Lectures on fluid dynamics

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ABSTRACT: Notes for lectures that introduce students of physics to fluid dynamics. Prepared for a course at Heidelberg university in the summer term 2016.

Contents

1 What is a fluid?

Organizational issues

- There is a webpage to accompany this lecture: `http://www.thphys.uni-heidelberg.de/~floerchinger/lectures/`
- A list of books on fluid dynamics is provided on the webpage. The standard (and maybe most important for this course) reference is Landau & Lifshitz. It has been translated to many languages, including German.
- The first lecture will follow partly the book by Faber.
- Lecture notes will be provided eventually but it may take some time.
- On May 16th there is no lecture because of Pentecost (Pfingsten).

Solid state *versus* fluid

- Apply a shear force to a solid body: It is not possible to deform it. For elastic bodies, there is a restoring force that is proportional to the displacement. Depending on the material, the body may break if the shear force becomes too strong.
- For fluids, even a small shear force leads to a deformation. There is no restoring force proportional to the displacement. (But there might be resistance or friction which is usually proportional to the velocity of the movement.)
- Microscopically, the constituents of a solid body have a strong binding force, whereas the constituents of a fluid (typically atoms or molecules) can move with respect to each other more or less freely.
- It can also depend on the strength and time-scale of the external shear whether a material acts more like a fluid or a solid state. For example, materials like pitch (Pech) or asphalt are in principle liquids but with such a high viscosity that they resemble also solids. See for example the Pitch drop experiment `http://smp.uq.edu.au/content/pitch-drop-experiment`.

Liquid *versus* gas

- The distinction between liquids (such as water) and gases (such as air) is less rigid. In a liquid, the constituents are closer together whereas in a gas there is a considerable mean free path between scatterings. For air at normal pressure one has for example a mean free path of about 7×10^{-8} m. For water this is about 2×10^{-10} m.
- Besides the density, also other properties such as viscosity and compressibility differ between liquids and gases. We will come to this later.
- Nevertheless, for sufficiently large observational length scale, liquids like water and gases like air can be described by similar equations as we will see. Both are fluids.
- Water and air may be the most important fluids for many applications of fluid dynamics in everyday life. But there are many more substances that can be described as fluids in one way or the other. For example blood, the outer earth's core, liquid helium, the quark-gluon plasma, interstellar dust, or the dark matter in the universe. More details in due course.

More on stress forces

- Consider a small cube of side length a (and volume $V = a^3$) filled with a fluid. We assume first that the fluid is not moving. The surface elements pointing to the outside can be written as
$$d\vec{A} = |dA|\vec{n},$$
where $\vec{n}$ with $\vec{n}^2 = 1$ is a normal vector orthogonal to the surface element.
- The momentum of the fluid cell can we written as
$$P_j = \int dV \mathscr{P}_j,$$
where $j = 1, 2, 3$ and $\mathscr{P}_j$ is a momentum density.
- Newton's law for the fluid in the cube can be written as
$$\frac{\partial}{\partial t} P_j = F_j,$$
where F_j is the sum of forces transmitted to the fluid in the cube from outside via the surface.

Figure 1. Some examples of fluid phenomena that can be described by the methods we will discuss in these lectures.

- One can write the force on the surface element in the following form

  $$-F_j = \int dA_i \, T_{ij},$$

  where the integral goes over the surface of the cube and where T_{ij} is the so-called stress tensor.

- The components of T_{ij} can be illustrated as the components of the force acting on the different surface elements from within the fluid. For example, for constant T_{ij} and a surface element where the normal vector $\vec{n}$ points in x_1-direction, $\vec{n} = (1, 0, 0)$, the force is given by $-F_j = a^2 T_{1j}$ and similar for the other surfaces of the cube.

- We now discuss the properties of T_{ij} in a little more detail. We assume that the fluid density ρ is smooth or for simplicity even constant within the fluid cube. If we make the cube smaller and smaller, $a \to 0$, the forces on the whole fluid must average to zero because otherwise a finite, non-zero force would accelerate a very small mass of fluid inside the cube (the fluid mass in the cube goes like $a^3 \to 0$). The acceleration would be very fast and lead to an instantaneous motion. This implies that T_{ij} must be a continuous function.

- Similarly, one must have $T_{12} = T_{21}$ and so on. Otherwise, a non-zero torque (Drehmoment) would result of order

  $$(T_{12} - T_{21})a^3,$$

  which would rotationally accelerate the fluid cube with moment of inertia (Trägheitsmoment) of order a^5 very quickly.

- Finally, consider a fluid in mechanical equilibrium. In that case one must have

  $$T_{11} = T_{22} = T_{33} = p, \qquad T_{12} = T_{23} = T_{31} = 0.$$

This is known as *Pascal's law.* In more general situation we will discuss the form of T_{ij} in more detail later on.

A first fluid dynamics problem: the syringe

- Before we dive deeper into the theory of fluids, let us look at a first problem. Consider a syringe (Spritze) as it is used for medical injections filled with a liquid resembling water. It has a barrel of cylindrical form with length significantly larger that its radius. At one end there is a piston of area A and at the other a small hole leading into a needle with length l and internal radius a .
- We assume that the surface of the barrel is smooth such that there is no resistance to of the liquid when it flows along it. For the needle we do not assume this and rather assume so-called no-slip boundary conditions.
- The question we will pose ourselves is: What is the discharge rate Q defined as volume per unit time of liquid that flows out of the needle, for a given force F applied to the piston. We are here interested in the steady state motion (not the initial acceleration phase).
- The force can be quantified in terms of the difference between the pressure directly in front of the piston p_P and atmospheric pressure p_A outside the needle as

  $$F = (p_P - p_A)A$$

  and we may therefore also study the relation between Q and $p^* = (p_P - p_A)$, the so-called excess pressure.
- It is useful to have a few numbers in mind. A typical syringe may have a capacity of 3 ml and a needle with inner radius $a = 0.2$ mm and length $L = 3$ cm. A full load of water can be discharged in about 10 seconds. That would give $Q = 0.3$ ml/s and a mean velocity along the needle of

  $$U = \frac{Q}{\pi a^2} \approx 2.5\,\text{m/s}.$$

What do we need to describe the fluid

- What are the parameters and fields that one may need to follow to describe a fluid? One, for sure, is the *fluid velocity field* which will denote by $\vec{v}(t, \vec{x})$. Obviously, in SI units one has $[v] = \text{m/s}$.
- Another relevant field is the density $\rho(t, \vec{x})$ and it is convenient to work with mass density instead of say particle number density. So one has $[\rho] = \text{kg/m}^3$. Under typical laboratory conditions, water has a density of about $\rho = 10^3$ kg/m^3, air has $\rho = 1.2$ kg/m^3.
- Of direct importance is also the pressure p in units $[p] = \text{Pa} = \text{N/m}^2$. Standard conditions correspond to $p = 10^5$ N/m^2 but due to gravity, the pressure is obviously usually higher under water.
- In principle, also the temperature $T(t, \vec{x})$ can be a relevant field. Units are $[T] = K$.
- One material property that might be relevant is the sound velocity c_s (measured in $[c_s] = \text{m/s}$) which is related to the compressibility. (More details later.) For water at normal conditions one has $c_s = 1500$ m/s, for air $c_s = 340$ m/s.

- Another one is the heat capacity density c_p that tells how much energy needs to be injected (at constant pressure) to heat a certain volume of fluid by one degree in temperature. Units are $[c_P] = \mathrm{J}/(\mathrm{m}^3\ \mathrm{K})$. Water at normal conditions has about $c_P = 4.2 \times 10^6\ \mathrm{J}/(\mathrm{m}^3\ \mathrm{K})$, air has about $c_P = 10^3\ \mathrm{J}/(\mathrm{m}^3\ \mathrm{K})$.
- Velocity of sound and heat capacity density are examples for *thermodynamic properties.*
- Another relevant fluid property is the shear viscosity η (with units $[\eta] = \mathrm{kg}/(\mathrm{m\ s})$) as well as the bulk viscosity ζ (equal units). Water at normal conditions has $\eta = 10^{-3}\ \mathrm{kg}/(\mathrm{m\ s})$ and the ratio of bulk viscosity to shear viscosity is $\zeta/\eta \approx 3$. The shear viscosity of air at normal conditions is about $\eta = 1.8 \times 10^{-5}\ \mathrm{kg}/(\mathrm{m\ s})$.
- Also heat conductivity κ (in units $[\kappa] = \mathrm{J}/(\mathrm{m\ s\ K})$) could play a role. Water at normal conditions has about $\kappa = 0.6\ \mathrm{J}/(\mathrm{m\ s\ K})$, air has $\kappa = 0.025\ \mathrm{J}/(\mathrm{m\ s\ K})$. Viscosities and conductivities are *transport properties.*
- More general, also chemical properties such as concentrations, reaction rates etc. could play a role or for charged fluids the charge density and electric conductivity.

Some estimates for the syringe problem

- The above quantities could in principle play a role for the dynamic properties of a fluid. But what is really important in a concrete situation, say for the syringe problem?
- Can temperature change substantially? The total amount of work done per unit time is $p^* Q$. Assume for example $p^* = 10^4\ \mathrm{N/m}^2$. If all the work would go into heat of the fluid, by how much would the temperature change over the length of the needle? One has

  $$p^* Q = c_P Q \Delta T$$

  Using the heat capacity density of water leads to a temperature change ΔT of a view mK. This is clearly very small compared to the temperature of water at normal conditions and can therefore be neglected. So we can safely assume $T = \mathrm{const.}$
- The importance of compressibility can be quantified by the Mach number, which is the ratio of a typical fluid velocity to the velocity of sound,

  $$\mathrm{Ma} = \frac{\text{characteristic flow velocity}}{\text{sound velocity}}.$$

  Here one would take

  $$\mathrm{Ma} = U/c_s.$$

  If the Mach number is small, changes in density are usually of minor importance for the flow problem at hand. In other words, density inhomogeneities are very quickly leveled out so that they do not affect the flow pattern itself. For the syringe problem one has $\mathrm{Ma} = 2.5/1500$ which is clearly quite small. To good approximation one can therefore assume $\rho = \mathrm{const.}$ A flow with small Mach number is called subsonic, one with large Ma is called supersonic.
- The importance of viscosity can be quantified in terms of the Reynolds number,

  $$\mathrm{Re} = \frac{2a\rho U}{\eta} = \frac{2aU}{\nu}.$$

In the second equation we have introduced the so-called kinematic viscosity $\nu = \eta/\rho$. Quite generally, one defines the Reynolds number for a given flow problem as

$$\mathrm{Re} = \frac{\text{characteristic size} \cdot \text{characteristic flow velocity}}{\text{kinematic viscosity}}.$$

The factor of 2 in the above expression is purely conventional and due to the fact that for pipe-like problems, the Reynolds number is usually defined by the diameter instead of the radius.

A large Reynolds number implies a large separation between the scale of the fluid motion (a in our case) and the scale of viscosity. Typically, if Re is small, flow patterns are laminar, while they become turbulent for very large Re. What small and large means depends on the situation, however.

A typical value for the kinematic viscosity of water is $\nu = 10^{-6}\mathrm{m}^2/\mathrm{s}$ (air would have $\nu = 1.5 \times 10^{-5}\ \mathrm{m}^2/\mathrm{s}$). Together with $a = 2 \times 10^{-4}$ m and $U = 2.5$ m/s this leads to $\mathrm{Re} = 1000$.

- Another interesting characteristic is the Knudsen number which is defined by

$$\mathrm{Kn} = \frac{\text{mean free path}}{\text{characteristic size}}.$$

The Knudsen number has to be small in order to use a fluid approximation (instead of kinetic theory which follows particle positions and momenta in a more explicit way). For water, the mean free path is about 2×10^{-10} m and taking $a = 2 \times 10^{-4}$ m as characteristic size gives $\mathrm{Kn} = 10^{-6}$, which is quite small, indeed.

Dimensional anaysis

- Every physics student knows that it is useful to check the dimensions of the variables in order to make a first guess for the answer to a specific problem. As an exercise, we will now do this for the syringe problem.
- First, a general remark on units is in order. In non-relativistic fluid dynamics, one is free to choose units for length, time, mass and temperature freely. All other quantities (e.g. energy) can be expressed in derived units (e. g. $1\mathrm{J} = 1\mathrm{kg}\ \mathrm{m}^2/\mathrm{s}^2$).
- Physical relations have to hold in every possible system of units, or units on both sides of an equation have to match.
- We want to know the discharge rate Q (with dimension $[\text{Length}]^3[\text{Time}]^{-1}$ or $\mathrm{m}^3\mathrm{s}^{-1}$ in SI units) as a function of the excess pressure p^* (with dimension kg $\mathrm{m}^{-1}\mathrm{s}^{-2}$) as well as on the length l and radius a of the needle.
- The answer could also depend on density ρ (with dimensions kg m^{-3}) as well as viscosity η and sound velocity c_s. The effects of changes in temperature are small so that nothing else is expected to enter.
- We can now build dimensionless combinations. One is obviously a/l, another one is

$$\frac{p^* a^4}{\rho Q^2}.$$

Viscosity η and sound velocity c_s could enter in terms of Re and Ma.

- There are actually no more independent, dimensionless combinations. It must therefore be possible to write a relation between Q and p^* in the form

$$\frac{p^* a^4}{\rho\, Q^2} = f\left(\frac{a}{l}, \mathrm{Re}, \mathrm{Ma}\right).$$

- Because Ma$= 2.5/1500 \ll 1$ it should be a good approximation to set Ma$= 0$. The viscosity can not be neglected here because it due to friction that Q has actually a limiting value for constant force or pressure p^*. It remains to determine

$$\tilde{f}\left(\frac{a}{l}, \mathrm{Re}\right) = f\left(\frac{a}{l}, \mathrm{Re}, 0\right).$$

  To do this we actually need to develop the theory of fluids as we shall do next.

2 Symmetries and conservation laws

We now develop the theory of fluid dynamics in a systematic way. The starting point are quite general conservation laws that follow ultimately from basic principles of quantum field theory and general relativity.

Origin of conservation laws

- Because of Noether's theorem, continuous symmetries of the microscopic laws imply conservation laws.
- For example the symmetry with respect to U(1) transformations of the Schrödinger wave function for an electron
$$\psi(t,\vec{x}) \to \psi'(t,\vec{x}) = e^{i\alpha}\psi(t,\vec{x})$$
is related to the conservation of charge. Similar symmetries imply for example the conservation of baryons in the Standard model.
- Conservation of momentum is a consequence of a symmetry with respect to spatial translations $\vec{x} \to \vec{x} + \Delta\vec{x}$.
- Conservation of energy is a consequence of symmetry under translations in time $t \to t + \Delta t$.
- Conservation of angular momentum is a consequence of symmetry with respect to rotations
$$x_i \to R_{ij}x_j,$$
where R_{ij} is an orthogonal matrix, $RR^T = \mathbb{1}$.
- In general relativity, space and time are curved and translation symmetry with respect to temporal and spatial translations is in general not given any more. However, a covariant form of energy- and momentum conservation still holds. It is due to general covariance which might be seen as the freedom to use arbitrary coordinate systems.

Particle number and mass conservation laws in local form

- Consider some volume V with boundary $A = \partial V$. The total number of particles in the volume is
$$N(t) = \int_V d^3x\, n(t,\vec{x}),$$
where $n(t,\vec{x})$ is the particle number density.
- For fluid dynamics one assumes that $N \gg 1$ such that the discreteness of particle number is not important.
- The conservation of particle number implies that N can only change if there is a a net influx or out-flux of particles. We describe this by R_{out} which might be positive or negative. It can be written as
$$R_{\text{out}}(t) = \int_{A=\partial V} dA_j n_j(t,\vec{x}) = \int_V d^3x\; \partial_j n_j(t,\vec{x}),$$
where $n_j(t,\vec{x})$ with $j = 1,2,3$ are the components of the particle number current density. For the second equation we have used Gauss's theorem. We also use here and below the Einstein summation convention implying that indices that appear twice have to be summed over. Moreover, we use the abbreviation $\partial_j = \frac{\partial}{\partial x_j}$ for the partial spatial derivative and sometime $\partial_t = \frac{\partial}{\partial t}$ for the partial derivative with respect to time.

- The conservation of particle number is
$$\frac{\partial}{\partial t}N(t) + R_{\text{out}}(t) = \int_V d^3x \left\{ \frac{\partial}{\partial t} n(t,\vec{x}) + \partial_j n_j(t,\vec{x}) \right\} = 0.$$
Because this holds for any volume V one must have
$$\frac{\partial}{\partial t} n(t,\vec{x}) + \partial_j n_j(t,\vec{x}) = 0.$$
This is the particle number conservation law in local form.

- In non-relativistic physics particle numbers may change by chemical reactions, but no mass can be created or destroyed. The mass density is denoted by $\rho(t,\vec{x})$. We will often assume a simple material where only one particle species with mass m is present and one has $\rho(t,\vec{x}) = m\, n(t,\vec{x})$. The mass density current will be denoted by $\rho_j(t,\vec{x}) = m\, n_j(t,\vec{x})$ and the conservation law for mass becomes
$$\partial_t \rho + \partial_j \rho_j = 0.$$

Energy and momentum conservation

- Energy conservation works very similar to mass conservation,
$$\partial_t \mathscr{E} + \partial_j \mathscr{E}_j = 0$$
except that we still need to specify the energy density $\mathscr{E}(t,\vec{x})$ and the corresponding current or energy flux density $\mathscr{E}_j(t,\vec{x})$.

- Momentum conservation can be written in terms of the momentum density $\mathscr{P}_k$ and the momentum flux density $\mathscr{P}_{jk}$
$$\frac{\partial}{\partial t}\int_V d^3x\, \mathscr{P}_k + \int_{\partial V} dA_j \mathscr{P}_{jk} = \int_V d^3x\, \{\partial_t \mathscr{P}_k + \partial_j \mathscr{P}_{jk}\} = 0.$$
Because this holds for any volume, one has the local momentum conservation law
$$\partial_t \mathscr{P}_k + \partial_j \mathscr{P}_{jk} = 0.$$
In the absence of centrifugal forces, the momentum flux density must be symmetric, $\mathscr{P}_{jk} = \mathscr{P}_{kj}$ because otherwise strong rotational forces would act on a small fluid cell. We will concentrate on this situation below.

Symmetry transformations

- The conservation laws are consequences of invariances with respect to certain symmetry transformations on the microscopic level. They will be the basis for developing the theory of fluid dynamics. For that purpose it is useful to study how the fields
$$\rho, \rho_j, \qquad \mathscr{E}, \mathscr{E}_j, \qquad \mathscr{P}_k, \mathscr{P}_{jk}$$
transform with resect to certain symmetry transformation.

