverse propagator at $\omega = \pm m$, corresponding to poles of the propagator. However, for $\zeta > 0$ there is instead a branch cut extending along the real frequency axis, except for the point $\omega = 0$ where $P_2 = 0$.

The inverse retarded propagator is obtained by eveluating this just above the real axis, i. e. for $s_\mathrm{I}(\omega) = \mathrm{sign}(\mathrm{Im}(\omega)) = 1$. This gives

$$1/\Delta^R(\omega) = -\omega^2 - 2i\zeta m\omega + m^2.$$

Interpreted as a field equation, this corresponds to

$$[\partial_t^2 + 2\zeta m\partial_t + m^2]\phi(t) = 0.$$

This is the equation of motion of a damped harmonic oscillator where ζ is known as the damping ratio! The sign of the damping term would have been opposite if we had considered the inverse advanced propagator instead. We conclude that brach cuts can be associated with dissipative behaviour.

Fluctuation-dissipation relation for damped harmonic oscillator

For the damped harmonic oscillator we find the spectral density

$$\Delta^\rho(\omega) = 2\,\mathrm{Im}\left(\frac{1}{-\omega^2 + m^2 - 2i\zeta m\omega} \right) = \frac{4\zeta m\omega}{(\omega^2 - m^2)^2 + 4\zeta^2 m^2\omega^2}.$$

Note that this vanishes indeed for $\omega \to 0$ as it should.

From the fluctuation-dissipation relation we find the statistical correlation function

$$\Delta^S(\omega) = \left[\frac{1}{2} + n_B(\omega) \right] \frac{4\zeta m\omega}{(\omega^2 - m^2)^2 + 4\zeta^2 m^2\omega^2}.$$

Exercise: Work out the corresponding correlation function as a function of time difference by Fourier transform and discuss their physical significance.

Linear response

Let us now discuss an experimental situation where we conider the reaction of an expectation value to a change in our field theory at an earlier time. When this reaction or response is small, as a result of a small change or perturbation, one can describe it by linear response theory, that we will develop now.

For concreteness, we assume that the expectation value we observe is $\langle \chi_a(x) \rangle$ where $\chi_a(x)$ is some bosonic field, that can be fundamental or composite. Typical examples are an order parameter field $\phi(x)$, the componets of an electromagnetic current $J^\mu(x)$, or of the energy-momentum tensor $T^{\mu\nu}(x)$. Without loss of generality we can assume that the expectation value vanishes in thermal equilibrium, and consider only the modification as a result of the perturbation.

Concerning the perturbation, we shall assume it is given by a change in the action

$$\Delta S[\phi] = \int d^d y \, \{ j_b(y) \chi_b(y) \} \,,$$

where $\chi_b(y)$ are also the components of some field (fundamental or composite) and the source $j_b(y)$ parametrises the strength of the perturbation. We are interested in a linear term, which can be of the form

$$\bar{\chi}_a(x) = \langle \chi_a(x) \rangle = \int d^d y \, \left\{ \Delta^R_{ab}(x,y) j_b(y) \right\} .$$

The index R is here for "response", but we will see later that $\Delta^R_{ab}(x,y)$ is in fact a retarded correlation function so that the notation is in agreement with notation introduced earlier.

The expectation value could be corrected by higher order terms in the source, but for a stable thermal equilibrium situations one can always make $j_b(y)$ small enough so that the linear term fully dominates.

Translational invariance, causality, Fourier representation

If the equilibrium state i question has translational symmetries in time and in space, the response function can be a function of the coordinate differences only, $\Delta_{ab}(x,y) = \Delta_{ab}(x-y)$. Moreover, by causality it must vanish for $y^0 > x^0$ or more generally whenever y is not in the past light cone of x.

It is convenient to introduce the Fourier representation

$$\Delta^R_{ab}(x-y) = \int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} \Delta^R_{ab}(p).$$

The causality condition implies that $\Delta_{ab}(p)$ must be an analytic function in the upper half of the complex frequency plane. One can write now

$$\bar{\chi}_a(\omega, \mathbf{p}) = \Delta^R_{ab}(\omega, \mathbf{p}) j_b(\omega, \mathbf{p}),$$

where $j_b(\omega, \mathbf{p})$ and $\bar{\chi}_a(\omega, \mathbf{p})$ are the source and signal in Fourier space, respectively. A periodic driving through the source induces a periodic signal of the same frequency. The generation of higher harmonics would correspond to non-linear response.

