Using non-Riemannian geometry to study relativistic fluids

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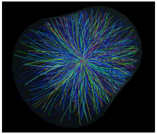
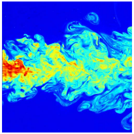
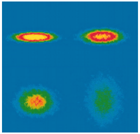
STRUCTURES CLUSTER OF EXCELLENCE

Fluid dynamics

- long distances, long times or strong enough interactions
- quantum fields form a fluid!
- needs **macroscopic** fluid properties
  - thermodynamic equation of state $p(T, \mu)$
  - shear + bulk viscosity $\eta(T, \mu)$, $\zeta(T, \mu)$
  - heat conductivity $\kappa(T, \mu)$, ...
  - relaxation times, ...
  - electrical conductivity $\sigma(T, \mu)$
- fixed by **microscopic** properties encoded in Lagrangian $\mathscr{L}_{\text{QCD}}$
- old dream of condensed matter physics: understand the fluid properties!

Relativistic fluid dynamics

Energy-momentum tensor and conserved current

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + (p + \pi_{\text{bulk}})\Delta^{\mu\nu} + \pi^{\mu\nu}$$
$$N^\mu = n\, u^\mu + \nu^\mu$$

- tensor decomposition using fluid velocity u^μ, $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$
- thermodynamic equation of state $p = p(T, \mu)$

Covariant **conservation laws** $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu N^\mu = 0$ imply

- equation for energy density ϵ
- equation for fluid velocity u^μ
- equation for particle number density n

Need **further evolution equations** [e.g Israel & Stewart]

- equation for shear stress $\pi^{\mu\nu}$
- equation for bulk viscous pressure π_{bulk}

$$\tau_{\text{bulk}}\, u^\mu \partial_\mu \pi_{\text{bulk}} + \ldots + \pi_{\text{bulk}} = -\zeta\, \nabla_\mu u^\mu$$

- equation for diffusion current ν^μ
- non-hydrodynamic degrees of freedom are needed for relativistic causality!

Remarks

- derivation from quantum effective action $\Gamma[\phi]$ wanted
  [Floerchinger, JHEP 1205, 021 (2012); JHEP 1609, 099 (2016)]
- expectation values *and* correlation functions of interest
- underlying principle: most excitations or modes relax quickly
  [Kadanoff & Martin (1963)]
- exception: conserved quantities like energy, momentum or particle density ("hydrodynamic modes")
- but: some non-hydrodynamic modes are needed for causality
- how to obtain additional equations of motion for them?

One-particle irreducible or quantum effective action

- partition function $Z[J]$, Schwinger functional $W[J]$

$$Z[J] = \int D\chi \; e^{iS[\chi] - i\int_x \{J(x)\chi(x)\}}$$

- **quantum effective action** $\Gamma[\phi]$ defined by Legendre transform

$$\Gamma[\phi] = \int_x J(x)\phi(x) - W[J]$$

with expectation values $\phi(x) = \delta W[J]/\delta J(x)$

- includes all quantum and statistical fluctuations !
- equation of motion for field expectation values

$$\frac{\delta}{\delta\phi(x)}\Gamma[\phi] = J(x)$$

- functional renormalization group: flow equation for $\Gamma[\phi]$
- can be used in and out of equilibrium

Covariant energy-momentum conservation

- quantum effective action $\Gamma[\phi, g]$ at stationary matter fields

$$\frac{\delta}{\delta\phi(x)}\Gamma[\phi, g] = 0$$

- energy-momentum tensor defined by

$$\delta\Gamma[\phi, g] = \frac{1}{2}\int d^d x \sqrt{g}\; T^{\mu\nu}(x)\delta g_{\mu\nu}(x)$$

- diffeomorphism is gauge transformation of metric

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \nabla_\mu \varepsilon_\nu(x) + \nabla_\nu \varepsilon_\mu(x)$$

- from invariance of $\Gamma[\phi, g]$ under diffeomorphisms

$$\nabla_\mu T^{\mu\nu}(x) = 0$$

- work here in Riemann geometry with Levi-Civita connection

$$\delta\Gamma_\mu{}^\rho{}_\nu = \frac{1}{2} g^{\rho\lambda}\left(\nabla_\mu \delta g_{\nu\lambda} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu}\right)$$

Why curved space?

- spacetime metric $g_{\mu\nu}(x)$ provides source for $T^{\mu\nu}(x)$
- metric is actually a gauge field
- full renormalized $T^{\mu\nu}(x)$ follows from variation of $\Gamma[\phi, g]$
- can still evaluate everything in flat space in the end

Why non-Riemannian geometry?

