# *Entropy and quantum field theory*

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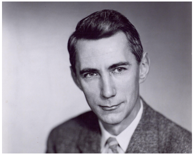
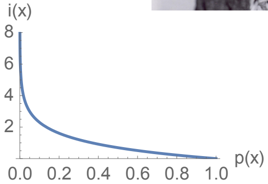
# *Entropy and information*

[Claude Shannon (1948)]



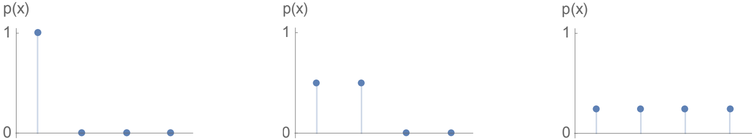
- random variable $x$ with probability distribution $p(x)$
- information content or "surprise" of outcome $x$

$$i(x) = -\ln p(x)$$



- Entropy is expectation value of information content

$$S = \langle i(x) \rangle = -\sum_x p(x) \ln p(x)$$



$S = 0$ $S = \ln(2)$ $S = 2\ln(2)$

# *Entropy in quantum theory*

[John von Neumann (1932)]

- based on the quantum density operator $\rho$

$$S = -\mathsf{Tr}\{\rho \ln \rho\}$$

- for pure states $\rho = |\psi\rangle\langle\psi|$ one has $S = 0$
- for mixed states $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ one has $S = -\sum_j p_j \ln p_j > 0$
- unitary time evolution conserves entropy

$$-\mathsf{Tr}(U\rho U^\dagger)\ln(U\rho U^\dagger) = -\mathsf{Tr}\rho\ln\rho \qquad \rightarrow \qquad S = \mathsf{const.}$$

- global characterization of quantum state

# *Entropy at thermal equilibrium*

- micro canonical ensemble: **maximal entropy** $S$ for given **conserved quantities** $E, N$ in given volume $V$
- **universality** at equilibrium
- starting point for development of thermodynamics

$$S(E,N,V), \qquad dS = \frac{1}{T}dE - \frac{\mu}{T}dN + \frac{p}{T}dV$$

- grand canonical ensemble with density operator

$$\rho = \frac{1}{Z}e^{-\frac{1}{T}(H-\mu N)}, \qquad Z = \mathrm{Tr}\left\{e^{-\frac{1}{T}(H-\mu N)}\right\}$$

# *Ideal fluid dynamics*

- thermal equilibrium

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p(u^\mu u^\nu + g^{\mu\nu}), \qquad N^\mu = n u^\mu, \qquad s^\mu = s u^\mu$$

- fluid velocity $u^\mu$
- thermodynamic equation of state $p(T, \mu)$ with $dp = s dT + n d\mu$
- local thermal equilibrium approximation: $u^\mu(x)$, $T(x)$, $\mu(x)$
- neglect gradients: lowest order of a derivative expansion
- evolution of $u^\mu(x)$, $T(x)$ and $\mu(x)$ from conservation laws

$$\nabla_\mu T^{\mu\nu}(x) = 0, \qquad \nabla_\mu N^\mu(x) = 0.$$

- entropy current also conserved

$$\nabla_\mu s^\mu(x) = 0.$$

# *Out-of-equilibrium*

- quantum field theory *out-of-equilibrium* is less well understood
- interesting topic of current research
- is non-equilibrium dynamics also governed by information?
- approach to equilibrium
- universality

# *Quantum entanglement*

[Einstein, Podolski, Rosen (1935)]

- two quantum systems $A$ and $B$ can be in a *product state*

$$|\psi_{\text{product}}\rangle = |\uparrow\rangle_A |\downarrow\rangle_B$$

- or in an *entangled state*

$$|\psi_{\text{entangled}}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\right)$$

- entanglement: measurement of system $B$ → prediction for system $A$
- Einstein: "*Spukhafte Fernwirkung*"

# *Reduced density matrix*

- quantum density matrix for system $A + B$ in pure state

$$\rho = |\psi_{AB}\rangle\langle\psi_{AB}|$$

- reduced density matrix for subsystem $A$

$$\rho_A = \mathsf{Tr}_B\{\rho\}$$

- product state $\rho = |\psi_{\mathsf{product}}\rangle\langle\psi_{\mathsf{product}}|$ leads to

$$\rho_A = |\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

- entangled state $\rho = |\psi_{\mathsf{entangled}}\rangle\langle\psi_{\mathsf{entangled}}|$ leads to

$$\rho_A = \frac{1}{2}|\uparrow\rangle\langle\uparrow| + \frac{1}{2}|\downarrow\rangle\langle\downarrow| = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

