Entropy and quantum field theory

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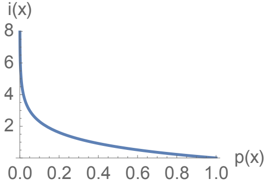
Entropy and information

[Claude Shannon (1948)]

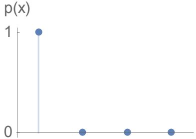
- consider a random variable x with probability distribution $p(x)$
- information content or "surprise" associated with outcome x

$$i(x) = -\ln p(x)$$

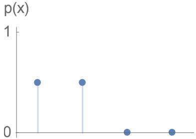


- Entropy is expectation value of information content

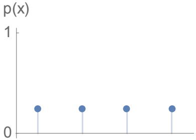
$$S = \langle i(x) \rangle = -\sum_x p(x) \ln p(x)$$



$S = 0$



$S = \ln(2)$



$S = 2\ln(2)$

Entropy at thermal equilibrium

- micro canonical ensemble: **maximal entropy** S for given **conserved quantities** E, N in given volume V
- **universality** at equilibrium
- starting point for development of thermodynamics ...

$$S(E,N,V), \qquad dS = \frac{1}{T}dE - \frac{\mu}{T}dN + \frac{p}{T}dV$$

- ... grand canonical ensemble with density operator ...

$$\rho = \frac{1}{Z}e^{-\frac{1}{T}(H-\mu N)}$$

- ... Matsubara formalism for quantum fields ...

Ideal fluid dynamics

- thermal equilibrium

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + p(u^\mu u^\nu + g^{\mu\nu}), \qquad N^\mu = n u^\mu, \qquad s^\mu = s u^\mu$$

- fluid velocity u^μ
- thermodynamic equation of state $p(T, \mu)$ with $dp = sdT + nd\mu$
- local thermal equilibrium approximation: $u^\mu(x)$, $T(x)$, $\mu(x)$
- neglect gradients: lowest order of a derivative expansion
- evolution of $u^\mu(x)$, $T(x)$ and $\mu(x)$ from conservation laws

$$\nabla_\mu T^{\mu\nu}(x) = 0, \qquad \nabla_\mu N^\mu(x) = 0.$$

- entropy current also conserved

$$\nabla_\mu s^\mu(x) = 0.$$

Out-of-equilibrium

- quantum field theory out-of-equilibrium is less well understood
- interesting topic of current research
- is non-equilibrium dynamics also governed by information?
- approach to equilibrium
- universality

Entropy in quantum theory

[John von Neumann (1932)]

$$S = -\mathsf{Tr}\rho \ln \rho$$

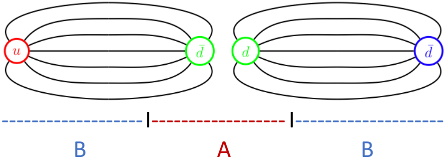
- based on the quantum density operator ρ
- for pure states $\rho = |\psi\rangle\langle\psi|$ one has $S = 0$
- for mixed states $\rho = \sum_j p_j |j\rangle\langle j|$ one has $S = -\sum_j p_j \ln p_j > 0$
- unitary time evolution conserves entropy

$$-\mathsf{Tr}(U\rho U^\dagger) \ln(U\rho U^\dagger) = -\mathsf{Tr}\rho \ln \rho \qquad \rightarrow \qquad S = \mathsf{const.}$$

- global characterization of quantum state

Entropy and entanglement

- consider a split of a quantum system into two $A + B$



- reduced density operator for system A

$$\rho_A = \mathsf{Tr}_B\{\rho\}$$

- entropy associated with subsystem A

$$S_A = -\mathsf{Tr}_A\{\rho_A \ln \rho_A\}$$

- pure **product** state $\rho = \rho_A \otimes \rho_B$ leads to $S_A = 0$
- pure **entangled** state $\rho \neq \rho_A \otimes \rho_B$ leads to $S_A > 0$
- S_A is called **entanglement entropy**

Classical statistics

- consider system of two random variables x and y
- joint probability $p(x,y)$, joint entropy

$$S = -\sum_{x,y} p(x,y) \ln p(x,y)$$

- reduced or marginal probability $p(x) = \sum_y p(x,y)$
- reduced or marginal entropy

$$S_x = -\sum_{x} p(x) \ln p(x)$$

- one can prove: **joint entropy is greater than** or equal to **reduced entropy**

$$S \geq S_x$$

- **globally pure** state $S = 0$ is also **locally pure** $S_x = 0$

Quantum statistics

- consider system with two subsystems A and B
- combined state ρ , combined or full entropy

$$S = -\mathsf{Tr}\{\rho \ln \rho\}$$

- reduced density matrix $\rho_A = \mathsf{Tr}_B\{\rho\}$
- reduced or entanglement entropy

$$S_A = -\mathsf{Tr}_A\{\rho_A \ln \rho_A\}$$

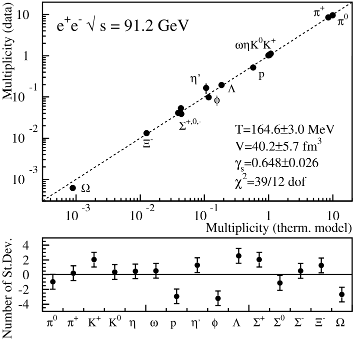
- for quantum systems **entanglement makes a difference**

$$S \not\geq S_A$$

- **coherent information** $I_{B\rangle A} = S_A - S$ can be **positive**!
- **globally pure** state $S = 0$ can be **locally mixed** $S_A > 0$

The thermal model puzzle

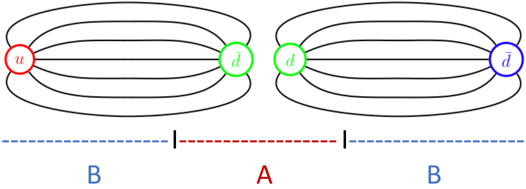
- elementary particle collision experiments such as $e^+ \, e^-$ collisions show thermal-like features
- particle multiplicities well described by thermal model



[Becattini, Casterina, Milov & Satz, EPJC 66, 377 (2010)]

- conventional thermalization by collisions unlikely
- alternative explanations needed

QCD strings



- particle production from QCD strings
- e. g. Lund model (Pythia)
- different regions in a string are entangled
- subinterval A is described by reduced density matrix of mixed form

$$\rho_A = \mathrm{Tr}_B \rho$$

- characterization by entanglement entropy

$$S_A = -\mathrm{Tr}\left\{\rho_A \ln(\rho_A)\right\}$$

- could this lead to thermal-like effects?