Gauge invariance and conservation laws

The source $j_b(y)$ is sometimes a gauge field and the responding field might be a conserved current. This has implications for the response function. For example, an electromagnetic current $J^\mu(x)$ might be induced as a response to a perturbation in the electromagnetic gauge field $A_\nu(y)$,

$$\delta J^\mu(x) = \int d^d y \, \{ \Delta^{\mu\nu}_R (x-y) \delta A_\nu(y) \} \,,$$

with electromagnetic retarded response function $\Delta_R^{\mu\nu}(x-y)$. The current should be unaffected by a gauge transformation of the form $A_\nu(y) \to A_\nu(y) + \partial_\nu \alpha(y)$, or

$$0 = \int d^d y \, \{\Delta_R^{\mu\nu}(x-y)\partial_\nu \delta\alpha(y)\} = \int d^d y \, \{\delta\alpha(y) \nabla_\nu \Delta_R^{\mu\nu}(x-y)\} \, .$$

Because $\alpha(y)$ is arbitrary this implies

$$\nabla_\nu \Delta_R^{\mu\nu}(x-y) = 0.$$

Similarly, the induced current should be conserved, $\nabla_\mu J^\mu(x) = 0$, which implies

$$\nabla_\mu \Delta_R^{\mu\nu}(x-y) = 0.$$

In a similar way one can work out implications of gauge invariance, or Ward-Takahashi identities, for many response functions.

Field theoretic discussion

Let us now investigate the response function from a quantum field theory point of view. We can write the expectation value as a functional integral along a Schwinger-Keldysh path from some initial time $t_{\text{in}} < y^0$ forwards, until the time x^0 (or later) then backwards to the initial time t_{in}, and then downwards in the complex time plane to the point $t_{\text{in}} - i\beta$ where the countour is closed.

This corresponds to

$$\langle \chi_a(x) \rangle = \frac{1}{Z} \int D\phi_+ D\phi_- D\phi_0 \left\{ \chi_a(x) e^{iS_{\text{M}}[\phi_+]} e^{-S_{\text{E}}[\phi_0]} e^{-iS^*_{\text{M}}[\phi_-]} \right\} ,$$

where $S_{\text{M}}[\phi_+]$ is a Minkoski space action for forward time evolution with the usual $i\epsilon$ terms, $S^*_{\text{M}}[\phi_-]$ its relative for backward time evolution where the sign if the $i\epsilon$ terms is reversed, and $S_{\text{E}}[\phi_0]$ is a Euclidean action that describes the thermal density matrix at time t_0 in the Matsubara formalism.

The functional integrals have to be clued with the following boundary conditions at the initial time t_0

$$\phi_+(t_0, \mathbf{x}) = \phi_0(t_0 - i\beta, \mathbf{x}), \qquad \phi_-(t_0, \mathbf{x}) = \phi_0(t_0, \mathbf{x}),$$

and at the time $t_{\text{f}} = x^0$ (or any time later that that) one must set

$$\phi_+(t_{\text{f}}, \mathbf{x}) = \phi_-(t_{\text{f}}, \mathbf{x}).$$

Taking now into account that

$$S[\phi] = S_0[\phi] + \int d^d y \, \{j_b(y) \chi_b(y)[\phi]\}$$

leads to a contribution from the expansion of the forward evolving action $S_{\text{M}}[\phi_+]$ proportional to $i\langle \chi_a(x)\chi_b(y)\rangle$ and one from the backward evolving action. Using the cyclic property of the trace it can be understood as $-i\langle \chi_b(y)\chi_a(x)\rangle$. Both contributions vanish for $y^0 > x^0$ by causality, or formally because the correspnding parts of the forward and backward evolution operators cancel. In summary we find

$$\Delta_{ab}(x-y) = i\theta(x^0 - y^0)\langle [\chi_a(x), \chi_b(y)] \rangle = \Delta_{ab}^R(x-y).$$

This is simply the retarded correlation function we have discussed previously! In other words, linear response experiments can directly access the retarded two-point correlation between the responding field $\chi_a(x)$ and the perturbation field $\chi_b(y)$ to which the source $j_b(y)$ couples.

