Dynamics of entanglement in expanding quantum fields

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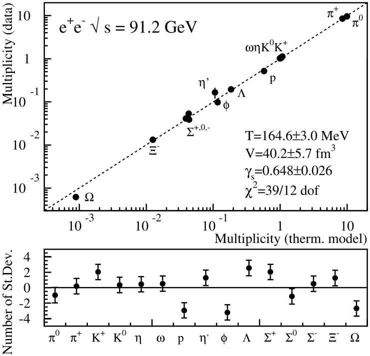
based on

- J. Berges, S. Floerchinger & R. Venugopalan, *Thermal excitation spectrum from entanglement in an expanding QCD string* [arxiv:1707.05338]
- J. Berges, S. Floerchinger & R. Venugopalan, *Dynamics of entanglement in expanding quantum fields* [arXiv:1712.09362]

Motivation

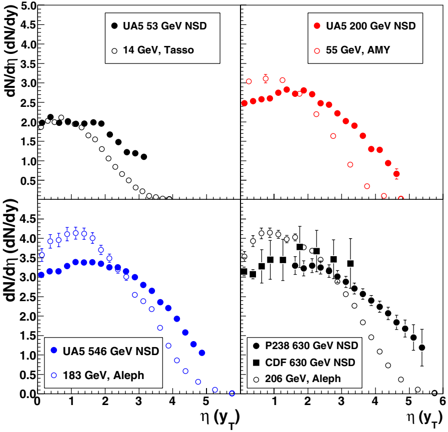
- Elementary particle collision experiments such as $e^+ \, e^-$ collisions show thermal-like features.
- Example: particle multiplicities



[Becattini, Casterina, Milov & Satz, EPJC 66, 377 (2010)]

- Conventional thermalization by collisions unlikely.
- Alternative explanations needed.

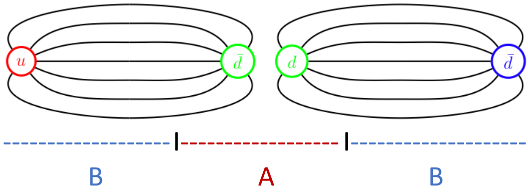
Rapidity distribution



[open (filled) symbols: e^+e^- (pp), Grosse-Oetringhaus & Reygers (2010)]

- Rapidity distribution $dN/d\eta$ has plateau around midrapidity
- Only logarithmic dependence on collision energy

QCD strings



- Particle production from QCD strings.
- e. g. Lund model (Pythia).
- Different regions in a string are entangled.
- Subinterval A is described by reduced density matrix

$$\rho_A = \mathrm{Tr}_B \rho.$$

- Reduced density matrix is of mixed state form.
- Could this lead to thermal-like effects?

Microscopic model

- QCD in 1+1 dimensions described by 't Hooft model

$$\mathscr{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - ig\mathbf{A}_\mu)\psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{2} \mathrm{tr}\, \mathbf{F}_{\mu\nu} \mathbf{F}^{\mu\nu}$$

- Fermionic fields ψ_i with sums over flavor species $i = 1, \ldots, N_f$
- SU(N_c) gauge fields $\mathbf{A}_\mu$ with field strength tensor $\mathbf{F}_{\mu\nu}$
- Gluons are not dynamical in two dimensions
- Gauge coupling g has dimension of mass
- Non-trivial, interacting theory, cannot be solved exactly
- Spectrum of excitations known for $N_c \to \infty$ with $g^2 N_c$ fixed ['t Hooft (1974)]

Schwinger model

- QED in 1+1 dimension

$$\mathscr{L} = -\bar{\psi}_i \gamma^\mu (\partial_\mu - iqA_\mu)\psi_i - m_i \bar{\psi}_i \psi_i - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- Geometric confinement
- U(1) charge related to string tension $q = \sqrt{2\sigma}$
- For single fermion one can **bosonize theory** exactly [Coleman, Jackiw, Susskind (1975)]

$$S = \int d^2x \sqrt{g} \Big\{ -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} M^2 \phi^2 - \frac{m\, q\, e^\gamma}{2\pi^{3/2}} \cos\left(2\sqrt{\pi}\phi + \theta\right) \Big\}$$