# *Entanglement entropy*

- consider system $A + B$ in globally pure state
- reduced density $\rho_A = \mathsf{Tr}_B\{\rho\}$ is *mixed* if $A$ and $B$ *entangled*
- reduced density $\rho_A = \mathsf{Tr}_B\{\rho\}$ is *pure* if $A$ and $B$ *not entangled*
- **Entanglement entropy** quantifies degree of entanglement

$$S_A = -\mathsf{Tr}_A\{\rho_A \ln \rho_A\}$$

- product state $S_A = 0$
- entangled state $S_A > 0$

# *Classical statistics*

- consider system of two random variables $x$ and $y$
- joint probability $p(x, y)$ , joint entropy

$$S = -\sum_{x,y} p(x, y) \ln p(x, y)$$

- reduced or marginal probability $p(x) = \sum_y p(x, y)$
- reduced or marginal entropy

$$S_x = -\sum_{x} p(x) \ln p(x)$$

- one can prove: **joint entropy is greater than** or equal to **reduced entropy**

$$S \geq S_x$$

- **globally pure** state $S = 0$ is also **locally pure** $S_x = 0$

# *Quantum statistics*

- consider system with two subsystems $A$ and $B$
- combined state $\rho$ , combined or full entropy

$$S = -\mathsf{Tr}\{\rho \ln \rho\}$$

- reduced density matrix $\rho_A = \mathsf{Tr}_B\{\rho\}$
- reduced or entanglement entropy

$$S_A = -\mathsf{Tr}_A\{\rho_A \ln \rho_A\}$$

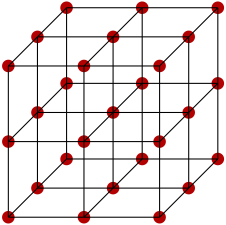
- for quantum systems **entanglement makes a difference**

$$S \ngeq S_A$$

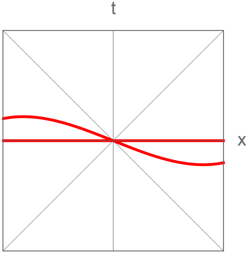
- **coherent information** $I_{B\rangle A} = S_A - S$ can be **positive** !
- **globally pure** state $S = 0$ can be **locally mixed** $S_A > 0$

# *Quantum field theory*

- field theory: one degree of freedom per space point $\phi(\mathbf{x})$



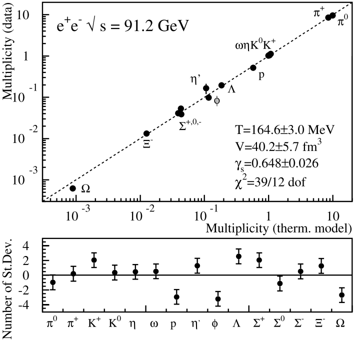
- states specified at constant time $t$ or on any Cauchy hypersurface



- fields in different *spatial regions* can be entangled

# *The thermal model puzzle*

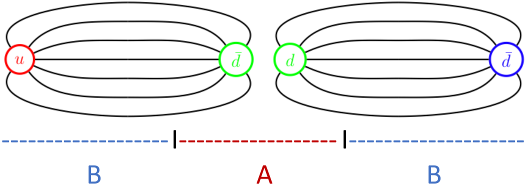
- $e^+ \, e^-$ collisions show thermal-like features
- particle multiplicities well described by thermal model



[Becattini, Casterina, Milov & Satz, EPJC 66, 377 (2010)]

- conventional thermalization by collisions unlikely
- more thermal-like features difficult to understand in PYTHIA
  [Fischer, Sjöstrand (2017)]
- alternative explanations needed

# *QCD strings*



- particle production from QCD strings
- e. g. Lund model (Pythia)
- different regions in a string are entangled
- subinterval $A$ is described by reduced density matrix of mixed form

$$\rho_A = \mathrm{Tr}_B\{\rho\}$$

- characterization by entanglement entropy

$$S_A = -\mathrm{Tr}\,\{\rho_A \ln \rho_A\}$$

- could this lead to thermal-like effects?

# *Microscopic model*

- QCD in 1+1 dimensions described by 't Hooft model

$$\mathscr{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - ig\mathbf{A}_\mu)\psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{2} \mathsf{tr}\, \mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}$$

- fermionic fields $\psi_i$ with sums over flavor species $i = 1, \ldots, N_f$
- SU($N_c$) gauge fields $\mathbf{A}_\mu$ with field strength tensor $\mathbf{F}_{\mu\nu}$
- gluons are not dynamical in two dimensions
- gauge coupling $g$ has dimension of mass
- non-trivial, interacting theory, cannot be solved exactly
- spectrum of excitations known for $N_c \to \infty$ with $g^2 N_c$ fixed ['t Hooft (1974)]