Microscopic model

- QCD in 1+1 dimensions described by 't Hooft model

$$\mathscr{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - ig\mathbf{A}_\mu)\psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{2} \mathsf{tr}\, \mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu}$$

- fermionic fields ψ_i with sums over flavor species $i = 1, \dots, N_f$
- SU(N_c) gauge fields $\mathbf{A}_\mu$ with field strength tensor $\mathbf{F}_{\mu\nu}$
- gluons are not dynamical in two dimensions
- gauge coupling g has dimension of mass
- non-trivial, interacting theory, cannot be solved exactly
- spectrum of excitations known for $N_c \to \infty$ with $g^2 N_c$ fixed
  ['t Hooft (1974)]

Schwinger model

- QED in 1+1 dimension

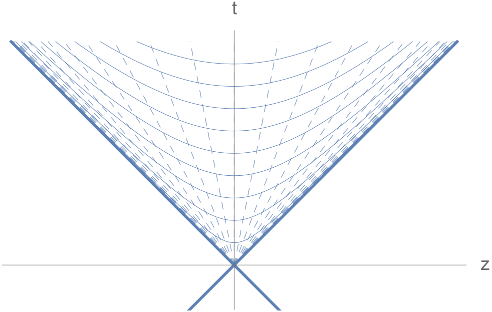
$$\mathscr{L} = -\bar{\psi}_i\gamma^\mu(\partial_\mu - iqA_\mu)\psi_i - m_i\bar{\psi}_i\psi_i - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}$$

- geometric confinement
- U(1) charge related to string tension $q = \sqrt{2\sigma}$
- for single fermion one can **bosonize theory** exactly
  [Coleman, Jackiw, Susskind (1975)]

$$S = \int d^2x\sqrt{g}\bigg\{ -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M^2\phi^2 \\ - \frac{m\,q\,e^\gamma}{2\pi^{3/2}}\cos\left(2\sqrt{\pi}\phi + \theta\right)\bigg\}$$

- Schwinger bosons are dipoles $\phi \sim \bar{\psi}\psi$
- mass is related to U(1) charge by $M = q/\sqrt{\pi} = \sqrt{2\sigma/\pi}$
- massless Schwinger model $m = 0$ leads to free bosonic theory

Expanding string solution



- external quark-anti-quark pair on trajectories $z = \pm t$
- coordinates: Bjorken time $\tau = \sqrt{t^2 - z^2}$, rapidity $\eta = \text{arctanh}(z/t)$
- metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$
- symmetry with respect to longitudinal boosts $\eta \to \eta + \Delta\eta$

Coherent field evolution

- Schwinger boson field depends only on τ

$$\bar{\phi} = \bar{\phi}(\tau)$$

- equation of motion

$$\partial_\tau^2 \bar{\phi} + \frac{1}{\tau}\partial_\tau \bar{\phi} + M^2 \bar{\phi} = 0.$$

- Gauss law: electric field $E = q\phi/\sqrt{\pi}$ must approach the U(1) charge of the external quarks $E \to q_\mathsf{e}$ for $\tau \to 0_+$

$$\bar{\phi}(\tau) \to \frac{\sqrt{\pi} q_\mathsf{e}}{q} \qquad (\tau \to 0_+)$$

- solution of equation of motion [Loshaj, Kharzeev (2011)]

$$\bar{\phi}(\tau) = \frac{\sqrt{\pi} q_\mathsf{e}}{q} J_0(M\tau)$$

Gaussian states

- theories with quadratic action typically have Gaussian density matrix
- fully characterized by field expectation values

  $$\bar{\phi}(x) = \langle \phi(x) \rangle, \qquad \bar{\pi}(x) = \langle \pi(x) \rangle$$

  and connected two-point correlation functions, e. g.

  $$\langle \phi(x)\phi(y) \rangle_c = \langle \phi(x)\phi(y) \rangle - \bar{\phi}(x)\bar{\phi}(y)$$

- if ρ is Gaussian, also reduced density matrix ρ_A is Gaussian

Entanglement entropy for Gaussian state

- entanglement entropy of Gaussian state in region A
  [Berges, Floerchinger, Venugopalan, 1712.09362]

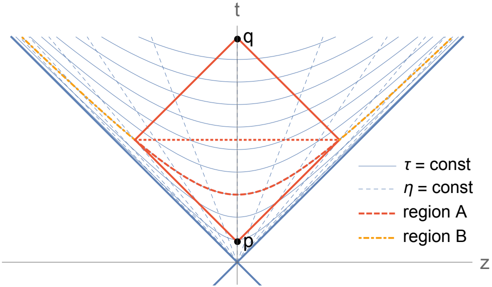
$$S_A = \frac{1}{2} \mathsf{Tr}_A \left\{ D \ln(D^2) \right\},$$

- operator trace over region A only
- matrix of correlation functions

$$D(x,y) = \begin{pmatrix} -i\langle \phi(x)\pi(y)\rangle_c & i\langle \phi(x)\phi(y)\rangle_c \\ -i\langle \pi(x)\pi(y)\rangle_c & i\langle \pi(x)\phi(y)\rangle_c \end{pmatrix}.$$

- involves connected correlation functions of field $\phi(x)$ and canonically conjugate momentum field $\pi(x)$
- expectation value $\bar{\phi}$ does not appear explicitly
- coherent states and vacuum have equal entanglement entropy S_A