- Schwinger bosons are dipoles $\phi \sim \bar{\psi}\psi$
- Mass is related to U(1) charge by $M = q/\sqrt{\pi} = \sqrt{2\sigma/\pi}$
- Massless Schwinger model $m = 0$ leads to free bosonic theory

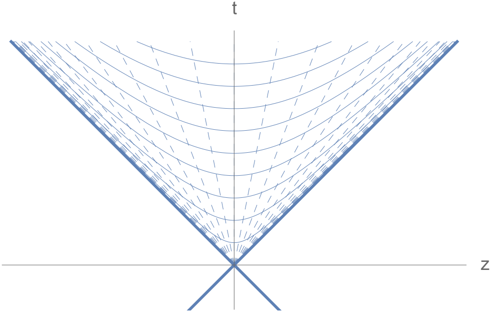
Transverse coordinates

- So far dynamics strictly confined to 1+1 dimensions
- Transverse coordinates may fluctuate, can be described by Nambu-Goto action ($h_{\mu\nu} = \partial_\mu X^m \partial_\nu X_m$)

$$S_{\mathsf{NG}} = \int d^2x \sqrt{-\det h_{\mu\nu}}\,\{-\sigma + \ldots\}$$
$$\approx \int d^2x \sqrt{g}\left\{-\sigma - \frac{\sigma}{2} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^i + \ldots\right\}$$

- Two additional, massless, bosonic degrees of freedom corresponding to transverse coordinates X^i with $i = 1, 2$.

Expanding string solution 1



- Consider string formed between (external) quark-anti-quark pair on trajectories

$$z = \pm t$$

- Coordinates: Bjorken time $\tau = \sqrt{t^2 - z^2}$, rapidity $\eta = \text{arctanh}(z/t)$
- Metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$
- Symmetry with respect to longitudinal boosts $\eta \to \eta + \Delta\eta$

Expanding string solution 2

- Schwinger boson field depends only on τ

$$\bar{\phi} = \bar{\phi}(\tau)$$

- Equation of motion

$$\partial_\tau^2 \bar{\phi} + \frac{1}{\tau}\partial_\tau \bar{\phi} + M^2 \bar{\phi} = 0.$$

- Gauss law: electric field $E = q\phi/\sqrt{\pi}$ must approach the U(1) charge of the external quarks $E \to q_\mathsf{e}$ for $\tau \to 0_+$

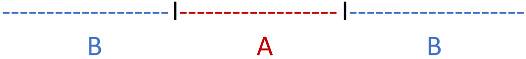
$$\bar{\phi}(\tau) \to \frac{\sqrt{\pi} q_\mathsf{e}}{q} \qquad (\tau \to 0_+)$$

- Solution of equation of motion [Loshaj, Kharzeev (2011)]

$$\bar{\phi}(\tau) = \frac{\sqrt{\pi} q_\mathsf{e}}{q} J_0(M\tau)$$

Reduced density matrix

- Consider now physical processes such as hadron formation
- Assume that these are local processes in some space region A



- Reduced density matrix, trace over complement region B

$$\rho_A = \text{Tr}_B\,\rho$$

- In general ρ_A mixed state density matrix even if ρ is pure
- Reason: entanglement between regions A and B
- Characterization by entanglement entropy

$$S_A = -\text{Tr}\left\{\rho_A \ln(\rho_A)\right\}$$

Gaussian states

- Theories with quadratic action typically have Gaussian density matrix
- Fully characterized by field expectation values

$$\bar{\phi}(x) = \langle \phi(x) \rangle, \qquad \bar{\pi}(x) = \langle \pi(x) \rangle$$

  and connected two-point correlation functions, e. g.

$$\langle \phi(x)\phi(y) \rangle_c = \langle \phi(x)\phi(y) \rangle - \bar{\phi}(x)\bar{\phi}(y)$$