- can one learn more by studying further deformations of geometry?
- some equations of motion for non-ideal fluids still missing
- can still evaluate everything in Riemannian geometry / flat space in the end

Local scaling or Weyl gauge transformations

- transforms matter fields

$$\phi(x) \to e^{-\Delta_\phi \zeta(x)} \phi(x)$$

- scales metric

$$g_{\mu\nu}(x) \to e^{2\zeta(x)} g_{\mu\nu}(x)$$

- Weyl gauge field (Abelian)

$$B_\mu(x) \to B_\mu(x) - \partial_\mu \zeta(x)$$

- variation of effective action with respect to $B_\mu(x)$ gives a current $W^\mu(x)$
- equation of motion from variation with respect to $\zeta(x)$

$$\nabla_\rho W^\rho(x) = \frac{2}{d} \left[T^\mu{}_\mu(x) - \mathscr{U}^\mu{}_\mu(x) \right]$$

- in general not conserved but right hand side can be calculated
- vanishes for conformal field theories in flat space

Non-Riemannian geometry

[Floerchinger & Grossi, Phys. Rev. D 105, 085015 (2022)]

- general connection

$$\Gamma_\mu{}^\rho{}_\sigma = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\sigma\lambda} + \partial_\sigma g_{\mu\lambda} - \partial_\lambda g_{\mu\sigma} \right) + C_\mu{}^\rho{}_\sigma$$
$$+ \hat{B}_\mu{}^\rho{}_\sigma + \hat{B}_{\sigma\mu}{}^\rho - \hat{B}^\rho{}_{\mu\sigma} + B_\mu \delta^\rho{}_\sigma + B_\sigma \delta_\mu{}^\rho - B^\rho g_{\mu\sigma}$$

- contorsion $C_\mu{}^\rho{}_\sigma$ = gauge field for local Lorentz transformations
- Weyl gauge field B_μ = gauge field for local dilatations
- proper non-metricity $\hat{B}_\mu{}^\rho{}_\sigma$ = gauge field for local shear transformations
- together form the group GL(d) of basis changes in tangent space / the frame bundle

Hypermomentum current

[von der Heyde, Kerlick & Hehl (1976)]

[Floerchinger & Grossi, Phys. Rev. D 105, 085015 (2022)]

- in Non-Riemannian geometry, affine connection $\Gamma_\mu{}^\rho{}_\sigma(x)$ can be varied independent of the metric $g_{\mu\nu}(x)$

$$\delta\Gamma = \int d^dx\sqrt{g}\left\{\frac{1}{2}\mathscr{U}^{\mu\nu}(x)\delta g_{\mu\nu}(x) - \frac{1}{2}\mathscr{S}^\mu{}_\rho{}^\sigma(x)\delta\Gamma_\mu{}^\rho{}_\sigma(x)\right\}$$

  with new symmetric tensor $\mathscr{U}^{\mu\nu}$ and *hypermomentum* current $\mathscr{S}^\mu{}_\rho{}^\sigma$
- hypermomentum current can be decomposed further

$$\mathscr{S}^\mu{}_\rho{}^\sigma = Q^\mu{}_\rho{}^\sigma + W^\mu\,\delta_\rho{}^\sigma + S^\mu{}_\rho{}^\sigma + S^{\sigma\mu}{}_\rho + S_\rho{}^{\mu\sigma}$$

  with
  - spin current $S^{\mu\rho\sigma} = -S^{\mu\sigma\rho}$
  - dilatation current W^μ
  - shear current $Q^{\mu\rho\sigma} = Q^{\mu\sigma\rho}, \qquad Q^{\mu\rho}{}_\rho = 0$

Extended symmetries

- local Lorentz transformations, local dilatations and local shear transformation are *extended symmetries*: they change the quantum effective action Γ but in a specific way
- *extended symmetries* $\Rightarrow$ *non-conserved* Noether currents
- example: Partial Conservation of Axial Current (PCAC) relations: not a symmetry in the presence of quark masses but still very useful

Extended symmetries 2

[Floerchinger & Grossi, Phys. Rev. D 105, 085015 (2022)]

- consider transformation of fields

$$\phi(x) \to \phi(x) + i d\xi^j(x)\, T_j \phi(x)$$

- might be non-Abelian with structure constants

$$[T_k, T_l] = i f_{kl}{}^{j}\, T_j$$

- introduce external gauge field and covariant derivative

$$D_\mu \phi(x) = \left(\nabla_\mu - i A^j_\mu(x)\, T_j\right) \phi(x)$$

- gauge field transforms as usual

$$A^j_\mu(x) \to A^j_\mu(x) + f_{kl}{}^{j} A^k_\mu(x)\, d\xi^l(x) + \nabla_\mu d\xi^j(x)$$

Extended symmetries 3

[Floerchinger & Grossi,Phys. Rev. D 105, 085015 (2022)]

- change of effective action $\Gamma[\phi, A]$

$$\Gamma[\phi + i d\xi^j T_j \phi, A^j_\mu + f_{kl}{}^j A^k_\mu d\xi^l + \nabla_\mu d\xi^j] = \Gamma[\phi] + \int d^d x \sqrt{g} \left\{ \mathcal{I}_j(x)\, d\xi^j(x) \right\}$$

- define current through

$$\mathscr{J}_j^{\,\mu}(x) = \frac{1}{\sqrt{g}} \frac{\delta \Gamma}{\delta A^j_\mu(x)}$$

- obtain conservation-type relation (for $\delta\Gamma/\delta\phi = 0$)

$$D_\mu \mathscr{J}_j^{\,\mu}(x) = \nabla_\mu \mathscr{J}_j^{\,\mu}(x) + f_{jk}{}^l A^k_\mu(x) \mathscr{J}_l^{\,\mu}(x) = -\mathcal{I}_j(x)$$