# *Schwinger model*

- QED in 1+1 dimension

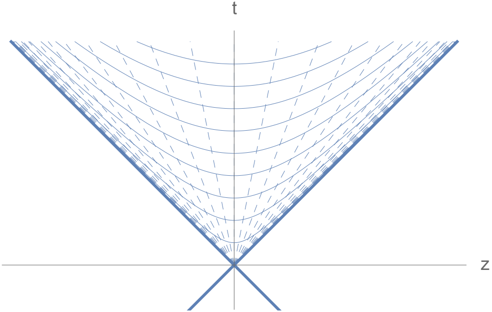
$$\mathscr{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - iqA_\mu)\psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- geometric confinement
- U(1) charge related to string tension $q = \sqrt{2\sigma}$
- for single fermion one can **bosonize theory** exactly
  [Coleman, Jackiw, Susskind (1975)]

$$S = \int d^2x \sqrt{g} \Big\{ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} M^2 \phi^2 - \frac{m\, q\, e^\gamma}{2\pi^{3/2}} \cos\left(2\sqrt{\pi}\phi + \theta\right) \Big\}$$

- Schwinger bosons are dipoles $\phi \sim \bar{\psi}\psi$
- mass is related to U(1) charge by $M = q/\sqrt{\pi} = \sqrt{2\sigma/\pi}$
- massless Schwinger model $m = 0$ leads to free bosonic theory

# *Expanding string solution*



- external quark-anti-quark pair on trajectories $z = \pm t$
- coordinates: Bjorken time $\tau = \sqrt{t^2 - z^2}$, rapidity $\eta = \text{arctanh}(z/t)$
- metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$
- symmetry with respect to longitudinal boosts $\eta \to \eta + \Delta\eta$

# *Coherent field evolution*

- Schwinger boson field depends only on $\tau$

$$\bar{\phi} = \bar{\phi}(\tau)$$

- equation of motion

$$\partial_\tau^2 \bar{\phi} + \frac{1}{\tau} \partial_\tau \bar{\phi} + M^2 \bar{\phi} = 0.$$

- Gauss law: electric field $E = q\bar{\phi}/\sqrt{\pi}$ must approach the U(1) charge of the quarks $E \to q$ for $\tau \to 0_+$

$$\bar{\phi}(\tau) \to \sqrt{\pi} \qquad\qquad (\tau \to 0_+)$$

- solution to equation of motion [Loshaj, Kharzeev (2011)]

$$\bar{\phi}(\tau) = \sqrt{\pi}\, J_0(M\tau)$$

# *Gaussian states*

- theories with quadratic action typically have Gaussian density matrix
- fully characterized by field expectation values

  $$\bar{\phi}(x) = \langle \phi(x) \rangle, \qquad \bar{\pi}(x) = \langle \pi(x) \rangle$$

  and connected two-point correlation functions, e. g.

  $$\langle \phi(x)\phi(y) \rangle_c = \langle \phi(x)\phi(y) \rangle - \bar{\phi}(x)\bar{\phi}(y)$$

- if $\rho$ is Gaussian, also reduced density matrix $\rho_A$ is Gaussian

# *Entanglement entropy for Gaussian state*

- entanglement entropy of Gaussian state in region $A$
  [Berges, Floerchinger, Venugopalan, JHEP 1804 (2018) 145]

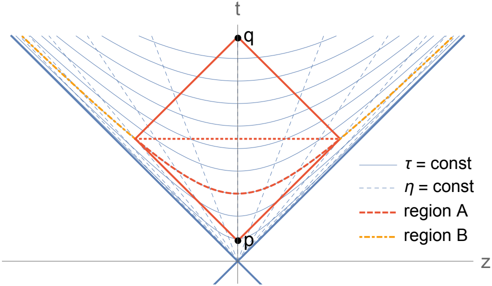
$$S_A = \frac{1}{2}\mathsf{Tr}_A\left\{D\ln(D^2)\right\},$$

- operator trace over region $A$ only
- matrix of correlation functions

$$D(x,y) = \begin{pmatrix} -i\langle\phi(x)\pi(y)\rangle_c & i\langle\phi(x)\phi(y)\rangle_c \\ -i\langle\pi(x)\pi(y)\rangle_c & i\langle\pi(x)\phi(y)\rangle_c \end{pmatrix}.$$

- involves connected correlation functions of field $\phi(x)$ and canonically conjugate momentum field $\pi(x)$
- expectation value $\bar{\phi}$ does not appear explicitly
- coherent states and vacuum have equal entanglement entropy $S_A$

