Thermalization and fluid dynamics in high energy collisions

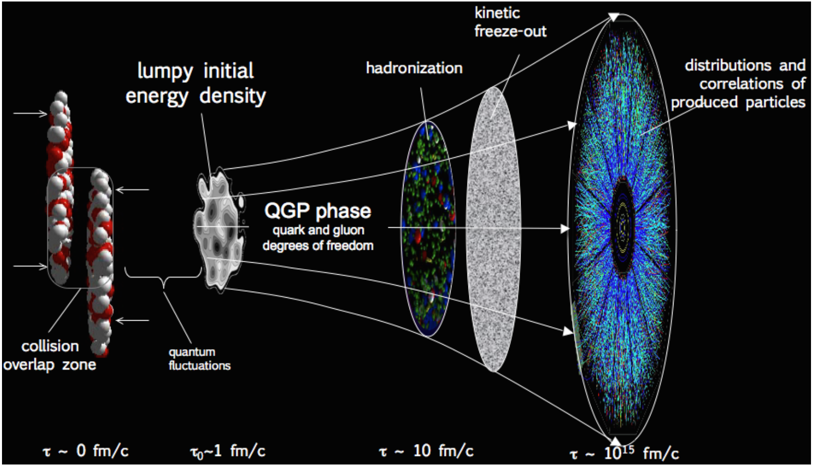
Stefan Floerchinger (Uni Heidelberg)

Quantum Paths, Erwin Schrödinger Institut Wien, 24.04.2018

UNIVERSITÄT HEIDELBERG ZUKUNFT SEIT 1386

ISOQUANT SFB1225

Little bangs in the laboratory



Microscopic description

Lagrangian

$$\mathscr{L} = -\frac{1}{2}\text{tr}\, \mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} - \sum_f \bar{\psi}_f \left(i\gamma^\mu \mathbf{D}_\mu - m_f\right)\psi_f$$

with

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - ig[\mathbf{A}_\mu, \mathbf{A}_\nu], \qquad \mathbf{D}_\mu = \partial_\mu - ig\mathbf{A}_\mu$$

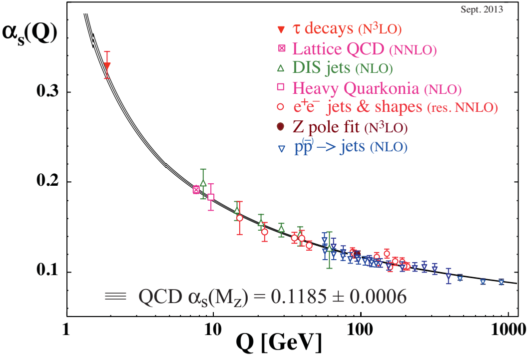
particle content

- $N_c^2 - 1 = 8$ real massless vector bosons: gluons
- $N_c \times N_f$ massive Dirac fermions: quarks

quark masses

Up	2.3 MeV	Charm	1275 MeV	Top	173 GeV
Down	4.8 MeV	Strange	95 MeV	Bottom	4180 MeV

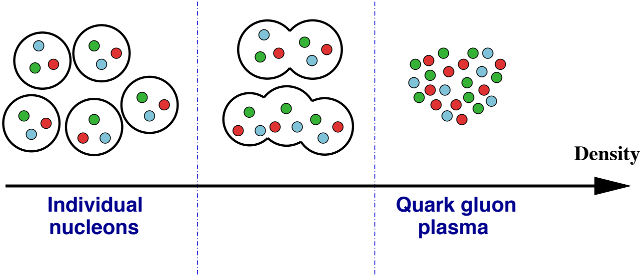
Asymptotic freedom



[Particle Data Group (2013)]

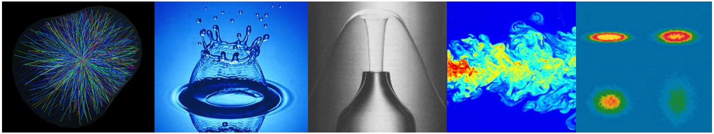
- coupling constant small at high momentum transfer / energy scale
- high-temperature QCD should be weakly coupled
- low-temperature QCD should be strongly coupled

Confinement - deconfinement



- for low temperature / density: quarks and gluons confined to hadrons
- for high temperature / density: deconfined quarks and gluons
- in between no sharp phase transition but continuous crossover