- We start with translations in space which we formulate as transformations for the fields $\rho(t,\vec{x})$ etc. If $\rho(t,\vec{x})$ has a maximum at some position $\vec{x}_0$, we want the transformed field to have a maximum at $\vec{x}_0 + \Delta\vec{x}$. The field transforms as
$$\rho(t,\vec{x}) \to \rho'(t,\vec{x}) = \rho(t,\vec{x} - \Delta\vec{x}).$$

- Translations in time work similar,

$$\rho(t,\vec{x}) \to \rho'(t,\vec{x}) = \rho(t-\Delta t,\vec{x}).$$

- For the current densities the transformation law is analogous. Doing the translations in space and time together means

$$\begin{aligned}\rho(t,\vec{x}) \to & \rho'(t,\vec{x}) = \rho(t-\Delta t,\vec{x}-\Delta\vec{x}),\\ \rho_j(t,\vec{x}) \to & \rho'_j(t,\vec{x}) = \rho_j(t-\Delta t,\vec{x}-\Delta\vec{x}),\end{aligned}$$

- The conservation law is invariant under translations in the sense that if (ρ,ρ_j) is a conserved field configuration, this is also the case for (ρ',ρ'_j),

$$\partial_t\rho'(t,\vec{x}) + \partial_j\rho'_j(t,\vec{x}) = \partial_t\rho(t-\Delta t,\vec{x}-\Delta\vec{x}) + \partial_j\rho_j(t-\Delta t,\vec{x}-\Delta\vec{x}) = 0.$$

  This works completely analogous for $(\mathscr{E},\, \mathscr{E}_j)$ and $(\mathscr{P}_k,\, \mathscr{P}_{jk})$.

- Consider now rotations. We transform the fields such that a configuration with maximum at $\vec{x}_0$ is mapped to one with maximum at $R\vec{x}$ where R with $RR^T = \mathbb{1}$ is the rotation matrix. The transformation for the mass density is

$$\rho(t,\vec{x}) \to \rho'(t,\vec{x}) = \rho(t,R^{-1}\vec{x}). \tag{2.1}$$

- The current density ρ_j has also a direction which needs to be rotated, as well. The transformation law is

$$\rho_j(t,\vec{x}) \to \rho'_j(t,\vec{x}) = R_{jk}\rho_k(t,R^{-1}\vec{x}).$$

- We also need to know how derivatives of fields transform. Consider for example

$$\partial_j\rho(t,\vec{x}) \to \partial_j\rho'(t,\vec{x}) = \partial_j\rho(t,R^{-1}\vec{x}) = (R^{-1})_{kj}(\partial_k\rho)(t,R^{-1}\vec{x}).$$

- The combination $\partial_j\rho_j$ transforms therefore like

$$\partial_j\rho_j(t,\vec{x}) \to (R^{-1})_{kj}R_{jm}(\partial_k\rho_m)(t,R^{-1}\vec{x}) = \delta_{km}(\partial_k\rho_m)(t,R^{-1}\vec{x}) = (\partial_k\rho_k)(t,R^{-1}\vec{x}).$$

  This implies, that the conservation law $\partial_t\rho+\partial_j\rho_j$ is also invariant under rotations. For energy density and energy current density $(\mathscr{E},\mathscr{E}_j)$, the argument is completely analogous.

- The momentum density $\mathscr{P}_k$ is itself a vector field. It transforms under rotations like

$$\mathscr{P}_k(t,\vec{x}) \to \mathscr{P}'_k(t,\vec{x}) = R_{km}\mathscr{P}_m(t,R^{-1}\vec{x}).$$

  The momentum flux density is a tensor with respect to rotations. It transforms like

$$\mathscr{P}_{jk}(t,\vec{x}) \to \mathscr{P}'_{jk}(t,\vec{x}) = R_{jm}R_{kn}\mathscr{P}_{mn}(t,R^{-1}\vec{x}).$$

  Exercise: Show that the momentum conservation law is invariant under rotations.

Galilei boosts

- The non-relativistic laws of physics are also invariant under Galilei boost transformations,

$$(t, \vec{x}) \to (t, \vec{x} + \Delta\vec{v}\, t),$$

where $\Delta\vec{v}$ is the velocity of the boost. The equations of motion for a non-relativistic fluid are supposed to invariant under these transformations, as well. The transformation behavior of the fields

$$\rho, \rho_j, \qquad \mathscr{E}, \mathscr{E}_j, \qquad \mathscr{P}_k, \mathscr{P}_{jk}$$

with respect to Galilei boosts is a little more involved than for translations and rotations.

- We start with mass density and mass density current. The density transforms like

$$\rho(t, \vec{x}) \to \rho'(t, \vec{x}) = \rho(t, \vec{x} - \Delta\vec{v}\, t).$$

Accordingly,

$$\partial_t \rho(t, \vec{x}) \to \partial_t \rho'(t, \vec{x}) = (\partial_t \rho)(t, \vec{x} - \Delta\vec{v}\, t) - \Delta v_j \partial_j \rho(t, \vec{x} - \Delta\vec{v} t).$$

In order for the mass conservation law to be invariant, one must have the following transformation behavior of the mass current

$$\rho_j(t, \vec{x}) \to \rho'_j(t, \vec{x}) = \rho_j(t, \vec{x} - \Delta\vec{v} t) + \Delta v_j \rho(t, \vec{x} - \Delta\vec{v}\, t).$$

This leads indeed to

$$\partial_j \rho_j(t, \vec{x}) \to \partial_j \rho'_j(t, \vec{x}) = \partial_j \rho_j(t, \vec{x} - \Delta\vec{v} t) + \Delta v_j \partial_j \rho(t, \vec{x} - \Delta\vec{v}\, t).$$

- Consider now the momentum density. The boost changes the momentum by the additional center of mass motion

$$\mathscr{P}_k(t, \vec{x}) \to \mathscr{P}'_k(t, \vec{x}) = \mathscr{P}_k(t, \vec{x} - \Delta\vec{v}\, t) + \Delta v_k\, \rho(t, \vec{x} - \Delta\vec{v}\, t)$$

Accordingly, one has

$$\begin{aligned} \partial_t \mathscr{P}_k(t, \vec{x}) \to \partial_t \mathscr{P}'_k(t, \vec{x}) =& (\partial_t \mathscr{P}_k)(t, \vec{x} - \Delta\vec{v}\, t) - \Delta v_j \partial_j \mathscr{P}_k(t, \vec{x} - \Delta\vec{v}\, t) \\ & + \Delta v_k\, (\partial_t \rho)(t, \vec{x} - \Delta\vec{v}\, t) - \Delta v_k \Delta v_j\, \partial_j \rho(t, \vec{x} - \Delta\vec{v}\, t) \\ =& (\partial_t \mathscr{P}_k)(t, \vec{x} - \Delta\vec{v}\, t) - \Delta v_j \partial_j \mathscr{P}_k(t, \vec{x} - \Delta\vec{v}\, t) \\ & - \Delta v_k\, \partial_j \rho_j(t, \vec{x} - \Delta\vec{v}\, t) - \Delta v_k \Delta v_j\, \partial_j \rho(t, \vec{x} - \Delta\vec{v}\, t) \end{aligned}$$

In the last equation, we have used the conservation law for mass in the form $\partial_t \rho = -\partial_j \rho_j$.

- In order for the conservation law for momentum to be invariant under Galilei transformations one needs the following transformation law of the momentum flux density

$$\begin{aligned} \mathscr{P}_{jk}(t, \vec{x}) \to \mathscr{P}'_{jk}(t, \vec{x}) =& \mathscr{P}_{jk}(t, \vec{x} - \Delta\vec{v}\, t) + \Delta v_j \mathscr{P}_k(t, \vec{x} - \Delta\vec{v}\, t) \\ & + \Delta v_k\, \rho_j(t, \vec{x} - \Delta\vec{v}\, t) + \Delta v_k \Delta v_j\, \rho(t, \vec{x} - \Delta\vec{v}\, t). \end{aligned} \tag{2.2}$$

It is, however, required that the momentum flux density is symmetric $\mathscr{P}_{jk} = \mathscr{P}_{kj}$. One can therefore conclude that mass current density and momentum density are actually equal,

$$\rho_j(t, \vec{x}) = \mathscr{P}_j(t, \vec{x}).$$

- One can actually define the fluid velocity field $v_j(t, \vec{x})$ such that

$$\rho_j(t, \vec{x}) = \mathscr{P}_j(t, \vec{x}) = \rho(t, \vec{x}) v_j(t, \vec{x}).$$

  This has the correct transformation behavior with respect to Galilei boost if the fluid velocity field transforms like

$$v_j(t, \vec{x}) \to v_j(t, \vec{x} - \Delta\vec{v}\, t) + \Delta v_j.$$

- For the momentum flux density it is actually useful to write

$$\mathscr{P}_{jk} = \rho\, v_i v_j + T_{ij}$$

  The transformation with respect to Galilei transformations comes out correctly if the stress tensor transforms simply like

$$T_{ij}(t, \vec{x}) \to T'_{ij}(t, \vec{x}) = T_{ij}(t, \vec{x} - \Delta\vec{v}\, t). \tag{2.3}$$

- It remains to discuss the energy density and energy flux density. The energy density has one part from the macroscopic kinetic motion of the fluid and one part from the fluids internal energy

$$\mathscr{E} = \frac{1}{2}\rho \vec{v}^2 + \varepsilon.$$

  The transformation behavior follows from the one of the fields ρ, $\vec{v}$ and

$$\varepsilon(t, \vec{x}) \to \varepsilon'(t, \vec{x}) = \varepsilon(t, \vec{x} - \Delta\vec{v}\, t).$$

  The energy flux density can be written as

$$\mathscr{E}_j = \left(\frac{1}{2}\rho \vec{v}^2 + \varepsilon\right) v_j + v_i T_{ij} + q_j.$$

  The first term is simply the transportation of kinetic and internal energy with the fluid motion. The second is an energy transport connected to momentum transport described by the stress tensor T_{ij}. The last term is also called the heat flow vector. Without proof, we state that the energy conservation law $\partial_t \mathscr{E} + \partial_j \mathscr{E}_j = 0$ is invariant with respect to Galilei boosts if the heat flow vector field has the simple transformation behavior

$$q_j(t, \vec{x}) \to q'_j(t, \vec{x}) = q_j(t, \vec{x} - \Delta\vec{v}\, t). \tag{2.4}$$

  Exercise: Check this.

The symmetry group of non-relativistic fluids

- Exercise: Show that the symmetry transformations of non-relativistic mechanics which consist of translations in space and time, spatial rotations and Galilei boosts constitute a group in the mathematical sense of the word.

3 Thermodynamics

Foundations of Thermodynamics

- We recall that thermodynamics gives an effective, statistical description of a system with many degrees of freedom.
- It is based on a maximization of entropy ($\hat{\rho}$ is the quantum mechanical density operator, Ω is the available phase-space volume)

$$S = -k_B \langle \ln \hat{\rho} \rangle = k_B \ln \Omega$$

  for given conserved quantities such as energy E, particle number N and for fixed volume V.
- A system in thermodynamic equilibrium is actually completely described by the conserved variables E, N and the volume and all other thermodynamic quantities can be expressed in terms of these, for example in the micro-canonical ensemble staring from the entropy $S(E, N, V)$ with the differential

$$dS = \frac{1}{T} dE - \frac{\mu}{T} dN + \frac{p}{T} dV, \tag{3.1}$$

  where T is the temperature, μ is the chemical potential and p is the pressure.
- The relations

$$\left(\frac{\partial S}{\partial E}\right)_{N,V} = \frac{1}{T}, \qquad \left(\frac{\partial S}{\partial N}\right)_{E,V} = -\frac{\mu}{T}, \qquad \left(\frac{\partial S}{\partial V}\right)_{E,N} = \frac{p}{T}$$

  can actually be seen as the *definitions* of the temperature, chemical potential and pressure.
- For a homogeneous fluid at rest it is useful to work with the internal energy density $\varepsilon = E/V$, particle number density $n = N/V$ and entropy density $s = S/V$. One finds from (3.1)

$$dS = V ds + s dV = \frac{V}{T} d\varepsilon + \frac{\varepsilon}{T} dV - \frac{\mu V}{T} dn - \frac{\mu n}{T} dV + \frac{p}{T} dV.$$

  For a homogeneous fluid, the change of V does not modify the densities s, ε and n. One obtains therefore two separate equations, the Gibbs-Duhem relation

$$\varepsilon + p = Ts + \mu n, \tag{3.2}$$

  and for the entropy density $s(\varepsilon, n)$ the differential

$$ds = \frac{1}{T} d\varepsilon - \frac{\mu}{T} dn. \tag{3.3}$$

Thermodynamic potentials and Legendre transforms

- The function $s(\varepsilon, n)$ can be inverted to give $\varepsilon(s, n)$ and the differential relation (3.3) implies

$$d\varepsilon = T ds + \mu dn.$$

- Other thermodynamic potentials can be obtained from the above relations by Legendre transforms. For example, the free energy density $f = \varepsilon - Ts$ has the differential

$$df = d\varepsilon - T ds - s dT = -s dT + \mu dn.$$

  It is a function of the variables T and n.

- Particularly important is the pressure because it corresponds to the thermodynamic potential density of the grand canonical ensemble. From (3.2) one obtains

$$dp = -d\varepsilon + Tds + sdT + \mu dn + nd\mu = sdT + nd\mu. \tag{3.4}$$

  The last relation is known as the differential Gibbs-Duhem relation. It states that for a homogeneous fluid in equilibrium, pressure cannot be varied independent of T and μ.

Equation of state

- Because for a macroscopic system in equilibrium, all thermodynamic ensembles are equivalent, one can actually construct all thermodynamic observables from the function $p(T,\mu)$ alone. This is useful because $p(T,\mu)$ is what one can best determine from microscopic calculations, for example in terms of quantum field theory.
- Because of this property, the function $p(T,\mu)$ or an equivalent relation are sometimes called *equation of state.*
- Entropy density and particle number density are directly obtained from $p(T,\mu)$ by

$$s = \left(\frac{\partial p}{\partial T}\right)_\mu, \qquad n = \left(\frac{\partial p}{\partial \mu}\right)_T.$$

- Once a thermodynamic quantity has been expressed in terms of $p(T,\mu)$ it is a function of T and μ only.
- It is possible to change the variables T and μ for two other independent variables, for example ε and n.
- If more then one particle number is conserved, there are several chemical potentials and accordingly more thermodynamic variables are needed.

Changes in thermodynamic variables and Jacobi determinant

- For manipulating expressions in thermodynamics, it is useful to work with the Jacobi determinant for a change of variables

$$\frac{\partial(x,y)}{\partial(a,b)} = \begin{vmatrix} (\frac{\partial x}{\partial a})_b & (\frac{\partial x}{\partial b})_a \\ (\frac{\partial y}{\partial a})_b & (\frac{\partial y}{\partial b})_a \end{vmatrix} = \left(\frac{\partial x}{\partial a}\right)_b \left(\frac{\partial y}{\partial b}\right)_a - \left(\frac{\partial x}{\partial b}\right)_a \left(\frac{\partial y}{\partial a}\right)_b.$$

- Note the symmetry properties

$$\frac{\partial(x,y)}{\partial(a,b)} = -\frac{\partial(x,y)}{\partial(b,a)} = -\frac{\partial(y,x)}{\partial(a,b)} = \frac{\partial(y,x)}{\partial(b,a)}.$$

- As a special case, one has

$$\frac{\partial(x,y)}{\partial(a,y)} = \left(\frac{\partial x}{\partial a}\right)_y$$

- Two subsequent changes of variables imply

$$\frac{\partial(x,y)}{\partial(a,b)}\frac{\partial(a,b)}{\partial(m,n)} = \frac{\partial(x,y)}{\partial(m,n)}$$

- Because of

$$\frac{\partial(x,y)}{\partial(x,y)} = 1$$

  the Jacobi determinant has in particular the property

$$\frac{\partial(x,y)}{\partial(a,b)} = \left(\frac{\partial(a,b)}{\partial(x,y)}\right)^{-1}.$$

Heat capacity densities

- The heat capacity densities c_V and c_P describe the amount of heat $dQ = TdS$ associated with a change in temperature at fixed volume and pressure, respectively.

- As an exercise, we derive the following relation relating c_V to $p(T, \mu)$,

$$\begin{aligned} c_V &= \frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_{V,N} = T\left(\frac{\partial s}{\partial T}\right)_n = T\frac{\partial(s,n)}{\partial(T,n)} = T\frac{\partial(s,n)}{\partial(T,\mu)}\frac{\partial(T,\mu)}{\partial(T,n)} \\ &= T\frac{\frac{\partial(s,n)}{\partial(T,\mu)}}{\frac{\partial(T,n)}{\partial(T,\mu)}} = \frac{T\left(\frac{\partial^2 p}{\partial T^2}\frac{\partial^2 p}{\partial \mu^2} - \frac{\partial^2 p}{\partial T \partial \mu}\frac{\partial^2 p}{\partial T \partial \mu}\right)}{\frac{\partial^2 p}{\partial \mu^2}}. \end{aligned} \tag{3.5}$$

- Similarly, one obtains (exercise)

$$c_P = \frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_{P,N} = \frac{T}{1/n}\left(\frac{\partial(s/n)}{\partial T}\right)_P = \frac{T}{n^2}\left(n^2\frac{\partial^2 p}{\partial T^2} - 2sn\frac{\partial^2 p}{\partial T\partial\mu} + s^2\frac{\partial^2 p}{\partial \mu^2}\right). \tag{3.6}$$

Compressibility

- The isothermal and adiabatic compressibilities describe changes in the volume or density associated with a change in pressure at fixed temperature or entropy per particle.

- As an exercise, derive the following relations

$$\begin{aligned} \kappa_T &= -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,N} = \frac{1}{n}\left(\frac{\partial n}{\partial p}\right)_T = \frac{1}{n^2}\frac{\partial^2 p}{\partial \mu^2}, \\ \kappa_S &= -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{S,N} = \frac{1}{n}\left(\frac{\partial n}{\partial p}\right)_{s/n} = \frac{\frac{\partial^2 p}{\partial T^2}\frac{\partial^2 p}{\partial \mu^2} - \frac{\partial^2 p}{\partial T\partial\mu}\frac{\partial^2 p}{\partial T\partial\mu}}{n^2\frac{\partial^2 p}{\partial T^2} - 2sn\frac{\partial^2 p}{\partial T\partial\mu} + s^2\frac{\partial^2 p}{\partial\mu^2}}. \end{aligned} \tag{3.7}$$

- Similarly, one can obtain for the thermal expansion coefficient

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{P,N} = -\frac{1}{n}\left(\frac{\partial n}{\partial T}\right)_P = \frac{1}{n^2}\left(s\frac{\partial^2 p}{\partial \mu^2} - n\frac{\partial^2 p}{\partial T\partial\mu}\right). \tag{3.8}$$

Sound velocity

- We will see later that the square of the sound velocity c_s is related to $\partial p/\partial\rho$ at fixed entropy per particle s/n.

