

*Dissipation from the analytically continued
1 PI effective action*

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[based on JHEP09 (2016) 099 (arxiv:1603.07148)]

Dissipation in quantum field theory

- Dissipation is **generation** of entropy
- Unitary evolution **conserves** entropy

- in practice often only incomplete information available
- expectation values of fundamental quantum fields and some composite operators
- quantum states with minimal information given some constraints
- use truncation of 1 PI effective action

Dissipation from integrating out fields

Example 1

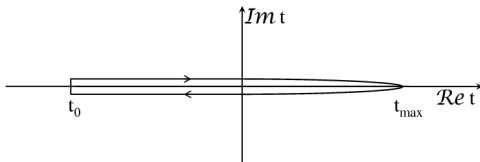
- consider muon with decay $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$
- full electroweak theory is unitary
- consider now effective action where fields for e^- , $\bar{\nu}_e$ and ν_μ have been integrated out
- effective action for μ^- contains decay width: appears as dissipative term

Example 2

- consider electromagnetic field A_μ
- field strength above Schwinger threshold: electron-positron pair production
- electron / positron field can be integrated out: dissipative term for electromagnetic field

Double time path formalism

- formalism for general, far-from-equilibrium situations: Schwinger-Keldysh double time path
- can be formulated with two fields $\Phi = \frac{1}{2}(\phi_+ + \phi_-)$, $\chi = \phi_+ - \phi_-$
- in principle for arbitrary initial density matrices, in praxis mainly Gaussian initial states
- allows to treat also dissipation
- useful also to treat initial state fluctuations or forced noise in classical statistical theories
- difficult to recover thermal equilibrium, in particular non-perturbatively
- formalism algebraically somewhat involved



Close-to-equilibrium situations

- out-of-equilibrium situations
- close-to-equilibrium: description by field expectation values and thermodynamic fields
- more complete description by following more fields explicitly
- example: Viscous fluid dynamics plus additional fields
- usually discussed in terms of
 - phenomenological constitutive relations
 - as a limit of kinetic theory
 - in AdS/CFT
- want non-perturbative formulation in terms of QFT concepts
- Analytic continuation as an alternative to Schwinger-Keldysh
- direct generalization of equilibrium formalism

Local equilibrium states

- Dissipation: energy and momentum get transferred to a heat bath
- Even if one starts with pure state $T = 0$ initially, dissipation will generate nonzero temperature
- Close-to-equilibrium situations: dissipation is local
- Convenient to use general coordinates with metric

$$g_{\mu\nu}(x)$$

- Need approximate **local** equilibrium description with temperature $T(x)$ and fluid velocity $u^\mu(x)$, will appear in combination

$$\beta^\mu(x) = \frac{u^\mu(x)}{T(x)}$$

- **Global** thermal equilibrium corresponds to β^μ Killing vector

$$\nabla_\mu \beta_\nu(x) + \nabla_\nu \beta_\mu(x) = 0$$

Local equilibrium

- Use similarity between local density matrix and translation operator

$$e^{\beta^\mu(x) \mathcal{P}_\mu} \longleftrightarrow e^{i\Delta x^\mu \mathcal{P}_\mu}$$

to represent partition function as functional integral with periodicity in imaginary direction such that

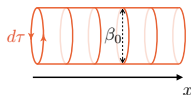
$$\phi(x^\mu - i\beta^\mu(x)) = \pm\phi(x^\mu)$$

- Partition function $Z[J]$, Schwinger functional $W[J]$ in Euclidean domain

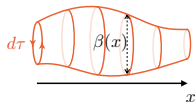
$$Z[J] = e^{W_E[J]} = \int D\phi e^{-S_E[\phi] + \int_x J\phi}$$

- First defined on **Euclidean manifold** $\Sigma \times M$ at constant time
- Approximate local equilibrium at all times: Hypersurface Σ can be shifted

(a) Global thermal equilibrium



(b) Local thermal equilibrium



Effective action

- Defined in euclidean domain by Legendre transform

$$\Gamma_E[\Phi] = \int_x J_a(x) \Phi_a(x) - W_E[J]$$

with expectation values

$$\Phi_a(x) = \frac{1}{\sqrt{g(x)}} \frac{\delta}{\delta J_a(x)} W_E[J]$$

- Euclidean field equation

$$\frac{\delta}{\delta \Phi_a(x)} \Gamma_E[\Phi] = \sqrt{g(x)} J_a(x)$$

resembles classical equation of motion for $J = 0$.

