

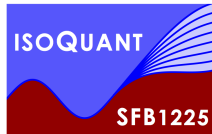
Entanglement in an expanding QCD string

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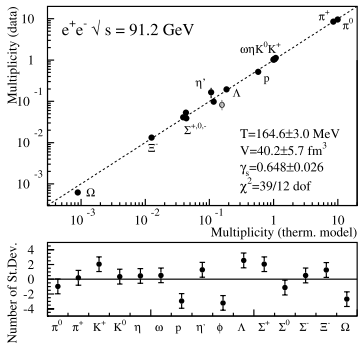


based on

- J. Berges, S. Floerchinger & R. Venugopalan, *Thermal excitation spectrum from entanglement in an expanding quantum string* [[Phys. Lett. B 778, 442 \(2018\)](#)]
- J. Berges, S. Floerchinger & R. Venugopalan, *Dynamics of entanglement in expanding quantum fields* [[arXiv:1712.09362](#)]

Motivation

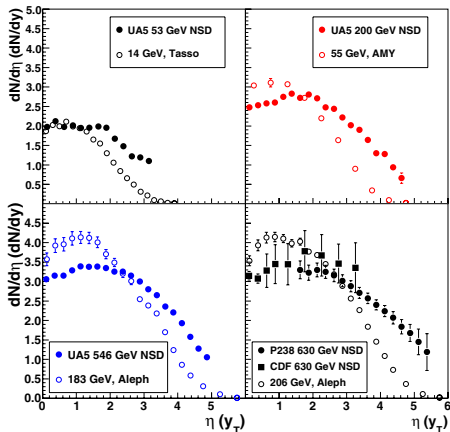
- elementary particle collision experiments such as $e^+ e^-$ collisions show some thermal-like features
- particle multiplicities well described by thermal model



[Becattini, Casterina, Milov & Satz, EPJC 66, 377 (2010)]

- conventional thermalization by collisions unlikely
- more thermal-like features difficult to understand in PYTHIA [Fischer, Sjöstrand (2017)]
- alternative explanations needed

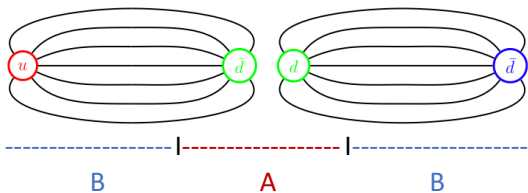
Rapidity distribution



[open (filled) symbols: e^+e^- (pp), Grosse-Oetringhaus & Reygers (2010)]

- rapidity distribution $dN/d\eta$ has plateau around midrapidity
- only logarithmic dependence on collision energy

QCD strings



- particle production from QCD strings
- Lund string model (e. g. PYTHIA)
- different regions in a string are entangled
- subinterval A is described by reduced density matrix

$$\rho_A = \text{Tr}_B \rho$$

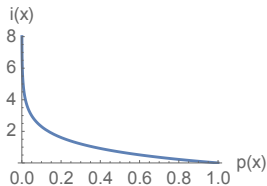
- reduced density matrix is of mixed state form
- could this lead to thermal-like effects?

Entropy and information

[Claude Shannon (1948)]

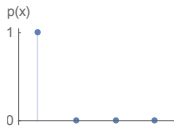
- consider a random variable x with probability distribution $p(x)$
- information content or “surprise” associated with outcome x

$$i(x) = -\ln p(x)$$

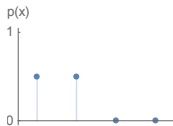


- Entropy is expectation value of information content

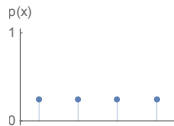
$$S = \langle i(x) \rangle = -\sum_x p(x) \ln p(x)$$



$$S = 0$$



$$S = \ln(2)$$



$$S = 2 \ln(2)$$

Classical statistics

- consider system of two random variables x and y
- joint probability $p(x, y)$, joint entropy