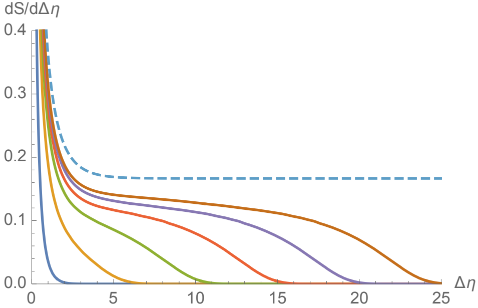
# *Rapidity interval*



- consider rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ at fixed Bjorken time $\tau$
- entanglement entropy does not change by unitary time evolution with endpoints kept fixed
- can be evaluated equivalently in interval $\Delta z = 2\tau \sinh(\Delta\eta/2)$ at fixed time $t = \tau \cosh(\Delta\eta/2)$
- need to solve eigenvalue problem with correct **boundary conditions**

# *Bosonized massless Schwinger model*

- entanglement entropy understood numerically for free massive scalars [Casini, Huerta (2009)]
- entanglement entropy density $dS/d\Delta\eta$ for bosonized massless Schwinger model ($M = \frac{q}{\sqrt{\pi}}$)



$M\tau = 1$, $10^{-1}$, $10^{-2}$, $10^{-3}$, $10^{-4}$, and $10^{-5}$

[Berges, Floerchinger, Venugopalan (2017)]

# *Conformal limit*

- for $M\tau \to 0$ one has conformal field theory limit
  [Holzhey, Larsen, Wilczek (1994)]

$$S(\Delta z) = \frac{c}{3} \ln \left( \Delta z / \epsilon \right) + \text{constant}$$

with small length $\epsilon$ acting as UV cutoff

- here this implies

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln \left( 2\tau \, \text{sinh}(\Delta\eta/2)/\epsilon \right) + \text{constant}$$

- conformal charge $c = 1$ for free massless scalars or Dirac fermions
- additive constant not universal but entropy density is

$$\begin{aligned} \frac{\partial}{\partial \Delta\eta} S(\tau, \Delta\eta) =& \frac{c}{6} \text{coth}(\Delta\eta/2) \\ \to& \frac{c}{6} \qquad\qquad (\Delta\eta \gg 1) \end{aligned}$$

- entropy becomes extensive in $\Delta\eta$ !

# *Universal entanglement entropy density*

- for very early times "Hubble" expansion rate dominates over masses and interactions

$$H = \frac{1}{\tau} \gg M = \frac{q}{\sqrt{\pi}}, m$$

- theory dominated by free, massless fermions
- universal entanglement entropy density

$$\frac{dS}{d\Delta\eta} = \frac{c}{6}$$

  with conformal charge $c$

- for QCD in 1+1 dimensions (gluons not dynamical)

$$c = N_c \times N_f$$

- from fluctuating transverse coordinates (Nambu-Goto action)

$$c = N_c \times N_f + 2 \approx 9 + 2 = 11$$

# Experimental access to entanglement ?

- could longitudinal entanglement be tested experimentally?
- entropy density $dS/d\eta$ not straight-forward to access
- measured is number of charged particles per unit rapidity
- typical values for collision energies $\sqrt{s} = 14 - 206$ GeV in the range

$$dN_{\text{ch}}/d\eta \approx 2 - 4$$

- entropy per particle $S/N$ can be estimated for a hadron resonance gas in thermal equilibrium $S/N_{\text{ch}} = 7.2$ would give

$$dS/d\eta \approx 14 - 28$$

- this is an upper bound: correlations beyond one-particle functions would lead to reduced entropy

# *Temperature and entanglement entropy*

- for conformal fields, entanglement entropy has also been calculated at non-zero temperature.
- for static interval of length $L$ [Korepin (2004); Calabrese, Cardy (2004)]

$$S(T,l) = \frac{c}{3} \ln \left( \frac{1}{\pi T \epsilon} \sinh(\pi L T) \right) + \mathsf{const}$$

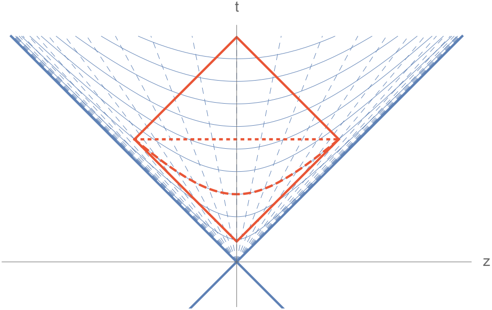
- compare this to our result in expanding geometry

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln \left( \frac{2\tau}{\epsilon} \mathsf{sinh}(\Delta\eta/2) \right) + \mathsf{const}$$