- One can obtain the following expression in terms of $p(T, \mu)$ (exercise)

$$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s/n} = \frac{n^2\frac{\partial^2 p}{\partial T^2} - 2sn\frac{\partial^2 p}{\partial T\partial\mu} + s^2\frac{\partial^2 p}{\partial\mu^2}}{\rho\left(\frac{\partial^2 p}{\partial T^2}\frac{\partial^2 p}{\partial \mu^2} - \frac{\partial^2 p}{\partial T\partial\mu}\frac{\partial^2 p}{\partial T\partial\mu}\right)}. \tag{3.9}$$

Fluid in global thermal equilibrium at rest

- A fluid at rest (i. e. with vanishing velocity $\vec{v} = 0$) and in global thermal equilibrium is characterized by translational symmetries in time and space as well as spatial rotation symmetry. Due to rotation symmetry, the mass current ρ_j and momentum density $\mathscr{P}_j$ as well as energy flux density $\mathscr{E}_j$ vanish in equilibrium. The momentum flux density $\mathscr{P}_{jk}$ can only have a term $\sim \delta_{jk}$. The conserved currents are

$$\rho = mn, \qquad \rho_j = 0, \qquad \mathscr{E} = \varepsilon, \qquad \mathscr{E}_j = 0, \qquad \mathscr{P}_k = 0, \qquad \mathscr{P}_{jk} = \delta_{jk} p.$$

- Only two thermodynamic variables are independent, however. One might take them to be T and μ and the other quantities $p(T,\mu)$, $n(T,\mu) = \frac{\partial}{\partial\mu}p(T,\mu)$ and

  $$\varepsilon(T,\mu) = -p(T,\mu) + T\frac{\partial p}{\partial T} + \mu\frac{\partial p}{\partial \mu}$$

  are dependent. They are related to T and μ by the equation of state.

- Alternatively, one could take ρ and ε as independent variables and use the equation of state to relate p to them.
  Exercise: Derive an equation for $p(\varepsilon,\rho)$ by starting from the entropy density in the microcanonic ensemble $s(\varepsilon,n)$.

Fluid in global thermal equilibrium in motion

- Let us now consider a fluid that is moving with a constant fluid velocity $\vec{v}$. It is assumed to be still in equilibrium in the sense that no entropy is being produced. We will obtain the description by making a Galilei transformation from the fluid rest frame.

- The transformation laws that have been discussed give for a boost with $\Delta\vec{v} = \vec{v}$ the transformed fields

  $$\rho = mn, \qquad \rho_j = \rho v_j, \qquad \mathscr{E} = \frac{1}{2}\rho\vec{v}^2 + \varepsilon, \qquad \mathscr{E}_j = \left(\frac{1}{2}\rho\vec{v}^2 + \varepsilon + p\right)v_j,$$
  $$\mathscr{P}_k = \rho v_j, \qquad \mathscr{P}_{jk} = \rho v_i v_j + \delta_{jk}p.$$

- Because all fields are constant in time and space, the conservation laws are obviously fulfilled.

4 Fluid dynamic equations of motion

Implications of the conservation laws

- We now consider a fluid in motion in the more realistic situation where the conserved fields

  $$\rho, \rho_j, \qquad \mathscr{E}, \mathscr{E}_j, \qquad \mathscr{P}_k, \mathscr{P}_{jk}$$

  are not homogeneous (i. e. constant) in space.

- We will derive equations of motion based on the conservation laws

  $$\partial_t \rho + \partial_j \rho_j = 0, \qquad \partial_t \mathscr{E} + \partial_j \mathscr{E}_j = 0, \qquad \partial_t \mathscr{P}_k + \partial_j \mathscr{P}_{jk} = 0.$$

  These are 1+1+3=5 independent differential equations.

- From Galilei invariance we saw that it is possible and useful to write the conserved fields as

  $$\begin{aligned} &\rho, && \rho_j = \rho\, v_j, \\ \mathscr{P}_k &= \rho\, v_j, & \mathscr{P}_{jk} &= \rho\, v_j v_k + T_{jk}, \\ \mathscr{E} &= \frac{1}{2}\rho\, \vec{v}^2 + \varepsilon, & \mathscr{E}_j &= \left(\frac{1}{2}\rho\, \vec{v}^2 + \varepsilon\right) v_j + v_i T_{ij} + q_j. \end{aligned} \tag{4.1}$$

  The fields ρ, v_j, T_{jk}, ε and q_j have $1 + 3 + 6 + 1 + 3 = 14$ components or field degrees of freedom. (Note that naively, the conserved fields have $4 + 4 + 9 = 17$ degrees of freedom. Three of them have already been fixed by the implications of Galilei invariance in the form $\rho_j = \mathscr{P}_j$.)

- We now study the implications of the conservation laws. The first is for conservation of mass,

  $$\partial_t \rho + v_j \partial_j \rho + \rho\, \partial_j v_j = 0. \tag{4.2}$$

  This equation is sometimes also called the *continuity equation.* It describes how the mass density evolves in time for given fluid velocity field v_j.

- It is interesting to observe that the time derivative enters in terms of the combination

  $$D_t = \partial_t + v_j \partial_j.$$

  This is actually an example for a *covariant derivative.* Under Galilei transformations one has

  $$\partial_t \rho(t, \vec{x}) \to \partial_t \rho'(t, \vec{x}) = (\partial_t \rho)(t, \vec{x} - \Delta\vec{v}t) - \Delta v_j \partial_j \rho(t, \vec{x} - \Delta\vec{v}t)$$

  and

  $$v_j(t, \vec{x}) \to v_j'(t, \vec{x}) = v(t, \vec{x}) + \Delta v_j$$

  such that $D_t \rho$ has the simple transformation law

  $$D_t \rho(t, \vec{x}) \to D_t \rho'(t, \vec{x}) = (D_t \rho)(t, \vec{x} - \Delta\vec{v}t).$$

  The covariant derivative describes a change along the flow lines of the fluid. Locally, i. e. for a given coordinate point $(t, \vec{x})$, one can boost to the fluid rest frame where $v_j(t, \vec{x}) = 0$ and therefore $D_t = \partial_t$.

- Let us now consider the momentum conservation law. It becomes

$$\partial_t \mathscr{P}_k + \partial_j \mathscr{P}_{jk} = \partial_t(\rho\, v_k) + \partial_j(\rho\, v_j v_k + T_{jk}) = 0,$$

  or

$$\rho\,(\partial_t + v_j\partial_j)\,v_k + \partial_j T_{jk} = 0. \tag{4.3}$$

  We have used in the last equation the mass conservation law (4.2). Observe that again a Galilei covariant derivative appears. It is instructive to check that (4.3) is invariant under Galilei transformations.

- The momentum conservation law in (4.3) describes the time evolution of the fluid velocity v_j for given mass density ρ and stress tensor T_{jk}.

- Finally, the energy conservation law can be brought to the following form with the help of the conservation laws for mass and momentum,

$$(\partial_t + v_j\partial_j)\,\varepsilon + \varepsilon\,\partial_j v_j + (\partial_j v_k)\,T_{jk} + \partial_j q_j = 0. \tag{4.4}$$

  This equation describes the time evolution of the energy density ε for given fluid velocity v_j, stress tensor T_{ij} and heat flux q_j.

Fluid dynamic derivative expansion

- Together, the conservation laws (4.2), (4.3) and (4.4) constitute 5 differential equations the describe how ρ, ε and v_j evolve in time. To solve them (for given initial conditions) we need 9 additional equations for the 6 components of the stress tensor T_{jk} and the 3 components of the heat flux q_j.

- For a homogeneous situation we know already that

$$T_{jk} = \delta_{jk}\,p, \qquad\qquad q_j = 0, \tag{4.5}$$

  where p is related to ε and ρ by the thermodynamic equilibrium relation, the equation of state $p = p(\varepsilon, \rho)$. Deviations from these relations must therefore involve *derivatives* of the fluid fields.

- Fluid dynamics as a physical theory is organized as an expansion in derivatives. It describes situations with a homogeneous thermal equilibrium as well as small deviations around that in the sense that the fluid fields must be varying slowly enough in time and space to make the derivative expansion justified.

- The lowest order of the derivative expansion consists in neglecting all terms with derivatives in the constitutive relations and in using the relations (4.5) also for a fluid which is not homogeneous. That leads to the equations of ideal fluid dynamics, which we discuss now first.

Ideal fluid equations of motion

- We now discuss the implications of the conservation laws for an ideal fluid, which is specified by the constitutive relations (4.5).

- The continuity equation is independent of any ideal fluid approximation. It remains of the form (4.2).

- Consider now the conservation of momentum (4.3). It becomes with (4.5)

  $$\rho\,\partial_t v_k + \rho\, v_j \partial_j v_k + \partial_k p = 0. \tag{4.6}$$

  This is known as the *Euler equation* (after Leonard Euler (1707-1783)).

- We have assumed here that no external force is acting on the fluid. If this is the case, the momentum conservation equation and accordingly Euler's equation are modified,

  $$\rho\,\partial_t v_k + \rho\, v_j \partial_j v_k + \partial_k p = \mathscr{F}_k,$$

  where $\mathscr{F}_k$ is a force density. For example, for a fluid in a gravitational field one has $\mathscr{F}_k = \rho\, g_k$ where g_k is the acceleration due to gravity.

- The energy conservation law (4.4) reads for an ideal fluid using (4.5) to

  $$\partial_t \varepsilon + v_j \partial_j \varepsilon + (\varepsilon + p)\partial_j v_j = 0. \tag{4.7}$$

- There is an alternative way of stating the energy conservation law. For that, one use the differential of entropy density

  $$ds = \frac{1}{T} d\varepsilon - \frac{\mu}{T} dn = \frac{1}{T} d\varepsilon - \frac{\mu}{mT} d\rho,$$

  to write

  $$\partial_t s = \frac{1}{T} \partial_t \varepsilon - \frac{\mu}{mT} \partial_t \rho \tag{4.8}$$

  and similar for the spatial derivative. One can then combine (4.7) and the continuity equation (4.2) to

  $$\partial_t s + v_j \partial_j s + \left(\frac{\varepsilon + p}{T} - \frac{\mu \rho}{mT} \right) \partial_j v_j = 0.$$

  The expression in the bracket can be written as

  $$\frac{\varepsilon + p}{T} - \frac{\mu\rho}{mT} = \frac{\varepsilon + p - \mu n}{T} = s$$

  where we have used the Gibbs-Duhem relation (3.2). In summary, one obtains

  $$\partial_t s + \partial_j (s v_j) = 0. \tag{4.9}$$

  The last equation has an interesting physical interpretation: For an ideal fluid, the entropy is conserved, in addition to mass, energy and momentum. One says: Ideal fluid dynamics is non-dissipative.

- Beyond the ideal fluid approximation, one does not have any reason to expect such a conservation law. In contrast, the second law of thermodynamics states that entropy can never decrease. In the presence of dissipation, the entropy will actually increase.

Dissipative fluid equations of motion

- We now consider corrections to the constitutive relations at first order in gradients. The stress tensor T_{jk} must transform as a tensor under rotations and according to (2.3) under Galilei transformations. Also, we are looking for a correction to the ideal fluid constitutive relation (4.5) that is of first order in derivatives.

- With respect to rotations, one can decompose a general symmetric tensor like T_{jk} into a traceless part $T_{jk} - \frac{1}{3}\delta_{jk}T_{ii}$ and the trace T_{ii} which transforms like a scalar. The only traceless and symmetric tensor of first order in derivatives of the fluid fields ρ, ε and v_j is proportional to
$$\sigma_{jk} = \frac{1}{2}\partial_j v_k + \frac{1}{2}\partial_k v_j - \frac{1}{3}\delta_{jk}\partial_i v_i.$$
For the trace, the following terms could in principle appear
$$\partial_i v_i, \qquad (\partial_t + v_j\partial_j)\varepsilon, \qquad (\partial_t + v_j\partial_j)\rho.$$
However, the conservation laws (4.2) and (4.4) imply
$$\begin{aligned}(\partial_t + v_j\partial_j)\rho &= -\rho\partial_j v_j,\\ (\partial_t + v_j\partial_j)\varepsilon &= -\varepsilon\partial_j v_j - (\partial_j v_k)\,T_{jk} - \partial_j q_j.\end{aligned}$$

- To lowest order in derivatives, one can therefore write
$$T_{jk} = \delta_{jk}p(\varepsilon,\rho) - 2\,\eta(\varepsilon,\rho)\sigma_{jk} - \delta_{jk}\,\zeta(\varepsilon,\rho)\partial_i v_i.$$
The coefficient η is the *shear viscosity* and ζ is the *bulk viscosity*. They are both functions of two independent thermodynamic variables, e. g. ε and ρ or T and μ.

- The momentum conservation law becomes now (in the absence of external forces)
$$\rho\left(\partial_t + v_j\partial_j\right)v_k + \partial_k p - \partial_j\left[\eta\left(\partial_j v_k + \partial_k v_j - \frac{2}{3}\delta_{jk}\partial_i v_i\right)\right] - \partial_k\left[\zeta\partial_i v_i\right] = 0. \tag{4.10}$$
This is the *Navier-Stokes equation* (named after Claude Louis Marie Henri Navier (1785 - 1836) and Georg Gabriel Stokes (1819 - 1903)). One can often assume that η and ζ are constant in space. In that case, the Navier-Stokes equation becomes
$$\rho\,\partial_t v_k + \rho\,v_j\partial_j v_k + \partial_k p - \eta\partial_j\partial_j v_k - \left(\zeta + \tfrac{1}{3}\eta\right)\partial_k\partial_j v_j = 0. \tag{4.11}$$

- The heat flux q_j must be a vector under rotations and transform with respect to Galilei transformations according to (2.4). To first order in derivatives of the fluid fields ρ, ε and v_j, two terms could appear
$$\partial_j\varepsilon \qquad \text{and} \qquad \partial_j\rho.$$
To see which combination of them is physically relevant, we have to consider the generalization of (4.9). For small deviations from an equilibrium situation we can still use the differential (4.8) and obtain
$$\partial_t s + \partial_j(s v_j) + \frac{1}{T}(\partial_j v_k)(T_{jk} - \delta_{jk}p(\varepsilon,\rho)) + \frac{1}{T}\partial_j q_j = 0.$$
The entropy current should actually have a part proportional to the heat flux (cf. $dQ = TdS$)
$$s_j = s v_j + \frac{q_j}{T}$$
and with $\partial_j\frac{q_j}{T} = \frac{1}{T}\partial_j q_j - q_j\frac{\partial_j T}{T^2}$ one obtains
$$\partial_t s + \partial_j s_j = -\frac{1}{T}(\partial_j v_k)(T_{jk} - \delta_{jk}p(\varepsilon,\rho)) - \frac{1}{T^2}q_j\partial_j T.$$

- Entropy production is positive, $\partial_t s + \partial_j s_j \geq 0$, for

  $$T_{jk} - \delta_{jk} p(\varepsilon, \rho) = -2\eta \sigma_{jk} - \delta_{jk} \zeta \partial_i v_i$$

  with $\eta \geq 0$ and $\zeta \geq 0$ and

  $$q_j = -\kappa(\varepsilon, \rho) \partial_j T$$

  where $\kappa(\varepsilon, q) \geq 0$ is the *heat conductivity* or *thermal conductivity.* Indeed, that results in

  $$\partial_t s + \partial_j s_j = \frac{1}{T} \left[2\eta \sigma_{jk} \sigma_{jk} + \zeta (\partial_j v_j)^2 \right] + \frac{\kappa}{T^2} \left[\partial_j T \right]^2 \geq 0.$$

- Note that the heat flux can be written as a linear combination of $\partial_j \varepsilon$ and $\partial_j \rho$, indeed. Exercise: show this explicitly.

- The energy conservation law becomes for the first order theory

  $$(\partial_t + v_j \partial_j)\, \varepsilon + (\varepsilon + p)\, \partial_j v_j - 2\eta\, \sigma_{jk} \sigma_{jk} - \zeta (\partial_j v_j)^2 - \partial_j (\kappa\, \partial_j T) = 0. \tag{4.12}$$

  The continuity equation (4.2), the Navier-Stokes equation (4.10) and the energy conservation equation in the form (4.12) provide a closed set of differential equations that can be used to propagate the fields ρ, v_j and ε in time. To solve them one also needs the thermodynamic equation of state $p(\varepsilon, \rho)$ and the transport coefficients η, ζ and κ as a function of the thermodynamic variables ε and ρ.

- For some purposes it is useful to have also an equation for the entropy per unit mass s/ρ. One finds by combining previous equations

  $$\partial_t (s/\rho) + v_j \partial_j (s/\rho) = \frac{1}{\rho T} \left[2\eta \sigma_{jk} \sigma_{jk} + \zeta (\partial_j v_j)^2 \right] + \frac{1}{\rho T} \partial_j \left[\kappa \partial_j T \right].$$

5 Ideal fluid flows

We discuss now solutions to the ideal fluid equations of motion. Of course, these solutions depend also on boundary conditions for different situations. The ideal fluid equations of motion describe situations where dissipative properties do not play any role, i. e. shear and bulk viscosity as well as heat conductivity can be neglected.

Isentropic flows

- The continuity equation (4.2) and the entropy conservation law (4.9) (valid for ideal fluids) imply also
$$\partial_t \left(\frac{s}{\rho}\right) + v_j \partial_j \left(\frac{s}{\rho}\right) = 0.$$
A particular simple solution to this equation is
$$\frac{s}{\rho} = \text{const}.$$
In particular, if s/ρ is spatially constant, it will also not change in time.
- Flows where this is a good approximation are called *isentropic*.
- To describe isentropic flows, the following thermodynamic identities are useful
$$d\left(\frac{\varepsilon + p}{\rho}\right) = \frac{1}{m} d\left(\frac{sT + \mu n}{n}\right) = \frac{1}{m}\left[T d\left(\frac{s}{n}\right) + \frac{s}{n} dT + d\mu\right] = T d\left(\frac{s}{\rho}\right) + \frac{1}{\rho} dp$$
where we have used the Gibbs-Duhem relation (3.2) and the differential of pressure (3.4). In particular, for isentropic flows one has
$$\frac{1}{\rho}\partial_k p = \partial_k \left(\frac{\varepsilon + p}{\rho}\right).$$
- Moreover it is useful to use the following vector identity
$$\left(\vec{v} \times \vec{\nabla} \times \vec{v}\right)_k = \epsilon_{kmn}\epsilon_{nrs} v_m \partial_r v_s = (\delta_{kr}\delta_{ms} - \delta_{ks}\delta_{mr}) v_m \partial_r v_s = \partial_k \left(\frac{\vec{v}^2}{2}\right) - v_m \partial_m v_k$$
- With this, Euler's equation (4.6) becomes
$$\partial_t v_k - (\vec{v} \times \vec{\nabla} \times \vec{v})_k + \partial_k \left(\frac{\varepsilon + p}{\rho} + \frac{\vec{v}^2}{2}\right) = 0. \tag{5.1}$$

Bernoulli's equation

- *Stationary* or *steady* flows describe situations where the fluid fields are independent of time such that e. g. $\partial_t v_k = 0$.
- Because the first term in (5.1) vanishes for stationary flows and the second term is orthogonal to v_k one obtains
$$v_k \partial_k \left(\frac{\varepsilon + p}{\rho} + \frac{\vec{v}^2}{2}\right) = 0.$$
This is Bernoulli's equation. It states that the expression in bracket is constant along the flow lines.