$$S = - \sum_{x,y} p(x, y) \ln p(x, y)$$

- reduced or marginal probability $p(x) = \sum_y p(x, y)$
- reduced or marginal entropy

$$S_x = - \sum_x p(x) \ln p(x)$$

- **joint entropy is greater than** or equal to **reduced entropy**

$$S \geq S_x$$

- **globally pure** state $S = 0$ is also **locally pure** $S_x = 0$

Entropy in quantum theory

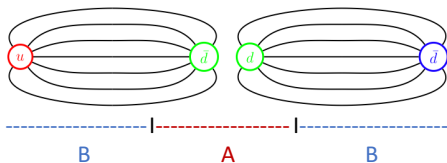
[John von Neumann (1932)]

$$S = -\text{Tr}\{\rho \ln \rho\}$$

- based on the quantum density operator ρ
- for pure states $\rho = |\psi\rangle\langle\psi|$ one has $S = 0$
- for mixed states $\rho = \sum_j p_j |j\rangle\langle j|$ one has $S = -\sum_j p_j \ln p_j > 0$
- unitary time evolution conserves entropy
- global characterization of quantum state

Entropy and entanglement

- consider a split of a quantum system into two $A + B$



- reduced density operator for system A

$$\rho_A = \text{Tr}_B\{\rho\}$$

- entropy associated with subsystem A : **entanglement entropy**

$$S_A = -\text{Tr}_A\{\rho_A \ln \rho_A\}$$

- globally pure** state $S = 0$ can be **locally mixed** $S_A > 0$
- pure **product** state $\rho = \rho_A \otimes \rho_B$ leads to $S_A = 0$
- pure **entangled** state $\rho \neq \rho_A \otimes \rho_B$ leads to $S_A > 0$
- coherent information** $I_{B|A} = S_A - S$ can be **positive**

Microscopic model

- QCD in 1+1 dimensions described by 't Hooft model

$$\mathcal{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - ig\mathbf{A}_\mu) \psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{2} \text{tr} \mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu}$$

- fermionic fields ψ_i with sums over flavor species $i = 1, \dots, N_f$
- $SU(N_c)$ gauge fields \mathbf{A}_μ with field strength tensor $\mathbf{F}_{\mu\nu}$
- gluons are not dynamical in two dimensions
- gauge coupling g has dimension of mass
- non-trivial, interacting theory, cannot be solved exactly
- spectrum of excitations known for $N_c \rightarrow \infty$ with $g^2 N_c$ fixed
['t Hooft (1974)]

Schwinger model

- QED in 1+1 dimension

$$\mathcal{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - iqA_\mu) \psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- geometric confinement
- U(1) charge related to string tension $q = \sqrt{2\sigma}$
- for single fermion one can **bosonize theory** exactly
[Coleman, Jackiw, Susskind (1975)]

$$S = \int d^2x \sqrt{g} \left\{ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} M^2 \phi^2 - \frac{m q e^\gamma}{2\pi^{3/2}} \cos(2\sqrt{\pi}\phi + \theta) \right\}$$

- Schwinger bosons are dipoles $\phi \sim \bar{\psi}\psi$
- scalar mass related to U(1) charge by $M = q/\sqrt{\pi} = \sqrt{2\sigma/\pi}$
- massless Schwinger model $m = 0$ leads to free bosonic theory

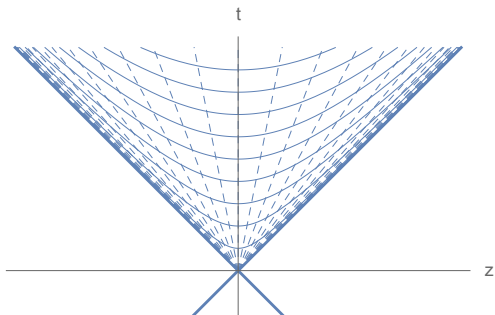
Transverse coordinates

- so far dynamics strictly confined to 1+1 dimensions
- transverse coordinates may fluctuate, can be described by Nambu-Goto action ($h_{\mu\nu} = \partial_\mu X^m \partial_\nu X_m$)

$$S_{\text{NG}} = \int d^2x \sqrt{-\det h_{\mu\nu}} \{-\sigma + \dots\}$$
$$\approx \int d^2x \sqrt{g} \left\{ -\sigma - \frac{\sigma}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i + \dots \right\}$$

