

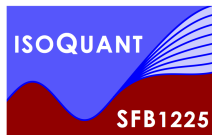
# *Entanglement in an expanding QCD string*

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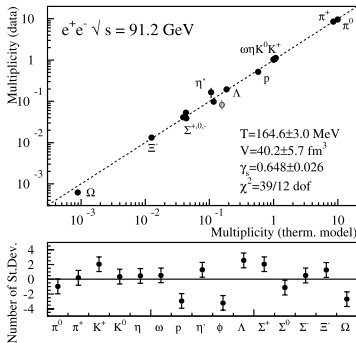


based on

- J. Berges, S. Floerchinger & R. Venugopalan, *Thermal excitation spectrum from entanglement in an expanding quantum string* [[Phys. Lett. B 778, 442 \(2018\)](#)]
- J. Berges, S. Floerchinger & R. Venugopalan, *Dynamics of entanglement in expanding quantum fields* [[arXiv:1712.09362](#)]

## Motivation

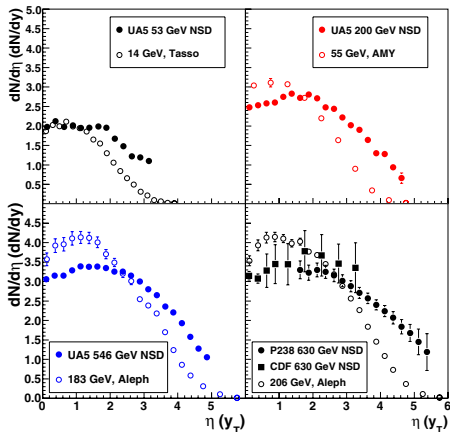
- elementary particle collision experiments such as  $e^+ e^-$  collisions show some thermal-like features
- particle multiplicities well described by thermal model



[Becattini, Casterina, Milov & Satz, EPJC 66, 377 (2010)]

- conventional thermalization by collisions unlikely
- more thermal-like features difficult to understand in PYTHIA [Fischer, Sjöstrand (2017)]
- alternative explanations needed

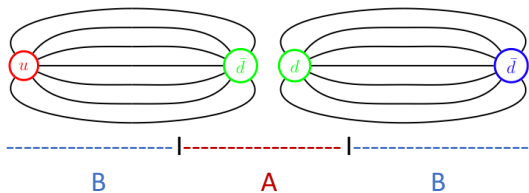
# Rapidity distribution



[open (filled) symbols:  $e^+e^-$  (pp), Grosse-Oetringhaus & Reygers (2010)]

- rapidity distribution  $dN/d\eta$  has plateau around midrapidity
- only logarithmic dependence on collision energy

## *QCD strings*



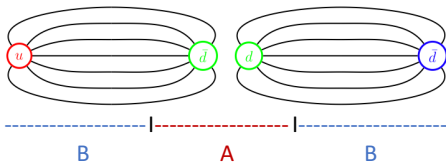
- particle production from QCD strings
- Lund string model (e. g. PYTHIA)
- different regions in a string are entangled
- subinterval  $A$  is described by reduced density matrix

$$\rho_A = \text{Tr}_B \rho$$

- reduced density matrix is of mixed state form
- could this lead to thermal-like effects?

## Entropy and entanglement

- consider a split of a quantum system into two  $A + B$



- reduced density operator for system  $A$

$$\rho_A = \text{Tr}_B\{\rho\}$$

- entropy associated with subsystem  $A$ : **entanglement entropy**

$$S_A = -\text{Tr}_A\{\rho_A \ln \rho_A\}$$

- globally pure** state  $S = 0$  can be **locally mixed**  $S_A > 0$
- coherent information**  $I_{B \rangle A} = S_A - S$  can be **positive**

## Microscopic model

- QCD in 1+1 dimensions described by 't Hooft model

$$\mathcal{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - ig\mathbf{A}_\mu) \psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{2} \text{tr} \mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu}$$

- fermionic fields  $\psi_i$  with sums over flavor species  $i = 1, \dots, N_f$
- $SU(N_c)$  gauge fields  $\mathbf{A}_\mu$  with field strength tensor  $\mathbf{F}_{\mu\nu}$
- gluons are not dynamical in two dimensions
- gauge coupling  $g$  has dimension of mass
- non-trivial, interacting theory, cannot be solved exactly
- spectrum of excitations known for  $N_c \rightarrow \infty$  with  $g^2 N_c$  fixed  
[ 't Hooft (1974) ]