- Need analytic continuation to obtain a viable equation of motion

Two-point functions

- Consider homogeneous background fields and global equilibrium

$$\beta^\mu = \left(\frac{1}{T}, 0, 0, 0 \right)$$

- Propagator and inverse propagator

$$\frac{\delta^2}{\delta J_a(-p)\delta J_b(q)} W_E[J] = G_{ab}(i\omega_n, \mathbf{p}) \delta(p - q)$$

$$\frac{\delta^2}{\delta \Phi_a(-p)\delta \Phi_b(q)} \Gamma_E[\Phi] = P_{ab}(i\omega_n, \mathbf{p}) \delta(p - q)$$

- From definition of effective action

$$\sum_b G_{ab}(p) P_{bc}(p) = \delta_{ac}$$

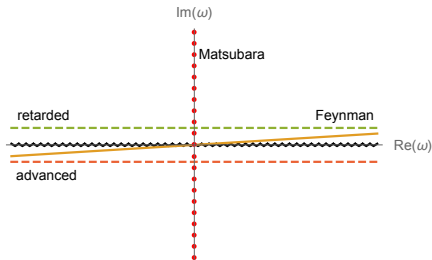
Spectral representation

- Källen-Lehmann spectral representation

$$G_{ab}(\omega, \mathbf{p}) = \int_{-\infty}^{\infty} dz \frac{\rho_{ab}(z^2 - \mathbf{p}^2, z)}{z - \omega}$$

with $\rho_{ab} \in \mathbb{R}$

- correlation functions can be analytically continued in $\omega = -u^\mu p_\mu$
- branch cut or poles on real frequency axis $\omega \in \mathbb{R}$ but nowhere else
- different propagators follow by evaluation of G_{ab} in different regions



$$\Delta_{ab}^M(p) = G_{ab}(i\omega_n, \mathbf{p})$$

$$\Delta_{ab}^R(p) = G_{ab}(p^0 + i\epsilon, \mathbf{p})$$

$$\Delta_{ab}^A(p) = G_{ab}(p^0 - i\epsilon, \mathbf{p})$$

$$\Delta_{ab}^F(p) = G_{ab}(p^0 + i\epsilon \text{ sign}(p^0), \mathbf{p})$$

Inverse propagator

- spectral representation for G_{ab} implies that *inverse propagator* $P_{ab}(\omega, \mathbf{p})$
 - can have zero-crossings for $\omega = p^0 \in \mathbb{R}$
 - has in general branch-cut for $\omega = p^0 \in \mathbb{R}$
- so far reference frame with $u^\mu = (1, 0, 0, 0)$
- more general: analytic continuation with respect to

$$\omega = -u^\mu p_\mu$$

- use **decomposition**

$$P_{ab}(p) = P_{1,ab}(p) - i s_1(-u^\mu p_\mu) P_{2,ab}(p)$$

with **sign function**

$$s_1(\omega) = \text{sign}(\text{Im } \omega)$$

- both functions $P_{1,ab}(p)$ and $P_{2,ab}(p)$ are regular (no discontinuities)

Sign operator in position space

- In position space, **sign function** becomes **operator**

$$\begin{aligned} s_I(-u^\mu p_\mu) &= \text{sign}(\text{Im}(-u^\mu p_\mu)) \\ &\rightarrow \text{sign}\left(\text{Im}\left(iu^\mu \frac{\partial}{\partial x^\mu}\right)\right) = \text{sign}\left(\text{Re}\left(u^\mu \frac{\partial}{\partial x^\mu}\right)\right) = s_R\left(u^\mu \frac{\partial}{\partial x^\mu}\right) \end{aligned}$$

- Geometric representation in terms of Lie derivative

$$s_R(\mathcal{L}_u) \quad \text{or} \quad s_R(\mathcal{L}_\beta)$$

- **Sign operator** appears also in analytically continued quantum effective action $\Gamma[\Phi]$