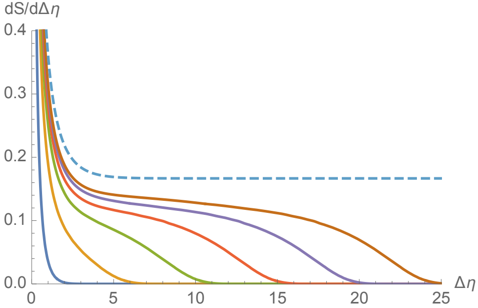
Rapidity interval



- consider rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ at fixed Bjorken time τ
- entanglement entropy does not change by unitary time evolution with endpoints kept fixed
- can be evaluated equivalently in interval $\Delta z = 2\tau \sinh(\Delta\eta/2)$ at fixed time $t = \tau \cosh(\Delta\eta/2)$
- need to solve eigenvalue problem with correct **boundary conditions**

Bosonized massless Schwinger model

- entanglement entropy understood numerically for free massive scalars [Casini, Huerta (2009)]
- entanglement entropy density $dS/d\Delta\eta$ for bosonized massless Schwinger model ($M = \frac{q}{\sqrt{\pi}}$)



$M\tau = 1$, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, and 10^{-5}

[Berges, Floerchinger, Venugopalan (2017)]

Conformal limit

- for $M\tau \to 0$ one has conformal field theory limit
  [Holzhey, Larsen, Wilczek (1994); Calabrese, Cardy (2004)]

$$S(\Delta z) = \frac{c}{3} \ln(\Delta z/\epsilon) + \text{constant}$$

  with small length ϵ acting as UV cutoff
- here this implies

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln\left(2\tau \sinh(\Delta\eta/2)/\epsilon\right) + \text{constant}$$

- conformal charge $c = 1$ for free massless scalars or Dirac fermions
- additive constant not universal but entropy density is

$$\begin{aligned} \frac{\partial}{\partial \Delta\eta} S(\tau, \Delta\eta) =& \frac{c}{6} \coth(\Delta\eta/2) \\ \to& \frac{c}{6} \qquad\qquad (\Delta\eta \gg 1) \end{aligned}$$

- entropy becomes extensive in $\Delta\eta$!

Universal entanglement entropy density

- for very early times "Hubble" expansion rate dominates over masses and interactions

$$H = \frac{1}{\tau} \gg M = \frac{q}{\sqrt{\pi}}, m$$

- theory dominated by free, massless fermions
- universal entanglement entropy density

$$\frac{dS}{d\Delta\eta} = \frac{c}{6}$$

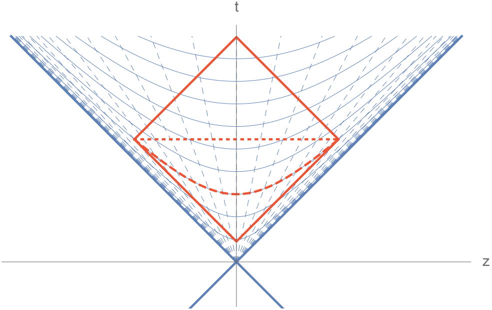
  with conformal charge c
- for QCD in 1+1 dimensions (gluons not dynamical)

$$c = N_c \times N_f$$

- from fluctuating transverse coordinates (Nambu-Goto action)

$$c = N_c \times N_f + 2 \approx 9 + 2 = 11$$

Modular or entanglement Hamiltonian



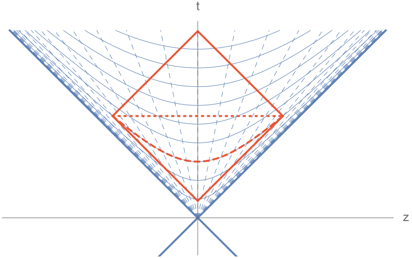
- conformal field theory [Casini, Huerta, Myers (2011), Arias, Blanco, Casini, Huerta (2017), see also Candelas, Dowker (1979)]

$$\rho_A = \frac{1}{Z_A} e^{-K}, \qquad Z_A = \mathsf{Tr}\, e^{-K}$$

- modular or entanglement Hamiltonian **local expression**

$$K = \int_\Sigma d\Sigma_\mu \, \xi_\nu(x) \, T^{\mu\nu}(x)$$

Time-dependent temperature



- energy-momentum of excitations around coherent field $T^{\mu\nu}(x)$
- combination of fluid velocity and temperature $\xi^\mu(x) = \frac{u^\mu(x)}{T(x)}$
- fluid velocity in τ-direction & time-dependent temperature [Berges, Floerchinger, Venugopalan (2017)]

$$T(\tau) = \frac{\hbar}{2\pi\tau}$$

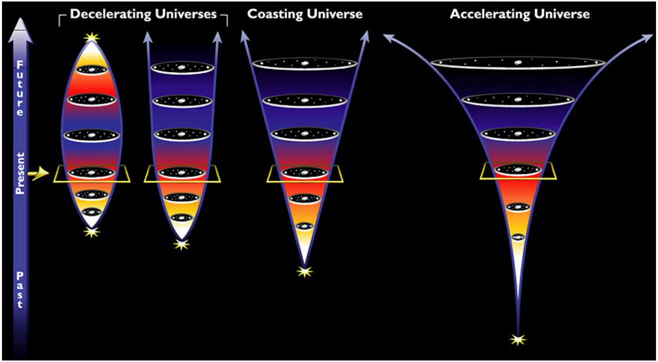
- **Entanglement between different rapidity intervals alone leads to local thermal density matrix at very early times !**
- Hawking-Unruh temperature in Rindler wedge $T(x) = \frac{\hbar c}{2\pi x}$

Physics picture

- alternative derivation via mode functions & Bogoliubov transforms [Berges, Floerchinger, Venugopalan, 1712.09362]
- coherent state vacuum at early time contains entangled pairs of quasi-particles with opposite wave numbers
- on finite rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ in- and out-flux of quasi-particles with thermal distribution via boundaries
- technically **limits** $\Delta\eta \to \infty$ and $M\tau \to 0$ **do not commute**
    - $\Delta\eta \to \infty$ for any finite $M\tau$ gives pure state
    - $M\tau \to 0$ for any finite $\Delta\eta$ gives thermal state with $T = 1/(2\pi\tau)$