## Schwinger model

- QED in 1+1 dimension

$$\mathcal{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - iqA_\mu) \psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

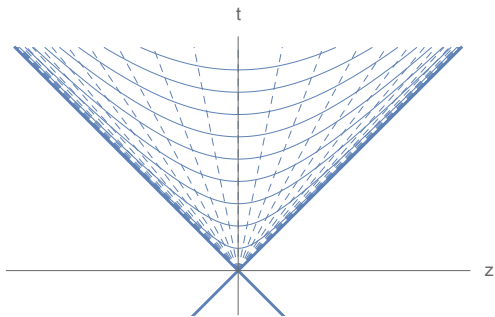
- geometric confinement
- U(1) charge related to string tension  $q = \sqrt{2\sigma}$
- for single fermion one can **bosonize theory** exactly  
[Coleman, Jackiw, Susskind (1975)]

$$S = \int d^2x \sqrt{g} \left\{ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} M^2 \phi^2 - \frac{m q e^\gamma}{2\pi^{3/2}} \cos(2\sqrt{\pi}\phi + \theta) \right\}$$

- Schwinger bosons are dipoles  $\phi \sim \bar{\psi}\psi$
- scalar mass related to U(1) charge by  $M = q/\sqrt{\pi} = \sqrt{2\sigma/\pi}$
- massless Schwinger model  $m = 0$  leads to free bosonic theory



## Expanding string solution 1



- external quark-anti-quark pair on trajectories  $z = \pm t$
- coordinates: Bjorken time  $\tau = \sqrt{t^2 - z^2}$ , rapidity  $\eta = \text{arctanh}(z/t)$
- metric  $ds^2 = -d\tau^2 + \tau^2 d\eta^2$
- symmetry with respect to longitudinal boosts  $\eta \rightarrow \eta + \Delta\eta$

## Expanding string solution 2

- Schwinger boson field depends only on  $\tau$

$$\bar{\phi} = \bar{\phi}(\tau)$$

- equation of motion

$$\partial_\tau^2 \bar{\phi} + \frac{1}{\tau} \partial_\tau \bar{\phi} + M^2 \bar{\phi} = 0.$$

- Gauss law: electric field  $E = q\phi/\sqrt{\pi}$  must approach the U(1) charge of the external quarks  $E \rightarrow q_e$  for  $\tau \rightarrow 0_+$

$$\bar{\phi}(\tau) \rightarrow \frac{\sqrt{\pi}q_e}{q} \quad (\tau \rightarrow 0_+)$$

- solution of equation of motion [Loshaj, Kharzeev (2011)]

$$\bar{\phi}(\tau) = \frac{\sqrt{\pi}q_e}{q} J_0(M\tau)$$

## *Gaussian states*

- theories with quadratic action often have Gaussian density matrix
- fully characterized by field expectation values

$$\bar{\phi}(x) = \langle \phi(x) \rangle, \quad \bar{\pi}(x) = \langle \pi(x) \rangle$$

and connected two-point correlation functions, e. g.

$$\langle \phi(x)\phi(y) \rangle_c = \langle \phi(x)\phi(y) \rangle - \bar{\phi}(x)\bar{\phi}(y)$$

- if  $\rho$  is Gaussian, also reduced density matrix  $\rho_A$  is Gaussian

## Entanglement entropy for Gaussian state

- entanglement entropy of Gaussian state in region  $A$   
[Berges, Floerchinger, Venugopalan, 1712.09362]

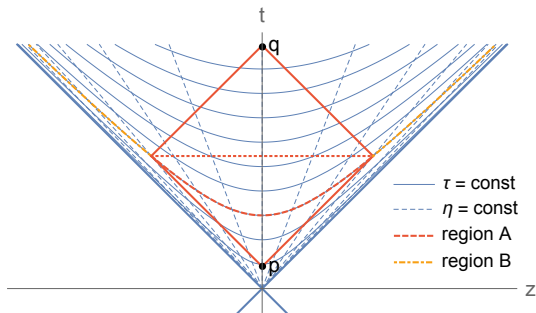
$$S_A = \frac{1}{2} \text{Tr}_A \{ D \ln(D^2) \}$$