Analytically continued 1 PI effective action

- Analytically continued quantum effective action defined by analytic continuation of correlation functions
- Quadratic part

$$\Gamma_2[\Phi] = \frac{1}{2} \int_{x,y} \Phi_a(x) \left[P_{1,ab}(x-y) + P_{2,ab}(x-y) s_R \left(u^\mu \frac{\partial}{\partial y^\mu} \right) \right] \Phi_b(y)$$

- Higher orders correlation functions less understood: no spectral representation
- Use inverse Hubbard-Stratonovich trick: terms quadratic in auxiliary field can be integrated out
- Allows to understand analytic structures of higher order terms
[Floerchinger, 1603.07148]

Equations of motion

- Can one obtain **causal** and **real** renormalized equations of motion from the 1 PI effective action?
- naively: time-ordered action / Feynman $i\epsilon$ prescription:

$$\frac{\delta}{\delta\Phi_a(x)} \Gamma_{\text{time ordered}}[\Phi] = \sqrt{g} J_a(x)$$

- This does not lead to causal and real equations of motion !
[e.g. Calzetta & Hu: *Non-equilibrium Quantum Field Theory* (2008)]

Retarded functional derivative

[Floerchinger, 1603.07148]

- **Real** and **causal dissipative field equations** follow from analytically continued effective action

$$\left. \frac{\delta\Gamma[\Phi]}{\delta\Phi_a(x)} \right|_{\text{ret}} = \sqrt{g}J(x)$$

- to calculate retarded variational derivative determine

$$\delta\Gamma[\Phi]$$

by varying the fields $\delta\Phi(x)$ including dissipative terms

- set signs according to

$$s_R(u^\mu \partial_\mu) \delta\Phi(x) \rightarrow -\delta\Phi(x), \quad \delta\Phi(x) s_R(u^\mu \partial_\mu) \rightarrow +\delta\Phi(x)$$

- proceed as usual
- opposite choice of sign: field equations for backward time evolution

Causality

- consider derivative of field equation (in flat space with $\sqrt{g} = 1$)

$$\left. \frac{\delta}{\delta\Phi_b(y)} \frac{\delta\Gamma}{\delta\Phi_a(x)} \right|_{\text{ret}} = \frac{\delta}{\delta\Phi_b(y)} J_a(x)$$

- inverting this equation gives retarded Green's function

$$\frac{\delta}{\delta J_b(y)} \Phi_a(x) = \Delta_{ab}^R(x, y)$$

- only non-zero for x future or null to y
- **Causality**: Field expectation value $\Phi_a(x)$ can only be influenced by the source $J_b(y)$ in or on the past light cone ✓

Damped harmonic oscillator 1

- Equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0$$

or

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x = 0$$

with $\omega_0 = \sqrt{k/m}$ and $\zeta = c/\sqrt{4mk}$

- What is action for damped oscillator? This does *not* work:

$$\int \frac{d\omega}{2\pi} \frac{m}{2} x^*(\omega) [\omega^2 + 2i\omega\zeta\omega_0 - \omega_0^2] x(\omega)$$

- Consider inverse propagator

$$\omega^2 + 2i s_1(\omega) \omega \zeta \omega_0 - \omega_0^2$$

with

$$s_1(\omega) = \text{sign}(\text{Im } \omega)$$

zero crossings (poles in the eff. propagator) are broadened to branch cut

Damped harmonic oscillator 2

- Take for effective action

$$\begin{aligned}\Gamma[x] &= \int \frac{d\omega}{2\pi} \frac{m}{2} x^*(\omega) [-\omega^2 - 2i s_1(\omega) \omega \zeta \omega_0 + \omega_0^2] x(\omega) \\ &= \int dt \left\{ -\frac{1}{2} m \dot{x}^2 + \frac{1}{2} c x s_R(\partial_t) \dot{x} + \frac{1}{2} k x^2 \right\}\end{aligned}$$

where the second line uses

$$s_1(\omega) = \text{sign}(\text{Im } \omega) \rightarrow \text{sign}(\text{Im } i\partial_t) = \text{sign}(\text{Re } \partial_t) = s_R(\partial_t)$$

- Variation gives up to boundary terms

$$\delta\Gamma = \int dt \left\{ m \ddot{x} \delta x + \frac{1}{2} c \delta x s_R(\partial_t) \dot{x} - \frac{1}{2} c \dot{x} s_R(\partial_t) \delta x + k x \delta x \right\}$$

Set now $s_R(\partial_t) \delta x \rightarrow -\delta x$ and $\delta x s_R(\partial_t) \rightarrow \delta x$. Defines $\frac{\delta\Gamma}{\delta x} \Big|_{\text{ret}}$.