- expressions agree for $L = \tau \Delta\eta$ (with metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$) and time-dependent temperature

$$T = \frac{1}{2\pi\tau}$$

# *Modular or entanglement Hamiltonian*



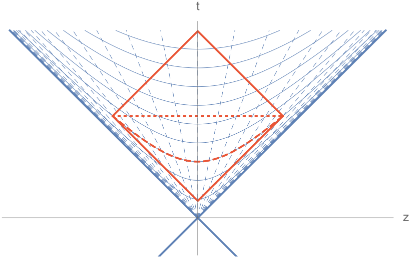
- conformal field theory [Casini, Huerta, Myers (2011), Arias, Blanco, Casini, Huerta (2017), see also Candelas, Dowker (1979)]

$$\rho_A = \frac{1}{Z_A} e^{-K}, \qquad Z_A = \mathsf{Tr}\, e^{-K}$$

- modular or entanglement Hamiltonian **local expression**

$$K = \int_\Sigma d\Sigma_\mu\, \xi_\nu(x)\, T^{\mu\nu}(x)$$

# *Time-dependent temperature*



- energy-momentum of excitations around coherent field $T^{\mu\nu}(x)$
- combination of fluid velocity and temperature $\xi^\mu(x) = \frac{u^\mu(x)}{T(x)}$
- fluid velocity in $\tau$-direction & time-dependent temperature
  [Berges, Floerchinger, Venugopalan (2017)]

$$T(\tau) = \frac{\hbar}{2\pi\tau}$$

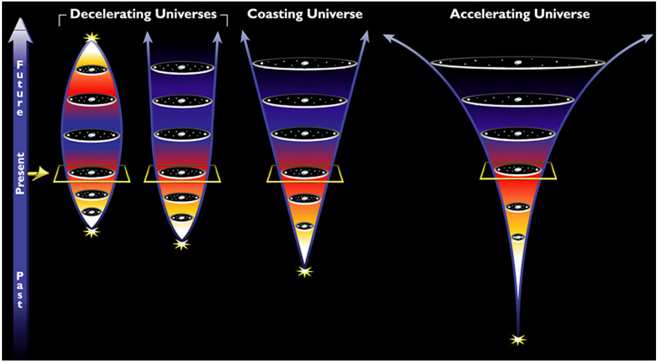
- **Entanglement between different rapidity intervals alone leads to local thermal density matrix at very early times !**
- Hawking-Unruh temperature in Rindler wedge $T(x) = \frac{\hbar c}{2\pi x}$

# *Physics picture*

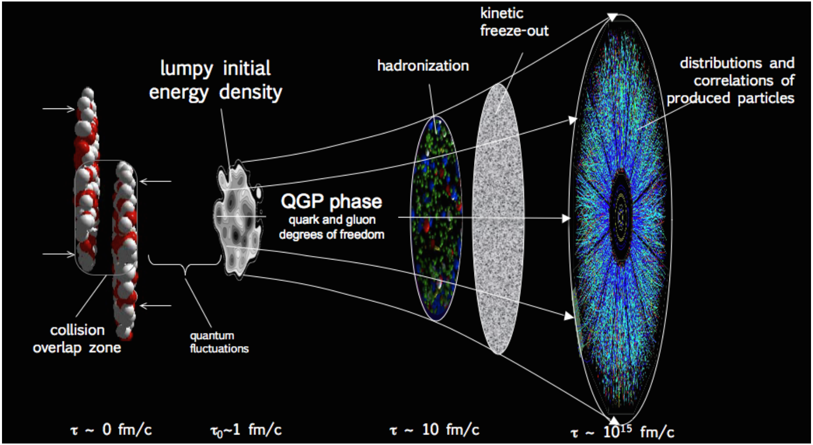
- alternative derivation via mode functions & Bogoliubov transforms [Berges, Floerchinger, Venugopalan, JHEP 1804 (2018) 145]
- coherent state vacuum at early time contains entangled pairs of quasi-particles with opposite wave numbers
- on finite rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ in- and out-flux of quasi-particles with thermal distribution via boundaries
- technically **limits** $\Delta\eta \to \infty$ and $M\tau \to 0$ **do not commute**
  - $\Delta\eta \to \infty$ for any finite $M\tau$ gives pure state
  - $M\tau \to 0$ for any finite $\Delta\eta$ gives thermal state with $T = 1/(2\pi\tau)$

# *Entanglement dynamics in cold atom experiments*

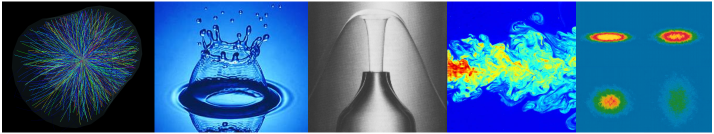
- entanglement can be directly accessed in cold atom experiments
  [Oberthaler group, Greiner group]
- expanding geometries could be realized by interplay of
  - longitudinal expansion
  - time dependent change of sound velocity $v_s(t)$