Static and dynamic susceptibilities

Consider a source function of the form

$$j_b(t', \mathbf{y}) = \theta(-t')e^{\epsilon t'} j_b(0, \mathbf{y}).$$

It grows very slowely or "adiabatically" in time, until it is switched off at time $t' = 0$. The response is given by

$$\begin{aligned}\bar{\chi}_a(t, \mathbf{x}) &= \int_{-\infty}^{0} dt' \int \frac{d\omega}{2\pi} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} e^{-i\omega(t-t')+\epsilon t'+i\mathbf{p}(\mathbf{x}-\mathbf{y})} \Delta^R_{ab}(\omega, \mathbf{p}) j_b(0, \mathbf{y}) \\ &= \int \frac{d\omega}{2\pi i} \int \frac{d^{d-1}p}{(2\pi)^{d-1}} e^{-i\omega t+i\mathbf{p}(\mathbf{x}-\mathbf{y})} \frac{1}{\omega - i\epsilon} \Delta^R_{ab}(\omega, \mathbf{p}) j_b(0, \mathbf{y}).\end{aligned}$$

Setting now also $t = 0$, the ω integration can be closed in the upper half of the complex plane. There is only a single pole at $\omega = i\epsilon$ because $\Delta^R_{ab}(\omega, \mathbf{p})$ must be analytic in the upper half plane for causality. Accordingly one finds (taking now $\epsilon \to 0$)

$$\bar{\chi}_a(0, \mathbf{x}) = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \Delta^R_{ab}(0, \mathbf{p}) j_b(0, \mathbf{y}).$$

This response can similarly be written in terms of the advanced Greens function, and we have previously seen that the difference between retarded and advanced Green functions, wich is the spectral density, must vanish for $\omega \to 0$.

The function

$$\chi_{ab}(\mathbf{p}) = \Delta^R_{ab}(0, \mathbf{p}) = \Delta^A_{ab}(0, \mathbf{p}),$$

is a static linear response function also known as static susceptibility. It can be determined directly in thermal equilibrium and describes the reponse of a thermal expectation value to a change in the Hamiltonian.

In contrast to this, the response function $\Delta^R_{ab}(\omega, \mathbf{p})$ at non-vanishing frequency ω contains information about the relaxation to equilibrium, or about thermalization dynamics, and needs a real time calculation or analytic continuation. It is also known as dynamic susceptibility.

Static and dynamic structure factors

Let us consider now the statistical correlation function for bosonic fields

$$\Delta^S_{ab}(x - y) = \frac{1}{2}\langle \chi_a(x)\chi_b(y) + \chi_b(y)\chi_a(x)\rangle = \int_p e^{ip(x-y)} \Delta^S_{ab}(p).$$

For commuting fields (no canonical conjugate momentum fields involved) this is simply the correlation function $\langle \chi_a(x)\chi_b(x)\rangle$. When the times agree, $x^0 = y^0$, this gives the equal-time correlation function,

$$\Delta^S_{ab}(0, \mathbf{x} - \mathbf{y}) = \int \frac{d^{d-1}p}{(2\pi)^{d-1}} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} S_{ab}(\mathbf{p}),$$

with the frequency integral

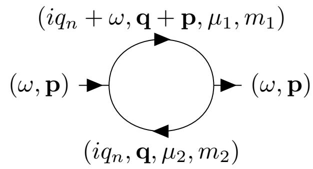
$$S_{ab}(\mathbf{p}) = \int \frac{d\omega}{2\pi} \Delta^S_{ab}(\omega, \mathbf{p}),$$

also known as the static structure factor. It can be experimentally accessed by recording values of the field $\chi_a(t, \mathbf{x})$ as a function of spatial position $\mathbf{x}$ (and the discrete index a when applicable), instanteneously at some time t. This is like "taking pictures". From many such recordings one can construct correlation functions. Similar to the static susceptibility, the equal-time correlation function or the static structure factor are thermal equilibrium properties and can be calculated in the Matsubara formalism without analytic continuation.

In contrast, the more general object $S_{ab}(\omega, \mathbf{p}) = \Delta^S_{ab}(\omega, \mathbf{p})$ is known as dynamic structure factor. Experimentally it can be accessed by inelastic scattering spectroscopy with the elastic limit corresponding to $\omega = 0$. Its Fourier transform, the statistical or structure correlation function $\Delta^S_{ab}(x^0 - y^0, \mathbf{x} - \mathbf{y})$ is also known as van Hove coorelation function.