- Equation of motion for forward time evolution

$$\frac{\delta\Gamma}{\delta x} \Big|_{\text{ret}} = m \ddot{x} + c \dot{x} + k x = 0$$

Scalar field with $O(N)$ symmetry

- Consider effective action (with $\rho = \frac{1}{2}\varphi_j\varphi_j$)

$$\Gamma[\varphi, g_{\mu\nu}, \beta^\mu] = \int d^d x \sqrt{g} \left\{ \frac{1}{2} Z(\rho, T) g^{\mu\nu} \partial_\mu \varphi_j \partial_\nu \varphi_j + U(\rho, T) + \frac{1}{2} C(\rho, T) [\varphi_j, s_R(u^\mu \partial_\mu)] \beta^\nu \partial_\nu \varphi_j \right\}$$

- Variation at fixed metric $g_{\mu\nu}$ and β^μ gives

$$\delta\Gamma = \int d^d x \sqrt{g} \left\{ Z(\rho, T) g^{\mu\nu} \partial_\mu \delta\varphi_j \partial_\nu \varphi_j + \frac{1}{2} Z'(\rho, T) \varphi_m \delta\varphi_m g^{\mu\nu} \partial_\mu \varphi_j \partial_\nu \varphi_j + U'(\rho, T) \varphi_m \delta\varphi_m + \frac{1}{2} C(\rho, T) [\delta\varphi_j, s_R(u^\mu \partial_\mu)] \beta^\nu \partial_\nu \varphi_j + \frac{1}{2} C(\rho, T) [\varphi_j, s_R(u^\mu \partial_\mu)] \beta^\nu \partial_\nu \delta\varphi_j + \frac{1}{2} C'(\rho, T) \varphi_m \delta\varphi_m [\varphi_j, s_R(u^\mu \partial_\mu)] \beta^\nu \partial_\nu \varphi_j \right\}$$

- set now $\delta\varphi_j s_R(u^\mu \partial_\mu) \rightarrow \delta\varphi_j$ and $s_R(u^\mu \partial_\mu) \delta\varphi_j \rightarrow -\delta\varphi_j$

Scalar field with $O(N)$ symmetry

- Field equation becomes

$$-\nabla_{\mu} [Z(\rho, T)\partial^{\mu}\varphi_j] + \frac{1}{2}Z'(\rho, T)\varphi_j\partial_{\mu}\varphi_m\partial^{\mu}\varphi_m \\ + U'(\rho, T)\varphi_j + C(\rho, T)\beta^{\mu}\partial_{\mu}\varphi_j = 0$$

- Generalized Klein-Gordon equation with additional damping term

Where do energy & momentum go?

- Modified variational principle leads to equations of motion with dissipation.
- But what happens to the dissipated energy and momentum?
- And other conserved quantum numbers?
- What about entropy production?

Energy-momentum tensor expectation value

- Analogous to field equation, obtain by retarded variation

$$\left. \frac{\delta \Gamma[\Phi, g_{\mu\nu}, \beta^\mu]}{\delta g_{\mu\nu}(x)} \right|_{\text{ret}} = -\frac{1}{2} \sqrt{g} \langle T^{\mu\nu}(x) \rangle$$

- Leads to Einstein's field equation when $\Gamma[\Phi, g_{\mu\nu}, \beta^\mu]$ contains Einstein-Hilbert term
- Useful to decompose

$$\Gamma[\Phi, g_{\mu\nu}, \beta^\mu] = \Gamma_R[\Phi, g_{\mu\nu}, \beta^\mu] + \Gamma_D[\Phi, g_{\mu\nu}, \beta^\mu]$$

where reduced action Γ_R contains no dissipative / discontinuous terms and Γ_D only dissipative terms