# *Little bangs in the laboratory*

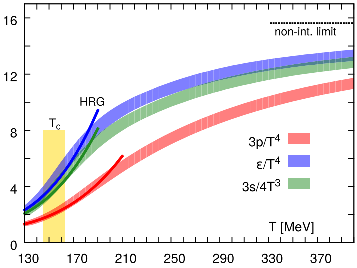


# *Fluid dynamics*

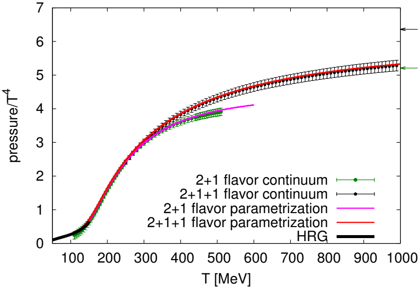


- long distances, long times or strong enough interactions
- matter or quantum fields form a fluid!
- needs **macroscopic** fluid properties
    - thermodynamic equation of state $p(T, \mu)$
    - shear viscosity $\eta(T, \mu)$
    - bulk viscosity $\zeta(T, \mu)$
    - heat conductivity $\kappa(T, \mu)$
    - relaxation times, …
- *ab initio* calculation of fluid properties difficult but fixed by **microscopic** properties in $\mathscr{L}_{\text{QCD}}$

# *Thermodynamics of QCD*



[Bazavov *et al.* (HotQCD) (2014)]



[Borsányi *et al.* (2016)]

- thermodynamic equation of state $p(T)$ rather well understood now
- also $\mu \neq 0$ is being explored
- progress in computing power

# *Transport coefficients*

- from perturbation theory / effective kinetic theory at leading order
  [Arnold, Moore, Yaffe (2003)]

$$\eta(T) = k \frac{T^3}{g^4 \log(1/g)} \,,$$

- next-to-leading order also understood now
  [Ghiglieri, Moore, Teaney (2015-2018)]
- form AdS/CFT correspondence (very strong coupling)
  [Kovtun, Son, Starinets (2003)]

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi}$$

- for viscous relativistic fluid (first order approximation)

$$\nabla_\mu s^\mu = 2\eta \, \sigma_{\rho\nu}\sigma^{\rho\nu} + \zeta(\nabla_\rho u^\rho)^2$$

# *Dissipation*

- dissipation is defined as entropy generation

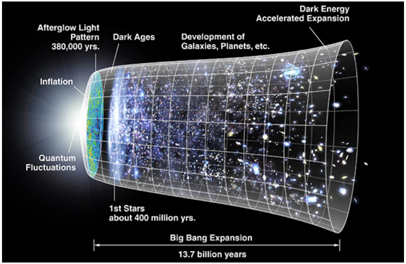
$$\frac{d}{dt}S > 0$$

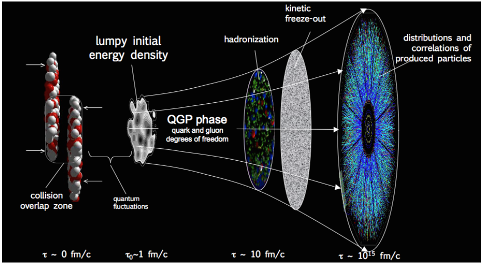
- or for extensive entropy $S = \int d\Sigma_\mu s^\mu$ locally

$$\nabla_\mu s^\mu > 0$$

- second law of thermodynamics
- effective loss of information
- local dissipation = entanglement generation ?

# *Big bang – little bang analogy*





- cosmol. scale: MPc$= 3.1 \times 10^{22}$ m
- Gravity + QED + Dark sector
- one big event

- nuclear scale: fm$= 10^{-15}$ m
- QCD
- very many events

- dynamical description as a fluid
- all information must be reconstructed from final state

# *The dark matter fluid*

- **high energy nuclear collisions**

$$\mathscr{L}_{\text{QCD}} \quad \rightarrow \quad \text{fluid properties}$$

- **late time cosmology**

$$\text{fluid properties} \quad \rightarrow \quad \mathscr{L}_{\text{dark matter}}$$

- until direct detection of dark matter it can only be observed via gravity

$$G^{\mu\nu} = 8\pi G_{\text{N}}\, T^{\mu\nu}$$

  so all we can access is

$$T^{\mu\nu}_{\text{dark matter}}$$

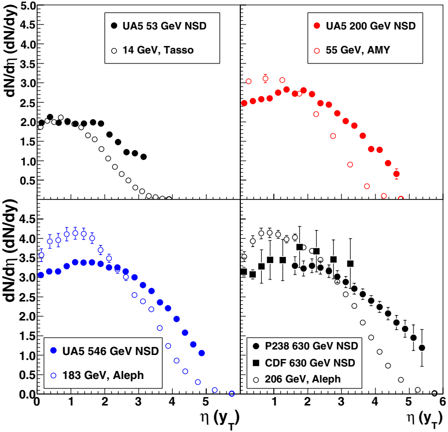
- strong motivation to study heavy ion collisions and cosmology together!