- One can in fact show using the continuity equation and entropy conservation for stationary flows that
$$v_j \partial_j \left(\frac{s}{\rho} \right) = 0,$$
which implies that that Bernoulli's equation does not need the assumption of an isentropic flow.

- For an incompressible fluid one has $\partial_j \rho = 0$. In that case one can directly show from Euler's equation (4.6) for a stationary flow
$$v_j \partial_j \left(\frac{1}{2} \rho \vec{v}^2 + p \right) = 0. \tag{5.2}$$
The combination $\frac{1}{2}\rho\vec{v}^2$ is sometimes called dynamic pressure. Along flow lines, the sum of thermodynamic pressure and dynamic pressure is constant. In regions of high fluid velocity, the pressure p must be smaller. This allows to understand the under-pressure on the upper side of aerofoils (Tragflächen) or how a simple water aspirator (Wasserstrahlpumpe) works.

Vorticity

- The field $\vec{w} = \vec{\nabla} \times \vec{v}$ or
$$w_j = \epsilon_{jkl} \partial_k v_l \tag{5.3}$$
is called *fluid vorticity.* Indeed it describes circular motion. Consider for example some area A with boundary $\partial A = C$. One has
$$\int_A w_j dA_j = \int_A (\epsilon_{jkl} \partial_k v_l) dA_j = \oint_C v_j dx_j,$$
where we have used Stokes' theorem. The integral along the line is a measure of circular motion.

- For isentropic flows one obtains by taking the curl of Euler's equation (5.1) an equation for the time evolution of vorticity
$$\partial_t w_k - (\vec{\nabla} \times (\vec{v} \times \vec{w}))_k = 0. \tag{5.4}$$

Kelvin's circulation theorem

- Another interesting property of ideal flows is the *conservation of circulation.* Consider the following line integral
$$\Gamma = \oint_C v_j(x) dx_j,$$
the so-called velocity circulation. The integral goes around some closed curve $C = \partial A$. We now consider how this quantity changes in time if the line elements are transported along the fluid with the fluid velocity.

- The time derivative is
$$\frac{d}{dt}\Gamma = \oint_C \left(\frac{d}{dt} v_j(x) \right) dx_j + \oint_C v_j(x) \left(\frac{d}{dt} dx_j \right).$$
In the first expression one can use
$$\frac{d}{dt} v_j = (\partial_t + v_k \partial_k)\, v_j = -\partial_j \left(\frac{\varepsilon + p}{\rho} \right),$$

in the second expression

$$\frac{d}{dt}dx_j = dv_j.$$

Using Stokes theorem gives

$$\frac{d}{dt}\Gamma = -\int_A \left[\epsilon_{jkl}\partial_k\partial_l\left(\frac{\varepsilon+p}{\rho}\right)\right]dA_j + \oint_C d\left(\frac{\vec{v}^2}{2}\right).$$

The first term vanishes as it is the curl of a gradient. The second term vanishes because a closed integral of a total differential vanishes.

- In summary one obtains

  $$\frac{d}{dt}\Gamma = 0$$

  which is known as Kelvin's theorem or theorem of circulation.

- One can apply Kelvin's theorem to an infinitesimal small contour $dC = \partial dA$ and finds

  $$\oint_{dC} v_j dx_j = \int_{dA} \left[\epsilon_{jkl}\partial_k v_l\right] dA_j = w_j dA_j = \text{const}.$$

  Intuitively speaking, the vorticity moves with the fluid.

- One might be tempted to think that vorticity cannot appear in the flow if it is absent on some surface upstream / in the past (see also (5.4)). However, tho points have to be taken into account:
  - Real fluids are not completely isentropic and inhomogeneous heating can produce vorticity.
  - The circulation theorem cannot be applied along boundary surfaces (the circle cannot be closed). Vorticity can be generated by boundary layers and be transported downstream.

Potential flows

- Potential flows are such that one can write the fluid velocity as a gradient

  $$v_j = \partial_j\phi, \tag{5.5}$$

  with some scalar function ϕ, the so-called *velocity potential.* Because $\vec{\nabla}\times\vec{\nabla}\phi = 0$, potential flows are vorticity-free. Also, the circulation along any curve vanishes,

  $$\oint v_j dx_j = 0$$

  and in particular there cannot be any closed flow lines.

- Euler's equation for an isentropic potential flow reads

  $$\partial_k\left(\partial_t\phi + \frac{\varepsilon+p}{\rho} + \frac{\vec{v}^2}{2}\right) = 0.$$

  By integration one finds

  $$\partial_t\phi + \frac{\varepsilon+p}{\rho} + \frac{\vec{v}^2}{2} = c(t)$$

  and in fact one can choose ϕ such that $c(t) = 0$. (There is a gauge symmetry $\phi \to \phi + f(t)$ which does not change $\vec{v} = \vec{\nabla}\phi$.)

- For stationary, isentropic potential flows one has a stronger form of Bernoulli's equation where

  $$\frac{\varepsilon+p}{\rho} + \frac{\vec{v}^2}{2} = \text{const},$$

  with a constant that is the same for all flow lines.

Incompressible flows

- For situations where the density ρ is constant in space and time one speaks of an *incompressible flow*. Usually density inhomogeneities propagate with the speed of sound c_S and if the flow pattern changes slowly compared to that one has often

$$\rho = \text{const.}$$

- The continuity equation (4.2) reduces to

$$\partial_j v_j = 0. \tag{5.6}$$

- Because of $\rho = \text{const}$ and $s = \text{const}$ one has only the three components of v_j left as variables that need to be determined. One can take as equations (5.6) and the equation for vorticity (5.4). Indeed, for given boundary conditions and vorticity w_k one can uniquely reconstruct the velocity from (5.3) and (5.6). This is analogous to the situation in magnetostatics where the magnetic field is fixed by

$$\partial_j B_j = 0, \qquad \epsilon_{kmn}\partial_m B_n = \mu_0 \, j_k.$$

The evolution equation (5.4) determines the change of vorticity with time. The pressure can be reconstructed *a posteriori* from Euler's equation.

Incompressible potential flows

- For isentropic, incompressible potential flows, the equations become particularly simple. Because of (5.5) and (5.6) one has

$$\partial_j \partial_j \phi = \Delta\phi = 0. \tag{5.7}$$

- The integrated version of Euler's equation reads for an incompressible potential flow

$$\partial_t \phi + \frac{p}{\rho} + \frac{1}{2}(\partial_j \phi)^2 = c(t). \tag{5.8}$$

- On a boundary given by the surface of some (moving) body, the fluid velocity must be parallel to the surface or the normal component has to vanish. This gives rise to von Neumann - type boundary conditions

$$n_j \partial_j \phi = 0.$$

- Consider some body moving through a fluid (isentropic, incompressible potential flow). The velocity potential ϕ is governed by the instantaneous equation (**??**) and depends via the boundary condition on the velocity of the body but not on the acceleration.

- It is illuminating to compare the equations for the idealized situation of an isentropic, incompressible and vorticity free fluid to those of electrostatics or magnetostatics in free space (no charges and currents)

$$\begin{aligned} &\vec\nabla \cdot \vec v = 0, \qquad &&\vec\nabla \cdot \vec E = 0, \qquad &&\vec\nabla \cdot \vec B = 0, \\ &\vec\nabla \times \vec v = 0, \qquad &&\vec\nabla \times \vec E = 0, \qquad &&\vec\nabla \times \vec B = 0. \end{aligned}$$

Solutions can therefore be found in a similar way as we will use below.

Potential flow past a sphere

- Consider the problem of a spherical ball falling through a fluid. If it goes too slow, viscosity will become important. If it goes too fast, whirls will develop and ultimately turbulence. For intermediate velocities, the assumptions we made for an incompressible potential flow match approximately.
- We will consider the problem in the reference frame of the sphere (which we take to have radius a). The equations we have to solve are

  $$\Delta\phi = 0 \qquad (r > a)$$

  with boundary condition

  $$\frac{\partial\phi}{\partial r} = 0 \qquad (r = a)$$

  corresponding to vanishing velocity component normal to the surface of the sphere. Very far away from the surface, the fluid velocity should be in z-direction

  $$\frac{\partial\phi}{\partial z} = v_0 \qquad (r \gg a).$$

- Without the sphere, the velocity potential would be

  $$\phi = v_0 z + \text{const.}$$

  This is obviously a solution to the Laplace equation. We will now add to this another solution (for $r > a$). We take over intuition and knowledge from electrostatics. In particular, a solution that fits to the axial symmetries of the present problem is a dipole in z direction. So let's try the following

  $$\phi = v_0 z + \frac{pz}{r^3} = v_0 r\cos\theta + \frac{p\cos\theta}{r^2}.$$

  By construction, this is a solution to the Laplace equation for $r > 0$. The boundary condition on the surface is

  $$\frac{\partial\phi}{\partial r} = v_0\cos\theta - 2\frac{p\cos\theta}{r^3} = 0 \qquad (r = a),$$

  which is fulfilled for $p = \frac{1}{2}v_0 a^3$.

- In summary, the velocity potential is

  $$\phi = v_0 z\left(1 + \frac{a^3}{2r^3}\right).$$

  The fluid velocity follows as a gradient

  $$\vec{v} = \vec{\nabla}\phi = v_0\begin{pmatrix} -\frac{3a^3xz}{2r^5} \\ -\frac{3a^3yz}{2r^5} \\ 1 - \frac{a^3(2z^2-x^2-y^2)}{2r^5}\end{pmatrix}.$$

6 Two-dimensional incompressible potential flows

Some problems can be described by two-dimensional fluid equations, for example if the fluid velocity in the third direction is simply constant (usually zero) or if the fluid is confined by some external potential. It is not always possible to neglect the effects of viscosity but we will assume so for the discussion in the present section. Also, we concentrate on flows with small Mach number which can be treated as incompressible.

Flow potential and topology

- The two components of the velocity (v_1, v_2) satisfy the equations

  $$\partial_1 v_1 + \partial_2 v_2 = 0, \qquad \partial_1 v_2 - \partial_2 v_1 = 0, \tag{6.1}$$

  where the first equation is the two-dimensional version of (5.6) and the second corresponds to the condition of vanishing vorticity $w_3 = \epsilon_{3kl}\partial_k v_l = 0$.

- Due to the second relation, the two-dimensional velocity can be written locally as the gradient of the velocity potential

  $$v_1 = \partial_1 \phi, \qquad v_2 = \partial_2 \phi.$$

  With this, the second relation is automatic and the first relation implies $\Delta\phi = 0$.

- The velocity potential is given by the line integral

  $$\phi(\vec{x}) = \int^{\vec{x}} \vec{v}(\vec{y}) \cdot d\vec{y}$$

  In a simply connected domain (where all curves can be contracted to a point), this gives a unique definition of ϕ (up to an irrelevant additive constant). In a multiply connected domain, the potential as defined by the integral above may depend on the path. Consider for example a domain with two holes. The value of a closed integral

  $$\Gamma = \oint \vec{v}(\vec{x}) \cdot d\vec{x}$$

  must be given by the velocity circulation Γ_1 around the first hole and Γ_2 around the second hole and the winding numbers w_1 and w_2 as

  $$\Gamma = w_1\Gamma_1 + w_2\Gamma_2.$$

  Winding numbers are topological properties of curves.

Stream function

- Because of the first relation in (6.1), one can also introduce a so-called stream function ψ with

  $$v_1 = \partial_2 \psi, \qquad v_2 = -\partial_1 \psi$$

  such that the first relation is automatic while the second relation implies $\Delta\psi = 0$.

- The fluid velocity is orthogonal to the gradient of the stream function,

  $$\vec{v} \cdot \vec{\nabla}\psi = v_1\partial_1\psi + v_2\partial_2\psi = -v_1 v_2 + v_2 v_1 = 0,$$

  which implies that lines of constant ψ correspond to the flow or streamlines. Moreover, because the equipotential lines are always orthogonal to the fluid velocity they are also orthogonal to the streamlines.

Complex potential

- The definitions of ϕ and ψ imply the relations

$$\partial_1\phi = \partial_2\psi, \qquad \partial_2\phi = -\partial_1\psi.$$

  These differential equations are actually equivalent to the so-called Cauchy-Riemann relations. As a consequence, the pair $(\phi(x), \psi(x))$ is not only a $\mathbb{R}^2$ valued function of the argument $(x_1, x_2) \in \mathbb{R}^2$. One can actually supplement them with a complex structure and define

$$w(z) = \phi(z) + i\psi(z), \qquad z = x_1 + ix_2.$$

  By virtue of the above differential equations, $w(z)$ is actually a well defined, analytic function of the complex variable z with derivative

$$\frac{dw}{dz} = \frac{\partial\phi}{\partial x_1} + i\frac{\partial\psi}{\partial x_1} = \frac{\partial\psi}{\partial x_2} - i\frac{\partial\phi}{\partial x_2} = v_1 - iv_2.$$

- The magnitude of the flow velocity is accordingly

$$|v| = \sqrt{v_1^2 + v_2^2} = \left|\frac{dw}{dz}\right|$$

- In other words, any solution to the equations that determine an incompressible potential flow in $d = 2$ dimensions defines an analytic function $w(z)$ and any analytic function $w(z)$ corresponds to a solution of these fluid equations. This is obviously a very powerful correspondence that we will use for some further discussion below.

Simple examples

- As a first example, consider for some $c = c_1 + ic_2 \in \mathbb{C}$ the linear function $w(z) = cz + \text{const}$. The velocity is homogeneous $v_1 = c_1$, $v_2 = -c_2$. In polar coordinates $c = |v|\, e^{-i\alpha}$ where α is the angle of the flow velocity to the x_1 axis.
- Consider $w(z) = \frac{1}{2}Az^2$ for some $A \in \mathbb{R}$. The flow profile $v_1 = Ax_1$, $v_2 = -Ax_2$ corresponds to a stagnation point at $z = 0$. It describes also a corner flow in a sector of angle $\pi/2$.
- Consider more general $w(z) = \frac{1}{n}Az^n$. The flow velocity is

$$v_1 - iv_2 = |v|\, e^{-i\alpha} = Az^{n-1} = Ar^{n-1}e^{i(n-1)\theta}$$

  where $z = x_1 + ix_2 = re^{i\theta}$. Obviously, the flow velocity is in positive radial direction if the complex phase of the flow field $e^{i(1-n)\theta}$ equals the complex phase of the position $e^{i\theta}$ i.e. for $e^{in\theta} = 1$ or $\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \ldots$ and in negative radial direction for $e^{in\theta} = -1$ or $\theta = \frac{\pi}{n}, \frac{3\pi}{n}, \ldots$. One can therefore describe by $w(z) = \frac{1}{n}Az^n$ an edge flow in the segment $0 < \theta < \pi/n$. For $n = 1/2$ one reaches the maximum sector angle and the flow is around a line (or a thin plate from the three-dimensional viewpoint).

Line integrals

- Consider the following complex line integral along some curve that is closed in the anti-clockwise sense,

$$\oint_C \frac{dw}{dz}dz = \oint_C (v_1 dx_1 + v_2 dx_2) + i\oint_C (v_1 dx_1 - v_2 dx_1) = \Gamma_C + iQ_C.$$

The first term or real part Γ_C is just the velocity circulation along C while the second term or imaginary term Q_C describes actually the fluid volume flow through the curve C. Indeed,

$$Q_C = \oint_C (v_1 dx_1 - v_2 dx_1) = \oint (v_1 n_1 + v_2 n_2) dl$$

where (n_1, n_2) is the outward pointing normal vector for a curve closed in the anti-clockwise sense in the complex plane. The residue theorem implies

$$\oint_C \frac{dw}{dz} dz = 2\pi i \sum \text{Res} \left(\frac{dw}{dz} \right)$$

where the sum on the right hand side goes over the residues of dw/dz inside the domain enclosed by C. For a function $w(z)$ that is analytic everywhere in that region, the right hand side vanishes. However, if the line encloses a hole in the domain of the flow, for example by some body, there can be non-zero contributions.

- Consider now the the example $w(z) = c \ln z$. This function is locally analytic but has a branch cut (e.g. on the negative x_1-axis) and a logarithmic singularity at $z = 0$. The derivative

  $$\frac{dw}{dz} = \frac{c}{z}$$

  has obviously a pole at the origin.

- For $c \in \mathbb{R}$, the flow field is

  $$v_1 = c \frac{x_1}{x_1^2 + x_2^2}, \qquad v_2 = c \frac{x_2}{x_1^2 + x_2^2},$$

  which corresponds to a source of mass at the origin. The volume outflow per unit time is given by the contour integral along any curve that encloses the origin and the residue theorem gives $Q_C = 2\pi c$.

- Consider now $c = -i\tilde{c}$ with $\tilde{c} \in \mathbb{R}$. In that case the flow field is

  $$v_1 = -\tilde{c} \frac{x_2}{x_1^2 + x_2^2}, \qquad v_2 = \tilde{c} \frac{x_1}{x_1^2 + x_2^2},$$

  which corresponds to a vortex line at the origin of the x_1-x_2-plane and along the third direction. This flow field has vanishing vorticity everywhere (except at the origin where it is singular) but the velocity circulation is non-zero for a loop that encloses the origin. Its value is measured by the real part of the contour integral and one finds by the residue theorem $\Gamma_c = 2\pi\tilde{c}$.

- The values of Γ_C and Q_C for the above two examples can also be evaluated directly from $w(z)$ according to

  $$\Gamma_C + iQ_C = \oint \frac{dw}{dz} dz = \oint dw.$$

  Because $w(z) = c \ln z$ has a discontinuity of value $2\pi i c$ one finds $\Gamma_C + iQ_C = 2\pi i c$.

Flow around a circular cylinder

- Let's consider a flow around a cylinder of radius a and with uniform velocity $v_1 = U, v_2 = 0$ at infinity. The boundary $|z| = a$ should correspond to a stream line, i. e. $\text{Im}\, w(z) = \text{const.}$ Moreover, for $|z| > a$, the function $w(z)$ should be analytic. A possibility is

  $$w(z) = U \left(z + \frac{a^2}{z} \right).$$

This is not unique, however. One can add a vortex term which has constant imaginary part for fixed $|z|$,

$$w(z) = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\ln z.$$

- Concentrate first on the simpler case of $\Gamma = 0$. In polar coordinates $z = re^{i\theta}$

  $$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta, \qquad \psi = U\left(r - \frac{a^2}{r}\right)\sin\theta.$$

  The fluid velocity in radial and azimuthal directions follow as

  $$u_r = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta.$$

- On the surface there is obviously *slip* (as expected for an ideal fluid)

  $$u_\theta = -2U\sin\theta.$$

  The slip vanishes at the stagnation points and has a maximum magnitude $2U$ in between.

- Taking now also non-vanishing circulation Γ into account gives for the velocities

  $$u_r = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}.$$

  and therefore at $r = a$

  $$u_\theta = -2U\sin\theta + \frac{\Gamma}{2\pi a}.$$

  Interestingly, nonzero Γ shifts the stagnation points and they merge to a single point for

  $$\frac{|\Gamma|}{4\pi a U} = 1.$$

  For even larger value, the stagnation point moves away from the cylinder surface.