- operator trace over region  $A$  only
- matrix of correlation functions

$$D(x, y) = \begin{pmatrix} -i\langle \phi(x)\pi(y) \rangle_c & i\langle \phi(x)\phi(y) \rangle_c \\ -i\langle \pi(x)\pi(y) \rangle_c & i\langle \pi(x)\phi(y) \rangle_c \end{pmatrix}$$

- involves connected correlation functions of field  $\phi(x)$  and canonically conjugate momentum field  $\pi(x)$
- expectation value  $\bar{\phi}$  does not appear explicitly
- coherent states and vacuum have equal entanglement entropy  $S_A$

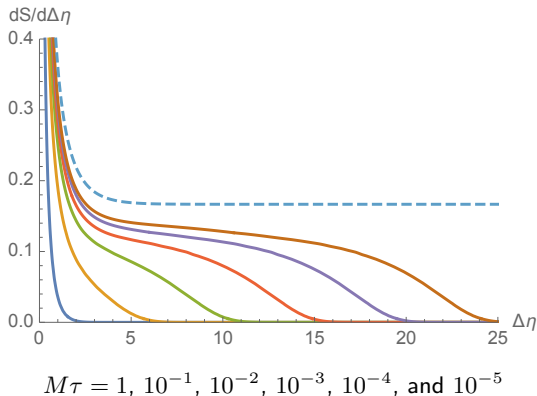
## Rapidity interval



- consider rapidity interval  $(-\Delta\eta/2, \Delta\eta/2)$  at fixed Bjorken time  $\tau$
- entanglement entropy does not change by unitary time evolution with endpoints kept fixed
- can be evaluated equivalently in interval  $\Delta z = 2\tau \sinh(\Delta\eta/2)$  at fixed time  $t = \tau \cosh(\Delta\eta/2)$
- need to solve eigenvalue problem with correct **boundary conditions**

## Bosonized massless Schwinger model

- entanglement entropy understood numerically for free massive scalars [Casini, Huerta (2009)]
- entanglement entropy density  $dS/d\Delta\eta$  for bosonized massless Schwinger model ( $M = \frac{g}{\sqrt{\pi}}$ )



## Conformal limit

- For  $M\tau \rightarrow 0$  one has conformal field theory limit  
[Holzhey, Larsen, Wilczek (1994)]

$$S(\Delta z) = \frac{c}{3} \ln(\Delta z/\epsilon) + \text{constant}$$

with small length  $\epsilon$  acting as UV cutoff.

- Here this implies

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln(2\tau \sinh(\Delta\eta/2)/\epsilon) + \text{constant}$$

- Conformal charge  $c = 1$  for free massless scalars or Dirac fermions.
- Additive constant not universal but entropy density is

$$\begin{aligned} \frac{\partial}{\partial \Delta\eta} S(\tau, \Delta\eta) &= \frac{c}{6} \coth(\Delta\eta/2) \\ &\rightarrow \frac{c}{6} \quad (\Delta\eta \gg 1) \end{aligned}$$

- Entropy becomes extensive in  $\Delta\eta$  !

## *Universal entanglement entropy density*

- for very early times “Hubble” expansion rate dominates over masses and interactions

$$H = \frac{1}{\tau} \gg M = \frac{q}{\sqrt{\pi}}, m$$

- theory dominated by free, massless fermions
- universal entanglement entropy density

$$\frac{dS}{d\Delta\eta} = \frac{c}{6}$$

with conformal charge  $c$

- for QCD in 1+1 D (gluons not dynamical, no transverse excitations)

$$c = N_c \times N_f$$

- from fluctuating transverse coordinates (Nambu-Goto action)

$$c = N_c \times N_f + 2 \approx 9 + 2 = 11$$



## Experimental access to entanglement ?

- could longitudinal entanglement be tested experimentally?
- unfortunately entropy density  $dS/d\eta$  not straight-forward to access
- measured in  $e^+e^-$  is the number of charged particles per unit rapidity  $dN_{\text{ch}}/d\eta$  (rapidity defined with respect to the thrust axis)
- typical values for collision energies  $\sqrt{s} = 14 - 206$  GeV in the range

$$dN_{\text{ch}}/d\eta \approx 2 - 4$$

- entropy per particle  $S/N$  can be estimated for a hadron resonance gas in thermal equilibrium  $S/N_{\text{ch}} = 7.2$  would give

$$dS/d\eta \approx 14 - 28$$

- this is an upper bound: correlations beyond one-particle functions would lead to reduced entropy