- two additional, massless, bosonic degrees of freedom corresponding to transverse coordinates X^i with $i = 1, 2$

Expanding string solution 1



- external quark-anti-quark pair on trajectories $z = \pm t$
- coordinates: Bjorken time $\tau = \sqrt{t^2 - z^2}$, rapidity $\eta = \text{arctanh}(z/t)$
- metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$
- symmetry with respect to longitudinal boosts $\eta \rightarrow \eta + \Delta\eta$

Expanding string solution 2

- Schwinger boson field depends only on τ

$$\bar{\phi} = \bar{\phi}(\tau)$$

- equation of motion

$$\partial_\tau^2 \bar{\phi} + \frac{1}{\tau} \partial_\tau \bar{\phi} + M^2 \bar{\phi} = 0.$$

- Gauss law: electric field $E = q\phi/\sqrt{\pi}$ must approach the U(1) charge of the external quarks $E \rightarrow q_e$ for $\tau \rightarrow 0_+$

$$\bar{\phi}(\tau) \rightarrow \frac{\sqrt{\pi}q_e}{q} \quad (\tau \rightarrow 0_+)$$

- solution of equation of motion [Loshaj, Kharzeev (2011)]

$$\bar{\phi}(\tau) = \frac{\sqrt{\pi}q_e}{q} J_0(M\tau)$$

Gaussian states

- theories with quadratic action often have Gaussian density matrix
- fully characterized by field expectation values

$$\bar{\phi}(x) = \langle \phi(x) \rangle, \quad \bar{\pi}(x) = \langle \pi(x) \rangle$$

and connected two-point correlation functions, e. g.

$$\langle \phi(x)\phi(y) \rangle_c = \langle \phi(x)\phi(y) \rangle - \bar{\phi}(x)\bar{\phi}(y)$$

- if ρ is Gaussian, also reduced density matrix ρ_A is Gaussian

Entanglement entropy for Gaussian state

- entanglement entropy of Gaussian state in region A
[Berges, Floerchinger, Venugopalan, 1712.09362]

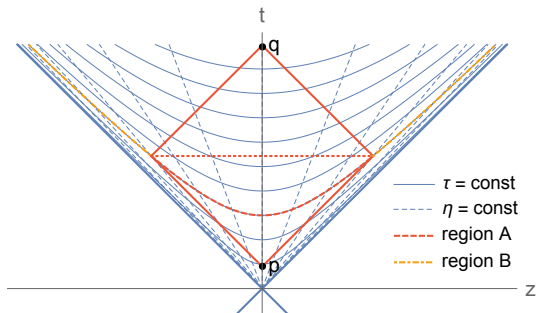
$$S_A = \frac{1}{2} \text{Tr}_A \{ D \ln(D^2) \}$$

- operator trace over region A only
- matrix of correlation functions

$$D(x, y) = \begin{pmatrix} -i\langle \phi(x)\pi(y) \rangle_c & i\langle \phi(x)\phi(y) \rangle_c \\ -i\langle \pi(x)\pi(y) \rangle_c & i\langle \pi(x)\phi(y) \rangle_c \end{pmatrix}$$

- involves connected correlation functions of field $\phi(x)$ and canonically conjugate momentum field $\pi(x)$
- expectation value $\bar{\phi}$ does not appear explicitly
- coherent states and vacuum have equal entanglement entropy S_A