- Energy-momentum tensor has two parts

$$\langle T^{\mu\nu} \rangle = (\bar{T}_R)^{\mu\nu} + (\bar{T}_D)^{\mu\nu}$$

General covariance

- Infinitesimal general coordinate transformations as a “gauge transformation” of the metric

$$\delta g_{\mu\nu}^G(x) = g_{\mu\lambda}(x) \frac{\partial \epsilon^\lambda(x)}{\partial x^\nu} + g_{\nu\lambda}(x) \frac{\partial \epsilon^\lambda(x)}{\partial x^\mu} + \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda} \epsilon^\lambda(x)$$

- Temperature / fluid velocity field transforms as vector

$$\delta \beta_G^\mu(x) = -\beta^\nu(x) \frac{\partial \epsilon^\mu(x)}{\partial x^\nu} + \frac{\partial \beta^\mu(x)}{\partial x^\nu} \epsilon^\nu(x)$$

- Also fields Φ_a transform in some representation, e. g. as scalars

$$\delta \Phi_a^G(x) = \epsilon^\lambda(x) \frac{\partial}{\partial x^\lambda} \Phi_a(x)$$

- Reduced action is invariant

$$\Gamma_R[\Phi + \delta \Phi^G, g_{\mu\nu} + \delta g_{\mu\nu}^G, \beta^\mu + \beta_G^\mu] = \Gamma_R[\Phi, g_{\mu\nu}, \beta^\mu]$$

Situation without dissipation

- Consider first situation **without dissipation** $\Gamma[\Phi, g_{\mu\nu}, \beta^\mu] = \Gamma_R[\Phi, g_{\mu\nu}]$
- Field equation implies (for $J = 0$)

$$\frac{\delta}{\delta\Phi_a(x)} \Gamma_R[\Phi, g_{\mu\nu}] = 0$$

- Gauge variation of the metric

$$\delta\Gamma_R = \int d^d x \sqrt{g} \epsilon^\lambda(x) \nabla_\mu \langle T^\mu{}_\lambda(x) \rangle$$

- General covariance $\delta\Gamma_R = 0$ and field equations imply covariant energy-momentum conservation

$$\nabla_\mu \langle T^\mu{}_\lambda(x) \rangle = 0$$

Situation with dissipation

- Consider now situation **with dissipation**. General covariance of Γ_R :

$$\delta\Gamma_R = \int d^d x \left\{ \frac{\delta\Gamma_R}{\delta\Phi_a} \delta\Phi_a^G + \sqrt{g} \epsilon^\lambda \nabla_\mu (\bar{T}_R)^\mu{}_\lambda + \frac{\delta\Gamma_R}{\delta\beta^\mu} \delta\beta^\mu \right\} = 0$$

- Reduced action **not stationary** with respect to field variations

$$\frac{\delta\Gamma_R}{\delta\Phi_a(x)} = - \left. \frac{\delta\Gamma_D}{\delta\Phi_a(x)} \right|_{\text{ret}} =: -\sqrt{g}(x) M_a(x)$$

- Reduced energy-momentum tensor **not conserved**

$$\nabla_\mu (\bar{T}_R)^\mu{}_\lambda(x) = -\nabla_\mu (\bar{T}_D)^\mu{}_\lambda(x)$$

- Dependence on $\beta^\mu(x)$ **cannot be dropped**

$$\frac{\delta\Gamma_R}{\delta\beta^\mu(x)} =: \sqrt{g}(x) K_\mu(x)$$

- General covariance implies **four additional differential equations** that determine β^μ

$$M_a \partial_\lambda \Phi_a + \nabla_\mu (\bar{T}_D)^\mu{}_\lambda = \nabla_\mu [\beta^\mu K_\lambda] + K_\mu \nabla_\lambda \beta^\mu$$

Entropy production

- Contraction of previous equation with β^λ gives

$$M_a \beta^\lambda \partial_\lambda \Phi_a + \beta^\lambda \nabla_\mu (\bar{T}_D)^\mu{}_\lambda = \nabla_\mu [\beta^\mu \beta^\lambda K_\lambda]$$