- global symmetry $\mathcal{I}_j(x) = 0 \Rightarrow$ conserved Noether current
- extended symmetry $\mathcal{I}_j(x) \neq 0$ but known at macroscopic level $\Rightarrow$ non-conserved Noether current

Equations of motion for dilatation and shear current

[Floerchinger & Grossi, Phys. Rev. D 105, 085015 (2022)]

- variation of connection contains Levi-Civita part and non-Riemannian part

$$\delta\Gamma_{\mu}{}^{\rho}{}_{\sigma} = \frac{1}{2} g^{\rho\lambda} \left(\nabla_\mu \delta g_{\sigma\lambda} + \nabla_\sigma \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\sigma}\right) + \delta C_{\mu}{}^{\rho}{}_{\sigma} + \delta D_{\mu}{}^{\rho}{}_{\sigma}$$

- variation at $\delta C_{\mu}{}^{\rho}{}_{\sigma} = \delta D_{\mu}{}^{\rho}{}_{\sigma} = 0$ gives energy-momentum tensor

$$T^{\mu\nu} = \mathscr{U}^{\mu\nu} + \frac{1}{2}\nabla_\rho \left(Q^{\rho\mu\nu} + W^\rho g^{\mu\nu}\right)$$

- new equation of motion for shear current

$$\nabla_\rho Q^{\rho\mu\nu} = 2\left[T^{\mu\nu} - \mathscr{U}^{\mu\nu} - \frac{g^{\mu\nu}}{d}\left(T^{\sigma}{}_{\sigma} - \mathscr{U}^{\sigma}{}_{\sigma}\right)\right]$$

- similar for Wey current

Spin current

[..., Floerchinger & Grossi, Phys. Rev. D 105, 085015 (2022)]

- tetrad formalism: vary tetrad ${V_\mu}^A$ and spin connection ${\Omega_\mu}^{AB}$

$$\delta\Gamma = \int d^dx\sqrt{g}\left\{ {\mathscr{T}^\mu}_A(x)\delta {V_\mu}^A(x) - \frac{1}{2}{S^\mu}_{AB}(x)\delta{\Omega_\mu}^{AB}(x)\right\}$$

  with

  - canonical energy-momentum tensor ${\mathscr{T}^\mu}_A$
  - spin current ${S^\mu}_{AB}$

- symmetric energy-momentum tensor in Belinfante-Rosenfeld form

$$T^{\mu\nu}(x) = \mathscr{T}^{\mu\nu}(x) + \frac{1}{2}\nabla_\rho\left[S^{\rho\mu\nu}(x) + S^{\mu\nu\rho}(x) + S^{\nu\mu\rho}(x)\right]$$

- equation of motion for spin current

$$\nabla_\mu S^{\mu\rho\sigma} = \mathscr{T}^{\sigma\rho} - \mathscr{T}^{\rho\sigma}$$

- non-conserved Noether current

Example: Scalar field

- action for scalar field in d spacetime dimensions

$$\Gamma = \int d^d x \sqrt{g} \left\{ -\frac{1}{2} g^{\mu\nu} \overline{\nabla}_\mu \varphi \overline{\nabla}_\nu \varphi - U(\varphi) - \frac{1}{2} \xi R \varphi^2 \right\},$$

- co-covariant derivative

$$\overline{\nabla}_\mu \varphi = (\partial_\mu - \Delta_\varphi B_\mu)\varphi = \left(\partial_\mu - \frac{d-2}{2} B_\mu \right) \varphi$$

- how does this extend to non-Riemannian space? Is R replaced by the generalized Riemann scalar $\overline{R}$?
- in that case one finds hypermomemtum current

$$\mathscr{S}^\mu{}_\rho{}^\sigma = -\frac{d-2}{2d} \delta^\sigma{}_\rho \, \partial^\mu \varphi^2 - \xi g^{\mu\sigma} \partial_\rho \varphi^2 + \xi \delta^\mu{}_\rho \partial^\sigma \varphi^2 .$$

- Weyl current (vanishes for conformal choice of ξ)

$$W^\mu = \left(\xi \frac{2d-2}{d} - \frac{d-2}{2d} \right) \partial^\mu \varphi^2$$

Implications for relativistic fluid dynamics

- dilatation current, shear current and spin current provide additional information about quantum fields out-of-equilibrium
- their contribution to energy-momentum tensor comes with derivatives and vanishes in equilibrium or for ideal fluids
- new equations of motion for “non-hydrodynamic” modes
- can one formulate non-ideal relativistic fluid dynamics on this basis?

Conclusions

- studying quantum field theory in non-Riemannian geometry can be useful
- coupling to general connection subtle, but can arise through renormalization effects
- new divergence-type equations of motion
- dilatation current, shear current and spin current
- extended symmetries $\Rightarrow$ non-conserved Noether currents
- fluid dynamics $\Leftrightarrow$ non-equilibrium quantum field theory
- to do: extension to fluids with additional charges