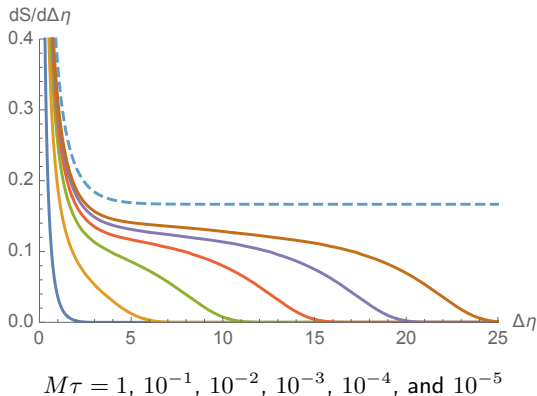
Rapidity interval



- consider rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ at fixed Bjorken time τ
- entanglement entropy does not change by unitary time evolution with endpoints kept fixed
- can be evaluated equivalently in interval $\Delta z = 2\tau \sinh(\Delta\eta/2)$ at fixed time $t = \tau \cosh(\Delta\eta/2)$
- need to solve eigenvalue problem with correct **boundary conditions**

Bosonized massless Schwinger model

- entanglement entropy understood numerically for free massive scalars [Casini, Huerta (2009)]
- entanglement entropy density $dS/d\Delta\eta$ for bosonized massless Schwinger model ($M = \frac{g}{\sqrt{\pi}}$)



Conformal limit

- For $M\tau \rightarrow 0$ one has conformal field theory limit
[Holzhey, Larsen, Wilczek (1994)]

$$S(\Delta z) = \frac{c}{3} \ln(\Delta z/\epsilon) + \text{constant}$$

with small length ϵ acting as UV cutoff.

- Here this implies

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln(2\tau \sinh(\Delta\eta/2)/\epsilon) + \text{constant}$$

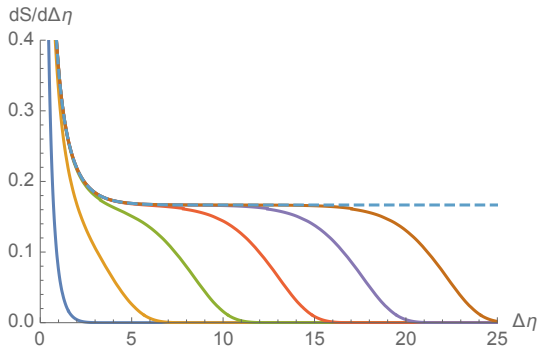
- Conformal charge $c = 1$ for free massless scalars or Dirac fermions.
- Additive constant not universal but entropy density is

$$\begin{aligned} \frac{\partial}{\partial \Delta\eta} S(\tau, \Delta\eta) &= \frac{c}{6} \coth(\Delta\eta/2) \\ &\rightarrow \frac{c}{6} \quad (\Delta\eta \gg 1) \end{aligned}$$

- Entropy becomes extensive in $\Delta\eta$!

Free massive fermions

- entanglement entropy for free Dirac fermions of mass m



$$m\tau = 1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, \text{ and } 10^{-5}$$

- same universal plateau $c/6$ with $c = 1$ at early time
- conformal limit corresponds to non-interacting fermions
- consistent with or without bosonization

Universal entanglement entropy density

- for very early times “Hubble” expansion rate dominates over masses and interactions

$$H = \frac{1}{\tau} \gg M = \frac{q}{\sqrt{\pi}}, m$$

- theory dominated by free, massless fermions
- universal entanglement entropy density

$$\frac{dS}{d\Delta\eta} = \frac{c}{6}$$

with conformal charge c

- for QCD in 1+1 D (gluons not dynamical, no transverse excitations)

$$c = N_c \times N_f$$

- from fluctuating transverse coordinates (Nambu-Goto action)

$$c = N_c \times N_f + 2 \approx 9 + 2 = 11$$

Experimental access to entanglement ?