- Consider special case

$$\sqrt{g} K_\mu(x) = \frac{\delta \Gamma_R}{\delta \beta^\mu(x)} = \frac{\delta}{\delta \beta^\mu(x)} \int d^d x \sqrt{g} U(T)$$

with grand canonical potential density $U(T) = -p(T)$ and temperature

$$T = \frac{1}{\sqrt{-g_{\mu\nu} \beta^\mu \beta^\nu}}$$

- Using $s = \partial p / \partial T$ gives entropy current

$$\beta^\mu \beta^\lambda K_\lambda = s^\mu = s u^\mu$$

- Local form of **second law of thermodynamics**

$$\nabla_\mu s^\mu = M_a \beta^\lambda \partial_\lambda \Phi_a + \beta^\lambda \nabla_\mu (\bar{T}_D)^\mu{}_\lambda \geq 0$$

Energy-momentum tensor for scalar field

- Analytic action

$$\Gamma[\varphi, g_{\mu\nu}, \beta^\mu] = \int d^d x \sqrt{g} \left\{ \frac{1}{2} Z(\rho, T) g^{\mu\nu} \partial_\mu \varphi_j \partial_\nu \varphi_j + U(\rho, T) \right. \\ \left. + \frac{1}{2} C(\rho, T) [\varphi_j, s_R(u^\mu \partial_\mu)] \beta^\nu \partial_\nu \varphi_j \right\}$$

- Energy-momentum tensor

$$\langle T^{\mu\nu}(x) \rangle = Z(\rho, T) \partial^\mu \varphi_j \partial^\nu \varphi_j \\ - \left(g^{\mu\nu} + u^\mu u^\nu T \frac{\partial}{\partial T} \right) \left\{ \frac{1}{2} Z(\rho, T) g^{\mu\nu} \partial_\mu \varphi_j \partial_\nu \varphi_j + U(\rho, T) \right\}$$

- Generalizes $T^{\mu\nu}$ for scalar field and $T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + g^{\mu\nu} p$ for ideal fluid with pressure $p = -U$ and enthalpy density $\epsilon + p = sT = -T \frac{\partial}{\partial T} U$.
- General covariance and covariant conservation law imply

$$\nabla_\mu \langle T^{\mu\nu}(x) \rangle = 0 \quad \implies \quad \text{Differential eqs. for } \beta^\mu(x)$$

Entropy production for scalar field

- Entropy current

$$s^\mu = \beta^\mu \beta^\lambda K_\lambda = -\beta^\mu T \frac{\partial}{\partial T} \left\{ \frac{1}{2} Z(\rho, T) g^{\alpha\beta} \partial_\alpha \varphi_j \partial_\beta \varphi_j + U(\rho, T) \right\}$$

- Generalized entropy density

$$s_G = -\frac{\partial}{\partial T} \left\{ \frac{1}{2} Z(\rho, T) g^{\alpha\beta} \partial_\alpha \varphi_j \partial_\beta \varphi_j + U(\rho, T) \right\}$$

- Entropy generation positive semi-definite for $C(\rho, T) \geq 0$

$$\nabla_\mu s^\mu = C(\rho, T) (\beta^\mu \partial_\mu \varphi_j) (\beta^\nu \partial_\nu \varphi_j) \geq 0$$

- For fluid at rest $u^\mu = (1, 0, 0, 0)$

$$\nabla_\mu s^\mu = \dot{s}_G = \frac{C(\rho, T)}{T^2} \dot{\varphi}_j \dot{\varphi}_j$$

entropy increases when φ_j oscillates. For example reheating after inflation.