- Let us now consider the pressure in order to determine the force per unit length acting on the cylinder. Bernoulli's theorem gives

  $$p + \frac{1}{2}\rho\vec{v}^2 = \text{const}.$$

  Using $\vec{v}^2 = u_r^2 + u_\theta^2$ gives

  $$\frac{p}{\rho} = \text{const} - 2U^2\sin^2\theta + \frac{U\Gamma}{\pi a}\sin\theta \qquad (r = a).$$

  Interestingly, the pressure is symmetric with respect to reflection about the vertical axis $\theta \to \pi - \theta$. This implies that there is no *drag force* (parallel to the flow velocity at infinity) and it is a pure *lift force* (orthogonal to the fluid velocity at infinity). This is an instance of *d'Alembert's paradox.*

- The contribution to y-component of the force by an infinitesimal surface element is $dF_L = -pa\sin\theta d\theta$ and the net force (per unit length) is

  $$F_L = \rho\int_0^{2\pi}\left(2U^2\sin^2\theta - \frac{U\Gamma}{\pi a}\sin\theta\right)a\sin\theta\,d\theta = -\rho U\Gamma.$$

  We will see that there is actually a general theorem which gives the same result.

Conformal maps

- Consider an analytic map between complex variables

  $$Z = f(z), \qquad z = f^{-1}(Z).$$

  One can use this to define a complex potential for the new variable Z

  $$W(Z) = w(f^{-1}(Z)).$$

  At points where $f^{-1}(Z)$ is analytic and where $w(z)$ is analytic, the function $W(Z)$ is analytic and gives a solution of the two-dimensional fluid equations.

- Because of $W(Z) = w(z)$, the potential and stream lines in the new variable are the images $Z = f(z)$ from the old potential and stream lines.

- The fluid velocity follows from

  $$V_1 - iV_2 = \frac{dW}{dZ} = \frac{dw/dz}{dZ/dz} = \frac{v_1 - iv_2}{f'(z)}.$$

- Consider two corresponding points z_0 and $Z_0 = f(z_0)$ and let $f^{(n)}(z_0)$ be the first non-vanishing derivative at that point. Small deviations from that point are related by

  $$\delta Z = \frac{(\delta z)^n}{n!} f^{(n)}(z_0),$$

  and the corresponding angles

  $$\arg \delta Z = n \arg \delta z + \arg f^{(n)}(z_0).$$

  Accordingly, angle differences are related by

  $$\arg \delta Z_1 - \arg \delta Z_2 = n(\arg \delta z_1 - \arg \delta z_2).$$

  Usually, $n = 1$ in which case angles between lines are preserved. That is actually the reason for the name conformal.

- Simple examples for conformal maps are linear transformations

  $$Z = az + b.$$

  They correspond to translations, rotations and dilatations of the flow profile.
  Exercise (simple): Work this out in more detail.

- A non-trivial example is the *Joukowski transformation*

  $$Z = z + \frac{c^2}{z}.$$

  When combined with a linear transformation, it allows to map the flow around a circular cylinder to the flow around elliptic cylinders, flat plates and more interesting aerofoil-like shapes.
  Exercise (advanced): Use the Joukowski transformation to obtain (with different angle of attack and circulation) the flow past an elliptic cylinder, past a flat plate, past symmetric aerofoils and past asymmetric aerofoils.

Kutta-Joukowski condition

- So far we have not specified what actually determines the circulation around some body in a flow field that is uniform at large distances. For circular cylinders we have found solutions of the ideal fluid equations for any value of Γ.
- In reality, the circulation is actually determined by physics beyond the ideal fluid approximation. For wing-like structures with a sharp trailing edge, the *Kutta-Joukowski condition* says that the circulation is such that the flow profile is regular (non-singular) at the trailing edge. This is not a theorem but a plausible hypothesis that is experimentally confirmed in many situations.

Blasius's theorem

- We consider now the force (per unit length) acting on a body in a two-dimensional flow profile. It is transmitted by the pressure on the surface so can be calculated by an integral along the boundary. The force transmitted via a small surface element ds with angle θ is $(dF_1, dF_2) = p(-\sin\theta, \cos\theta)ds$ and therefore

$$F_1 - iF_2 = \int p(-\sin\theta - i\cos\theta)ds = -i\int pe^{-i\theta}ds = -i\oint pe^{-2i\theta}dz$$

  where we have used $dz = e^{i\theta}ds$.

- Use now Bernoulli's equation $p = -\frac{1}{2}\rho|v|^2$ and $|v|e^{-i\theta} = dw/dz$. That gives

$$F_1 - iF_2 = \frac{i\rho}{2}\oint\left(\frac{dw}{dz}\right)^2 dz,$$

  a result known as *Blasius's theorem.*

Kutta-Joukowski lift theorem

- We now consider a body of arbitrary shape in a flow that is uniform with velocity U (in x_1 direction) very far away from the body. Use a coordinate system such that the origin is inside the body. One can expand

$$\frac{dw}{dz} = U + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

  which is a *Laurent series.* We use that no singularities can appear outside the body and that dw/dz approaches U for $z \to \infty$.

- Blasius's theorem gives for the force

$$F_1 - iF_2 = \frac{i\rho}{2}\oint\left(U + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right)^2 dz.$$

  The closed integral can be evaluated with the residue theorem giving

$$F_1 - iF_2 = -2\pi\rho U a_1.$$

- Calculate now also the velocity circulation for the above expansion

$$\Gamma = \oint\frac{dw}{dz}dz = \oint\left(U + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots\right)dz = 2\pi i a_1.$$

- In summary, one finds $F_1 - iF_2 = i\rho U\Gamma$ which shows again the the drag force vanishes, $F_1 = 0$, while the lift force
$$F_2 = -\rho\, U\, \Gamma$$
is the product of the distant velocity U, the mass density ρ and the velocity circulation Γ.

It is important to keep in mind that all the mathematical results in the present section have been derived under the strongly simplifying assumption of an isentropic, incompressible, vorticity-free two-dimensional flow profile. It is a different (and important) question whether these assumptions are justified to answer a specific question in a specific situation about a physical phenomenon.

7 Laminar viscous flows

We will now discuss some simple solutions of the fluid equations with viscosity. You should be able to find them yourself.

Shear flow between moving plates

- Exercise: Consider a fluid with constant density and viscosities between two infinite plates in the x-y-plane at $z = 0$ and $z = h$. The plate at $z = h$ is moving in the x-direction with velocity U. Assuming no-slip boundary conditions at the two plates and a stationary flow, what is the fluid velocity profile? What force per unit area that is needed on the plates to keep the stationary situation?

Flow between plates with pressure gradients

- Exercise: Consider again the situation with two infinite plates at $z = 0$ and $z = h$ but now assume that they are not moving. Assume that there is a pressure gradient in the x-direction but approximate the mass density ρ and viscosities as constant. Find the stationary flow profile corresponding to no-slip boundary conditions at the plates and determine the average fluid velocity as a function of the pressure gradient.

Pipe flow

- Exercise: Consider now the flow though a pipe with radius R with a pressure gradient along the pipe $\Delta p/\Delta l$. Assuming constant density and viscosities, and no-slip boundary conditions at $r = R$, what is the flow profile in the pipe? Determine the volume of fluid per unit time Q that flows though the pipe and show that is satisfies the law of *Hagen-Poiseulle*,

$$Q = \frac{\pi R^4}{8\eta}\frac{\Delta p}{\Delta l}.$$

8 Small perturbations

In this section we will consider realistic fluids with realist thermodynamic equation of state and non-vanishing shear and bulk viscosity as well as heat conductivity. We will be interested in small perturbations around static fluid configurations.

Stability *versus* instability

- We have constructed solutions to the equations that govern a fluid, namely continuity equation, Navier-Stokes equation and the energy conservation law. Besides finding such solutions, it is important to know whether they are stable or unstable.

- For solutions that are *linearly stable*, arbitrarily small (linear) perturbations around them decrease in time and / or space. For *linearly unstable* solutions, such small perturbations grow in time or space, whereas for *marginally stable* solutions linear perturbations remain constant.

- It can also happen that a solution is stable with respect to very small (linear) perturbations, but unstable with respect to perturbations with somewhat larger amplitude.

- If a solution is unstable, small fluctuations around it grow and lead to deviations from this solution. Sometimes this leads to another laminar solution, sometimes also to turbulence.

- To investigate linear stability, one writes the fluid fields in the form

$$\rho = \bar{\rho} + \delta\rho, \qquad \varepsilon = \bar{\varepsilon} + \delta\varepsilon, \qquad v_j = \bar{v}_j + \delta v_j,$$

  where $\bar{\rho}$, $\bar{\varepsilon}$ and $\bar{v}_j$ describe the background solution to be investigated and $\delta\rho$, $\delta\varepsilon$ and δv_j the small perturbations around it. Because the perturbations are small and because $\bar{\rho}$, $\bar{\varepsilon}$ and $\bar{v}_j$ are a solution to the fluid equations, one can use linearized equations for the perturbations. These linearized equations depend on the background solution, of course. We will see for specific examples how this goes.

Background in static, homogeneous and isotropic equilibrium

- We consider as a background a static fluid in a homogeneous and isotropic equilibrium configuration. These symmetries will be helpful in solving the equations for the perturbations. For simplicity, we work in the reference frame where the fluid is at rest $\bar{v}_j = 0$.

- The background configuration is characterized by a (constant) mass density $\bar{\rho}$ and internal energy density $\bar{\varepsilon}$. The pressure $\bar{p}$ and all other thermodynamic variables are related via the thermodynamic equation of state.

- Viscosities and heat conductivity are also determined by the background energy and mass density,

$$\bar{\eta} = \eta(\bar{\varepsilon}, \bar{\rho}), \qquad \bar{\zeta} = \zeta(\bar{\varepsilon}, \bar{\rho}), \qquad \bar{\kappa} = \kappa(\bar{\varepsilon}, \bar{\rho}).$$

Small perturbations

- We now consider small perturbations in the fluid fields, i. e. we write

$$\rho(t, \vec{x}) = \bar{\rho} + \delta\rho(t, \vec{x}), \qquad \varepsilon(t, \vec{x}) = \bar{\varepsilon} + \delta\varepsilon(t, \vec{x}), \qquad v_j(t, \vec{x}) = \delta v_j(t, \vec{x}).$$

  and we assume that the perturbations are so small that only terms of linear order need to be considered in their equation of motion.

- The equations of motion for the perturbations are given by the linearized continuity equation

  $$\partial_t \delta\rho + \bar\rho \partial_j \delta v_j = 0,$$

  the linearized equation for energy density

  $$\partial_t \delta\varepsilon + (\bar\varepsilon + \bar p)\partial_j \delta v_j - \bar\kappa \left(\tfrac{\partial T}{\partial \rho}\right)_\varepsilon \partial_j^2 \delta\rho - \bar\kappa \left(\tfrac{\partial T}{\partial \varepsilon}\right)_\rho \partial_j^2 \delta\varepsilon = 0,$$

  and the linearized Navier-Stokes equation

  $$\bar\rho \partial_t \delta v_k + \left(\tfrac{\partial p}{\partial \rho}\right)_\varepsilon \partial_k \delta\rho + \left(\tfrac{\partial p}{\partial \varepsilon}\right)_\rho \partial_k \delta\varepsilon - \bar\eta \partial_j^2 \delta v_k - (\bar\zeta + \tfrac{1}{3}\bar\eta)\partial_k \partial_j \delta v_j = 0.$$

Symmetries and decomposition of perturbations

- The background configuration that we have chosen is invariant under translations in time $t \to t + \Delta t$, translations in space $\vec x \to \vec x + \Delta \vec x$ and rotations $x_i \to R_{ij} x_j$. (It is not invariant under Galilei boosts.) These symmetries are also helpful to solve the equations for the perturbations. More specific, it is useful to work with a decomposition into eigenfunctions with respect to these transformations.

- Consider translations in time and space. The perturbations transform like

  $$\delta\rho(t, \vec x) \to \delta\rho'(t, \vec x) = \delta\rho(t - \Delta t, \vec x - \Delta \vec x).$$

  Now consider the Fourier decomposition

  $$\delta\rho(t, \vec x) = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3}\, \delta\rho(\omega, \vec k)\, e^{-i\omega t + i\vec k \vec x}.$$

  (Because of $\delta\rho(t, \vec x) \in \mathbb{R}$ one has $\delta\rho^*(\omega, \vec k) = \delta\rho(-\omega, -\vec k)$.) Translations in space and time become

  $$\delta\rho(\omega, \vec k) \to \delta\rho'(\omega, \vec k) = e^{i\omega\Delta t - i\vec k \Delta \vec x} \delta\rho(\omega, \vec k).$$

  This shows that the Fourier modes are eigenfunctions of translations.

- Consider now rotations. While $\delta\rho$ and $\delta\varepsilon$ transform as scalars, the fluid velocity $\delta\vec v$ transforms as a vector and in a similar way gradients like $\vec\nabla\rho$ which in Fourier space are proportional to the wavevector $\vec k$. However, the scalar product $\vec k \cdot \delta\vec v$ transforms again as a scalar. It is therefore useful to decompose the fluid velocity according to

  $$\delta v_j(t, \vec x) = \partial_j \delta\phi(t, \vec x) + \delta v_j^T(t, \vec x), \qquad \delta v_j(\omega, \vec k) = i k_j \delta\phi(\omega, \vec k) + \delta v_j^T(\omega, \vec k)$$

  where the longitudinal part (the velocity component parallel to $\vec k$ in Fourier space) is characterized by the velocity potential ϕ and the transverse part (the velocity component orthogonal to $\vec k$ in Fourier space) by $v_j^T(t, \vec x)$ which is a solenoidal field, $\partial_j v_j^T(t, \vec x) = 0$.

- The decomposition into eigenfunctions with respect to translations and rotations has the advantage that linear equations can be solved for each mode separately. Indeed, the different modes can be characterized by the wave-numbers ω, $\vec k$ (which characterize temporal and spatial translations) and by two polarizations for the transverse vectors (characterization of behavior with respect to rotations). Modes with different wave-numbers do not mix in a sense that becomes more clear below. A manifestation of this principle is that the partial differential equations for $\delta\rho(t, \vec x)$ etc will become algebraic equations in the Fourier domain.

Linear equations in Fourier space

- After Fourier transform and using the decomposition for δv_j one obtains for the linear equations of motion in the scalar sector ($k^2 = \vec{k}^2$)

$$\begin{aligned} -i\omega\,\delta\rho - \bar{\rho}\,k^2\delta\phi &= 0, \\ -i\omega\,\delta\varepsilon - (\bar{\varepsilon}+\bar{p})k^2\delta\phi + \bar{\kappa}\left(\tfrac{\partial T}{\partial\rho}\right)_\varepsilon k^2\delta\rho + \bar{\kappa}\left(\tfrac{\partial T}{\partial\varepsilon}\right)_\rho k^2\delta\varepsilon &= 0, \\ -i\omega\bar{\rho}\,\delta\phi + \left(\tfrac{\partial p}{\partial\rho}\right)_\varepsilon\delta\rho + \left(\tfrac{\partial p}{\partial\varepsilon}\right)_\rho\delta\varepsilon + (\bar{\zeta}+\tfrac{4}{3}\bar{\eta})k^2\delta\phi &= 0. \end{aligned}$$

  In the vector sector one obtains

$$-i\omega\bar{\rho}\,\delta v_k^T + \bar{\eta}k^2\delta v_k^T = 0.$$

  Observe that the scalar and vector sectors decouple, indeed.

- Let us first discuss the vector sector because it is quite simple. The dispersion relation is

$$\omega = -i\frac{\bar{\eta}}{\bar{\rho}}k^2.$$

  This the dispersion relation of diffusion type with a diffusion constant $D = \bar{\nu} = \bar{\eta}/\bar{\rho}$. The frequency is imaginary, corresponding to an exponential decay in time $\sim e^{-\bar{\nu}k^2t}$ of the modes with wavenumber $\vec{k}$. Note that in the ideal fluid limit one has $\bar{\nu} \to 0$ and the transverse velocity modes have the dispersion relation $\omega = 0$ corresponding to no time evolution.

- The equation that govern the scalar sector can be written as

$$\begin{pmatrix} -i\omega & 0 & -\bar{\rho}k^2 \\ \bar{\kappa}\left(\frac{\partial T}{\partial\rho}\right)_\varepsilon k^2 & -i\omega + \bar{\kappa}\left(\frac{\partial T}{\partial\varepsilon}\right)_\rho k^2 & -(\bar{\varepsilon}+\bar{p})k^2 \\ \left(\frac{\partial p}{\partial\rho}\right)_\varepsilon & \left(\frac{\partial p}{\partial\varepsilon}\right)_\rho & -i\omega\bar{\rho} + (\bar{\zeta}+\frac{4}{3}\bar{\eta})k^2 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta\varepsilon \\ \delta\phi \end{pmatrix} = 0.$$

  Non-trivial solutions correspond to a vanishing determinant of the matrix on the left hand side. In other words: solutions are eigenvectors with eigenvalue zero. They only exist for certain values of ω which fulfill the so-called dispersion relation. The latter is obtained precisely by setting the determinant to zero. One obtains thus (we drop the bars)

$$\begin{aligned} \omega^3 + i\omega^2k^2\left[\tfrac{\zeta+\frac{4}{3}\eta}{\rho} + \kappa\left(\tfrac{\partial T}{\partial\varepsilon}\right)_\rho\right] - \omega k^4\left[\tfrac{\zeta+\frac{4}{3}\eta}{\rho}\kappa\left(\tfrac{\partial T}{\partial\varepsilon}\right)_\rho\right] \\ -\omega k^2\left[\frac{\varepsilon+p}{\rho}\left(\frac{\partial p}{\partial\varepsilon}\right)_\rho + \left(\frac{\partial p}{\partial\rho}\right)_\varepsilon\right] - ik^4\kappa\left[\left(\tfrac{\partial T}{\partial\varepsilon}\right)_\rho\left(\frac{\partial p}{\partial\rho}\right)_\varepsilon - \left(\frac{\partial T}{\partial\rho}\right)_\varepsilon\left(\frac{\partial p}{\partial\varepsilon}\right)_\rho\right] = 0 \end{aligned} \tag{8.1}$$

- The following thermodynamic identity is useful

$$c_s^2 = \left(\frac{\partial p}{\partial\rho}\right)_{s/n} = \frac{\varepsilon+p}{\rho}\left(\frac{\partial p}{\partial\varepsilon}\right)_\rho + \left(\frac{\partial p}{\partial\rho}\right)_\varepsilon$$

  Exercise: use $ds = \frac{1}{T}d\varepsilon - \frac{\mu}{T}dn$ and $\varepsilon + p = Ts + \mu n$ to derive this.