- could longitudinal entanglement be tested experimentally?
- unfortunately entropy density $dS/d\eta$ not straight-forward to access
- measured in e^+e^- is the number of charged particles per unit rapidity $dN_{\text{ch}}/d\eta$ (rapidity defined with respect to the thrust axis)
- typical values for collision energies $\sqrt{s} = 14 - 206$ GeV in the range

$$dN_{\text{ch}}/d\eta \approx 2 - 4$$

- entropy per particle S/N can be estimated for a hadron resonance gas in thermal equilibrium $S/N_{\text{ch}} = 7.2$ would give

$$dS/d\eta \approx 14 - 28$$

- this is an upper bound: correlations beyond one-particle functions would lead to reduced entropy

Temperature and entanglement entropy

- for conformal fields, entanglement entropy has also been calculated at non-zero temperature.
- for static interval of length L [Korepin (2004); Calabrese, Cardy (2004)]

$$S(T, l) = \frac{c}{3} \ln \left(\frac{1}{\pi T \epsilon} \sinh(\pi L T) \right) + \text{const}$$

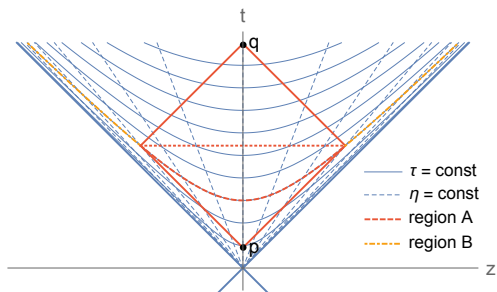
- compare this to our result in expanding geometry

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln \left(\frac{2\tau}{\epsilon} \sinh(\Delta\eta/2) \right) + \text{const}$$

- expressions agree for $L = \tau\Delta\eta$ (with metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$) and time-dependent temperature

$$T = \frac{1}{2\pi\tau}$$

Modular or entanglement Hamiltonian 1



- conformal field theory
- hypersurface Σ with boundary on the intersection of two light cones
- reduced density matrix [Casini, Huerta, Myers (2011), Arias, Blanco, Casini, Huerta (2017), see also Candelas, Dowker (1979)]

$$\rho_A = \frac{1}{Z_A} e^{-K}, \quad Z_A = \text{Tr} e^{-K}$$

- modular or entanglement Hamiltonian K

Modular or entanglement Hamiltonian 2

- modular or entanglement Hamiltonian is **local expression**

$$K = \int_{\Sigma} d\Sigma_{\mu} \xi_{\nu}(x) T^{\mu\nu}(x).$$

- energy-momentum tensor $T^{\mu\nu}(x)$ of excitations
- vector field

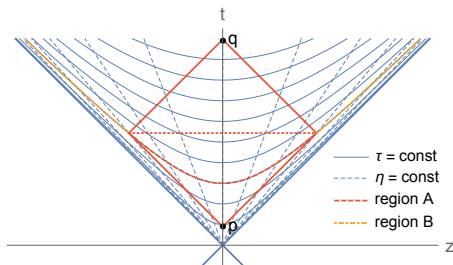
$$\xi^{\mu}(x) = \frac{2\pi}{(q-p)^2} [(q-x)^{\mu}(x-p)(q-p) \\ + (x-p)^{\mu}(q-x)(q-p) - (q-p)^{\mu}(x-p)(q-x)]$$

end point of future light cone q , starting point of past light cone p

- inverse temperature and fluid velocity

$$\xi^{\mu}(x) = \beta^{\mu}(x) = \frac{u^{\mu}(x)}{T(x)}$$

Modular or entanglement Hamiltonian 3



- for $\Delta\eta \rightarrow \infty$: fluid velocity in τ -direction, τ -dependent temperature

$$T(\tau) = \frac{\hbar}{2\pi\tau}$$

- **Entanglement between different rapidity intervals alone leads to local thermal density matrix at very early times !**
- Hawking-Unruh temperature in Rindler wedge $T(x) = \hbar c / (2\pi x)$