## Temperature and entanglement entropy

- for conformal fields, entanglement entropy has also been calculated at non-zero temperature.
- for static interval of length  $L$  [Korepin (2004); Calabrese, Cardy (2004)]

$$S(T, l) = \frac{c}{3} \ln \left( \frac{1}{\pi T \epsilon} \sinh(\pi L T) \right) + \text{const}$$

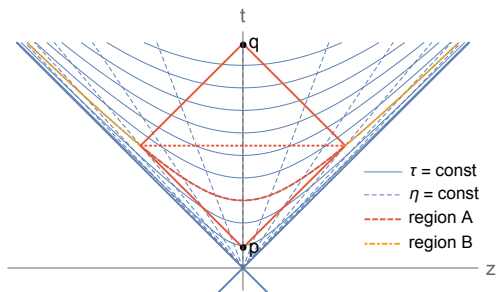
- compare this to our result in expanding geometry

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln \left( \frac{2\tau}{\epsilon} \sinh(\Delta\eta/2) \right) + \text{const}$$

- expressions agree for  $L = \tau\Delta\eta$  (with metric  $ds^2 = -d\tau^2 + \tau^2 d\eta^2$ ) and time-dependent temperature

$$T = \frac{1}{2\pi\tau}$$

# Modular or entanglement Hamiltonian 1



- conformal field theory
- hypersurface  $\Sigma$  with boundary on the intersection of two light cones
- reduced density matrix [Casini, Huerta, Myers (2011), Arias, Blanco, Casini, Huerta (2017), see also Candelas, Dowker (1979)]

$$\rho_A = \frac{1}{Z_A} e^{-K}, \quad Z_A = \text{Tr} e^{-K}$$

- modular or entanglement Hamiltonian  $K$

## Modular or entanglement Hamiltonian 2

- modular or entanglement Hamiltonian is **local expression**

$$K = \int_{\Sigma} d\Sigma_{\mu} \xi_{\nu}(x) T^{\mu\nu}(x).$$

- energy-momentum tensor  $T^{\mu\nu}(x)$  of excitations
- vector field

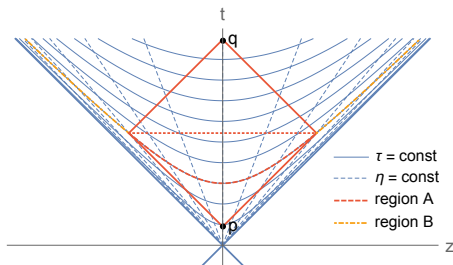
$$\xi^{\mu}(x) = \frac{2\pi}{(q-p)^2} [(q-x)^{\mu}(x-p)(q-p) \\ + (x-p)^{\mu}(q-x)(q-p) - (q-p)^{\mu}(x-p)(q-x)]$$

end point of future light cone  $q$ , starting point of past light cone  $p$

- inverse temperature and fluid velocity

$$\xi^{\mu}(x) = \beta^{\mu}(x) = \frac{u^{\mu}(x)}{T(x)}$$

## Modular or entanglement Hamiltonian 3



- for  $\Delta\eta \rightarrow \infty$ : fluid velocity in  $\tau$ -direction,  $\tau$ -dependent temperature

$$T(\tau) = \frac{\hbar}{2\pi\tau}$$

- **Entanglement between different rapidity intervals alone leads to local thermal density matrix at very early times !**
- Hawking-Unruh temperature in Rindler wedge  $T(x) = \hbar c/(2\pi x)$

## Alternative derivation: mode functions

- fluctuation field  $\varphi = \phi - \bar{\phi}$  has equation of motion

$$\partial_\tau^2 \varphi(\tau, \eta) + \frac{1}{\tau} \partial_\tau \varphi(\tau, \eta) + \left( M^2 - \frac{1}{\tau^2} \frac{\partial^2}{\partial \eta^2} \right) \varphi(\tau, \eta) = 0$$