- If ρ is Gaussian, also reduced density matrix ρ_A is Gaussian

Entanglement entropy for Gaussian state

- Entanglement entropy of Gaussian state in region A
  [Berges, Floerchinger, Venugopalan, 1712.09362]

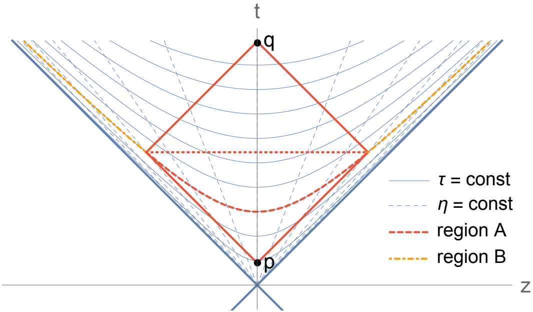
$$S_A = \frac{1}{2}\mathsf{Tr}_A \left\{ D \ln(D^2) \right\},$$

- Operator trace over region A only
- Matrix of correlation functions

$$D(x,y) = \begin{pmatrix} -i\langle\phi(x)\pi(y)\rangle_c & i\langle\phi(x)\phi(y)\rangle_c \\ -i\langle\pi(x)\pi(y)\rangle_c & i\langle\pi(x)\phi(y)\rangle_c \end{pmatrix}.$$

- Involves connected correlation functions of field $\phi(x)$ and canonically conjugate momentum field $\pi(x)$
- Expectation value $\bar{\phi}$ does not appear explicitly
- Coherent states and vacuum have equal entanglement entropy S_A

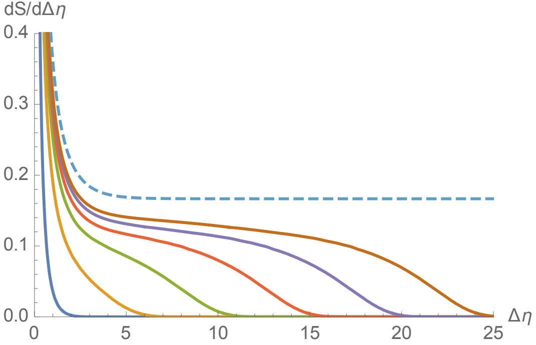
Rapidity interval



- Consider rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ at fixed Bjorken time τ
- Entanglement entropy does not change by unitary time evolution with endpoints kept fixed
- Can be evaluated equivalently in interval $\Delta z = 2\tau \sinh(\Delta\eta/2)$ at fixed time $t = \tau \cosh(\Delta\eta/2)$
- Need to solve eigenvalue problem with correct **boundary conditions**

Bosonized massless Schwinger model

- Entanglement entropy understood numerically for free massive scalars [Casini, Huerta (2009)]
- Entanglement entropy density $dS/d\Delta\eta$ for bosonized massless Schwinger model ($M = \frac{q}{\sqrt{\pi}}$)



$M\tau = 1,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4}$, and 10^{-5}

Conformal limit

- For $M\tau \to 0$ one has conformal field theory limit
[Holzhey, Larsen, Wilczek (1994)]

$$S(\Delta z) = \frac{c}{3} \ln\left(\Delta z/\epsilon\right) + \text{constant}$$

with small length ϵ acting as UV cutoff.

- Here this implies

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln\left(2\tau \,\mathsf{sinh}(\Delta\eta/2)/\epsilon\right) + \text{constant}$$

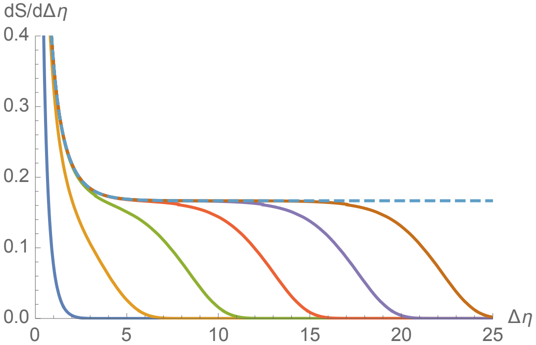
- Conformal charge $c = 1$ for free massless scalars or Dirac fermions.
- Additive constant not universal but entropy density is

$$\begin{aligned} \frac{\partial}{\partial \Delta\eta} S(\tau, \Delta\eta) =& \frac{c}{6} \mathsf{coth}(\Delta\eta/2) \\ \to& \frac{c}{6} \qquad\qquad (\Delta\eta \gg 1) \end{aligned}$$

- Entropy becomes extensive in $\Delta\eta$!