Fluid dynamics



- long distances, long times or strong enough interactions
- matter or quantum fields form a fluid!
- needs **macroscopic** fluid properties
    - thermodynamic equation of state $p(T, \mu)$
    - shear viscosity $\eta(T, \mu)$
    - bulk viscosity $\zeta(T, \mu)$
    - heat conductivity $\kappa(T, \mu)$
    - relaxation times, ...
- *ab initio* calculation of fluid properties difficult but fixed by **microscopic** properties in $\mathscr{L}_{\text{QCD}}$

Relativistic fluid dynamics

Energy-momentum tensor and conserved current

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + (p + \pi_{\mathsf{bulk}})\Delta^{\mu\nu} + \pi^{\mu\nu}$$
$$N^\mu = n\, u^\mu + \nu^\mu$$

- tensor decomposition using fluid velocity u^μ, $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$
- thermodynamic equation of state $p = p(T, \mu)$

Covariant **conservation laws** $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu N^\mu = 0$ imply

- equation for **energy density** ϵ

$$u^\mu \partial_\mu \epsilon + (\epsilon + p + \pi_{\mathsf{bulk}})\nabla_\mu u^\mu + \pi^{\mu\nu}\nabla_\mu u_\nu = 0$$

- equation for **fluid velocity** u^μ

$$(\epsilon + p + \pi_{\mathsf{bulk}})u^\mu \nabla_\mu u^\nu + \Delta^{\nu\mu}\partial_\mu(p + \pi_{\mathsf{bulk}}) + \Delta^\nu{}_\alpha \nabla_\mu \pi^{\mu\alpha} = 0$$

- equation for **particle number density** n

$$u^\mu \partial_\mu n + n\nabla_\mu u^\mu + \nabla_\mu \nu^\mu = 0$$

Constitutive relations

Second order relativistic fluid dynamics:

- equation for **shear stress** $\pi^{\mu\nu}$

$$\tau_{\text{shear}}\, P^{\rho\sigma}{}_{\alpha\beta}\, u^{\mu}\nabla_{\mu}\pi^{\alpha\beta} + \pi^{\rho\sigma} + 2\eta\, P^{\rho\sigma\alpha}{}_{\beta}\,\nabla_{\alpha}u^{\beta} + \ldots = 0$$

  with **shear viscosity** $\eta(T,\mu)$

- equation for **bulk viscous pressure** π_{bulk}

$$\tau_{\text{bulk}}\, u^{\mu}\partial_{\mu}\pi_{\text{bulk}} + \pi_{\text{bulk}} + \zeta\, \nabla_{\mu}u^{\mu} + \ldots = 0$$

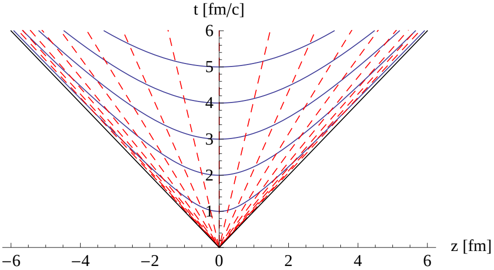
  with **bulk viscosity** $\zeta(T,\mu)$

- equation for **baryon diffusion current** ν^{μ}

$$\tau_{\text{heat}}\, \Delta^{\alpha}{}_{\beta}\, u^{\mu}\nabla_{\mu}\nu^{\beta} + \nu^{\alpha} + \kappa \left[\frac{nT}{\epsilon + p}\right]^2 \Delta^{\alpha\beta}\partial_{\beta}\left(\frac{\mu}{T}\right) + \ldots = 0$$

  with **heat conductivity** $\kappa(T,\mu)$

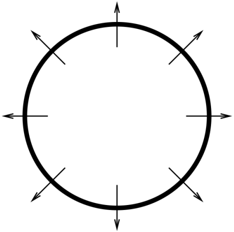
Bjorken boost invariance



How does the fluid velocity look like?

- Bjorkens guess: $v_z(t, x, y, z) = z/t$
- leads to an invariance under Lorentz-boosts in the z-direction
- use coordinates $\tau = \sqrt{t^2 - z^2}$, x, y, $\eta = \text{arctanh}(z/t)$
- Bjorken boost symmetry is reasonably accurate close to mid-rapidity $\eta \approx 0$

Transverse expansion



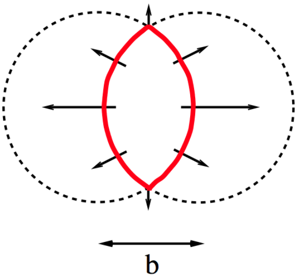
- for central collisions ($r = \sqrt{x^2 + y^2}$)

$$\epsilon = \epsilon(\tau, r)$$

- initial pressure gradient leads to **radial flow**

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \ f(\tau, r)$$

Non-central collisions



- pressure gradients larger in reaction plane
- leads to larger fluid velocity in this direction
- more particles fly in this direction
- can be quantified in terms of elliptic flow v_2
- particle distribution

$$\frac{dN}{d\phi} = \frac{N}{2\pi}\left[1 + 2\sum_m v_m \cos\left(m\left(\phi - \psi_R\right)\right)\right]$$