Self energy loop diagram

We now consider a frequency and momentum dependent loop diagram in a finite temperature situation, and allow also for chemical potential terms. The diagram involves two propagators, and it appears in this or a related form in many circumstances, for example as a correction to the inverse propagator when there are three point vertices, or as a subdiagram of a vertex correction.



We take the incoming frequency and momentum to be $(\omega, \mathbf{p})$ where ω is a real frequency which we should consider to be analytically continued from a Matsubara feequency $\omega = i2\pi Tm$. To cover the general case, we allow the two propagators in the loop to have chemical potentials μ_1 and μ_2, and masses m_1 and m_2, respectively.

We find for the core of the diagram

$$S(\omega, \mathbf{p}) = -T \sum_{n=-\infty}^{\infty} \int_{\mathbf{q}} \frac{1}{[-(iq_n + \omega + \mu_1)^2 + (\mathbf{q} + \mathbf{p})^2 + m_1^2]\,[-(iq_n + \mu_2)^2 + \mathbf{q}^2 + m_2^2]}.$$

The minus sign has been added because it will typically appear like this in a concrete application, e. g. in perturbation theory.

Matsubara propagators in time domain

The Matsubara propagators appearing here can be written in the form

$$\frac{1}{(q_n - i\mu)^2 + \mathbf{q}^2 + m^2} = \int_0^\beta d\tau e^{iq_n\tau} J(\tau, \mu, \sqrt{\mathbf{q}^2 + m^2}).$$

This uses the Matsubara sum we have calculated previously

$$\begin{aligned} J(\Delta\tau, \mu, x) =& T \sum_{n=-\infty}^{\infty} \frac{e^{-i2\pi Tn\Delta\tau}}{(2\pi Tn - i\mu)^2 + x^2} \\ =& \frac{1}{2x}\left[\theta(\Delta\tau) + n_B(x - \mu)\right] e^{-(x-\mu)\Delta\tau} + \frac{1}{2x}[\theta(-\Delta\tau) + n_B(x + \mu)] e^{(x+\mu)\Delta\tau}, \\ =& \sum_{s=\pm 1} \frac{s}{2x}\left[\theta(\Delta\tau) + n_B(sx - \mu)\right] e^{-(sx-\mu)\Delta\tau} \end{aligned}$$

as well as the identity

$$T \int_0^\beta d\tau e^{i2\pi Tn\tau} = \delta_{0n}.$$

Note that the precise integration bounds are somewhat arbitrary here, and one could equally well integrate from some point σ to $\sigma + \beta$. The symmetric choice we made simplifies some of the following steps.

We find thus

$$S(\omega,\mathbf{p}) = -T\sum_{n=-\infty}^{\infty}\int_{\mathbf{q}}\int_0^\beta d\tau_1 d\tau_2\, e^{iq_n(\tau_1+\tau_2)+\omega\tau_1}$$
$$\times J\Big(\tau_1,\mu_1,\sqrt{(\mathbf{q}+\mathbf{p})^2+m_1^2}\Big)J\Big(\tau_2,\mu_2,\sqrt{\mathbf{q}^2+m_2^2}\Big).$$

Peforming sums and integrals

At this point one can perform the sum over n using

$$T\sum_{n=-\infty}^{\infty} e^{i2\pi Tn(\tau_1+\tau_2)} = \sum_{k=-\infty}^{\infty}\delta(\tau_1+\tau_2-k\beta).$$

For our choice of integration interval for Matsubara frequencies only one Dirac delta with $k=1$ actually contributes. Using this one can perform one of the time integrals, resulting in

$$S(\omega,\mathbf{p}) = -\int_{\mathbf{q}}\int_0^\beta d\tau\, e^{\omega\tau} J\Big(\tau,\mu_1,\sqrt{(\mathbf{q}+\mathbf{p})^2+m_1^2}\Big)J\Big(\beta-\tau,\mu_2,\sqrt{\mathbf{q}^2+m_2^2}\Big).$$

Using the concrete expression for J one can now perform also the integral over the remaining Matsubara time τ. One finds integrals of the form

$$\int_0^\beta d\tau \exp(\omega\tau + x_1\tau + x_2(\beta-\tau)) = \frac{\exp((\omega+x_1)/T)-\exp(x_2/T)}{\omega+x_1-x_2}$$
$$\to \frac{\exp(x_1/T)-\exp(x_2/T)}{\omega+x_1-x_2},$$

where x_1 and x_2 are combinations of energies and chemical potentials. In going from the first to the second line we have used that ω is analytically continued from points on the Matsubara axis where $\omega = i2\pi Tm$. On all these points one has $\exp(\omega/T)=1$.