Free massive fermions

- Entanglement entropy can also be calculated for free Dirac fermions of mass m



$m\tau = 1,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-4}$, and 10^{-5}

- Same universal plateau $c/6$ with $c = 1$ at early time
- Conformal limit corresponds to non-interacting fermions
- Consistent with or without bosonization

Universal entanglement entropy density

- For very early times "Hubble" expansion rate dominates over masses and interactions

$$H = \frac{1}{\tau} \gg M = \frac{q}{\sqrt{\pi}}, m$$

- Theory dominated by free, massless fermions
- Universal entanglement entropy density

$$\frac{dS}{d\Delta\eta} = \frac{c}{6}$$

  with conformal charge c
- For QCD in 1+1 dimensions (gluons not dynamical)

$$c = N_c \times N_f$$

- From fluctuating transverse coordinates (Nambu-Goto action)

$$c = N_c \times N_f + 2 \approx 9 + 2 = 11$$

Experimental access to entanglement ?

- Could longitudinal entanglement be tested experimentally?
- Unfortunately entropy density $dS/d\eta$ not straight-forward to access.
- Measured in e^+e^- is the number of charged particles per unit rapidity $dN_{\text{ch}}/d\eta$ (rapidity defined with respect to the thrust axis)
- Around mid-rapidity logarithmic dependence on the collision energy.
- Typical values for collision energies $\sqrt{s} = 14 - 206$ GeV in the range

$$dN_{\text{ch}}/d\eta \approx 2 - 4$$

- Entropy per particle S/N can be estimated for a hadron resonance gas in thermal equilibrium $S/N_{\text{ch}} = 7.2$ would give

$$dS/d\eta \approx 14 - 28$$

- This is an upper bound: correlations beyond one-particle functions would lead to reduced entropy.

Temperature and entanglement entropy

- For conformal fields, entanglement entropy has also been calculated at non-zero temperature.
- For static interval of length L [Korepin (2004); Calabrese, Cardy (2004)]

$$S(T,l) = \frac{c}{3} \ln \left(\frac{1}{\pi T \epsilon} \sinh(\pi L T) \right) + \text{const}$$

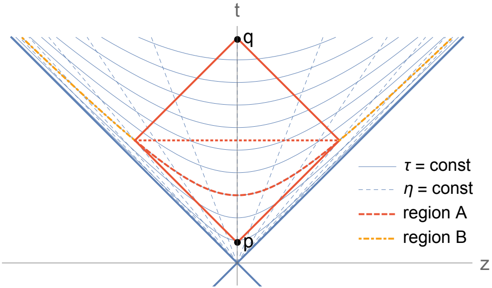
- Compare this to our result in expanding geometry

$$S(\tau, \Delta\eta) = \frac{c}{3} \ln \left(\frac{2\tau}{\epsilon} \,\text{sinh}(\Delta\eta/2) \right) + \text{constant}$$

- Expressions agree for $L = \tau \Delta\eta$ (with metric $ds^2 = -d\tau^2 + \tau^2 d\eta^2$) and time-dependent temperature

$$T = \frac{1}{2\pi\tau}$$

Modular or entanglement Hamiltonian 1



- Conformal field theory
- Hypersurface Σ with boundary on the intersection of two light cones
- Reduced density matrix
  [Casini, Huerta, Myers (2011), Arias, Blanco, Casini, Huerta (2017), see also Candelas, Dowker (1979)]

$$\rho_A = \frac{1}{Z_A} e^{-K}, \qquad Z_A = \mathrm{Tr}\, e^{-K},$$

- Modular or entanglement Hamiltonian K.