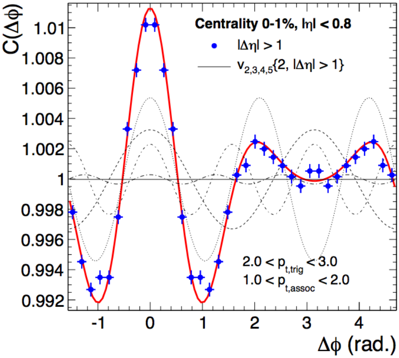
- symmetry $\phi \to \phi + \pi$ implies $v_1 = v_3 = v_5 = \ldots = 0$.

Two-particle correlation function

- normalized two-particle correlation function

$$C(\phi_1, \phi_2) = \frac{\left\langle \frac{dN}{d\phi_1} \frac{dN}{d\phi_2} \right\rangle_{\text{events}}}{\left\langle \frac{dN}{d\phi_1} \right\rangle_{\text{events}} \left\langle \frac{dN}{d\phi_2} \right\rangle_{\text{events}}} = 1 + 2 \sum_m v_m^2 \cos(m\,(\phi_1 - \phi_2))$$

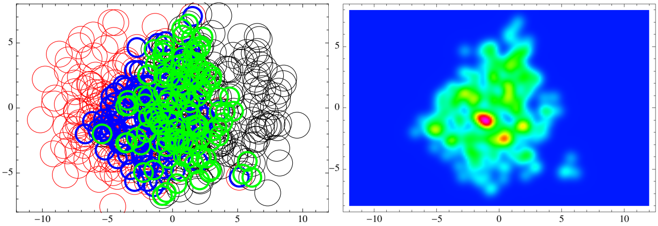
- surprisingly v_2, v_3, v_4, v_5 and v_6 are all non-zero!



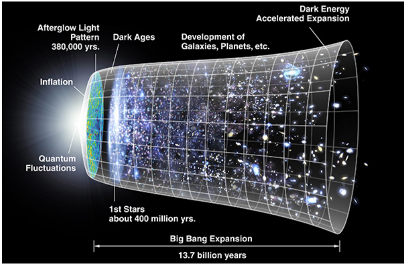
[ALICE 2011, similar results from CMS, ATLAS, Phenix, Star]

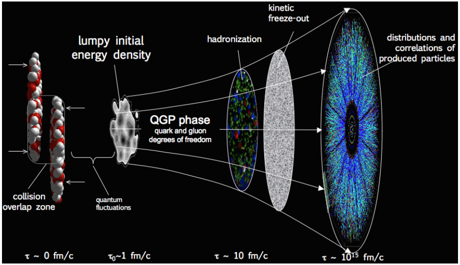
Event-by-event fluctuations

- deviations from symmetric initial energy density distribution from event-by-event fluctuations
- one example is Glauber model



Big bang – little bang analogy



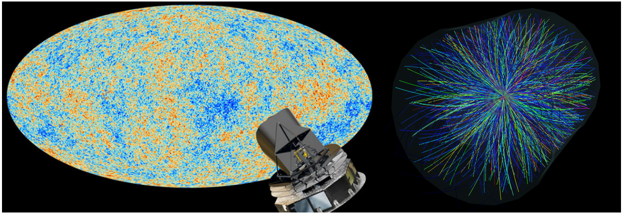


- cosmol. scale: MPc$= 3.1 \times 10^{22}$ m
- Gravity + QED + Dark sector
- one big event

- nuclear scale: fm$= 10^{-15}$ m
- QCD
- very many events

- initial conditions not directly accessible
- all information must be reconstructed from final state
- dynamical description as a fluid
- fluctuating initial state

Similarities to cosmological fluctuation analysis



- fluctuation spectrum contains info from early times
- many numbers can be measured and compared to theory
- can lead to detailed understanding of evolution

Cosmological perturbation theory

[Lifshitz, Peebles, Bardeen, Kosama, Sasaki, Ehler, Ellis, Hawking, Mukhanov, Weinberg, ...]