Entanglement dynamics in cold atom experiments

- entanglement can be directly accessed in cold atom experiments
  [Oberthaler group, Greiner group]
- expanding geometries can be realized by interplay of
  - longitudinal expansion
  - time dependent change of sound velocity $v_s(t)$
  - time dependent gap or mass $M^2(t)$



Dissipation

- dissipation can be defined as (effective) entropy generation

$$\frac{d}{dt}S > 0$$

- for extensive entropy $S = \int_\Sigma d\Sigma_\mu s^\mu$ one has locally

$$\nabla_\mu s^\mu > 0$$

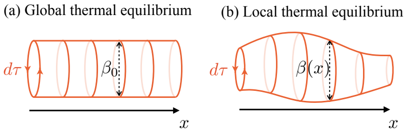
- related to effective loss of information
- second law of thermodynamics: entropy gets produced, not destroyed
- local dissipation - entanglement generation (?)

Dissipation and the quantum effective action

- dissipation usually discussed on the level of equations of motion
- one would like to have a formulation in terms of an effective action
  - fluctuations & correlation functions
  - renormalization
  - effective field theories
  - coupling to gravity
- one possibility: Schwinger-Keldysh double time path formalism
- another possibility: analytic continuation of the 1PI effective action
  [Floerchinger, JHEP 1609, 099 (2016)]

Local equilibrium & partition function

[Floerchinger, JHEP 1609, 099 (2016)]



- local equilibrium with $T(x)$ and $u^\mu(x)$

$$\beta^\mu(x) = \frac{u^\mu(x)}{T(x)}$$

- similarity between local density matrix and translation operator

$$e^{\beta^\mu(x)\mathscr{P}_\mu} \qquad \longleftrightarrow \qquad e^{i\Delta x^\mu \mathscr{P}_\mu}$$

- represent partition function as functional integral with periodicity

$$\phi(x^\mu - i\beta^\mu(x)) = \pm\phi(x^\mu)$$

- partition function $Z[J]$, Schwinger functional $W[J]$ in Euclidean

$$Z[J] = e^{W_E[J]} = \int D\phi \, e^{-S_E[\phi] + \int_x J\phi}$$

One-particle irreducible or quantum effective action

- in Euclidean domain $\Gamma[\phi]$ defined by Legendre transform

$$\Gamma_E[\Phi] = \int_x J_a(x)\Phi_a(x) - W_E[J]$$

with expectation values

$$\Phi_a(x) = \frac{1}{\sqrt{g}(x)} \frac{\delta}{\delta J_a(x)} W_E[J]$$

- **Euclidean** field equation

$$\frac{\delta}{\delta \Phi_a(x)} \Gamma_E[\Phi] = \sqrt{g}(x)\, J_a(x)$$

resembles classical equation of motion for $J = 0$

- need **analytic continuation** to obtain a viable equation of motion

Two-point functions

- homogeneous background field and global equilibrium

$$\beta^\mu = \left(\frac{1}{T}, 0, 0, 0\right)$$

- propagator and inverse propagator

$$\frac{\delta^2}{\delta J_a(-p)\delta J_b(q)} W_E[J] = G_{ab}(p)\ (2\pi)^4 \delta^{(4)}(p-q)$$

$$\frac{\delta^2}{\delta \Phi_a(-p)\delta \Phi_b(q)} \Gamma_E[\Phi] = P_{ab}(p)\ (2\pi)^4 \delta^{(4)}(p-q)$$

- from definition of effective action

$$\sum_b G_{ab}(p) P_{bc}(p) = \delta_{ac}$$

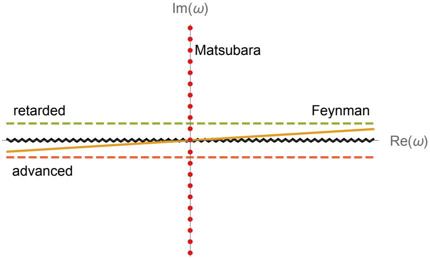
Spectral representation

- Källen-Lehmann spectral representation

$$G_{ab}\left(\omega, \mathbf{p}\right) = \int_{-\infty}^{\infty} dz \, \frac{\rho_{ab}(z^2 - \mathbf{p}^2, z)}{z - \omega}$$

  with $\rho_{ab} \in \mathbb{R}$

- correlation functions can be analytically continued in $\omega = -u^\mu p_\mu$
- branch cut or poles on real frequency axis $\omega \in \mathbb{R}$ but nowhere else
- different propagators follow by evaluation of G_{ab} in different regions



$$\Delta_{ab}^{M}(p) = G_{ab}\left(i\omega_n, \mathbf{p}\right)$$

$$\Delta_{ab}^{R}(p) = G_{ab}\left(p^0 + i\epsilon, \mathbf{p}\right)$$

$$\Delta_{ab}^{A}(p) = G_{ab}\left(p^0 - i\epsilon, \mathbf{p}\right)$$

$$\Delta_{ab}^{F}(p) = G_{ab}\left(p^0 + i\epsilon \,\text{sign}\left(p^0\right), \mathbf{p}\right)$$

Variational principle with effective dissipation

[Floerchinger, JHEP 1609, 099 (2016)]

- decompose inverse two-point function

$$P_{ab}(p) = P_{1,ab}(p) - i s_{\mathsf{I}}(-u^\mu p_\mu)\, P_{2,ab}(p)$$

  with $s_{\mathsf{I}}(\omega) = \text{sign}(\text{Im}\ \omega)$

- in position space, replace

$$\begin{aligned} & s_{\mathsf{I}}\left(-u^\mu p_\mu\right) = \text{sign}\left(\text{Im}(-u^\mu p_\mu)\right) \\ & \to \text{sign}\left(\text{Im}\left(i u^\mu \tfrac{\partial}{\partial x^\mu}\right)\right) = \text{sign}\left(\text{Re}\left(u^\mu \tfrac{\partial}{\partial x^\mu}\right)\right) = s_{\mathsf{R}}\left(u^\mu \tfrac{\partial}{\partial x^\mu}\right) \end{aligned}$$