Modular or entanglement Hamiltonian 2

- Modular or entanglement Hamiltonian is **local expression**

$$K = \int_{\Sigma} d\Sigma^{\mu}\, \xi^{\nu}(x)\, T_{\mu\nu}(x).$$

- Energy-momentum tensor $T_{\mu\nu}(x)$ and $\xi^{\nu}(x)$ is a vector field

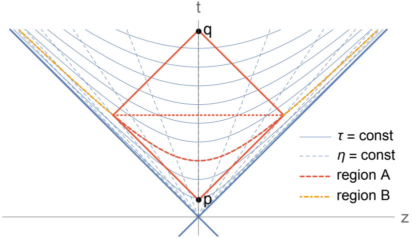
$$\xi^{\mu}(x) = \tfrac{2\pi}{(k-p)^2}\left[(k-x)^{\mu}(x-p)(k-p) + (x-p)^{\mu} \right.$$
$$\left. \times (k-x)(k-p) - (k-p)^{\mu}(x-p)(k-x)\right]$$

with end point of the future light cone k and starting point of the past light cone p.

- Inverse temperature and fluid velocity

$$\xi^{\mu}(x) = \beta^{\mu}(x) = \frac{u^{\mu}(x)}{T(x)}$$

Modular or entanglement Hamiltonian 3



- For k very far in the future $\xi^\mu(x) \to 2\pi\, x^\mu$
- Fluid velocity in τ-direction & time-dependent temperature

$$T(\tau) = \frac{\hbar}{2\pi\tau}$$

- **Entanglement between different rapidity intervals alone leads to local thermal density matrix at very early times !**
- Hawking-Unruh temperature in Rindler wedge $T(x) = \hbar c/(2\pi x)$

Alternative derivation: mode functions

- Fluctuation field $\varphi = \phi - \bar{\phi}$ has equation of motion

$$\partial_\tau^2 \varphi(\tau,\eta) + \frac{1}{\tau}\partial_\tau \varphi(\tau,\eta) + \left(M^2 - \frac{1}{\tau^2}\frac{\partial^2}{\partial \eta^2} \right) \varphi(\tau,\eta) = 0$$

- Solution in terms of plane waves

$$\varphi(\tau,\eta) = \int \frac{dk}{2\pi} \left\{ a(k) f(\tau,|k|) e^{ik\eta} + a^\dagger(k)\, f^*(\tau,|k|) e^{-ik\eta} \right\}$$

- Mode functions as Hankel functions

$$f(\tau,k) = \frac{\sqrt{\pi}}{2} e^{\frac{k\pi}{2}} H_{ik}^{(2)}(M\tau)$$

or alternatively as Bessel functions

$$\bar{f}(\tau,k) = \frac{\sqrt{\pi}}{\sqrt{2\sinh(\pi k)}} J_{-ik}(M\tau)$$

Bogoliubov transformation

- Mode functions are related

$$\bar{f}(\tau,k) = \alpha(k) f(\tau,k) + \beta(k) f^*(\tau,k)$$
$$f(\tau,k) = \alpha^*(k)\bar{f}(\tau,k) - \beta(k)\bar{f}^*(\tau,k)$$

- Creation and annihilation operators are related by

$$\bar{a}(k) = \alpha^*(k) a(k) - \beta^*(k) a^\dagger(k)$$
$$a(k) = \alpha(k)\bar{a}(k) + \beta(k)\bar{a}^\dagger(k)$$

- Bogoliubov coefficients

$$\alpha(k) = \sqrt{\frac{e^{\pi k}}{2\sinh(\pi k)}} \qquad \beta(k) = \sqrt{\frac{e^{-\pi k}}{2\sinh(\pi k)}}$$

- Vacuum $|\Omega\rangle$ with respect to $a(k)$ such that $a(k)|\Omega\rangle = 0$ contains excitations with respect to $\bar{a}(k)$ such that $\bar{a}(k)|\Omega\rangle \neq 0$ and *vice versa*