- Consider first the simpler case $\eta = \zeta = \kappa = 0$. One has simply

$$\omega\left(\omega^2 - c_s^2k^2\right) = 0,$$

  with the solutions $\omega = 0$ and $\omega = \pm c_s k$. The latter are the two sound modes and show that c_s is indeed the propagation velocity for a compression wave in an ideal fluid. The solutions are plane waves without damping $e^{-i(\pm c_s kt - \vec{k}\vec{x})}$.

- Consider now $\kappa = 0$ but non-vanishing viscosities. The dispersion relation becomes

$$\omega \left(\omega^2 + i\omega k^2 \tfrac{\zeta+\frac{4}{3}\eta}{\rho} - c_s^2 k^2 \right) = 0$$

There is still one solution with $\omega = 0$ and the two sound modes become

$$\omega = -i \tfrac{\zeta+\frac{4}{3}\eta}{2\rho} k^2 \pm \sqrt{c_s^2 - \tfrac{\zeta+\frac{4}{3}\eta}{4\rho}}\, k.$$

In this form, the dispersion relation shows that a plane wave gets damped by an imaginary part in the frequency proportional to a combination of shear and bulk viscosity divided by mass density. The imaginary part of ω is always negative for $c_s^2 \geq 0$ and $\zeta, \eta \geq 0$. These are the conditions that sound modes are damped and the equilibrium configuration is stable with respect to density perturbations.

- Consider now non-vanishing viscosities and heat conductivity but assume conditions of constant pressure $p =$const. The dispersion relation becomes

$$\omega \left[\omega + i \frac{\zeta+\frac{4}{3}\eta}{\rho} k^2 \right] \left[\omega + i\kappa \left(\tfrac{\partial T}{\partial \varepsilon} \right)_\rho k^2 \right] = 0.$$

The mode with $\omega = 0$ and $\omega = -i\frac{\zeta+\frac{4}{3}\eta}{\rho} k^2$ are remnants of the sound modes for $p =$const., while the third mode with $\omega = -i\kappa \left(\frac{\partial T}{\partial \varepsilon} \right)_\rho k^2$ describe heat conduction. Because of $d\varepsilon = Tds + \mu dn$ one has

$$\left(\tfrac{\partial \varepsilon}{\partial T} \right)_\rho = T \left(\tfrac{\partial s}{\partial T} \right) = c_p$$

which is the heat capacity density at constant pressure.

- The dispersion relation for heat conduction under conditions of constant pressure is therefore

$$\omega = -i \frac{\kappa}{c_p} k^2$$

and describes the diffusive transport of heat. The solution is of the form $e^{-\frac{\kappa}{c_p} k^2 t}$. Stability requires $c_p \geq 0$ for $\kappa \geq 0$.

- For the most general case where heat conductivity and viscosities are non-vanishing and pressure inhomogeneities are present, the requirement of linear stability is that (8.1) has only solutions ω with negative imaginary parts. We will not go into further details here.

Kelvin-Helmholtz instability

- We consider now a situation with a proper instability within the ideal fluid approximation. Consider a flow that is homogeneous everywhere except for a discontinuity along some surface. More specific, we take the discontinuity to be along the plane with $z = 0$ and assume $\bar{v}_j = 0$ for $z < 0$ and $\bar{v}_j = (\bar{v}, 0, 0)$ for $z > 0$. The pressure is assumed to be constant $\bar{p} =$ const and the density to be $\bar{\rho} = \bar{\rho}_+$ for $z > 0$ and $\bar{\rho} = \bar{\rho}_-$ for $z < 0$. One can check easily that this is indeed a solution to the continuity equation (4.2) and Euler equation (4.6).

- Consider now a small perturbation of the fluid velocity δv_j and pressure δp on the upper side. The linearized continuity equation for an incompressible fluid gives

$$\partial_j \delta v_j = 0,$$

Figure 2. Kelvin-Helmholtz clouds that formed due to a velocity shear between layers in the atmosphere.

the linearized Euler equation gives

$$\bar{\rho}_+ \left(\frac{\partial}{\partial t} \delta v_j + \bar{v} \frac{\partial}{\partial x} \delta v_j \right) + \partial_j \delta p = 0.$$

Taking the divergence of this equation and using the linearized continuity equation gives

$$\Delta \delta p = 0.$$

In the region $z < 0$ the equations are analogous, except that there is no background velocity.

- We make for the perturbation in pressure the ansatz

$$\delta p = \hat{p}_\pm e^{-i\omega t + ikx \mp kz},$$

where the upper (lower) sign corresponds to $z > 0$ ($z < 0$) respectively. This solves the Laplace equation, indeed, and falls off exponentially for large $|z|$

- Denote the horizontal position of the boundary surface by $\delta h(t, x)$. One has for the fluid velocity at $z > 0$

$$\frac{\partial}{\partial t} \delta h = \delta v_z - \bar{v} \frac{\partial}{\partial x} \delta h$$

and for $\delta h \sim e^{-i\omega t + ikx}$ this becomes

$$\delta v_z = -i(\omega - \bar{v}k)\delta h.$$

This can now be used in the z-component of the linearized Euler equation to give

$$\bar{\rho}_+ \left(\omega - \bar{v}k\right)^2 \delta h + k \delta p_+ = 0.$$

In a similar way one obtains for $z < 0$

$$\bar{\rho}_- \omega^2 \delta h - k \delta p_- = 0.$$

- The pressure on both sides of the discontinuity must be equal, $p_+ = p_-$. That leads to

  $$\left[\bar{\rho}_- \omega^2 + \bar{\rho}_+ \left(\omega - \bar{v}k\right)^2\right] \delta h = 0.$$

  The solutions to this equation have frequencies

  $$\omega = \bar{v}k \frac{\rho_+ \pm i\sqrt{\bar{\rho}_+ \bar{\rho}_-}}{\bar{\rho}_+ + \bar{\rho}_-}$$

  which shows that there is always a mode with positive imaginary part corresponding to an exponential growth. This indicates an instability known as Kelvin-Helmholtz instability. The effect is responsible e. g. for the (strong) water waves excited by (strong) winds or interesting patterns one can sometimes observe at boundary layers in the atmosphere.

9 Fluids in a gravitational field

We discuss here what happens when fluids are subject to gravity or when the gravitational equations and the fluid equations are solved together.

Fluid in external gravitational field

- First we discuss the conservation equations for fluids in a gravitational field in the approximation where the fluid is subject to the gravitational acceleration but does not change the gravitational field in any noticeable way itself. This is a good approximation for water in the ocean or air in the atmosphere, for example.
- The gravitational acceleration is
$$g_j = -\partial_j \varphi,$$
where φ is the gravitational or Newton potential.
- Because of the gravitational acceleration, the momentum in a fluid cell is now not conserved any more and one has
$$\partial_t \mathscr{P}_k + \partial_j \mathscr{P}_{jk} = \rho g_k = -\rho \partial_j \varphi.$$
- There is now in addition to the kinetic and internal energy of the fluid also a potential energy of the fluid which is locally $\rho\varphi$. Note, however, that the absolute value of φ does not have a physical significance in Newtonian gravity. In particular, if we would add to φ a time-dependent but spatially homogeneous term, the energy would be changed but the forces would not be modified. Accordingly, the energy conservation law becomes
$$\partial_t \mathscr{E} + \partial_j \mathscr{E}_j = \rho \partial_t \varphi.$$
- The mass conservation law is not modified by gravity.
- With these modifications one can now go through the derivation of the fluid equations again. The decomposition of (ρ, ρ_j) and $(\mathscr{P}_k, \mathscr{P}_{jk})$ in (4.1) remains unchanged. For energy density and energy flux density one has now
$$\mathscr{E} = \frac{1}{2}\rho \vec{v}^2 + \varepsilon + \rho\varphi, \qquad \mathscr{E}_j = \left(\frac{1}{2}\rho \vec{v}^2 + \varepsilon + \rho\varphi\right) v_j + v_i T_{ij} + q_j.$$
- The mass conservation equation (4.2) is unchanged and the momentum conservation law becomes
$$\rho\left(\partial_t v_k + v_j \partial_j v_k\right) + \partial_j T_{jk} = \rho g_k.$$
For an ideal fluid this leads to Euler's equation with the force term ρg_k on the right hand side and similar in the first order viscous approximation for the Navier-Stokes equation.
- The energy conservation law can be brought to the form (4.4). Additional terms involving φ all cancel out in the final expression. This implies also that the entropy evolution remains unchanged.

Hydrostatic equilibrium

- We now discuss solutions of the fluid equations in a gravitational field in a static situation with vanishing fluid velocity. From the Navier-Stokes equation and (4.4) one finds
$$\partial_k p = \rho g_k = -\rho \partial_k \varphi, \qquad \partial_k T = 0.$$
The second equation implies $T = \text{const}$ and one can use the differential of pressure $dp = sdT + nd\mu = sdT + \rho d(\mu/m)$ to obtain
$$\partial_k \left(\frac{\mu}{m} + \varphi \right) = 0.$$
The gravitational potential acts like a position dependent chemical potential. For example, if $\varphi = gz$ one has a thermal ensemble described by
$$\mu = \mu_0 - gz, \qquad T = T_0$$
and all thermodynamic quantities can be related to this for a given equation of state.
- The condition of constant temperature is only given for a complete mechanical and thermal equilibrium. One may, however, also have situations where temperature is not constant but there is nevertheless a mechanical equilibrium in the sense that the fluid is not moving macroscopically. In the presence of a temperature gradient and non-vanishing heat conductivity, there will be microscopic processes of heat conduction that counteract the temperature gradient, for example by collisional or radiative heat conduction.
- A fluid with a temperature gradient pointing against the gravitational acceleration can be unstable. The basic reason is that density usually decreases increasing temperature. A macroscopic motion can result from the buoyancy effect which is called *convection*.

Gravity surface waves

- A fluid in a gravitational field has an equilibrium configuration such that μ decreases with height z. For fluids with a first order phase transition such as water, the density jumps for some value μ_c (and corresponding height z) from a dense state (liquid) to a less dense state (vapor / gas). We now consider excitations of such a surface.
- We concentrate on the incompressible fluid approximation and on a potential flow. The equation (5.8) gets modified by the gravitational field term and reads
$$\partial_t \phi + \frac{p}{\rho} + \frac{1}{2}\vec{v}^2 + gz = \text{const}.$$
Here we are interested in small amplitudes so we can drop $\frac{1}{2}\vec{v}^2$. Denote by $z = h(t, x, y)$ the position of the surface. On that surface we assume constant pressure. The term $p/\rho = \text{const}$ can be eliminated by absorbing a position independent term. That leads to the following equation describing the surface,
$$\partial_t \rho + gh = 0.$$
- To linear order one has also for the vertical component of the fluid velocity
$$v_z = \partial_t h = \frac{\partial}{\partial z}\phi.$$
One can combine this to the following equation describing the surface
$$\left(\partial_t^2 \phi + g \frac{\partial}{\partial z} \phi \right)_{z=0} = 0.$$
In addition to this, for an incompressible flow one has $\Delta \phi = 0$.

- Solutions are given by linear compositions of plane waves of the form

$$\phi = Ae^{-i\omega t + ik_1 x + ik_2 y \pm kz}$$

It is easy to see that this satisfies the Laplace equation for $k = \sqrt{k_1^2 + k_2^2}$. In deep water we need to take the solution that decays exponentially for $z \to -\infty$ which corresponds to the upper sign. From the surface condition one obtains the dispersion relation

$$\omega = \sqrt{gk}.$$

- The surface gravity waves are dispersive, which means that the *phase velocity*

$$v_{\text{ph}} = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}}$$

depends on the wavelength λ.

Group velocity

- The velocity of wave packets is actually given by the group velocity

$$(v_{\text{gr}})_j = \frac{\partial \omega}{\partial k_j}.$$

To show this, consider a configuration

$$h(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\omega(\vec{k})t + i\vec{k}\vec{x}}\, \tilde{h}(\vec{k}),$$

where $\tilde{h}(\vec{k})$ is centered around some wave vector $\vec{k}_0$ so that one can expand

$$\omega = \omega(\vec{k}_0) + (v_{\text{gr}})_j (k - k_0)_j + \dots.$$

One has then

$$h(t, \vec{x}) = e^{-i\omega(\vec{k}_0)t + i\vec{k}_0\vec{x}} \int \frac{d^3k}{(2\pi)^3} e^{-i(k-k_0)_j((v_{\text{gr}})_j t - x_j)}\, \tilde{h}(\vec{k}).$$

The term before the integral is a pure phase factor and the integral depends only on

$$(v_{\text{gr}})_j t - x_j$$

which shows that indeed, the wave packet moves with velocity v_{gr}.

- For the surface gravity waves one has

$$v_{\text{gr}} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2} v_{\text{ph}}.$$

Surface tension

- Fluids like water have also a surface tension which plays a role for short wavelength surface excitations. To take it into account we add a corresponding term to the equation that determines the surface position

$$\partial_t \phi + gh - \frac{\sigma}{\rho}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) h = 0.$$

One can understand this as a surface tension energy term which adds to the potential energy gh (both divided by ρ).

- The two equations for ϕ become

$$\left[\partial_t^2\phi + g\frac{\partial}{\partial z}\phi - \frac{\sigma}{\rho}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial}{\partial z}\phi\right]_{z=0} = 0, \qquad\qquad \Delta\phi = 0.$$

  Solutions can be obtained similar as above. The dispersion relation becomes

$$\omega = \sqrt{gk + \frac{\sigma}{\rho}k^3}.$$

- The phase velocity

$$v_{\rm ph} = \frac{\omega}{k} = \sqrt{\frac{g}{k} + \frac{\sigma}{\rho}k}$$

  equals the group velocity

$$v_{\rm gr} = \frac{\partial\omega}{\partial k} = \frac{g + 3\frac{\sigma}{\rho}k^2}{2\sqrt{gk + \frac{\sigma}{\rho}k^3}}$$

  for $g = \frac{\sigma}{\rho}k^2$ or $\lambda^* = 2\pi/k = 2\pi\sqrt{\frac{g\rho}{\sigma}}$. For water, this wavelength is about 1.7 cm. Waves with smaller wavelengths $\lambda \ll \lambda^*$ are essentially dominated by the surface tension and called *capillary waves* while those with $\lambda \gg \lambda^*$ are *surface gravity waves*.

- Exercise: Discuss the effect of gravity and surface tension for the Kevin-Helmholtz instability.

- Exercise: Discuss gravity waves in a fluid with finite average vertical depth Δz.

10 Newtonian cosmology

A non-relativistic fluid coupled to Newtonian gravitational fields can be used to construct a simple model for the cosmological expansion. For a more realistic and more detailed description one needs general relativity and some understanding of relativistic fluid dynamics, but some interesting elements of cosmology can already be discussed with the equations we derived so far.

Self-gravitating fluids

- We now consider a fluid that is not only moving in the gravitational field due to some external source but that is actually gravitating and therefore the source of a gravitational field, itself.
- We need one more equation in addition to the fluid equations we have derived before which is for the gravitational field itself. It can be written

  $$\partial_k g_k = -\partial_k \partial_k \varphi = -4\pi G_\mathrm{N}\, \rho \tag{10.1}$$

  where $G_\mathrm{N} \approx 6.67 \times 10^{-11} \mathrm{m}^3/(\mathrm{kg\ s}^2)$ is Newtons gravitational constant.

Cosmological expansion

- Consider now the following ansatz for the fluid and gravitational fields

  $$\begin{aligned} \rho(t,\vec{x}) =& \rho_0 \left[\frac{a_0}{a(t)}\right]^3, \\ v_j(t,\vec{x}) =& x_j \left[\frac{\dot{a}(t)}{a(t)}\right], \\ g_j(t,\vec{x}) =& -x_j \left[\frac{4\pi G_\mathrm{N}}{3}\rho\right], \end{aligned} \tag{10.2}$$

  together with spatially constant pressure $p(t,\vec{x}) = p(t)$ and inner energy density $\varepsilon(t,\vec{x}) = \varepsilon(t)$.
- Equation (10.2) describes a fluid velocity that points away from the origin everywhere. Two arbitrary points move away from each other with a velocity that is proportional to their distance. The factor of proportionality

  $$H(t) = \frac{\dot{a}(t)}{a(t)}$$

  is the *Hubble parameter* or *Hubble rate* of expansion. The mass density gets diluted by this expansion and falls off with the third power of the scale factor $a(t)$. The latter is a dimensionless function of time and its overall size is usually chosen such that $a(t_0) = a_0 = 1$ for some time t_0 corresponding to today.
- It is easy to see that (10.2) satisfies the the mass conservation equation and Newton's equation (10.1). From the Navier-Stokes equation one finds

  $$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G_\mathrm{N}}{3}\rho. \tag{10.3}$$

  This can also be written as

  $$\dot{H} + H^2 = -\frac{4\pi G_\mathrm{N}}{3}\rho,$$

  which shows that this equation should be understood as an evolution equation for the Hubble parameter. This equation is known as the second Friedmann equation for a matter dominated universe.

- One can rewrite (10.4)

$$\frac{d}{dt}\dot{a}^2 = 2\ddot{a}\dot{a} = -\frac{8\pi G_\mathrm{N}}{3}\rho\, a^2\frac{\dot{a}}{a} = \frac{d}{dt}\frac{8\pi G_\mathrm{N}}{3}\rho\, a^2.$$

  Integration leads to

$$\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{8\pi G_\mathrm{N}}{3}\rho, \tag{10.4}$$

  which is the first Friedmann equation for a matter dominated universe. The combination K/a^2 is the spatial curvature. It arises here as an integration constant. Observations have shown that $k = 0$ to good approximation and we will specialize to this case in the following.

- For $K = 0$ and using $\rho \sim 1/a^3$ one can directly integrate (10.4) and finds

$$a(t) = a_0 \left(\frac{t - t_s}{t_0 - t_s}\right)^{2/3}.$$

  At the time $t = t_s$ the scale factor vanishes and according to (10.2) the mass density diverges. If the universe was always matter dominated, that would correspond to the time of the big bang. One may choose units such that $t_s = 0$.

- The Hubble parameter is for the above solution for $t_s = 0$

$$H = \frac{\dot{a}}{a} = \frac{2}{3t}$$

  and can be used to estimate the age of the universe. The so-called deceleration parameter is

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -\frac{\dot{H}}{H^2} - 1 = \frac{1}{2}$$

  and because $q > 0$ one says that a matter dominated universe is decelerating. Recent measurements suggest $q < 0$, however, which is one of the strongest indications that our present universe contains in addition to non-relativistic matter another component, so-called dark energy.

- The evolution equation for inner energy becomes

$$\partial_t\varepsilon(t) + 3(\varepsilon(t) + p(t))\left[\frac{\dot{a}(t)}{a(t)}\right] - 9\zeta(t)\left[\frac{\dot{a}(t)}{a(t)}\right]^2 = 0.$$

  For given equation of state and bulk viscosity this can be solved and determines then the temperature as a function of time. Usually one has for $T = 0$ also $\varepsilon = p = 0$ and $\zeta = 0$ so that $T = 0$ is a solution (e.g. cold dark matter).