- solves evolution equations for fluid + gravity
- expands in perturbations around homogeneous background
- detailed understanding how different modes evolve
- very simple equations of state $p = w\,\epsilon$
- viscosities usually neglected $\eta = \zeta = 0$
- photons and neutrinos are free streaming

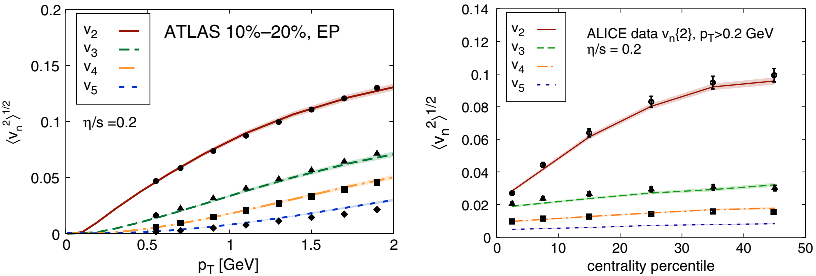
Fluid dynamic perturbation theory for heavy ion collisions

[Floerchinger & Wiedemann, PLB 728, 407 (2014)]

- solves evolution equations for relativistic QCD fluid
- expands in perturbations around event-averaged solution
- leads to linear + non-linear response formalism
- good convergence properties
  [Floerchinger *et al.*, PLB 735, 305 (2014), Brouzakis *et al.* PRD 91, 065007 (2015)]
- comparison to cosmology rather direct

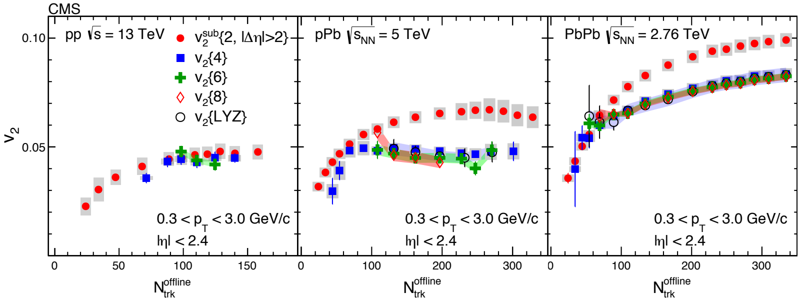
Fluid dynamic simulations

- second order relativistic fluid dynamics simulated numerically
- fluctuating initial conditions
- η/s is varied to find experimentally favored value



[Gale, Jeon, Schenke, Tribedy, Venugopalan (2013)]

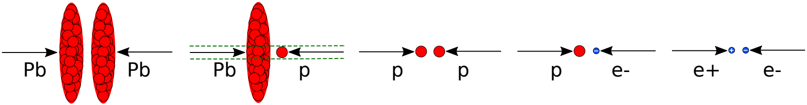
Collective behavior in large and small systems



- flow coefficients from higher order cumulants $v_2\{n\}$ agree: $\rightarrow$ collective behavior
- elliptic flow signals also in **pPb** and **pp**!
- can fluid approximation work for pp collisions?

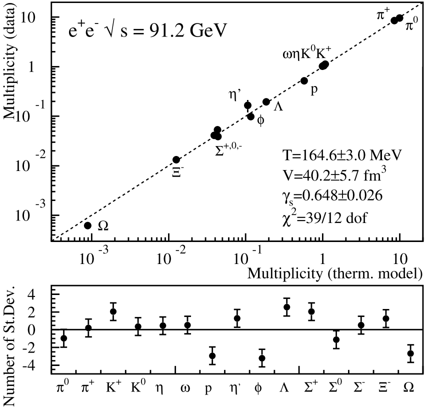
Questions and puzzles

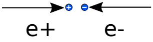
- how universal are collective flow and fluid dynamics?
- what determines density distribution of a proton?
- do we really understand elementary particle collision physics?
- multi-parton interactions?
- more elementary systems such as ep or e^+e^-?



The thermal model puzzle

- elementary e^+e^- collision experiments show thermal-like features
- particle multiplicities well described by thermal model



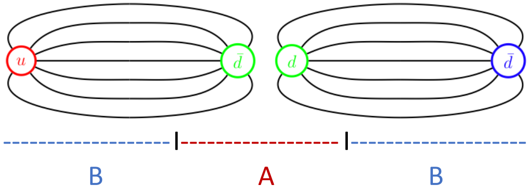


[Becattini, Casterina, Milov & Satz, EPJC 66, 377 (2010)]

- conventional thermalization by final state interactions unlikely
- alternative explanations needed

QCD strings and entanglement

[Berges, Floerchinger, Venugopalan (2017)]