Proceeding this way gives with $E_1 = \sqrt{(\mathbf{q}+\mathbf{p})^2+m_1^2}$ and $E_2 = \sqrt{\mathbf{q}^2+m_2^2}$

$$S(\omega,\mathbf{p}) = -\int_{\mathbf{q}}\frac{1}{4E_1E_2}\Bigg\{\frac{[1+n_B(E_1-\mu_1)][1+n_B(E_2-\mu_2)]\Big[\exp(-\frac{E_1-\mu_1}{T})-\exp(-\frac{E_2-\mu_2}{T})\Big]}{\omega-E_1+\mu_1+E_2-\mu_2}$$
$$+\frac{[1+n_B(E_1-\mu_1)]n_B(E_2+\mu_2)\Big[\exp(-\frac{E_1-\mu_1}{T})-\exp(\frac{E_2+\mu_2}{T})\Big]}{\omega-E_1+\mu_1-E_2+\mu_2}$$
$$+\frac{n_B(E_1+\mu_1)[1+n_B(E_2-\mu_2)]\Big[\exp(\frac{E_1+\mu_1}{T})-\exp(-\frac{E_2-\mu_2}{T})\Big]}{\omega+E_1+\mu_1+E_2-\mu_2}$$
$$+\frac{n_B(E_1+\mu_1)n_B(E_2+\mu_2)\Big[\exp(\frac{E_1+\mu_1}{T})-\exp(\frac{E_2+\mu_2}{T})\Big]}{\omega+E_1+\mu_1-E_2-\mu_2}\Bigg\}.$$

Here one can use the identities

$$\exp(z/T)n_B(z) = 1+n_B(z),$$

and

$$\exp(-z/T)[1+n_B(z)] = n_B(z).$$

Interpreting the branch cut through loss minus gain

Combining terms we find

$$S(\omega,\mathbf{p}) = \int_{\mathbf{q}} \frac{1}{4E_1E_2}\left\{\frac{[1+n_B(E_1-\mu_1)][1+n_B(E_2+\mu_2)] - n_B(E_1-\mu_1)n_B(E_2+\mu_2)}{\omega - E_1+\mu_1-E_2-\mu_2}\right.$$
$$+\frac{n_B(E_1+\mu_1)n_B(E_2-\mu_2) - [1+n_B(E_1+\mu_1)][1+n_B(E_2-\mu_2)]}{\omega+E_1+\mu_1+E_2-\mu_2}$$
$$+\frac{[1+n_B(E_1-\mu_1)]n_B(E_2-\mu_2) - n_B(E_1-\mu_1)[1+n_B(E_2-\mu_2)]}{\omega-E_1+\mu_1+E_2-\mu_2}$$
$$\left.+\frac{n_B(E_1+\mu_1)[1+n_B(E_2+\mu_2)] - [1+n_B(E_1+\mu_1)]n_B(E_2+\mu_2)}{\omega+E_1+\mu_1-E_2-\mu_2}\right\}.$$

This can be simplified further, but as it stands the different terms have a very nice physical interpretation.

Specifically, the branch cut part of $S(\omega,\mathbf{p})$ is obtained by replacing

$$\frac{1}{\omega-E_1+\mu_1-E_2-\mu_2} \to -i\pi\, s_{\mathrm{I}}(\omega)\,\delta(\omega-E_1+\mu_1-E_2-\mu_2),$$

and so on. It is indeed on the real frequency axis. The resulting Dirac delta implies energy conservation. The physical interpretation of the imaginary part is a difference between loss and gain processes.

Specifically, the term in the first line describes the transition from the incoming state with frequency ω into a two-particle state with energy E_1+E_2 minus the transition from such a two-particle state into the outgoing state. For the decay parts one has Bose enhancement of the vaccum amplitudes when the modes are already occupied. The gain process comes from thermally occupied modes.

Similarly, the second line describes a transition of the incoming particle with energy ω together with two anti-particles from the bath into an empty state minus the gain process which is a decay into two anti-particles and the outgoing state.