Role of different mode functions

- Hankel functions $f(\tau, k)$ are superpositions of *positive* frequency modes with respect to Minkowski time t
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to Minkowski time t
- At very early time $1/\tau \gg M$ conformal symmetry

$$ds^2 = \tau^2 \left[-d\ln(\tau)^2 + d\eta^2 \right]$$

- Hankel functions $f(\tau, k)$ are superpositions of *positive and negative* frequency modes with respect to conformal time $\ln(\tau)$
- Bessel functions $\bar{f}(\tau, k)$ are superpositions of *positive* frequency modes with respect to conformal time $\ln(\tau)$

Occupation numbers

- Minkowski space coherent states have two-point functions

$$\langle \bar{a}^\dagger(k)\bar{a}(k')\rangle_c = \bar{n}(k)\, 2\pi\, \delta(k-k') = |\beta(k)|^2\, 2\pi\, \delta(k-k')$$
$$\langle \bar{a}(k)\bar{a}(k')\rangle_c = \bar{u}(k)\, 2\pi\, \delta(k+k') = -\alpha^*(k)\beta^*(k)\, 2\pi\, \delta(k+k')$$
$$\langle \bar{a}^\dagger(k)\bar{a}^\dagger(k')\rangle_c = \bar{u}^*(k)\, 2\pi\, \delta(k+k') = -\alpha(k)\beta(k)\, 2\pi\, \delta(k+k')$$

- Occupation number

$$\bar{n}(k) = |\beta(k)|^2 = \frac{1}{e^{2\pi k} - 1}$$

- Bose-Einstein distribution with excitation energy $E = |k|/\tau$ and temperature

$$T = \frac{1}{2\pi\tau}$$

- Off-diagonal occupation number $\bar{u}(k) = -1/(2\sinh(\pi k))$ make sure we still have pure state

Local description

- Consider now rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$
- Fourier expansion becomes discrete

$$\varphi(\eta) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \varphi_n \, e^{in\pi \frac{\eta}{\Delta\eta}}$$

$$\varphi_n = \int_{-\Delta\eta/2}^{\Delta\eta/2} d\eta \, \varphi(\eta) \, \frac{1}{2} \left[e^{-in\pi \frac{\eta}{\Delta\eta}} + (-1)^n e^{in\pi \frac{\eta}{\Delta\eta}} \right]$$

- Relation to continuous momentum modes by integration kernel

$$\varphi_n = \int \frac{dk}{2\pi} \sin(\tfrac{k\Delta\eta}{2} - \tfrac{n\pi}{2}) \left[\frac{1}{k - \frac{n\pi}{\Delta\eta}} + \frac{1}{k + \frac{n\pi}{\Delta\eta}} \right] \varphi(k)$$

- Local density matrix determined by correlation functions

$$\langle \varphi_n \rangle, \qquad \langle \pi_n \rangle, \qquad \langle \varphi_n \varphi_m \rangle_c, \qquad \text{etc.}$$

Emergence of locally thermal state

- Mode functions at early time

$$\bar{f}(\tau, k) = \frac{1}{\sqrt{2k}} e^{-ik\ln(\tau) - i\theta(k,M)}$$

- Phase varies strongly with k for $M \to 0$

$$\theta(k, M) = k\ln(M/2) + \arg(\Gamma(1 - ik))$$

- Off-diagonal term $\bar{u}(k)$ have factors strongly oscillating with k

$$\langle\varphi(\tau, k)\varphi^*(\tau, k')\rangle_c = 2\pi\delta(k - k')\frac{1}{|k|} \times \left\{\left[\tfrac{1}{2} + \bar{n}(k)\right] + \cos\left[2k\ln(\tau) + 2\theta(k, M)\right]\bar{u}(k)\right\}$$

cancel out when going to finite interval !