- Instructive is also the equation for entropy per unit mass. One finds (exercise)

$$\partial_t(s/\rho) = \frac{9H^2\zeta}{\rho T}.$$

  For an expanding fluid with non-zero bulk viscosity, the entropy per unit mass increases due to the expansive motion. Interestingly, because the expansion rate $H = \dot{a}/a$ enters in square, this is actually the same for a contractive motion.

Symmetries

- The solution (10.2) is obviously invariant with respect to rotations about the origin. One says: it is *isotropic*. It is also invariant under a modified form of spatial translations

  $$\rho(t,\vec{x}) \to \rho(t,\vec{x}-\Delta x), \qquad v_j(t,\vec{x}) \to v_j(t,\vec{x}-\Delta x) + \Delta x_j \left[\frac{\dot{a}}{a}\right],$$

  and similar for the other fields.

Small perturbations

- Take now the above discussed solution as a background and consider small perturbations around it. Write

  $$\rho = \bar{\rho} + \delta\rho, \qquad v_j = \bar{v}_j + \delta v_j, \qquad g_j = \bar{g}_j + \delta g_j, \qquad p = \bar{p} + \delta p,$$

  and obtain the linearized equations of motion

  $$\begin{aligned} \partial_t \delta\rho + 3H\delta\rho + Hx_j\partial_j\delta\rho + \bar{\rho}\,\partial_j\delta v_j &= 0, \\ \partial_t \delta v_k + H\delta v_k + Hx_j\partial_j\delta v_k + \frac{1}{\bar{\rho}}\partial_k\delta p &= \delta g_k, \\ \partial_j\delta g_j = -4\pi G_{\mathrm{N}}\delta\rho, \qquad \epsilon_{jkl}\partial_k\delta g_l &= 0. \end{aligned} \tag{10.5}$$

- We concentrate on an ideal fluid with $\eta = \zeta = \kappa = 0$ and assume it to be isentropic, $s/\rho = \text{const}$. One can then write

  $$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_{s/\rho} \delta\rho = c_s^2 \delta\rho,$$

  with adiabatic sound velocity c_s.

- We use now Fourier transform in the spatial domain and write

  $$\delta\rho(t,\vec{x}) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}(a_0/a(t))} \delta\rho(t,\vec{q}).$$

  The factor $a_0/a(t)$ in the exponent has been introduced in order to get rid of the apparent explicit $\vec{x}$-dependence of eq. (10.5). It describes a stretching of waves with the expansion or a conventional Fourier transform with respect to comoving coordinates $\vec{x}(a_0/a(T))$. One finds now

  $$\begin{aligned} \partial_t\delta\rho + 3H\delta\rho + i\left(\frac{a_0}{a}\right)\bar{\rho}\,q_j\delta v_j &= 0, \\ \partial_t\delta v_k + H\delta v_k + i\left(\frac{a_0}{a}\right)\frac{c_s^2}{\bar{\rho}}q_k\delta\rho &= \delta g_k, \\ i\left(\frac{a_0}{a}\right)q_j\delta g_j = -4\pi G_{\mathrm{N}}\delta\rho, \qquad \epsilon_{jkl}q_k\delta g_l &= 0. \end{aligned} \tag{10.6}$$

- The two equations for δg_j have the obvious solution

  $$\delta g_j = \left(\frac{a}{a_0}\right)\frac{i4\pi G_{\mathrm{N}} q_j}{\vec{q}^2}\delta\rho.$$

- To solve the other equations decompose velocity into transverse vector and scalar part
$$\delta v_j(t,\vec{x}) = \partial_j \delta\phi(t,\vec{x}) + \delta v_j^T(t,\vec{x}), \qquad \delta v_j(\omega,\vec{q}) = i\left(\frac{a_0}{a}\right) q_j \delta\phi(\omega,\vec{q}) + \delta v_j^T(\omega,\vec{q}),$$
and one introduces the density contrast
$$\delta(t,\vec{q}) = \frac{\delta\rho(t,\vec{q})}{\bar{\rho}(t)}.$$
- The transverse velocity modes decouple
$$\partial_t \delta v_j^T(t,\vec{q}) + H(t)\delta v_j^T(t,\vec{q}) = 0,$$
while the velocity potential is governed by the equation
$$\partial_t \delta\phi + H\delta\phi = \left[-c_s^2 + \frac{4\pi G_{\rm N}\bar{\rho}}{\vec{q}^2}\left(\frac{a}{a_0}\right)^2\right]\delta.$$
The equation for the density perturbation becomes
$$\partial_t \delta = \vec{q}^2 \left(\frac{a_0}{a}\right)^2 \delta\phi.$$
- The transverse or rotational modes decay simply with the scale factor,
$$\delta v_j^T(t,\vec{q}) = \left(\frac{a_0}{a(t)}\right)\delta v_j^T(t_0,\vec{q}).$$
In contract, the compressional modes have a more interesting behavior. By combing equations, one finds
$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} + \left[c_s^2\vec{q}^2\left(\frac{a_0}{a}\right)^2 - 4\pi G_{\rm N}\bar{\rho}\right]\delta = 0. \tag{10.7}$$

Jeans analysis

- We first analyze (10.7) with the artificial simplification $a = a_0 = \text{const}$,
$$\ddot{\delta} + \left[c_s^2\vec{q}^2 - 4\pi G_{\rm N}\bar{\rho}\right]\delta = 0.$$
Because c_s and $\bar{\rho}$ are now also assumed to be independent of time, one can solve this equation by plane waves $\delta \sim e^{-i\omega t}$ and finds the dispersion relation
$$\omega^2 = c_s^2\vec{q}^2 - 4\pi G_{\rm N}\bar{\rho}.$$
- For large wave vectors $\vec{q}^2 \gg 4\pi G_{\rm N}\bar{\rho}/c_s^2$ this is the conventional dispersion relation of sound waves. However, in the opposite limit $\vec{q}^2 \to 0$, the two frequencies are
$$\omega = \pm i\sqrt{4\pi G_{\rm N}\bar{\rho}}.$$
One of them is exponentially growing, while the other is exponentially damped. The boundary between oscillating and growing/decaying behavior is given by the Jeans wave number or corresponding wavelength
$$q_J = \sqrt{\frac{4\pi G_{\rm N}\bar{\rho}}{c_s^2}}, \qquad \lambda_J = \sqrt{\frac{\pi c_s^2}{G_{\rm N}\bar{\rho}}}.$$
The growth rate or "e-folding rate" for $\vec{q}^2 < q_J^2$ is given by
$$\operatorname{Im}\omega = c_s\left(q_J^2 - \vec{q}^2\right)^{1/2}.$$
In reality the Jeans wavelength changes during the evolution of the universe and marks the boundary between oscillations due to effects of pressure at small scales and gravitational collapse on larger scales.

Pressure-less expansion

- We now consider the full evolution equation for the density contrast (10.7) for the special case of vanishing sound velocity $c_s = 0$. This corresponds to a pressure-less non-relativistic gas wich is often called "dust". Using $H = \dot{a}/a = 2/(3t)$ and $4\pi G_{\text{N}}\bar{\rho} = 3H^2/2 = 2/(3t^2)$ one finds
$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0.$$

- This differential equation of second order has two independent solutions,
$$\delta \sim t^{2/3}, \qquad \delta \sim t^{-1},$$
one of them growing, the other decaying. This shows that the density contrast does in reality not grow exponentially as the Jeans approximation suggests, but only algebraically.

- In the present approximation of an ideal, pressure-less fluid, the density contrast grows uniformly, independent of the wave-number. This changes when a non-zero viscosity of sound is taken into account or dissipative terms such as shear and bulk viscosity and heat conductivity.

- We have concentrated here on a linear analysis. For the real cosmological fluid that is a good approximation at early times, when the deviations from a homogeneous and isotropic background solution are small. However, at late times, structures form in the Universe such as Galaxy clusters and Galaxies (so-called *large scale structure*). The density contrast grows then locally large and a linear treatment looses validity. By formulating a perturbative approach, one can take some non-linear effects into account or one can use numerical techniques, so-called N-body simulations. It is one of the current research challenges in cosmology to understand non-linear structure formation with semi-analytic techniques in order to better understand late-time cosmology and the physics of dark matter and dark energy.

11 Kinetic theory

Fluid dynamics is an effective theory which arises as a particular limit of another, more microscopic theoretical description. This can be for example many-body quantum theory or quantum field theory at non-vanishing temperature and density. However, for certain situations the quantum specific aspects of the microscopic theory might be less important and in can be described in terms of classical concepts. We now discuss the description of classical particles coupled to gravity in the framework of kinetic theory as an example.

The Boltzmann and Vlasov equations

- We consider a set of particles with mass m that interact gravitationally and via collisions. We describe them in terms of the phase-space distribution function $f(\vec{p}; t, \vec{x})$. From the latter one can obtain for example the mass density as

$$\rho(t, \vec{x}) = \int_{\vec{p}} m\, f(\vec{p}; t, \vec{x}), \tag{11.1}$$

  with $\int_{\vec{p}} = \int \frac{d^3p}{(2\pi)^3}$.

- Each particle moves between collisions on trajectories that are governed by the classical equations of motion, i. e.

$$\frac{d}{dt} p_j = -m \partial_j \varphi$$

  where φ is the gravitational potential.

- The distribution function $f(\vec{p}; t, \vec{x})$ changes in time because particle move with the velocity $v_j = p_j/m$, because particle momenta change according to the equation above, and because of collisions. That can be expressed in terms of the equation

$$\frac{d}{dt} f(\vec{p}; t, \vec{x}) = \frac{\partial}{\partial t} f(\vec{p}; t, \vec{x}) + \frac{p_j}{m} \frac{\partial}{\partial x_j} f(\vec{p}; t, \vec{x}) - m(\partial_j \varphi) \frac{\partial}{\partial p_j} f(\vec{p}; t, \vec{x}) = (\partial_t f(\vec{p}; t, \vec{x}))_{\text{collisions}} \tag{11.2}$$

  This is the equation named after *Ludwig Eduard Boltzmann.* In situations where the collision term can be neglected and the right hand side vanishes, it is called *Vlasov* equation.

- The gravitational field is governed by the Poisson equation

$$\partial_j \partial_j \varphi(t, \vec{x}) = 4\pi G_{\text{N}} \rho(t, \vec{x}) = 4\pi G_{\text{N}} \int_{\vec{p}} f(\vec{p}; t, \vec{x}).$$

  Because of this constraint, the Vlasov equation is non-linear, already.

- The collision term can in general be rather complicated because it depends on all the details of scattering physics. When only elastic two-to-two collisions play a role, it can be written as

$$(\partial_t f(\vec{p}; t, \vec{x}))_{\text{collisions}} = \int_{\vec{q}_2, \vec{q}_3 \vec{q}_4} w(\vec{p}, \vec{q}_2; \vec{q}_3, \vec{q}_4) \left[f(\vec{q}_3; t, \vec{x}) f(\vec{q}_4; t, \vec{x}) - f(\vec{p}; t, \vec{x}) f(\vec{q}_2; t, \vec{x}) \right]$$

  where $w(\vec{q}_1, \vec{q}_2; \vec{q}_3, \vec{q}_4)$ is the transition probability per unit time for incoming particles with momenta $\vec{q}_1$ and $\vec{q}_2$ to outgoing particles with momenta $\vec{q}_3$ and $\vec{q}_4$.

- The transition probability has the symmetry properties

$$w(\vec{q}_1, \vec{q}_2; \vec{q}_3, \vec{q}_4) = w(\vec{q}_2, \vec{q}_1; \vec{q}_4, \vec{q}_3) = w(\vec{q}_3, \vec{q}_4; \vec{q}_1, \vec{q}_2).$$

Conservation laws

- Energy and momentum conservation imply that one can write

$$w(\vec{p},\vec{q}_2;\vec{q}_3,\vec{q}_4)=\sigma(\vec{p},\vec{q}_2;\vec{q}_3,\vec{q}_4)\;\delta^{(3)}(\vec{q}_1+\vec{q}_2-\vec{q}_3-\vec{q}_4)\;\delta\left(\tfrac{\vec{q}_1^{\,2}}{2m}+\tfrac{\vec{q}_2^{\,2}}{2m}-\tfrac{\vec{q}_3^{\,2}}{2m}-\tfrac{\vec{q}_4^{\,2}}{2m}\right),$$

  where $\sigma(\vec{p},\vec{q}_2;\vec{q}_3,\vec{q}_4)$ contains the details of the scattering process.

- The symmetry and conservation properties of $w(\vec{q}_1,\vec{q}_2;\vec{q}_3,\vec{q}_4)$ imply (exercise)

$$\int_{\vec{p}}\left(\partial_t f(\vec{p};t,\vec{x})\right)_{\text{collisions}}=0,\qquad\qquad \int_{\vec{p}}p_j\left(\partial_t f(\vec{p};t,\vec{x})\right)_{\text{collisions}}=0,$$

$$\int_{\vec{p}}\frac{\vec{p}^{\,2}}{2m}\left(\partial_t f(\vec{p};t,\vec{x})\right)_{\text{collisions}}=0.$$

Moments of the distribution function

- Moments of the distribution function with respect to $\vec{p}$ yield the conserved fields. The mass density is given in (11.1), the momentum density or mass current is

$$\mathscr{P}_k(t,\vec{x})=\rho_k(t,\vec{x})=\rho(t,\vec{x})v_j(t,\vec{x})=\int_{\vec{p}}p_k\,f(\vec{p};t,\vec{x}).$$

  The momentum flux density is given by

$$\mathscr{P}_{jk}(t,\vec{x})=\rho(t,\vec{x})v_j(t,\vec{x})v_k(t,\vec{x})+T_{jk}(t,\vec{x})=\int_{\vec{p}}\frac{p_jp_k}{m}\,f(\vec{p};t,\vec{x}),$$

  and the energy and energy flux densities are given by

$$\begin{aligned}\mathscr{E}(t,\vec{x})&=\int_{\vec{p}}\left(\frac{\vec{p}^{\,2}}{2m}+\varphi(t,\vec{x})m\right)f(\vec{p};t,\vec{x}),\\ \mathscr{E}_j(t,\vec{x})&=\int_{\vec{p}}\left(\frac{\vec{p}^{\,2}}{2m}+\varphi(t,\vec{x})m\right)\frac{p_j}{m}f(\vec{p};t,\vec{x}).\end{aligned}$$

- By taking moments of the Boltzmann equation, one recovers the conservation laws. For example, the integral of (11.2) gives

$$\frac{\partial}{\partial t}\int_{\vec{p}}mf(\vec{p};t,\vec{x})+\frac{\partial}{\partial x_j}\int_{\vec{p}}p_jf(\vec{p};t,\vec{x})-m\partial_j\varphi\int_{\vec{p}}\frac{\partial}{\partial p_j}f(\vec{p};t,\vec{x})=m\int_{\vec{p}}\left(\partial_t f(\vec{p};t,\vec{x})\right)_{\text{collisions}}.$$

  The third term is a boundary term in momentum space and vanishes because $f(\vec{p};t,\vec{x})$ vanishes for large $|\vec{p}|$. The right hand side vanishes as argued above. One recovers therefore

$$\partial_t\rho(t,\vec{x})+\partial_j\rho_j(t,\vec{x})=0.$$

- In a similar way one recovers

$$\partial_t\mathscr{P}_k+\partial_j\mathscr{P}_{jk}=-\rho\partial_k\varphi,\qquad\qquad \partial_t\mathscr{E}+\partial_j\mathscr{E}_j=\rho\partial_t\varphi.$$

Transport properties from Boltzmann equation

- From a kinetic theory description, one can not only recover the macroscopic conservation laws, but one can also derive expressions for transport properties like shear and bulk viscosity or heat conductivity. To that end one first studies the solution of Boltzmann's equation corresponding to thermal equilibrium (given by the Boltzmann distribution) and then small perturbations around it that correspond to non-vanishing shear stress, bulk viscous pressure or heat flux. Solving the linearized Boltzmann equation leads then to expressions for the transport terms for a given form of the scattering probability $w(\vec{q}_1, \vec{q}_2; \vec{q}_3, \vec{q}_4)$.

- The actual calculations are somewhat involved and we do not have the space here to go into the details. We just quote the results for a simple, non-relativistic gas with elastic two-to-two cross section σ_el (a billiard ball model), and mean thermal particle velocity $\bar{v} = \sqrt{3T/m}$ (obtained from $\langle \frac{1}{2} m \vec{v}^2 \rangle = \frac{3}{2} T$) with respect to the fluid rest frame and mean free time between collisions
$$\tau = \frac{1}{\sigma_\text{el}\, \bar{v}\, n}.$$
The shear and bulk viscosities are
$$\eta = n\, T\, \tau = \frac{m\bar{v}}{3\, \sigma_\text{el}}, \qquad\qquad \zeta = 0,$$
and the heat conductivity is
$$\kappa = \frac{5}{2} n\, T\, \tau = \frac{5 m\bar{v}}{6\, \sigma_\text{el}}.$$
We use here units where $k_B = 1$. Note that η is independent of density n. This result was first obtained by Maxwell. He was surprised that η does not depend on density and started to perform some measurements himself to confirm this (see `http://www-outreach.phy.cam.ac.uk/camphy/museum/area1/exhibit1.htm`).

- The fact that viscosity and heat conductivity grow large when σ_el becomes small has the following reason. As we have discussed before, viscosity describes the (diffusive) transport of momentum while heat conductivity describes the transport of energy. This transport is more efficient when the particles that carry the momentum can travel large distances between the scatterings where they change direction of motion.

Moments of the Vlasov equation

- The properties of conventional fluids such as air or water are to a large extend dominated by the collision term in Boltzmann's equation. There are, however, also situations where it can be neglected. We have so far only included a gravitational force term but one can also consider charged particles where the electromagnetic force would appear in Vlasov's equation (more general all long-range forces). One can thereby describe an electro-magnetic plasma.

- A fluid that is presumably described rather well by the collision-less Boltzmann or Vlasov equation with gravitational force is dark matter during late time cosmology. We now derive some equations by taking moments of the Vlasov's equation to describe this.

- By multiplying the Vlasov equation subsequently by m, p_j and $\frac{\vec{p}^2}{2m} + m\varphi$, one obtains the conservation equations for mass, momentum and energy, as before. The mass density $\rho(t, \vec{x})$ and fluid velocity $v_j(t, \vec{x})$ can be defined via the mass current and momentum density as in section (4). The conservation laws lead to the same continuity equation (4.2), the momentum conservation law can be written in the form (4.3) with an additional acceleration term from the gravitational force and the energy conservation law in the form (4.4).

- In summary, the first few moments of the Vlasov equation imply

$$\begin{aligned}\partial_t\rho + v_j\partial_j\rho + \rho\partial_j v_j =&0,\\ \rho\left(\partial_t + v_j\partial_j\right)v_k + \partial_j T_{jk} =&\rho g_k,\\ \left(\partial_t + v_j\partial_j\right)\varepsilon + \varepsilon\,\partial_j v_j + \left(\partial_j v_k\right)T_{jk} + \partial_j q_j =&0.\end{aligned} \tag{11.3}$$

  Together with the Poisson equation for $g_k = -\partial_k\varphi$, these equations specify how ρ, v_k and ε change in time. However, in order to solve them, one needs to know also the stress tensor T_{jk} and the heat current q_j.