- particle production from QCD strings
- Lund model (Pythia)
- different regions in a string are **entangled**
- subinterval A has reduced density matrix of **mixed form** even if ρ is pure

$$\rho_A = \mathrm{Tr}_B\{\rho\}$$

- could this lead to thermal-like effects?
- characterization by **entanglement entropy**

$$S_A = -\mathrm{Tr}_A\,\{\rho_A \ln(\rho_A)\}$$

- **globally pure** state $S = 0$ can be **locally mixed** $S_A > 0$
- **coherent information** $I_{B\rangle A} = S_A - S$ can be **positive**

Schwinger model

- QED in 1+1 dimension

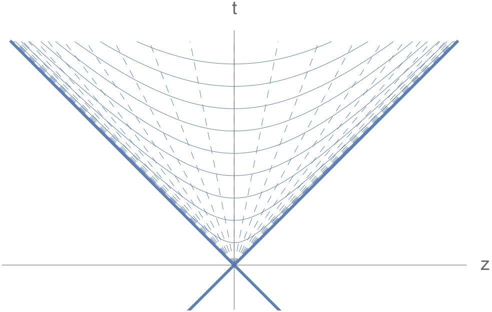
$$\mathscr{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - iqA_\mu)\psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

- geometric confinement
- U(1) charge related to string tension $q = \sqrt{2\sigma}$
- for single fermion one can **bosonize theory** exactly
  [Coleman, Jackiw, Susskind (1975)]

$$S = \int d^2x \sqrt{g} \Big\{ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} M^2 \phi^2 - \frac{m\, q\, e^\gamma}{2\pi^{3/2}} \cos\left(2\sqrt{\pi}\phi + \theta\right) \Big\}$$

- Schwinger bosons are dipoles $\phi \sim \bar{\psi}\psi$
- mass is related to U(1) charge by $M = q/\sqrt{\pi} = \sqrt{2\sigma/\pi}$
- massless Schwinger model $m = 0$ leads to free bosonic theory

Expanding string solution



- quark-anti-quark pair on trajectories $z = \pm t$
- coordinates: Bjorken time $\tau = \sqrt{t^2 - z^2}$, rapidity $\eta = \text{arctanh}(z/t)$
- Bjorken boost symmetry $\eta \to \eta + \Delta\eta$

Coherent field evolution

- Schwinger boson field expectation value depends only on τ

$$\bar{\phi} = \langle \phi \rangle = \bar{\phi}(\tau)$$

- equation of motion

$$\partial_\tau^2 \bar{\phi} + \frac{1}{\tau} \partial_\tau \bar{\phi} + M^2 \bar{\phi} = 0$$

- Gauss law: electric field $E = q\phi/\sqrt{\pi}$ must approach U(1) charge

$$\bar{\phi}(\tau) \to \sqrt{\pi} \qquad (\text{for } \tau \to 0_+)$$

- solution of equation of motion [Loshaj, Kharzeev (2011)]

$$\bar{\phi}(\tau) = \sqrt{\pi} J_0(M\tau)$$

Gaussian states

- theories with quadratic action often have Gaussian density matrix
- fully characterized by field expectation values

$$\bar{\phi}(x) = \langle \phi(x) \rangle, \qquad \bar{\pi}(x) = \langle \pi(x) \rangle$$

and connected two-point correlation functions, e. g.

$$\langle \phi(x)\phi(y) \rangle_c = \langle \phi(x)\phi(y) \rangle - \bar{\phi}(x)\bar{\phi}(y)$$

- if ρ is Gaussian, also reduced density matrix ρ_A is Gaussian

Entanglement entropy for Gaussian state

- entanglement entropy of Gaussian state in region A
  [Berges, Floerchinger, Venugopalan, 1712.09362]

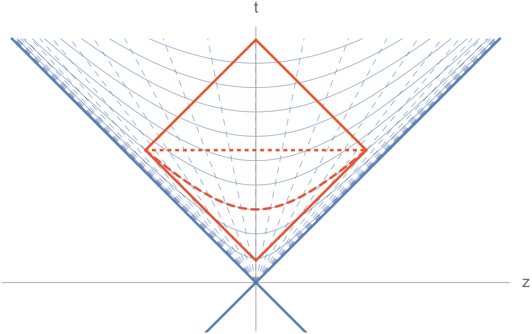
$$S_A = \frac{1}{2} \mathsf{Tr}_A \left\{ D \ln(D^2) \right\}$$