- solution in terms of plane waves

$$\varphi(\tau, \eta) = \int \frac{dk}{2\pi} \{ a(k) f(\tau, |k|) e^{ik\eta} + a^\dagger(k) f^*(\tau, |k|) e^{-ik\eta} \}$$

- mode functions as Hankel functions

$$f(\tau, k) = \frac{\sqrt{\pi}}{2} e^{\frac{k\pi}{2}} H_{ik}^{(2)}(M\tau)$$

or alternatively as Bessel functions

$$\bar{f}(\tau, k) = \frac{\sqrt{\pi}}{\sqrt{2 \sinh(\pi k)}} J_{-ik}(M\tau)$$

## Bogoliubov transformation

- mode functions are related

$$\begin{aligned}\bar{f}(\tau, k) &= \alpha(k)f(\tau, k) + \beta(k)f^*(\tau, k) \\ f(\tau, k) &= \alpha^*(k)\bar{f}(\tau, k) - \beta(k)\bar{f}^*(\tau, k)\end{aligned}$$

- creation and annihilation operators are related by

$$\begin{aligned}\bar{a}(k) &= \alpha^*(k)a(k) - \beta^*(k)a^\dagger(k) \\ a(k) &= \alpha(k)\bar{a}(k) + \beta(k)\bar{a}^\dagger(k)\end{aligned}$$

- Bogoliubov coefficients

$$\alpha(k) = \sqrt{\frac{e^{\pi k}}{2 \sinh(\pi k)}} \quad \beta(k) = \sqrt{\frac{e^{-\pi k}}{2 \sinh(\pi k)}}$$

- vacuum  $|\Omega\rangle$  with respect to  $a(k)$  such that  $a(k)|\Omega\rangle = 0$  contains excitations with respect to  $\bar{a}(k)$  such that  $\bar{a}(k)|\Omega\rangle \neq 0$  and *vice versa*

## *Role of different mode functions*

- Hankel functions  $f(\tau, k)$  are superpositions of *positive* frequency modes with respect to Minkowski time  $t$
- Bessel functions  $\bar{f}(\tau, k)$  are superpositions of *positive and negative* frequency modes with respect to Minkowski time  $t$
- at very early time  $1/\tau \gg M, m$  conformal symmetry

$$ds^2 = \tau^2 [-d \ln(\tau)^2 + d\eta^2]$$

- Hankel functions  $f(\tau, k)$  are superpositions of *positive and negative* frequency modes with respect to conformal time  $\ln(\tau)$
- Bessel functions  $\bar{f}(\tau, k)$  are superpositions of *positive* frequency modes with respect to conformal time  $\ln(\tau)$



## Occupation numbers

- Minkowski space coherent states have two-point functions

$$\langle \bar{a}^\dagger(k) \bar{a}(k') \rangle_c = \bar{n}(k) 2\pi \delta(k - k') = |\beta(k)|^2 2\pi \delta(k - k')$$

$$\langle \bar{a}(k) \bar{a}(k') \rangle_c = \bar{u}(k) 2\pi \delta(k + k') = -\alpha^*(k) \beta^*(k) 2\pi \delta(k + k')$$

$$\langle \bar{a}^\dagger(k) \bar{a}^\dagger(k') \rangle_c = \bar{u}^*(k) 2\pi \delta(k + k') = -\alpha(k) \beta(k) 2\pi \delta(k + k')$$

- occupation number

$$\bar{n}(k) = |\beta(k)|^2 = \frac{1}{e^{2\pi k} - 1}$$

- Bose-Einstein distribution with excitation energy  $E = |k|/\tau$  and temperature

$$T = \frac{1}{2\pi\tau}$$

- off-diagonal occupation number  $\bar{u}(k) = -1/(2 \sinh(\pi k))$  make sure we still have pure state

## Local description

- consider now rapidity interval  $(-\Delta\eta/2, \Delta\eta/2)$
- Fourier expansion becomes discrete

$$\varphi(\eta) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \varphi_n e^{in\pi \frac{\eta}{\Delta\eta}}$$

$$\varphi_n = \int_{-\Delta\eta/2}^{\Delta\eta/2} d\eta \varphi(\eta) \frac{1}{2} \left[ e^{-in\pi \frac{\eta}{\Delta\eta}} + (-1)^n e^{in\pi \frac{\eta}{\Delta\eta}} \right]$$