- The stress tensor can be related to the distribution function $f(\vec{p};t,\vec{x})$ by the following expression

$$T_{jk}(t,\vec{x}) = m\int_{\vec{p}}\left(\frac{p_j}{m} - v_j(t,\vec{x})\right)\left(\frac{p_k}{m} - v_k(t,\vec{x})\right)f(\vec{p};t,\vec{x}).$$

  When all the particle velocities equal the local fluid velocity, the stress tensor vanishes. That would be the case for a fluid at vanishing temperature $T = 0$. (Note, however, that we have assumed a classical point of view here. In reality, for a fluid / gas at very small temperature, quantum corrections become important. Bosonic particles would form a condensate, whereas fermionic particles would form a Fermi sphere.)

- The inner energy and heat current have the following expressions in terms of distribution functions

$$\begin{aligned}\varepsilon(t,\vec{x}) &= \int_{\vec{p}}\frac{m}{2}\left(\frac{\vec{p}}{m} - \vec{v}(t,\vec{x})\right)^2 f(\vec{p};t,\vec{x}),\\ q_j(t,\vec{x}) &= \int_{\vec{p}}\frac{m}{2}\left(\frac{\vec{p}}{m} - \vec{v}(t,\vec{x})\right)^2\frac{p_j}{m}\,f(\vec{p};t,\vec{x}),\end{aligned}$$

  and they correspond to the kinetic energy density and kinetic energy current in the local fluid rest frame.

- An approximation that is often investigated for the description of dark matter is the one of a vanishing stress tensor $T_{jk} = 0$. This corresponds formally to a fluid of cold particles without any velocity dispersion. For dark matter at early times, i.e. before non-linear structures form, this should be a good approximation. The equations for $\rho(t,\vec{x})$ and $v_j(t,\vec{x})$ are then the ones of an ideal, pressure-less fluid.

- Beyond that approximation, let us derive an evolution equation for the stress tensor. From the definition, we obtain

$$\begin{aligned}\partial_t T_{jk}(t,\vec{x}) =&m\int_{\vec{p}}\Big\{\left(\frac{p_j}{m} - v_j(t,\vec{x})\right)\left(\frac{p_k}{m} - v_k(t,\vec{x})\right)\partial_t f(t,\vec{x})\\ &- (\partial_t v_j)\left(\frac{p_k}{m} - v_k\right)f(t,\vec{x}) - \left(\frac{p_j}{m} - v_j\right)(\partial_t v_k)f(t,\vec{x})\Big\}.\end{aligned}$$

  The terms in the second line vanish by the definition of the fluid velocity $\vec{v}$. In the first line one can use Vlasov's equation to replace $\partial_t f$. Using then also partial integration leads to

$$\begin{aligned}\partial_t T_{jk}(t,\vec{x}) =& -m(\partial_l\varphi)\int_{\vec{p}}\frac{\partial}{\partial p_l}\left[\left(\frac{p_j}{m} - v_j(t,\vec{x})\right)\left(\frac{p_k}{m} - v_k(t,\vec{x})\right)\right]f(t,\vec{x})\\ &- m\int_{\vec{p}}\left(\frac{p_j}{m} - v_j(t,\vec{x})\right)\left(\frac{p_k}{m} - v_k(t,\vec{x})\right)\frac{p_l}{m}\frac{\partial}{\partial x_l}f(t,\vec{x}).\end{aligned}$$

After performing the derivatives with respect to p_l one finds that the contribution from the first term vanishes - again by the definition of $\vec{v}$. For the second term we wite

$$\begin{aligned}
\partial_t T_{jk}(t,\vec{x}) =& - m \int_{\vec{p}} \left(\frac{p_j}{m} - v_j(t,\vec{x})\right)\left(\frac{p_k}{m} - v_k(t,\vec{x})\right)\left(\frac{p_l}{m} - v_l(t,\vec{x})\right)\frac{\partial}{\partial x_l} f(t,\vec{x}) \\
& - v_l m \int_{\vec{p}} \left(\frac{p_j}{m} - v_j(t,\vec{x})\right)\left(\frac{p_k}{m} - v_k(t,\vec{x})\right)\frac{\partial}{\partial x_l} f(t,\vec{x}) \\
=& - \frac{\partial}{\partial x_l} T_{jkl}(t,\vec{x}) + m \int_{\vec{p}} \frac{\partial}{\partial x_l}\left[\left(\frac{p_j}{m} - v_j(t,\vec{x})\right)\left(\frac{p_k}{m} - v_k(t,\vec{x})\right)\left(\frac{p_l}{m} - v_l(t,\vec{x})\right)\right] f(t,\vec{x}) \\
& - v_l \frac{\partial}{\partial x_l} T_{jk}(t,\vec{x}) + m \int_{\vec{p}} \frac{\partial}{\partial x_l}\left[\left(\frac{p_j}{m} - v_j(t,\vec{x})\right)\left(\frac{p_k}{m} - v_k(t,\vec{x})\right)\right] f(t,\vec{x})
\end{aligned}$$

where we have used the definition

$$T_{jkl}(t,\vec{x}) = m \int_{\vec{p}} \left(\frac{p_j}{m} - v_j(t,\vec{x})\right)\left(\frac{p_k}{m} - v_k(t,\vec{x})\right)\left(\frac{p_l}{m} - v_l(t,\vec{x})\right) f(t,\vec{x}).$$

The remaining derivatives with respect to the fluid velocity can be performed and one finds the final equation

$$\partial_t T_{jk} + \frac{\partial}{\partial x_l} T_{jkl} + \left(\frac{\partial}{\partial x_l} v_j\right) T_{kl} + \left(\frac{\partial}{\partial x_l} v_k\right) T_{jl} + \left(\frac{\partial}{\partial x_l} v_l\right) T_{jk} + v_l \frac{\partial}{\partial x_l} T_{jk} = 0.$$

- Interestingly, for a situation without any velocity dispersion one has $T_{jk} = 0$ and $T_{jkl} = 0$ and as a consequence $\partial_t T_{kl} = 0$. However, it is unclear whether that is a stable solution.

12 Superfluids

Most substances become solid when they are cooled down to very small temperatures. There are, however, also some noticeable exceptions. In particular, ^{4}He consists of bosonic atoms and has at low temperatures a phase transition from a normal fluid state to a superfluid. The reason is the formation of a Bose-Einstein type condensate, as we will discuss below.

Macroscopic wavefunction

- At very small temperatures, a macroscopically large number of bosonic particles can occupy a single wave function which thereby becomes important also for the collective, fluid dynamic motion.
- At vanishing temperature, the macroscopic wave-function $\psi(t,\vec{x})$ has an equation of motion of the form
  $$i\hbar\partial_t\psi(t,\vec{x}) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(t,\vec{x}) + \lambda|\psi(t,\vec{x})|^2\psi(t,\vec{x}). \tag{12.1}$$
  This is known as the *Gross-Pitaevskii* equation. While the first and second term are kinetic terms, the third cubic is describing interactions among particles. We have assumed for simplicity a point-like interaction as it is approximately realized in cold atomic gases.
- Under translations in time and space, the wave function transforms simply as
  $$\psi(t,\vec{x}) \to \psi'(t,\vec{x}) = \psi(t-\Delta t, \vec{x}-\Delta x),$$
  and it is easy to see that this is a symmetry of (12.1).
- Another interesting symmetry is the global U(1) symmetry
  $$\psi(t,\vec{x}) \to e^{i\alpha}\psi(t,\vec{x}).$$
- Galilei transformations are slightly more involved,
  $$\psi(t,\vec{x}) \to \psi'(t,\vec{x}) = \psi(t,\vec{x}-\Delta\vec{v}\,t)\exp\left[\frac{i}{\hbar}\left(m\Delta\vec{v}\cdot\vec{x} - \frac{1}{2}m(\Delta\vec{v})^2 t\right)\right],$$
  but one easily checks that this is a symmetry of (12.1), indeed.
- A homogeneous condensate at rest is described by a constant wave function
  $$\psi(t,\vec{x}) = \sqrt{n_0}\,e^{-i\mu t/\hbar}. \tag{12.2}$$
  It is clear that this breaks the U(1) symmetry but also Galilei symmetry. (The condensate is at rest only in one reference frame.) This is an example for spontaneous symmetry breaking where the equations of motion remain symmetric but the symmetry breaking in on the level of the solution.

Superfluid flow

- In the reference frame where the condensate moves with velocity $\vec{v}$, the wave function is of the form
  $$\psi(t,\vec{x}) = \sqrt{n_0}e^{iS} = \sqrt{n_0}\exp\left[\frac{i}{\hbar}\left(m\vec{v}\cdot\vec{x} - \left(\tfrac{1}{2}m\vec{v}^2+\mu\right)t\right)\right].$$
  The velocity is given by the gradient of the phase S,
  $$\vec{v} = \frac{\hbar}{m}\vec{\nabla}S.$$

Moreover, the phase S satisfies the equation

$$\hbar\,\partial_t S = -\left(\frac{1}{2}m\vec{v}^2 + \mu\right).$$

- The above equation for S holds *a priori* only in equilibrium. However, according to the general principles of fluid dynamics, it should also hold locally to lowest order in a derivative expansion. By taking the divergence, one obtains

  $$\partial_t v_j + \partial_j\left(\frac{1}{2}\vec{v}^2 + \frac{\mu}{m}\right) = 0.$$

  This is the evolution equation for the fluid velocity of an ideal potential flow at constant temperature. Indeed, using $dp = sdT + \rho d(\mu/m)$ one can rewrite the equation

  $$\rho\left[\partial_t v_k + v_j \partial_j v_k\right] + (\partial_k p)_T = 0$$

  which is Euler's equation.

- We found that the condensate ψ follows the evolution equation of an ideal fluid, i.e. it seems to have no viscosity. There is a more detailed criterion which we discuss below.

Fluid equations from Gross-Pitaevskii equation

- We have obtained the equation of motion for v_k by boosting the equilibrium solution to the Gross-Pitaevskii equation. Now we discuss another way to derive it.

- One can show easily that it follows from (12.1) that the following mass density and current are conserved

  $$\rho = m\psi^*\psi, \qquad \rho_j = \frac{\hbar}{2i}\left[\psi^*\partial_j\psi - (\partial_j\psi^*)\psi\right],$$

  i.e. one has $\partial_t\rho + \partial_j\rho_j = 0$.

- One can also write the macroscopic wave function as

  $$\psi = \sqrt{\frac{\rho}{m}}e^{i\phi\frac{m}{\hbar}}$$

  and one finds

  $$\rho_j = \rho\,\partial_j\phi.$$

  This shows that ϕ plays the role of a velocity potential.

- One can also use this decomposition directly in (12.1) which gives then two real equations for ρ and ϕ. One of them is the conservation law for mass, the other one is

  $$\partial_t\phi + \frac{1}{2}(\vec{\nabla}\phi)^2 - \frac{\hbar^2}{2m^2}\frac{1}{\sqrt{\rho}}\vec{\nabla}^2\sqrt{\rho} + \frac{\lambda}{m^2}\rho.$$

  This equation determines the phase or superfluid velocity potential $\phi(t,\vec{x})$.

- Comparison to the discussion above shows that the descriptions can be matched, except for the third term in the above equation which is sometimes called quantum pressure term. It is $\sim\hbar^2$ and of higher orders in spatial derivatives. Fluid dynamics is organized as a gradient expansion and one would have to take this term into account only at higher order in the expansion.

Excitations

- We now investigate the condensate and small perturbations around it in more detail. First, the ansatz (12.2) in (12.1) gives
$$\mu = \lambda n_0.$$
In the next step, we take small perturbations into account by writing
$$\psi(t,\vec{x}) = \left[\sqrt{n_0} + \delta\psi(t,\vec{x})\right] e^{-i\mu t/\hbar}.$$
and expanding (12.1) to linear order. That gives
$$i\hbar\partial_t \delta\psi(t,\vec{x}) = -\frac{\hbar^2}{2m}\vec{\nabla}^2\delta\psi(t,\vec{x}) + \lambda n_0 \delta\psi(t,\vec{x}) + \lambda n_0 \delta\psi^*(t,\vec{x}).$$

- We now search for solutions in Fourier space by expanding (now in units with $\hbar = 1$)
$$\delta\psi(t,\vec{x}) = \int \frac{d\omega}{2\pi}\int\frac{d^3p}{(2\pi)^3}\left\{u(\omega,\vec{p})e^{-i\omega t + i\vec{p}\vec{x}} + v^*(\omega,\vec{p})e^{i\omega t + i\vec{p}\vec{x}}\right\}.$$
One obtains the system of equations
$$\begin{pmatrix} -\omega + \frac{\vec{p}^2}{2m} + \lambda n_0 & \lambda n_0 \\ \lambda n_0 & \omega + \frac{\vec{p}^2}{2m} + \lambda n_0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0.$$
The dispersion relation is obtained via the determinant as
$$-\omega^2 + \left(\frac{\vec{p}^2}{2m}\right)^2 + \frac{\vec{p}^2}{2m}2\lambda n_0 = 0.$$
Note that for small momentum $|\vec{p}|$, the frequency is essentially linear in $|\vec{p}|$.

- The excitations around a Bose-Einstein condensate are bosonic quasi-particles called phonons. For small momentum they have a dispersion relation like sound waves with velocity of sound
$$c_s = \frac{d\omega}{dp}\bigg|_{p=0} = \sqrt{\lambda n_0/m}.$$
For large momentum p, the dispersion relation approach that of conventional free particles, $\omega = \vec{p}^2/(2m)$.

Landau's criterion of superfluidity

- We now discuss a situation where a certain quantity of the fluid described by the condensate ψ flows with a velocity $\vec{v}$ past some body or through a capillary. If the energy and momentum of the fluid are $E = E_0$ and $\vec{P} = \vec{P}_0 = 0$ in the fluid rest frame, they are
$$E' = E + \vec{P}\vec{v} + \frac{1}{2}M\vec{v}^2 = E_0 + \frac{1}{2}M\vec{v}^2, \qquad \vec{P}' = \vec{P} + M\vec{v} = M\vec{v},$$
in the rest frame of the body. We used here first the general transformation of some anergy E and momentum $\vec{P}$ under Galilei transformations and then the particular values for the homogeneous state.

- Consider now an excitation of some type of the fluid with energy $\epsilon(\vec{p})$ and momentum $\vec{p}$. In the fluid rest frame we have now $E = E_0 + \epsilon(\vec{p})$ and $\vec{P} = \vec{p}$. The energy and momentum in the rest frame of the capillary are then
$$E' = E_0 + \epsilon(\vec{p}) + \vec{p}\cdot\vec{v} + \frac{1}{2}M\vec{v}^2, \qquad \vec{P}' = \vec{p} + M\vec{v}.$$
Comparison to the corresponding relation for the homogeneous state shows that the energy and momentum associated to the excitation are $\epsilon(\vec{p}) + \vec{p}\cdot\vec{v}$ and $\vec{p}$, respectively.

- Now, the point is that at small temperature, excitations will only be created in the fluid in appreciable numbers when it is energetically favorable, i.e. for
$$\epsilon(\vec{p}) + \vec{p}\cdot\vec{v} < 0.$$
If this relation is not fulfilled for any momentum $\vec{p}$, no excitations that could transport momentum out of a local fluid cell will be created. This means that there is no viscosity and the flow is superfluid.

- Usually, the above relation is fulfilled for some velocities $|\vec{v}| > v_c$ for some direction $\vec{p}$. The critical velocity v_c describes up to which fluid velocity the fluid remains a superfluid. For larger velocities, excitations get created and superfluidity breaks down.

- Landau's criterion is rather general and can be applied for many situations with different dispersion relation $\epsilon(\vec{p})$. As an example, consider the dispersion relation for cold atomic gases with repulsive contact interaction $\lambda > 0$. One has
$$\epsilon(\vec{p}) = \sqrt{\left(\frac{\vec{p}^2}{2m}\right)^2 + \frac{\vec{p}^2}{2m}2\lambda n_0}$$
and the critical velocity follows as
$$v_c = \sqrt{\lambda n_0/m}.$$
For this present model one has $v_c = c_s$ but that is not the general case. The sound velocity is determined by $d\omega/dp$ at vanishing momentum, while the critical velocity is determined via a global condition.

Quantized vortices

- An interesting feature of a superfluid is that velocity circulation is quantized. As we have seen, the velocity potential equals the complex phase of the macroscopic wave function. Consider a line integral which may be topologically non-trivial (i. e. not necessarily be reducible to a point). One has
$$\Gamma = \oint_C (v_s)_j dx_j = \oint_C \partial_j\phi\, dx_j = k\frac{2\pi\hbar}{m} = k\frac{h}{m}$$
where $k \in \mathbb{Z}$ is some integrer. This is because the phase of the macroscopic wave function ψ must change by a multiple of 2π.

- We have found that the superfluid velocity circulation is quantized (*Onsager* 1947). It can only change by discrete units
$$\Delta\Gamma = \frac{h}{m}.$$

- In an otherwise homogeneous condensate one can have vortex lines with velocity circulation quantized as discussed above. At the center of such a vortex line, the density ρ goes to zero which avoids a singularity in the wave function.

- Let us determine the angular momentum (within a certain volume) of such a vortex line
$$L = \int_V d^3x\{\rho r v\},$$
where r is the distance from the core and v is the azimuthal component of the superfluid velocity. Using $v = k\hbar/(mr)$ gives
$$L = \int_V d^3x\frac{\rho}{m}k\hbar = Nk\hbar, .$$
In the last step we assumed that ρ is essentially constant, $\rho = mN/V$. Each particle carries on average an angular momentum $k\hbar$.

Two fluid model

- At zero temperature the fluid is entirely superfluid and does not carry any quasiparticle excitations or any other form of entropy. However, at non-zero temperature this changes.
- One can model the situation at temperatures that are non-vanishing but small enough such that there is still a condensate, as a fluid with two components: a superfluid and a normal one. The entropy is carried entirely by the normal component, while the superfluid component describes the motion of the condensate.
- The mass density and current are decomposed into two components

  $$\begin{aligned}\rho(t,\vec{x}) =& \rho_s(t,\vec{x}) + \rho_n(t,\vec{x}),\\ \rho_j(t,\vec{x}) =& \rho_s(t,\vec{x})\,(v_s)_j(t,\vec{x}) + \rho_n(t,\vec{x})\,(v_n)_j(t,\vec{x}),\end{aligned}$$

  and the mass conservation law reads

  $$\partial_t\rho + \partial_j\rho_j = \partial_t\rho_s + \partial_t\rho_n + \rho_s\partial_j(v_s)_j + \rho_n\partial_j(v_n)_j + (v_s)_j\partial_j\rho_s + (v_n)_j\partial_j\rho_n = 0.$$

- In a similar way one can now write the momentum flux density, energy density, energy flux density etc. and derive the fluid dynamic equations of motion. The equations are somewhat involved because there are now two velocities $\vec{v}_s$ and $\vec{v}_n$. We will not go further into the details here.