- this symbol appears also in $\Gamma[\Phi]$

Retarded functional derivative

[Floerchinger, JHEP 1609, 099 (2016)]

- **real** and **causal dissipative field equations** follow from analytically continued effective action

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi_a(x)}\Big|_{\mathsf{ret}} = \sqrt{g}J(x)$$

- to calculate retarded variational derivative determine

$$\delta\Gamma[\Phi]$$

  by varying the fields $\delta\Phi(x)$ including dissipative terms
- set signs according to

$$s_{\mathsf{R}}(u^\mu\partial_\mu)\,\delta\Phi(x) \to -\delta\Phi(x), \qquad\qquad \delta\Phi(x)\,s_{\mathsf{R}}(u^\mu\partial_\mu) \to +\delta\Phi(x)$$

- proceed as usual
- opposite choice of sign: field equations for backward time evolution

Damped harmonic oscillator 1

- equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0$$

or with $\omega_0 = \sqrt{k/m}$ and $\zeta = c/\sqrt{4mk}$

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0$$

- is there an action for damped oscillator? This does *not* work:

$$\int \frac{d\omega}{2\pi}\,\frac{m}{2} x^*(\omega)\left[\omega^2 \;\; +2i\,\omega\,\zeta\omega_0 \;\; -\omega_0^2\right] x(\omega)$$

- consider inverse propagator

$$\omega^2 + 2i\,s_{\mathsf{I}}(\omega)\,\omega\,\zeta\omega_0 - \omega_0^2$$

with sign function

$$s_{\mathsf{I}}(\omega) = \mathsf{sign}\,(\mathsf{Im}\,\omega)$$

Damped harmonic oscillator 2

- effective action

$$\Gamma[x] = \int \frac{d\omega}{2\pi}\ \frac{m}{2} x^*(\omega)\left[-\omega^2 - 2i\, s_{\mathsf{I}}(\omega)\,\omega\,\zeta\omega_0 + \omega_0^2\right] x(\omega)$$

$$= \int dt \left\{ -\frac{1}{2} m\dot{x}^2 + \frac{1}{2} c\, x\, s_{\mathsf{R}}(\partial_t)\dot{x} + \frac{1}{2} k x^2 \right\}$$

where the second line uses

$$s_{\mathsf{I}}(\omega) = \mathsf{sign}(\mathsf{Im}\,\omega) \to \mathsf{sign}(\mathsf{Im}\, i\partial_t) = \mathsf{sign}(\mathsf{Re}\,\partial_t) = s_{\mathsf{R}}(\partial_t)$$

- variation gives up to boundary terms

$$\delta\Gamma = \int dt \left\{ m\ddot{x}\,\delta x + \frac{1}{2} c\,\delta x\, s_{\mathsf{R}}(\partial_t)\dot{x} - \frac{1}{2} c\,\dot{x}\, s_{\mathsf{R}}(\partial_t)\delta x + kx\,\delta x \right\}$$

- set now $s_{\mathsf{R}}(\partial_t)\delta x \to -\delta x$ and $\delta x\, s_{\mathsf{R}}(\partial_t) \to \delta x$. Defines $\frac{\delta\Gamma}{\delta x}\big|_{\mathsf{ret}}$.
- equation of motion for forward time evolution

$$\frac{\delta\Gamma}{\delta x}\bigg|_{\mathsf{ret}} = m\ddot{x} + c\dot{x} + kx = 0$$

Entropy production

[Floerchinger, JHEP 1609, 099 (2016)]

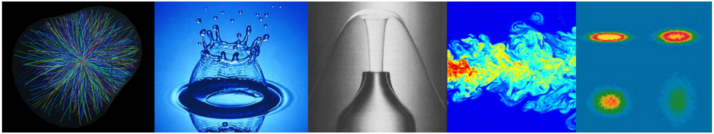
- analysis of general covariance leads to entropy production law

$$\nabla_\mu s^\mu = \frac{1}{\sqrt{g}} \frac{\delta \Gamma_D}{\delta \Phi_a}\Big|_{\text{ret}} \beta^\lambda \partial_\lambda \Phi_a + \beta_\mu \nabla_\nu \left(-\frac{2}{\sqrt{g}} \frac{\delta \Gamma_D}{\delta g_{\mu\nu}}\Big|_{\text{ret}} \right)$$

- should be positive by second law of thermodynamics
- so far only understood close-to-equilibrium
- e.g. for viscous fluid

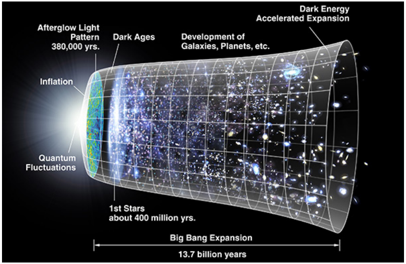
$$\nabla_\mu s^\mu = \frac{1}{T}\left[2\eta \sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta (\nabla_\rho u^\rho)^2\right]$$

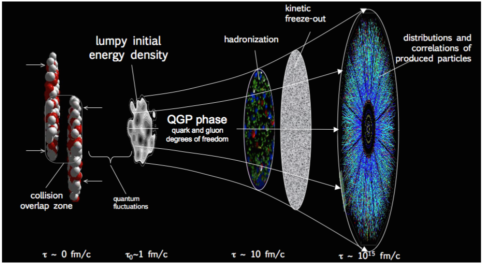
Fluid dynamics



- long distances, long times or strong enough interactions
- quantum fields form a fluid!
- needs **macroscopic** fluid properties
  - equation of state $p(T, \mu)$
  - shear viscosity $\eta(T, \mu)$
  - bulk viscosity $\zeta(T, \mu)$
  - heat conductivity $\kappa(T, \mu)$
  - relaxation times, ...
- *ab initio* calculation of transport properties difficult but in principle fixed by **microscopic** properties encoded in lagrangian
- standard model of high energy nuclear collisions based on relativistic dissipative fluid dynamics
- ongoing experimental and theoretical effort to understand this better