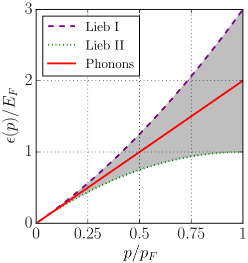
- Only Bose-Einstein occupation numbers $\bar{n}(k)$ remain

Physics picture

- Coherent state vacuum at early time contains entangled pairs of quasi-particles with opposite wave numbers
- On finite rapidity interval $(-\Delta\eta/2, \Delta\eta/2)$ in- and out-flux of quasi-particles with thermal distribution via boundaries
- Technically limits $\Delta\eta \to \infty$ and $M\tau \to 0$ do not commute
  - $\Delta\eta \to \infty$ for any finite $M\tau$ gives pure state
  - $M\tau \to 0$ for any finite $\Delta\eta$ gives thermal state with $T = 1/(2\pi\tau)$

Testing the mechanism with cold atoms

- Lieb-Liniger model for interacting bosonic atoms in $D = 1$ dimensions has linear dispersion at small momenta $\omega = v_s\, p$
  - strong interaction $\gamma \gg 1$ sound velocity $v_s = v_F = \pi n/m$
  - weak interaction $\gamma \ll 1$ sound velocity $v_s = \sqrt{\gamma} n/m = \sqrt{gnm}$



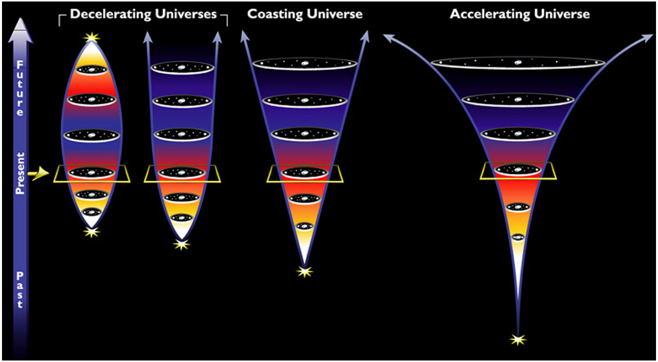
strong interactios $\gamma \gg 1$ [De Rosi et al. (2017)]

- Effective metric for phonons

$$ds^2 = -v_s^2 dt^2 + dx^2$$

Expanding geometries in cold atom experiments

- Expanding geometries can be realized by interplay of
  - longitudinal expansion
  - time dependent change of sound velocity $v_s(t)$
  - time dependent gap or mass $M^2(t)$



Entanglement and deep inelastic scattering

- How strongly entangled is the nuclear wave function?
- What is the entropy of quasi-free partons and can it be understood as a result of entanglement? [Kharzeev, Levin (2017)]

$$S = \ln[xG(x)]$$

- Does saturation at small Bjorken-x have an entropic meaning?
- Entanglement entropy and entropy production in the color glass condensate [Kovner, Lublinsky (2015)]
- Could entanglement entropy help for a non-perturbative extension of the parton model?
- Entropy of perturbative and non-perturbative Pomeron descriptions [Shuryak, Zahed (2017)]

Conclusions

- Rapidity intervals in an expanding string are entangled
- Entanglement comes in via boundary terms
- At very early times theory effectively conformal

$$\frac{1}{\tau} \gg m, q$$

- Entanglement entropy extensive in rapidity $\frac{dS}{d\Delta\eta} = \frac{c}{6}$
- Determined by conformal charge $c = N_c \times N_f + 2$
- Reduced density matrix for conformal field theory is of locally thermal form with temperature

$$T = \frac{\hbar}{2\pi\tau}$$

- Entanglement could be important ingredient to understand apparent "thermal effects" in e^+e^- and other collider experiments

Backup

Bosonization out-of-equilibrium

[Gutman, Grefen, Mirlin, PRB 81, 085436 (2010)]

Bosonization consists of several steps

1. mapping between Hilbert space of fermions and bosons
2. construction of bosonic Hamiltonian H_B by expressing fermionic Hamiltonian H_F in terms of bosonic operators
3. expressing fermionic operators in bosonic language
4. evaluate observables (e. g. Greens functions) with respect to many-body bosonic density matrix ρ_B

First three steps are independent of state, only last step depends on ρ_B.

- States with general fermionic occupation numbers $n_F(\vec{p})$ lead to non-local & higher order Greens-functions in bosonized theory
- For confining theories only bosonic excitations are asymptotic states.