- operator trace over region A only
- matrix of correlation functions

$$D(x,y) = \begin{pmatrix} -i\langle\phi(x)\pi(y)\rangle_c & i\langle\phi(x)\phi(y)\rangle_c \\ -i\langle\pi(x)\pi(y)\rangle_c & i\langle\pi(x)\phi(y)\rangle_c \end{pmatrix}$$

- involves connected correlation functions of field $\phi(x)$ and canonically conjugate momentum field $\pi(x)$
- expectation value $\bar{\phi}$ does not appear explicitly
- coherent states and vacuum have equal entanglement entropy S_A

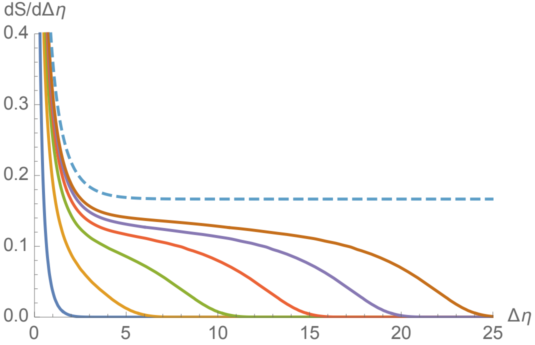
Rapidity interval



- consider rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ at fixed Bjorken time τ
- entanglement entropy does not change by unitary time evolution with endpoints kept fixed
- can be evaluated equivalently in interval $\Delta z = 2\tau \sinh(\Delta\eta/2)$ at fixed time $t = \tau \cosh(\Delta\eta/2)$
- need to solve eigenvalue problem with correct **boundary conditions**

Bosonized massless Schwinger model

- entanglement entropy understood numerically for free massive scalars [Casini, Huerta (2009)]
- entanglement entropy density $dS/d\Delta\eta$ for bosonized massless Schwinger model ($M = \frac{q}{\sqrt{\pi}}$)



$M\tau = 1$, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, and 10^{-5}

Conformal limit

- For $M\tau \to 0$ one has conformal field theory limit
  [Holzhey, Larsen, Wilczek (1994)]

$$S(\Delta z) = \frac{c}{3} \ln \left(\Delta z / \epsilon \right) + \text{constant}$$

with small length ϵ acting as UV cutoff.

- Here this implies

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln \left(2\tau \, \text{sinh}(\Delta\eta/2)/\epsilon \right) + \text{constant}$$

- Conformal charge $c = 1$ for free massless scalars or Dirac fermions.
- Additive constant not universal but entropy density is

$$\begin{aligned} \frac{\partial}{\partial \Delta\eta} S(\tau, \Delta\eta) = & \frac{c}{6} \text{coth}(\Delta\eta/2) \\ \to & \frac{c}{6} \qquad\qquad (\Delta\eta \gg 1) \end{aligned}$$

- Entropy becomes extensive in $\Delta\eta$!

Universal entanglement entropy density

- for very early times "Hubble" expansion rate dominates over masses and interactions

$$H = \frac{1}{\tau} \gg M = \frac{q}{\sqrt{\pi}}, m$$

- theory dominated by free, massless fermions
- universal entanglement entropy density

$$\frac{dS}{d\Delta\eta} = \frac{c}{6}$$

  with conformal charge c
- for QCD in 1+1 D (gluons not dynamical, no transverse excitations)

$$c = N_c \times N_f$$

- from fluctuating transverse coordinates (Nambu-Goto action)

$$c = N_c \times N_f + 2 \approx 9 + 2 = 11$$

Temperature and entanglement entropy

- for conformal fields, entanglement entropy has also been calculated at non-zero temperature.
- for static interval of length L [Korepin (2004); Calabrese, Cardy (2004)]

$$S(T,l) = \frac{c}{3} \ln \left(\frac{1}{\pi T \epsilon} \sinh(\pi L T) \right) + \text{const}$$

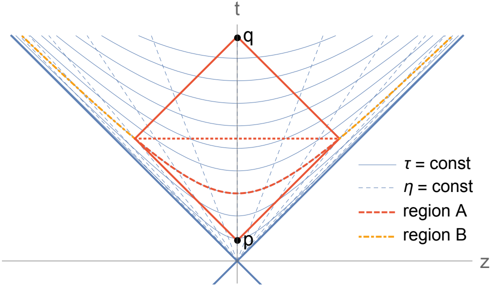
- compare this to our result in expanding geometry

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln \left(\frac{2\tau}{\epsilon} \sinh(\Delta\eta/2) \right) + \text{const}$$