Alternative derivation: mode functions

- fluctuation field $\varphi = \phi - \bar{\phi}$ has equation of motion

$$\partial_\tau^2 \varphi(\tau, \eta) + \frac{1}{\tau} \partial_\tau \varphi(\tau, \eta) + \left(M^2 - \frac{1}{\tau^2} \frac{\partial^2}{\partial \eta^2} \right) \varphi(\tau, \eta) = 0$$

- solution in terms of plane waves

$$\varphi(\tau, \eta) = \int \frac{dk}{2\pi} \{ a(k) f(\tau, |k|) e^{ik\eta} + a^\dagger(k) f^*(\tau, |k|) e^{-ik\eta} \}$$

- mode functions as Hankel functions

$$f(\tau, k) = \frac{\sqrt{\pi}}{2} e^{\frac{k\pi}{2}} H_{ik}^{(2)}(M\tau)$$

or alternatively as Bessel functions

$$\bar{f}(\tau, k) = \frac{\sqrt{\pi}}{\sqrt{2 \sinh(\pi k)}} J_{-ik}(M\tau)$$

Bogoliubov transformation

- mode functions are related

$$\begin{aligned}\bar{f}(\tau, k) &= \alpha(k)f(\tau, k) + \beta(k)f^*(\tau, k) \\ f(\tau, k) &= \alpha^*(k)\bar{f}(\tau, k) - \beta(k)\bar{f}^*(\tau, k)\end{aligned}$$

- creation and annihilation operators are related by

$$\begin{aligned}\bar{a}(k) &= \alpha^*(k)a(k) - \beta^*(k)a^\dagger(k) \\ a(k) &= \alpha(k)\bar{a}(k) + \beta(k)\bar{a}^\dagger(k)\end{aligned}$$

- Bogoliubov coefficients

$$\alpha(k) = \sqrt{\frac{e^{\pi k}}{2 \sinh(\pi k)}} \quad \beta(k) = \sqrt{\frac{e^{-\pi k}}{2 \sinh(\pi k)}}$$

- vacuum $|\Omega\rangle$ with respect to $a(k)$ such that $a(k)|\Omega\rangle = 0$ contains excitations with respect to $\bar{a}(k)$ such that $\bar{a}(k)|\Omega\rangle \neq 0$ and *vice versa*

Role of different mode functions

- Hankel functions $f(\tau, k)$ are superpositions of *positive* frequency modes with respect to Minkowski time t
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to Minkowski time t
- at very early time $1/\tau \gg M, m$ conformal symmetry

$$ds^2 = \tau^2 [-d \ln(\tau)^2 + d\eta^2]$$

- Hankel functions $f(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to conformal time $\ln(\tau)$
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive* frequency modes with respect to conformal time $\ln(\tau)$

Occupation numbers

- Minkowski space coherent states have two-point functions

$$\langle \bar{a}^\dagger(k) \bar{a}(k') \rangle_c = \bar{n}(k) 2\pi \delta(k - k') = |\beta(k)|^2 2\pi \delta(k - k')$$

$$\langle \bar{a}(k) \bar{a}(k') \rangle_c = \bar{u}(k) 2\pi \delta(k + k') = -\alpha^*(k) \beta^*(k) 2\pi \delta(k + k')$$

$$\langle \bar{a}^\dagger(k) \bar{a}^\dagger(k') \rangle_c = \bar{u}^*(k) 2\pi \delta(k + k') = -\alpha(k) \beta(k) 2\pi \delta(k + k')$$

- occupation number

$$\bar{n}(k) = |\beta(k)|^2 = \frac{1}{e^{2\pi k} - 1}$$

- Bose-Einstein distribution with excitation energy $E = |k|/\tau$ and temperature

$$T = \frac{1}{2\pi\tau}$$

- off-diagonal occupation number $\bar{u}(k) = -1/(2 \sinh(\pi k))$ make sure we still have pure state