Big bang – little bang analogy





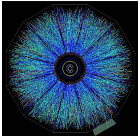
- cosmol. scale: MPc$= 3.1 \times 10^{22}$ m
- Gravity + QED + Dark sector
- one big event

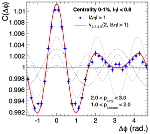
- nuclear scale: fm$= 10^{-15}$ m
- QCD
- very many events

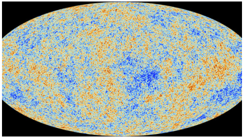
- dynamical description as a fluid
- all information must be reconstructed from final state

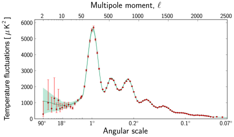
Fluid dynamic perturbation theory for heavy ions

[Floerchinger & Wiedemann, PLB 728, 407 (2014)]
[ongoing work with E. Grossi, J. Lion, A. Mazeliauskas]









- goal: determine QCD fluid properties from experiments
- so far: numerical fluid simulations e.g. [Heinz & Snellings (2013)]
- new idea: solve fluid equations for smooth and symmetric background and order-by-order in perturbations
- less numerical effort – more systematic studies
- good convergence properties [Floerchinger *et al.*, PLB 735, 305 (2014), Brouzakis *et al.* PRD 91, 065007 (2015)]
- similar to cosmological perturbation theory

Dissipation in cosmology

[Floerchinger, Tetradis & Wiedemann, PRL 114, 091301 (2015)]

Evolution of energy density in first order viscous fluid dynamics

$$u^\mu \partial_\mu \epsilon + (\epsilon + p)\nabla_\mu u^\mu - \zeta \Theta^2 - 2\eta \sigma^{\mu\nu}\sigma_{\mu\nu} = 0$$

with

- bulk viscosity ζ
- shear viscosity η

For $\vec{v}^2 \ll c^2$ and Newtonian potentials $\Phi, \Psi \ll 1$

$$\begin{aligned}&\dot{\epsilon} + \vec{v}\cdot\vec{\nabla}\epsilon + (\epsilon + p)\left(3\tfrac{\dot{a}}{a} + \vec{\nabla}\cdot\vec{v}\right)\\ &= \tfrac{\zeta}{a}\left[3\tfrac{\dot{a}}{a} + \vec{\nabla}\cdot\vec{v}\right]^2 + \tfrac{\eta}{a}\left[\partial_i v_j \partial_i v_j + \partial_i v_j \partial_j v_i - \tfrac{2}{3}(\vec{\nabla}\cdot\vec{v})^2\right]\end{aligned}$$

Fluid dynamic backreaction

[Floerchinger, Tetradis & Wiedemann, PRL 114, 091301 (2015)]

Expectation value of energy density $\bar{\epsilon} = \langle \epsilon \rangle$

$$\frac{1}{a}\dot{\bar{\epsilon}} + 3H\left(\bar{\epsilon} + \bar{p} - 3\bar{\zeta}H\right) = D$$

with dissipative backreaction term

$$D = \frac{1}{a^2}\langle \eta \left[\partial_i v_j \partial_i v_j + \partial_i v_j \partial_j v_i - \frac{2}{3}\partial_i v_i \partial_j v_j\right]\rangle$$
$$+ \frac{1}{a^2}\langle \zeta [\vec{\nabla}\cdot\vec{v}]^2\rangle + \frac{1}{a}\langle \vec{v}\cdot\vec{\nabla}\left(p - 6\zeta H\right)\rangle$$

- D vanishes for unperturbed homogeneous and isotropic universe
- D has contribution from shear & bulk viscous dissipation and thermodynamic work done by contraction against pressure gradients
- dissipative terms in D are positive semi-definite
- for spatially constant viscosities and scalar perturbations only

$$D = \frac{\bar{\zeta} + \frac{4}{3}\bar{\eta}}{a^2}\int d^3q \; P_{\theta\theta}(q)$$

Dissipation of perturbations

[Floerchinger, Tetradis & Wiedemann, PRL 114, 091301 (2015)]

- Dissipative backreaction does not need negative effective pressure

$$\tfrac{1}{a}\dot{\bar{\epsilon}} + 3H\left(\bar{\epsilon} + \bar{p}_{\text{eff}}\right) = D$$

- D is an integral over perturbations, could become large at late times.
- Can it potentially accelerate the universe?
- Need additional equation for scale parameter a
- Use trace of Einstein's equations $R = 8\pi G_{\text{N}} T^{\mu}{}_{\mu}$

$$\tfrac{1}{a}\dot{H} + 2H^2 = \tfrac{4\pi G_{\text{N}}}{3}\left(\bar{\epsilon} - 3\bar{p}_{\text{eff}}\right)$$

  does not depend on unknown quantities like $\langle(\epsilon + p_{\text{eff}})u^{\mu}u^{\nu}\rangle$
- To close the equations one needs equation of state $\bar{p}_{\text{eff}} = \bar{p}_{\text{eff}}(\bar{\epsilon})$ and dissipation parameter D

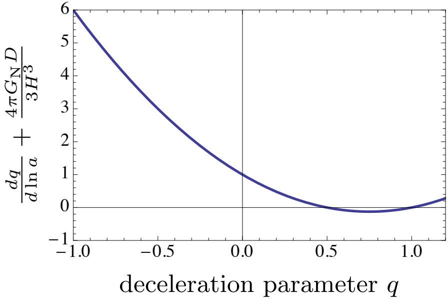
Deceleration parameter

[Floerchinger, Tetradis & Wiedemann, PRL 114, 091301 (2015)]

- assume now vanishing effective pressure $\bar{p}_{\text{eff}} = 0$
- obtain for deceleration parameter $q = -1 - \frac{\dot{H}}{aH^2}$

$$-\frac{dq}{d\ln a} + 2(q-1)\left(q - \tfrac{1}{2}\right) = \frac{4\pi G_{\text{N}} D}{3H^3}$$

- for $D = 0$ attractive fixed point at $q_* = \frac{1}{2}$ (deceleration)
- for $D > 0$ fixed point shifted towards $q_* < 0$ (acceleration)



Conclusions

- quantum field theory & information theory are entangled !
- could be essential element for universal non-equilibrium theory
- entanglement helps to understand “thermal effects” in e^+e^- and other collider experiments
  - at very early times theory effectively conformal $\frac{1}{\tau} \gg m, q$
  - entanglement entropy extensive in rapidity $\frac{dS}{d\Delta\eta} = \frac{c}{6}$
  - reduced density matrix for excitations at early times thermal $T = \frac{\hbar}{2\pi\tau}$
- experiments with cold atoms could allow to investigate entanglement directly
- effectively dissipative dynamics can have interesting consequences for cosmology

BACKUP

Coarse graining etc.