Ideal fluid

- Consider effective action

$$\Gamma[g_{\mu\nu}, \beta^\mu] = \Gamma_R[g_{\mu\nu}, \beta^\mu] = \int d^d x \sqrt{g} U(T)$$

with effective potential $U(T) = -p(T)$ and temperature

$$T = \frac{1}{\sqrt{-g_{\mu\nu} \beta^\mu \beta^\nu}}$$

- Variation of $g_{\mu\nu}$ at fixed β^μ leads to

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu}$$

where $\epsilon + p = Ts = T \frac{\partial}{\partial T} p$ is the enthalpy density

- Describes ideal fluid. General covariance of covariant conservation $\nabla_\mu T^{\mu\nu} = 0$ leads to ideal fluid equations

$$u^\mu \partial_\mu \epsilon + (\epsilon + p) \nabla_\mu u^\mu = 0 \qquad (\epsilon + p) u^\mu \nabla_\mu u^\nu + \Delta^{\nu\mu} \partial_\mu p = 0$$

Viscous fluid

- Analytic action

$$\Gamma[g_{\mu\nu}, \beta^\mu] = \int_x \left\{ U(T) + \frac{1}{4} [g_{\mu\nu}, s_R(\mathcal{L}_u)] (2\eta(T)\sigma^{\mu\nu} + \zeta(T)\Delta^{\mu\nu}\nabla_\rho u^\rho) \right\}$$

with projector

$$\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$$

and

$$\sigma^{\mu\nu} = \left(\frac{1}{2} \Delta^{\mu\alpha} \Delta^{\mu\beta} + \frac{1}{2} \Delta^{\mu\beta} \Delta^{\mu\alpha} - \frac{1}{d-1} \Delta^{\mu\nu} \Delta^{\alpha\beta} \right) \nabla_\alpha u_\beta$$

leads to

$$\langle T^{\mu\nu} \rangle = -\frac{2}{\sqrt{g}} \frac{\delta\Gamma[g_{\mu\nu}, \beta^\mu]}{\delta g_{\mu\nu}} \Big|_{\text{ret}} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu} - 2\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\nabla_\rho u^\rho$$

- Describes viscous fluid with shear viscosity $\eta(T)$ and bulk viscosity $\zeta(T)$
- Entropy production

$$\nabla_\mu s^\mu = \frac{1}{T} [2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta(\nabla_\rho u^\rho)^2]$$

Conclusions

- A variational principle for theories with dissipation can be based on analytic continuation.
- Needs a local equilibrium setup: Generalized Gibbs ensemble with $T(x)$ and $u^\mu(x)$.
- Works at least for close-to-equilibrium situations, e. g. fluid dynamics coupled to additional fields.
- General covariance and energy-momentum conservation lead to equations for fluid velocity and entropy production.
- Local form of second law of thermodynamics is implemented on the level of the effective action $\Gamma[\Phi]$.
- Many potential applications.

BACKUP

Equations of motion from the Feynman action ?

- Consider damped harmonic oscillator as example. Time-ordered or Feynman action is obtained from analytic action by replacing $s_1(\omega) \rightarrow \text{sign}(\omega)$

$$\Gamma_{\text{time ordered}}[x] = \int \frac{d\omega}{2\pi} \frac{m}{2} x^*(\omega) [-\omega^2 - 2i|\omega| \zeta\omega_0 + \omega_0^2] x(\omega)$$

- Field equation $\frac{\delta}{\delta x(t)} \Gamma_{\text{time ordered}}[x] = J(t)$ would give

$$[-\omega^2 - 2i|\omega| \zeta\omega_0 + \omega_0^2] x(\omega) = J(\omega)$$

- Violates reality constraint $x^*(\omega) = x(-\omega)$ for $J^*(\omega) = J(-\omega)$
- Solution not causal

$$x(t) = \int_{t'} \Delta_F(t - t') J(t')$$

because Feynman propagator $\Delta_F(t - t')$ not causal.

- In contrast, retarded variation of analytic action leads to real and causal equation of motion

Tree-like structures

- Discontinuous terms in analytic action could be of the form

$$\Gamma_{\text{Disc}}[\Phi] = \int d^d x \sqrt{g} \left\{ f[\Phi](x) s_R(u^\mu(x) \frac{\partial}{\partial x^\mu}) g[\Phi](x) \right\}$$