Local description

- consider now rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$
- Fourier expansion becomes discrete

$$\varphi(\eta) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \varphi_n e^{in\pi \frac{\eta}{\Delta\eta}}$$

$$\varphi_n = \int_{-\Delta\eta/2}^{\Delta\eta/2} d\eta \varphi(\eta) \frac{1}{2} \left[e^{-in\pi \frac{\eta}{\Delta\eta}} + (-1)^n e^{in\pi \frac{\eta}{\Delta\eta}} \right]$$

- relation to continuous momentum modes by integration kernel

$$\varphi_n = \int \frac{dk}{2\pi} \sin\left(\frac{k\Delta\eta}{2} - \frac{n\pi}{2}\right) \left[\frac{1}{k - \frac{n\pi}{\Delta\eta}} + \frac{1}{k + \frac{n\pi}{\Delta\eta}} \right] \varphi(k)$$

- local density matrix determined by correlation functions

$$\langle \varphi_n \rangle, \quad \langle \pi_n \rangle, \quad \langle \varphi_n \varphi_m \rangle_c, \quad \text{etc.}$$

Emergence of locally thermal state

- mode functions at early time

$$\bar{f}(\tau, k) = \frac{1}{\sqrt{2k}} e^{-ik \ln(\tau) - i\theta(k, M)}$$

- phase varies strongly with k for $M \rightarrow 0$

$$\theta(k, M) = k \ln(M/2) + \arg(\Gamma(1 - ik))$$

- off-diagonal term $\bar{u}(k)$ have factors strongly oscillating with k

$$\begin{aligned} \langle \varphi(\tau, k) \varphi^*(\tau, k') \rangle_c &= 2\pi \delta(k - k') \frac{1}{|k|} \\ &\times \left\{ \left[\frac{1}{2} + \bar{n}(k) \right] + \cos [2k \ln(\tau) + 2\theta(k, M)] \bar{u}(k) \right\} \end{aligned}$$

cancel out when going to finite interval !

- only Bose-Einstein occupation numbers $\bar{n}(k)$ remain

Physics picture

- coherent state vacuum at early time contains entangled pairs of quasi-particles with opposite wave numbers
- on finite rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ in- and out-flux of quasi-particles with thermal distribution via boundaries
- technically limits $\Delta\eta \rightarrow \infty$ and $M\tau \rightarrow 0$ do not commute
 - $\Delta\eta \rightarrow \infty$ for any finite $M\tau$ gives pure state
 - $M\tau \rightarrow 0$ for any finite $\Delta\eta$ gives thermal state with $T = 1/(2\pi\tau)$

Entanglement and QCD physics

- how strongly entangled is the nuclear wave function?
- what is the entropy of quasi-free partons and can it be understood as a result of entanglement? [Kharzeev, Levin (2017)]

$$S = \ln[xG(x)]$$

- does saturation at small Bjorken- x have an entropic meaning?
- entanglement entropy and entropy production in the color glass condensate [Kovner, Lublinsky (2015)]
- could entanglement entropy help for a non-perturbative extension of the parton model?
- entropy of perturbative and non-perturbative Pomeron descriptions [Shuryak, Zahed (2017)]

Conclusions

- rapidity intervals in an expanding string are entangled
- at very early times theory effectively conformal

$$\frac{1}{\tau} \gg m, q$$

- entanglement entropy extensive in rapidity $\frac{dS}{d\Delta\eta} = \frac{c}{6}$
- determined by conformal charge $c = N_c \times N_f + 2$
- reduced density matrix for conformal field theory is of locally thermal form with temperature

$$T = \frac{\hbar}{2\pi\tau}$$

- entanglement could be important ingredient to understand apparent “thermal effects” in e^+e^- and other collider experiments