# *Conclusions*

- quantum field theory & information theory are entangled !
- could be essential element for universal non-equilibrium theory
- entanglement helps to understand “thermal effects” in $e^+e^-$ and other collider experiments
    - at very early times theory effectively conformal $\frac{1}{\tau} \gg m, q$
    - entanglement entropy extensive in rapidity $\frac{dS}{d\Delta\eta} = \frac{c}{6}$
    - reduced density matrix for excitations at early times thermal $T = \frac{\hbar}{2\pi\tau}$
- experiments with cold atoms could allow to investigate entanglement directly
- interesting relations to black hole physics and cosmology

# BACKUP

# *Rapidity distribution*



[open (filled) symbols: $e^+e^-$ (pp), Grosse-Oetringhaus & Reygers (2010)]

- rapidity distribution $dN/d\eta$ has plateau around midrapidity
- only logarithmic dependence on collision energy

# *Transverse coordinates*

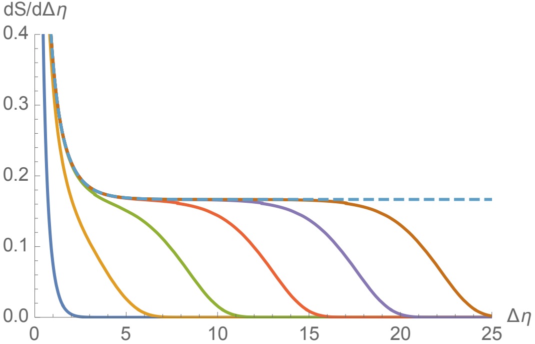
- So far dynamics strictly confined to 1+1 dimensions
- Transverse coordinates may fluctuate, can be described by Nambu-Goto action ($h_{\mu\nu} = \partial_\mu X^m \partial_\nu X_m$)

$$\begin{aligned} S_{\mathsf{NG}} &= \int d^2x \sqrt{-\mathsf{det}\, h_{\mu\nu}}\, \{-\sigma + \dots\} \\ &\approx \int d^2x \sqrt{g} \left\{-\sigma - \frac{\sigma}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i + \dots\right\} \end{aligned}$$

- Two additional, massless, bosonic degrees of freedom corresponding to transverse coordinates $X^i$ with $i = 1, 2$.

# *Free massive fermions*

- Entanglement entropy can also be calculated for free Dirac fermions of mass $m$



$m\tau = 1$, $10^{-1}$, $10^{-2}$, $10^{-3}$, $10^{-4}$, and $10^{-5}$

- Same universal plateau $c/6$ with $c = 1$ at early time
- Conformal limit corresponds to non-interacting fermions
- Consistent with or without bosonization

# *Alternative derivation: mode functions*

- Fluctuation field $\varphi = \phi - \bar{\phi}$ has equation of motion

$$\partial_\tau^2 \varphi(\tau,\eta) + \frac{1}{\tau}\partial_\tau \varphi(\tau,\eta) + \left( M^2 - \frac{1}{\tau^2}\frac{\partial^2}{\partial \eta^2} \right) \varphi(\tau,\eta) = 0$$

- Solution in terms of plane waves

$$\varphi(\tau,\eta) = \int \frac{dk}{2\pi} \left\{ a(k) f(\tau,|k|) e^{ik\eta} + a^\dagger(k)\, f^*(\tau,|k|) e^{-ik\eta} \right\}$$

- Mode functions as Hankel functions

$$f(\tau,k) = \frac{\sqrt{\pi}}{2} e^{\frac{k\pi}{2}} H_{ik}^{(2)}(M\tau)$$

or alternatively as Bessel functions

$$\bar{f}(\tau,k) = \frac{\sqrt{\pi}}{\sqrt{2\sinh(\pi k)}} J_{-ik}(M\tau)$$

# *Bogoliubov transformation*

- Mode functions are related

$$\begin{aligned}\bar{f}(\tau,k) =& \alpha(k) f(\tau,k) + \beta(k) f^*(\tau,k)\\ f(\tau,k) =& \alpha^*(k)\bar{f}(\tau,k) - \beta(k)\bar{f}^*(\tau,k)\end{aligned}$$

- Creation and annihilation operators are related by

$$\begin{aligned}\bar{a}(k) =& \alpha^*(k) a(k) - \beta^*(k) a^\dagger(k)\\ a(k) =& \alpha(k)\bar{a}(k) + \beta(k)\bar{a}^\dagger(k)\end{aligned}$$

- Bogoliubov coefficients

$$\alpha(k) = \sqrt{\frac{e^{\pi k}}{2\sinh(\pi k)}} \qquad \beta(k) = \sqrt{\frac{e^{-\pi k}}{2\sinh(\pi k)}}$$