- entropy in quantum system can emerge when
    - system is divided into pieces with reduced density matrix
    - subsystems are composed again as mixed states
- cuts may divide
    - different regions
    - high-momentum and low-momentum
    - "system" and "bath"
- entropy in classical systems from coarse graining phase space
- entropy in kinetic theory from neglecting two-particle correlations (Boltzmann's "Stosszahlansatz")

Transverse coordinates

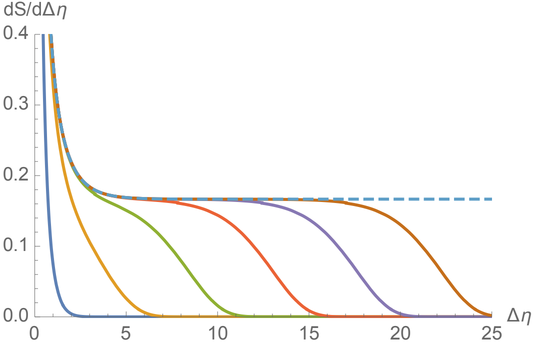
- So far dynamics strictly confined to 1+1 dimensions
- Transverse coordinates may fluctuate, can be described by Nambu-Goto action ($h_{\mu\nu} = \partial_\mu X^m \partial_\nu X_m$)

$$\begin{aligned} S_{\mathsf{NG}} &= \int d^2x \sqrt{-\mathsf{det}\, h_{\mu\nu}}\, \{-\sigma + \ldots\} \\ &\approx \int d^2x \sqrt{g} \left\{-\sigma - \frac{\sigma}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i + \ldots\right\} \end{aligned}$$

- Two additional, massless, bosonic degrees of freedom corresponding to transverse coordinates X^i with $i = 1, 2$.

Free massive fermions

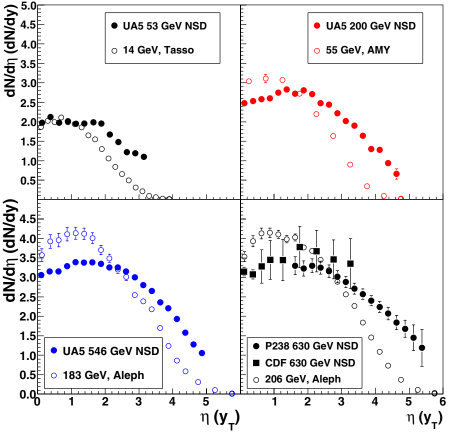
- Entanglement entropy can also be calculated for free Dirac fermions of mass m



$m\tau = 1$, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, and 10^{-5}

- Same universal plateau $c/6$ with $c = 1$ at early time
- Conformal limit corresponds to non-interacting fermions
- Consistent with or without bosonization

Rapidity distribution



[open (filled) symbols: e^+e^- (pp), Grosse-Oetringhaus & Reygers (2010)]

- Rapidity distribution $dN/d\eta$ has plateau around midrapidity
- Only logarithmic dependence on collision energy

Experimental access to entanglement ?

- Could longitudinal entanglement be tested experimentally?
- Unfortunately entropy density $dS/d\eta$ not straight-forward to access.
- Measured in e^+e^- is the number of charged particles per unit rapidity $dN_{\text{ch}}/d\eta$ (rapidity defined with respect to the thrust axis)
- Around mid-rapidity logarithmic dependence on the collision energy.
- Typical values for collision energies $\sqrt{s} = 14 - 206$ GeV in the range

$$dN_{\text{ch}}/d\eta \approx 2 - 4$$

- Entropy per particle S/N can be estimated for a hadron resonance gas in thermal equilibrium $S/N_{\text{ch}} = 7.2$ would give

$$dS/d\eta \approx 14 - 28$$

- This is an upper bound: correlations beyond one-particle functions would lead to reduced entropy.

Temperature and entanglement entropy

- For conformal fields, entanglement entropy has also been calculated at non-zero temperature.
- For static interval of length l [Calabrese, Cardy (2004)]

$$S(T,l) = \frac{c}{3}\ln\left(\frac{1}{\pi T \epsilon}\sinh(\pi l T)\right) + \text{const}$$

- Compare this to our result in expanding geometry

$$S(\tau,\Delta\eta) = \frac{c}{3}\ln\left(\frac{2\tau}{\epsilon}\,\mathsf{sinh}(\Delta\eta/2)\right) + \text{constant}$$

- Expressions agree for $l = \tau\Delta\eta$ (with metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$) and time-dependent temperature

$$T = \frac{1}{2\pi\tau}$$

Alternative derivation: mode functions

- Fluctuation field $\varphi = \phi - \bar{\phi}$ has equation of motion

$$\partial_\tau^2 \varphi(\tau,\eta) + \frac{1}{\tau}\partial_\tau \varphi(\tau,\eta) + \left(M^2 - \frac{1}{\tau^2}\frac{\partial^2}{\partial \eta^2} \right) \varphi(\tau,\eta) = 0$$

- Solution in terms of plane waves

$$\varphi(\tau,\eta) = \int \frac{dk}{2\pi} \left\{ a(k) f(\tau,|k|) e^{ik\eta} + a^\dagger(k)\, f^*(\tau,|k|) e^{-ik\eta} \right\}$$