- expressions agree for $L = \tau\Delta\eta$ (with metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$) and time-dependent temperature

$$T = \frac{1}{2\pi\tau}$$

Modular or entanglement Hamiltonian



- conformal field theory [Casini, Huerta, Myers (2011), Arias, Blanco, Casini, Huerta (2017), see also Candelas, Dowker (1979)]

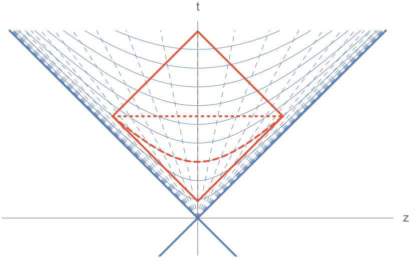
$$\rho_A = \frac{1}{Z_A} e^{-K}, \qquad Z_A = \mathrm{Tr}\, e^{-K}$$

- modular or entanglement Hamiltonian **local expression**

$$K = \int_\Sigma d\Sigma_\mu \, \xi_\nu(x) \, T^{\mu\nu}(x)$$

- energy-momentum of excitations around coherent field $T^{\mu\nu}(x)$

Time-dependent temperature



- combination of fluid velocity and temperature $\xi^\mu(x) = \frac{u^\mu(x)}{T(x)}$
- for $\Delta\eta \to \infty$: fluid velocity in τ-direction & time-dependent temperature [Berges, Floerchinger, Venugopalan (2017)]

$$T(\tau) = \frac{\hbar}{2\pi\tau}$$

- **Entanglement between rapidity intervals leads to local thermal density matrix at very early times !**
- Hawking-Unruh temperature in Rindler wedge $T(x) = \frac{\hbar c}{2\pi x}$

Physics picture

- coherent state vacuum at early time contains entangled pairs of quasi-particles with opposite wave numbers
- on finite rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ in- and out-flux of quasi-particles with thermal distribution via boundaries
- technically limits $\Delta\eta \to \infty$ and $M\tau \to 0$ do not commute
    - $\Delta\eta \to \infty$ for any finite $M\tau$ gives pure state
    - $M\tau \to 0$ for any finite $\Delta\eta$ gives thermal state with $T = 1/(2\pi\tau)$

Conclusions

- high energy nuclear collisions produce a relativistic QCD fluid!
- interesting parallels between cosmology and heavy ion collisions
- similar physics of evolving fluid fluctuations
- experimental hints for collective flow also in pPb and pp collisions
- expanding QCD strings: entanglement between rapidity intervals can lead to thermal-like effects!

Backup slides

QCD in two dimensions

- QCD in 1+1 dimensions described by 't Hooft model

$$\mathscr{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - ig\mathbf{A}_\mu)\psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{2}\mathrm{tr}\,\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}$$

- fermionic fields ψ_i with sums over flavor species $i = 1, \ldots, N_f$
- SU(N_c) gauge fields $\mathbf{A}_\mu$ with field strength tensor $\mathbf{F}_{\mu\nu}$
- gluons are not dynamical in two dimensions
- gauge coupling g has dimension of mass
- non-trivial, interacting theory, cannot be solved exactly
- spectrum of excitations known for $N_c \to \infty$ with $g^2 N_c$ fixed
  ['t Hooft (1974)]

Alternative derivation: mode functions

- fluctuation field $\varphi = \phi - \bar{\phi}$ has equation of motion

$$\partial_\tau^2 \varphi(\tau,\eta) + \frac{1}{\tau}\partial_\tau \varphi(\tau,\eta) + \left(M^2 - \frac{1}{\tau^2}\frac{\partial^2}{\partial \eta^2} \right) \varphi(\tau,\eta) = 0$$

- solution in terms of plane waves

$$\varphi(\tau,\eta) = \int \frac{dk}{2\pi} \left\{ a(k) f(\tau,|k|) e^{ik\eta} + a^\dagger(k)\, f^*(\tau,|k|) e^{-ik\eta} \right\}$$

- mode functions as Hankel functions

$$f(\tau,k) = \frac{\sqrt{\pi}}{2} e^{\frac{k\pi}{2}} H_{ik}^{(2)}(M\tau)$$

or alternatively as Bessel functions

$$\bar{f}(\tau,k) = \frac{\sqrt{\pi}}{\sqrt{2\sinh(\pi k)}} J_{-ik}(M\tau)$$

Bogoliubov transformation

- mode functions are related

$$\begin{aligned}\bar{f}(\tau,k) =& \alpha(k) f(\tau,k) + \beta(k) f^*(\tau,k)\\ f(\tau,k) =& \alpha^*(k)\bar{f}(\tau,k) - \beta(k)\bar{f}^*(\tau,k)\end{aligned}$$