- relation to continuous momentum modes by integration kernel

$$\varphi_n = \int \frac{dk}{2\pi} \sin\left(\frac{k\Delta\eta}{2} - \frac{n\pi}{2}\right) \left[ \frac{1}{k - \frac{n\pi}{\Delta\eta}} + \frac{1}{k + \frac{n\pi}{\Delta\eta}} \right] \varphi(k)$$

- local density matrix determined by correlation functions

$$\langle \varphi_n \rangle, \quad \langle \pi_n \rangle, \quad \langle \varphi_n \varphi_m \rangle_c, \quad \text{etc.}$$

## *Emergence of locally thermal state*

- mode functions at early time

$$\bar{f}(\tau, k) = \frac{1}{\sqrt{2k}} e^{-ik \ln(\tau) - i\theta(k, M)}$$

- phase varies strongly with  $k$  for  $M \rightarrow 0$

$$\theta(k, M) = k \ln(M/2) + \arg(\Gamma(1 - ik))$$

- off-diagonal term  $\bar{u}(k)$  have factors strongly oscillating with  $k$

$$\begin{aligned} \langle \varphi(\tau, k) \varphi^*(\tau, k') \rangle_c &= 2\pi \delta(k - k') \frac{1}{|k|} \\ &\times \left\{ \left[ \frac{1}{2} + \bar{n}(k) \right] + \cos [2k \ln(\tau) + 2\theta(k, M)] \bar{u}(k) \right\} \end{aligned}$$

cancel out when going to finite interval !

- only Bose-Einstein occupation numbers  $\bar{n}(k)$  remain

## *Physics picture*

- coherent state vacuum at early time contains entangled pairs of quasi-particles with opposite wave numbers
- on finite rapidity interval  $(-\Delta\eta/2, \Delta\eta/2)$  in- and out-flux of quasi-particles with thermal distribution via boundaries
- technically limits  $\Delta\eta \rightarrow \infty$  and  $M\tau \rightarrow 0$  do not commute
  - $\Delta\eta \rightarrow \infty$  for any finite  $M\tau$  gives pure state
  - $M\tau \rightarrow 0$  for any finite  $\Delta\eta$  gives thermal state with  $T = 1/(2\pi\tau)$

## *Entanglement and QCD physics*

- how strongly entangled is the nuclear wave function?
- what is the entropy of quasi-free partons and can it be understood as a result of entanglement? [Kharzeev, Levin (2017)]

$$S = \ln[xG(x)]$$

- does saturation at small Bjorken- $x$  have an entropic meaning?
- entanglement entropy and entropy production in the color glass condensate [Kovner, Lublinsky (2015)]
- could entanglement entropy help for a non-perturbative extension of the parton model?
- entropy of perturbative and non-perturbative Pomeron descriptions [Shuryak, Zahed (2017)]

## Conclusions

- rapidity intervals in an expanding string are entangled
- at very early times theory effectively conformal

$$\frac{1}{\tau} \gg m, q$$

- entanglement entropy extensive in rapidity  $\frac{dS}{d\Delta\eta} = \frac{c}{6}$
- determined by conformal charge  $c = N_c \times N_f + 2$
- reduced density matrix for conformal field theory is of locally thermal form with temperature

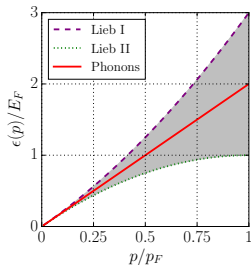
$$T = \frac{\hbar}{2\pi\tau}$$

- entanglement could be important ingredient to understand apparent “thermal effects” in  $e^+e^-$  and other collider experiments

*Backup*

## Testing the mechanism with cold atoms

- Lieb-Liniger model for interacting bosonic atoms in  $D = 1$  dimensions has linear dispersion at small momenta  $\omega = v_s p$ 
  - strong interaction  $\gamma \gg 1$  sound velocity  $v_s = v_F = \pi n/m$
  - weak interaction  $\gamma \ll 1$  sound velocity  $v_s = \sqrt{\gamma}n/m = \sqrt{gnm}$



strong interactions  $\gamma \gg 1$  [De Rosi et al. (2017)]