- More general, tree-like structure are possible such as

$$\Gamma_{\text{Disc}}[\Phi] = \int_{x,y} \left\{ f[\Phi](x) s_R(u^\mu(x) \frac{\partial}{\partial x^\mu}) g[\Phi](x,y) s_R(u^\mu(y) \frac{\partial}{\partial y^\mu}) h[\Phi](y) \right\}$$

or

$$\Gamma_{\text{Disc}}[\Phi] = \int_{x,y,z} \left\{ f[\Phi](x) s_R(u^\mu(x) \frac{\partial}{\partial x^\mu}) g[\Phi](x,y,z) s_R(u^\mu(y) \frac{\partial}{\partial y^\mu}) h[\Phi](y) \right. \\ \left. \times s_R(u^\mu(z) \frac{\partial}{\partial z^\mu}) j[\Phi](z) \right\}$$

- For retarded variation calculate $\delta\Gamma$ and set $s_R(u^\mu \partial_\mu) \rightarrow -1$ if derivative operator points towards node that is varied and $s_R(u^\mu \partial_\mu) \rightarrow 1$ if derivative operator points in opposite direction

Analytic continuation of FRG equations

[Floerchinger, JHEP 1205 (2012) 021]

- Consider a point $p_0^2 - \vec{p}^2 = m^2$ where $P_1(m^2) = 0$.
- One can expand around this point

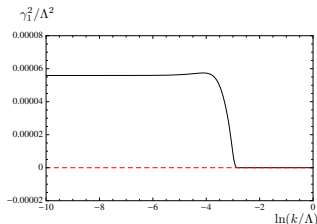
$$P_1 = Z(-p_0^2 + \vec{p}^2 + m^2) + \dots$$

$$P_2 = Z\gamma^2 + \dots$$

- Leads to Breit-Wigner form of propagator (with $\gamma^2 = m\Gamma$)

$$G(p) = \frac{1}{Z} \frac{-p_0^2 + \vec{p}^2 + m^2 + i s(p_0) m\Gamma}{(-p_0^2 + \vec{p}^2 + m^2)^2 + m^2\Gamma^2}.$$

- A few flowing parameters describe efficiently the singular structure of the propagator.



Truncation for relativistic scalar $O(N)$ theory

$$\Gamma_k = \int_{t, \vec{x}} \left\{ \sum_{j=1}^N \frac{1}{2} \bar{\phi}_j \bar{P}_\phi(i\partial_t, -i\vec{\nabla}) \bar{\phi}_j + \frac{1}{4} \bar{\rho} \bar{P}_\rho(i\partial_t, -i\vec{\nabla}) \bar{\rho} + \bar{U}_k(\bar{\rho}) \right\}$$

with $\bar{\rho} = \frac{1}{2} \sum_{j=1}^N \bar{\phi}_j^2$.

- Goldstone propagator massless, expanded around $p_0 - \vec{p}^2 = 0$

$$\bar{P}_\phi(p_0, \vec{p}) \approx \bar{Z}_\phi (-p_0^2 + \vec{p}^2)$$

- Radial mode is massive, expanded around $p_0^2 - \vec{p}^2 = m_1^2$

$$\begin{aligned} \bar{P}_\phi(p_0, \vec{p}) + \bar{\rho}_0 \bar{P}_\rho(p_0, \vec{p}) + \bar{U}'_k + 2\bar{\rho} \bar{U}''_k \\ \approx \bar{Z}_\phi Z_1 \left[(-p_0^2 + \vec{p}^2 + m_1^2) - is(p_0) \gamma_1^2 \right] \end{aligned}$$

Flow of the effective potential

$$\partial_t U_k(\rho) \Big|_{\bar{p}} = \frac{1}{2} \int_{p_0=i\omega_n, \bar{p}} \left\{ \frac{(N-1)}{\bar{p}^2 - p_0^2 + U' + \frac{1}{Z_\phi} R_k} + \frac{1}{Z_1 [(\bar{p}^2 - p_0^2) - i s(p_0) \gamma_1^2] + U' + 2\rho U'' + \frac{1}{Z_\phi} R_k} \right\} \frac{1}{Z_\phi} \partial_t R_k.$$

- Summation over Matsubara frequencies $p_0 = i2\pi T n$ can be done using contour integrals.
- Radial mode has non-zero decay width since it can decay into Goldstone excitations.
- Use Taylor expansion for numerical calculations

$$U_k(\rho) = U_k(\rho_{0,k}) + m_k^2(\rho - \rho_{0,k}) + \frac{1}{2} \lambda_k(\rho - \rho_{0,k})^2$$