The third line describes a reaction of the incoming particle with frequeny ω together with an anti-particle with energy E_2 into a particle with energy E_1 minus the corresponding gain term.

Finally, the last line stands for a reaction of the incoming particle with an anti-particle of energy E_1 into a particle of energy E_2 minus the gain term.

We observe that the entire brach cut part of $S(\omega,\mathbf{p})$ can be written as loss minus gain, where the ratio of loss over gain is $\exp(\omega/T)$, following detailed balance.

Vacuum self energy and decay rate

In the vaccum limit $T=\mu_1=\mu_2=0$ one finds

$$S(\omega,\mathbf{p}) = \int_{\mathbf{q}} \frac{1}{4E_1E_2}\left\{\frac{1}{\omega-E_1-E_2} - \frac{1}{\omega+E_1+E_2}\right\}.$$

Assuming now $m_1=m_2=m$, the brach cut starts here at $\omega^2-\mathbf{p}^2=4m^2$ or for $\mathbf{p}=0$ at $|\omega|=2m$, which corresponds to the energy threshold for the production of two particles with mass m. It is simplest to calculate this for $\omega>0$, $\mathbf{p}=0$ where $E_1=E_2=\sqrt{\mathbf{q}^2+m^2}$. Evaluating the integrand for $\mathrm{Im}(\omega)=\pm\epsilon$ gives for the branch cut part of $S(\omega,\mathbf{p})$

$$\mathrm{Disc}\; S(\omega,\mathbf{p}) = -i\pi s_{\mathrm{I}}(\omega)\int_{\mathbf{q}} \frac{1}{4E^2}[\delta(\omega-2E)-\delta(\omega+2E)].$$

The remaining integral can be done using the Dirac delta. It is an integral over the phase space for the decay into the two particles. Using that the result must be Lorentz invariant gives

$$-i\pi s(\omega)\theta(\omega^2-\mathbf{p^2}-4m^2)\frac{\Omega_{d-1}}{(2\pi)^{d-1}2^d}\frac{(\omega^2-\mathbf{p}^2-4m^2)^{\frac{d-2}{2}}}{\sqrt{\omega^2-\mathbf{p}^2}},$$

with

$$s(\omega)=\operatorname{sign}(\operatorname{Im}(\omega))\operatorname{sign}(\operatorname{Re}(\omega)).$$

This is the imaginary part that corresponds to the continuum of two-particle states. It could also been calculated with the help of Cutkosky cutting rules. We see here a manifestation of the optical theorem, telling that the imaginary part of the forward amplitude must be propational to the decay propability into a continuum of scattering states!

Vanishing chemical potentials but finite temperature

At vanishing chemical potentials $\mu_1=\mu_2=0$ we obtain for the branch cut

$$\begin{aligned}\text{Disc } S(\omega,\mathbf{p}) = -is_{\mathrm{I}}(\omega)\pi\int_{\mathbf{q}}\frac{1}{4E_1E_2}\Big\{ &[[1+n_B(E_1)][1+n_B(E_2)]-n_B(E_1)n_B(E_2)]\,\delta(\omega-E_1-E_2)\\ &+[n_B(E_1)n_B(E_2)-[1+n_B(E_1)][1+n_B(E_2)]]\,\delta(\omega+E_1+E_2)\\ &+[[1+n_B(E_1)]n_B(E_2)-n_B(E_1)[1+n_B(E_2)]]\,\delta(\omega-E_1+E_2)\\ &+[n_B(E_1)[1+n_B(E_2)]-[1+n_B(E_1)]n_B(E_2)]\,\delta(\omega+E_1-E_2)\Big\}.\end{aligned}$$

As mentioned before, all terms decompose naturally into a loss minus a gain term which differ by a factor $\exp(\omega/T)$. This allows to write

$$\text{Disc } S(\omega,\mathbf{p})=-is_{\mathrm{I}}(\omega)\pi\left[e^{\frac{\omega}{2T}}-e^{-\frac{\omega}{2T}}\right][H(\omega,\mathbf{p})+H(-\omega,\mathbf{p})]$$

where

$$H(\omega,\mathbf{p})=\int_{\mathbf{q}}\frac{1}{4E_1E_2}\left[n_B(E_1)e^{\frac{E_1}{2T}}n_B(E_2)e^{\frac{E_2}{2T}}\right][\delta(\omega-E_1-E_2)+\delta(\omega-E_1+E_2)].$$