- Mode functions as Hankel functions

$$f(\tau,k) = \frac{\sqrt{\pi}}{2} e^{\frac{k\pi}{2}} H_{ik}^{(2)}(M\tau)$$

or alternatively as Bessel functions

$$\bar{f}(\tau,k) = \frac{\sqrt{\pi}}{\sqrt{2\sinh(\pi k)}} J_{-ik}(M\tau)$$

Bogoliubov transformation

- Mode functions are related

$$\begin{aligned}\bar{f}(\tau,k) =&\alpha(k)f(\tau,k)+\beta(k)f^*(\tau,k)\\ f(\tau,k) =&\alpha^*(k)\bar{f}(\tau,k)-\beta(k)\bar{f}^*(\tau,k)\end{aligned}$$

- Creation and annihilation operators are related by

$$\begin{aligned}\bar{a}(k) =&\alpha^*(k)a(k)-\beta^*(k)a^\dagger(k)\\ a(k) =&\alpha(k)\bar{a}(k)+\beta(k)\bar{a}^\dagger(k)\end{aligned}$$

- Bogoliubov coefficients

$$\alpha(k)=\sqrt{\frac{e^{\pi k}}{2\sinh(\pi k)}}\qquad\qquad \beta(k)=\sqrt{\frac{e^{-\pi k}}{2\sinh(\pi k)}}$$

- Vacuum $|\Omega\rangle$ with respect to $a(k)$ such that $a(k)|\Omega\rangle=0$ contains excitations with respect to $\bar{a}(k)$ such that $\bar{a}(k)|\Omega\rangle\neq 0$ and *vice versa*

Role of different mode functions

- Hankel functions $f(\tau, k)$ are superpositions of *positive* frequency modes with respect to Minkowski time t
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to Minkowski time t
- At very early time $1/\tau \gg M$ conformal symmetry

$$ds^2 = \tau^2 \left[-d\ln(\tau)^2 + d\eta^2 \right]$$

- Hankel functions $f(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to conformal time $\ln(\tau)$
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive* frequency modes with respect to conformal time $\ln(\tau)$

Occupation numbers

- Minkowski space coherent states have two-point functions

$$\begin{aligned}\langle \bar{a}^\dagger(k)\bar{a}(k')\rangle_c &= \bar{n}(k)\,2\pi\,\delta(k-k') = |\beta(k)|^2\,2\pi\,\delta(k-k')\\ \langle \bar{a}(k)\bar{a}(k')\rangle_c &= \bar{u}(k)\,2\pi\,\delta(k+k') = -\alpha^*(k)\beta^*(k)\,2\pi\,\delta(k+k')\\ \langle \bar{a}^\dagger(k)\bar{a}^\dagger(k')\rangle_c &= \bar{u}^*(k)\,2\pi\,\delta(k+k') = -\alpha(k)\beta(k)\,2\pi\,\delta(k+k')\end{aligned}$$

- Occupation number

$$\bar{n}(k) = |\beta(k)|^2 = \frac{1}{e^{2\pi k}-1}$$

- Bose-Einstein distribution with excitation energy $E = |k|/\tau$ and temperature

$$T = \frac{1}{2\pi\tau}$$

- Off-diagonal occupation number $\bar{u}(k) = -1/(2\sinh(\pi k))$ make sure we still have pure state

Local description

- Consider now rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$
- Fourier expansion becomes discrete

$$\varphi(\eta) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \varphi_n \, e^{in\pi \frac{\eta}{\Delta\eta}}$$

$$\varphi_n = \int_{-\Delta\eta/2}^{\Delta\eta/2} d\eta \, \varphi(\eta) \, \frac{1}{2} \left[e^{-in\pi \frac{\eta}{\Delta\eta}} + (-1)^n e^{in\pi \frac{\eta}{\Delta\eta}} \right]$$

- Relation to continuous momentum modes by integration kernel

$$\varphi_n = \int \frac{dk}{2\pi} \sin(\tfrac{k\Delta\eta}{2} - \tfrac{n\pi}{2}) \left[\frac{1}{k - \frac{n\pi}{\Delta\eta}} + \frac{1}{k + \frac{n\pi}{\Delta\eta}} \right] \varphi(k)$$

- Local density matrix determined by correlation functions

$$\langle \varphi_n \rangle, \qquad \langle \pi_n \rangle, \qquad \langle \varphi_n \varphi_m \rangle_c, \qquad \text{etc.}$$

Emergence of locally thermal state

- Mode functions at early time

$$\bar{f}(\tau,k) = \frac{1}{\sqrt{2k}} e^{-ik\ln(\tau) - i\theta(k,M)}$$

- Phase varies strongly with k for $M \to 0$

$$\theta(k,M) = k\ln(M/2) + \arg(\Gamma(1-ik))$$

- Off-diagonal term $\bar{u}(k)$ have factors strongly oscillating with k

$$\begin{aligned}\langle\varphi(\tau,k)\varphi^*(\tau,k')\rangle_c = {} & 2\pi\delta(k-k')\frac{1}{|k|} \\ & \times \left\{ \left[\tfrac{1}{2} + \bar{n}(k)\right] + \cos\left[2k\ln(\tau) + 2\theta(k,M)\right]\bar{u}(k) \right\}\end{aligned}$$

  cancel out when going to finite interval !

- Only Bose-Einstein occupation numbers $\bar{n}(k)$ remain

Entanglement and deep inelastic scattering

- How strongly entangled is the nuclear wave function?
- What is the entropy of quasi-free partons and can it be understood as a result of entanglement? [Kharzeev, Levin (2017)]

$$S = \ln[xG(x)]$$

- Does saturation at small Bjorken-x have an entropic meaning?
- Entanglement entropy and entropy production in the color glass condensate [Kovner, Lublinsky (2015)]
- Could entanglement entropy help for a non-perturbative extension of the parton model?
- Entropy of perturbative and non-perturbative Pomeron descriptions [Shuryak, Zahed (2017)]