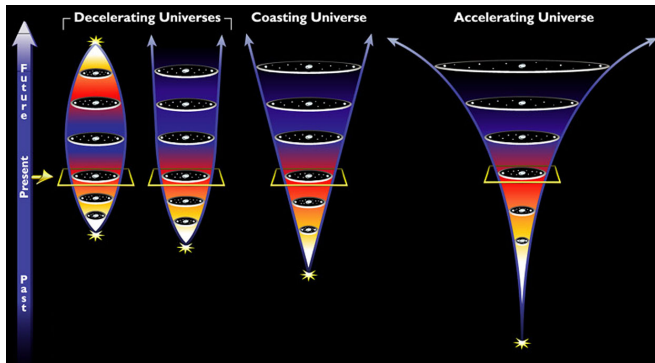
- effective metric for phonons

$$ds^2 = -v_s^2 dt^2 + dx^2$$



# Expanding geometries in cold atom experiments

- expanding geometries can be realized by interplay of
  - longitudinal expansion
  - time dependent change of sound velocity  $v_s(t)$
  - time dependent gap or mass  $M^2(t)$

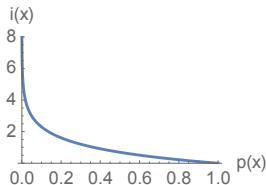


# Entropy and information

[Claude Shannon (1948)]

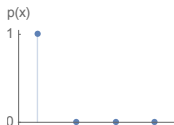
- consider a random variable  $x$  with probability distribution  $p(x)$
- information content or “surprise” associated with outcome  $x$

$$i(x) = -\ln p(x)$$

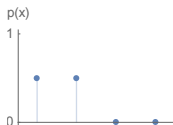


- Entropy is expectation value of information content

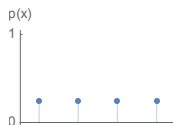
$$S = \langle i(x) \rangle = -\sum_x p(x) \ln p(x)$$



$$S = 0$$



$$S = \ln(2)$$



$$S = 2 \ln(2)$$

## Classical statistics

- consider system of two random variables  $x$  and  $y$
- joint probability  $p(x, y)$  , joint entropy

$$S = - \sum_{x,y} p(x, y) \ln p(x, y)$$

- reduced or marginal probability  $p(x) = \sum_y p(x, y)$
- reduced or marginal entropy

$$S_x = - \sum_x p(x) \ln p(x)$$

- **joint entropy is greater than** or equal to **reduced entropy**

$$S \geq S_x$$

- **globally pure** state  $S = 0$  is also **locally pure**  $S_x = 0$

## *Transverse coordinates*

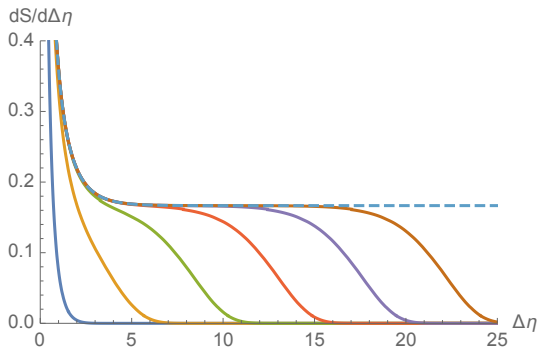
- so far dynamics strictly confined to 1+1 dimensions
- transverse coordinates may fluctuate, can be described by Nambu-Goto action ( $h_{\mu\nu} = \partial_\mu X^m \partial_\nu X_m$ )

$$S_{\text{NG}} = \int d^2x \sqrt{-\det h_{\mu\nu}} \{-\sigma + \dots\}$$
$$\approx \int d^2x \sqrt{g} \left\{ -\sigma - \frac{\sigma}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i + \dots \right\}$$

- two additional, massless, bosonic degrees of freedom corresponding to transverse coordinates  $X^i$  with  $i = 1, 2$

## Free massive fermions

- entanglement entropy for free Dirac fermions of mass  $m$



$m\tau = 1, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, \text{ and } 10^{-5}$

- same universal plateau  $c/6$  with  $c = 1$  at early time
- conformal limit corresponds to non-interacting fermions
- consistent with or without bosonization