- creation and annihilation operators are related by

$$\begin{aligned}\bar{a}(k) =& \alpha^*(k) a(k) - \beta^*(k) a^\dagger(k)\\ a(k) =& \alpha(k)\bar{a}(k) + \beta(k)\bar{a}^\dagger(k)\end{aligned}$$

- Bogoliubov coefficients

$$\alpha(k) = \sqrt{\frac{e^{\pi k}}{2\sinh(\pi k)}} \qquad \beta(k) = \sqrt{\frac{e^{-\pi k}}{2\sinh(\pi k)}}$$

- vacuum $|\Omega\rangle$ with respect to $a(k)$ such that $a(k)|\Omega\rangle = 0$ contains excitations with respect to $\bar{a}(k)$ such that $\bar{a}(k)|\Omega\rangle \neq 0$ and *vice versa*

Role of different mode functions

- Hankel functions $f(\tau, k)$ are superpositions of *positive* frequency modes with respect to Minkowski time t
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to Minkowski time t
- at very early time $1/\tau \gg M, m$ conformal symmetry

$$ds^2 = \tau^2 \left[-d\ln(\tau)^2 + d\eta^2\right]$$

- Hankel functions $f(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to conformal time $\ln(\tau)$
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive* frequency modes with respect to conformal time $\ln(\tau)$

Occupation numbers

- Minkowski space coherent states have two-point functions

$$\langle \bar{a}^\dagger(k)\bar{a}(k')\rangle_c = \bar{n}(k)\,2\pi\,\delta(k-k') = |\beta(k)|^2\,2\pi\,\delta(k-k')$$
$$\langle \bar{a}(k)\bar{a}(k')\rangle_c = \bar{u}(k)\,2\pi\,\delta(k+k') = -\alpha^*(k)\beta^*(k)\,2\pi\,\delta(k+k')$$
$$\langle \bar{a}^\dagger(k)\bar{a}^\dagger(k')\rangle_c = \bar{u}^*(k)\,2\pi\,\delta(k+k') = -\alpha(k)\beta(k)\,2\pi\,\delta(k+k')$$

- occupation number

$$\bar{n}(k) = |\beta(k)|^2 = \frac{1}{e^{2\pi k}-1}$$

- Bose-Einstein distribution with excitation energy $E = |k|/\tau$ and temperature

$$T = \frac{1}{2\pi\tau}$$

- off-diagonal occupation number $\bar{u}(k) = -1/(2\sinh(\pi k))$ make sure we still have pure state

Local description

- consider now rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$
- Fourier expansion becomes discrete

$$\varphi(\eta) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \varphi_n \, e^{in\pi \frac{\eta}{\Delta\eta}}$$

$$\varphi_n = \int_{-\Delta\eta/2}^{\Delta\eta/2} d\eta \, \varphi(\eta) \, \frac{1}{2} \left[e^{-in\pi \frac{\eta}{\Delta\eta}} + (-1)^n e^{in\pi \frac{\eta}{\Delta\eta}} \right]$$

- relation to continuous momentum modes by integration kernel

$$\varphi_n = \int \frac{dk}{2\pi} \sin(\tfrac{k\Delta\eta}{2} - \tfrac{n\pi}{2}) \left[\frac{1}{k - \frac{n\pi}{\Delta\eta}} + \frac{1}{k + \frac{n\pi}{\Delta\eta}} \right] \varphi(k)$$

- local density matrix determined by correlation functions

$$\langle \varphi_n \rangle, \qquad \langle \pi_n \rangle, \qquad \langle \varphi_n \varphi_m \rangle_c, \qquad \text{etc.}$$

Emergence of locally thermal state

- mode functions at early time

$$\bar{f}(\tau, k) = \frac{1}{\sqrt{2k}} e^{-ik\ln(\tau) - i\theta(k,M)}$$

- phase varies strongly with k for $M \to 0$

$$\theta(k, M) = k\ln(M/2) + \arg(\Gamma(1 - ik))$$

- off-diagonal term $\bar{u}(k)$ have factors strongly oscillating with k

$$\langle\varphi(\tau, k)\varphi^*(\tau, k')\rangle_c = 2\pi\delta(k - k')\frac{1}{|k|} \times \left\{\left[\tfrac{1}{2} + \bar{n}(k)\right] + \cos\left[2k\ln(\tau) + 2\theta(k, M)\right]\bar{u}(k)\right\}$$

  cancel out when going to finite interval !

- only Bose-Einstein occupation numbers $\bar{n}(k)$ remain