- Vacuum $|\Omega\rangle$ with respect to $a(k)$ such that $a(k)|\Omega\rangle = 0$ contains excitations with respect to $\bar{a}(k)$ such that $\bar{a}(k)|\Omega\rangle \neq 0$ and *vice versa*

# *Role of different mode functions*

- Hankel functions $f(\tau, k)$ are superpositions of *positive* frequency modes with respect to Minkowski time $t$
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to Minkowski time $t$
- At very early time $1/\tau \gg M$ conformal symmetry

$$ds^2 = \tau^2 \left[ -d\ln(\tau)^2 + d\eta^2 \right]$$

- Hankel functions $f(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to conformal time $\ln(\tau)$
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive* frequency modes with respect to conformal time $\ln(\tau)$

# *Occupation numbers*

- Minkowski space coherent states have two-point functions

$$
\begin{aligned}
\langle \bar{a}^\dagger(k)\bar{a}(k')\rangle_c &= \bar{n}(k)\,2\pi\,\delta(k-k') = |\beta(k)|^2\,2\pi\,\delta(k-k') \\
\langle \bar{a}(k)\bar{a}(k')\rangle_c &= \bar{u}(k)\,2\pi\,\delta(k+k') = -\alpha^*(k)\beta^*(k)\,2\pi\,\delta(k+k') \\
\langle \bar{a}^\dagger(k)\bar{a}^\dagger(k')\rangle_c &= \bar{u}^*(k)\,2\pi\,\delta(k+k') = -\alpha(k)\beta(k)\,2\pi\,\delta(k+k')
\end{aligned}
$$

- Occupation number

$$\bar{n}(k) = |\beta(k)|^2 = \frac{1}{e^{2\pi k}-1}$$

- Bose-Einstein distribution with excitation energy $E = |k|/\tau$ and temperature

$$T = \frac{1}{2\pi\tau}$$

- Off-diagonal occupation number $\bar{u}(k) = -1/(2\sinh(\pi k))$ make sure we still have pure state

# *Local description*

- Consider now rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$
- Fourier expansion becomes discrete

$$\varphi(\eta) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \varphi_n \, e^{in\pi \frac{\eta}{\Delta\eta}}$$

$$\varphi_n = \int_{-\Delta\eta/2}^{\Delta\eta/2} d\eta \, \varphi(\eta) \, \frac{1}{2} \left[ e^{-in\pi \frac{\eta}{\Delta\eta}} + (-1)^n e^{in\pi \frac{\eta}{\Delta\eta}} \right]$$

- Relation to continuous momentum modes by integration kernel

$$\varphi_n = \int \frac{dk}{2\pi} \sin(\tfrac{k\Delta\eta}{2} - \tfrac{n\pi}{2}) \left[ \frac{1}{k - \frac{n\pi}{\Delta\eta}} + \frac{1}{k + \frac{n\pi}{\Delta\eta}} \right] \varphi(k)$$

- Local density matrix determined by correlation functions

$$\langle \varphi_n \rangle, \qquad \langle \pi_n \rangle, \qquad \langle \varphi_n \varphi_m \rangle_c, \qquad \text{etc.}$$

# *Emergence of locally thermal state*

- Mode functions at early time

$$\bar{f}(\tau,k) = \frac{1}{\sqrt{2k}} e^{-ik\ln(\tau) - i\theta(k,M)}$$

- Phase varies strongly with $k$ for $M \to 0$

$$\theta(k,M) = k\ln(M/2) + \arg(\Gamma(1-ik))$$

- Off-diagonal term $\bar{u}(k)$ have factors strongly oscillating with $k$

$$\begin{aligned}\langle\varphi(\tau,k)\varphi^*(\tau,k')\rangle_c =\ & 2\pi\delta(k-k')\frac{1}{|k|} \\ & \times \left\{ \left[\tfrac{1}{2} + \bar{n}(k)\right] + \cos\left[2k\ln(\tau) + 2\theta(k,M)\right]\bar{u}(k)\right\}\end{aligned}$$

cancel out when going to finite interval !

- Only Bose-Einstein occupation numbers $\bar{n}(k)$ remain

# *Entanglement and deep inelastic scattering*

- How strongly entangled is the nuclear wave function?
- What is the entropy of quasi-free partons and can it be understood as a result of entanglement? [Kharzeev, Levin (2017)]

$$S = \ln[xG(x)]$$

- Does saturation at small Bjorken-$x$ have an entropic meaning?
- Entanglement entropy and entropy production in the color glass condensate [Kovner, Lublinsky (2015)]
- Could entanglement entropy help for a non-perturbative extension of the parton model?
- Entropy of perturbative and non-perturbative Pomeron descriptions [Shuryak, Zahed (2017)]