Variational principle for theories with dissipation from analytic continuation

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MITP Workshop *Relativistic Hydrodynamics: Theory and Modern Applications*, Mainz, October 12, 2016.

[based on JHEP09 (2016) 099]

Dissipation in quantum field theory

- Dissipation is **generation** of entropy
- Unitary evolution **conserves** entropy

- in practice often only incomplete information available
- expectation values of fundamental quantum fields and some composite operators
- quantum states with minimal information given some constraints
- use truncation of 1 PI effective action

Dissipation from integrating out fields

Example 1

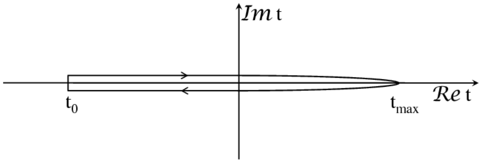
- consider muon with decay $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$
- full electroweak theory is unitary
- consider now effective action where fields for e^-, $\bar{\nu}_e$ and ν_μ have been integrated out
- effective action for μ^- contains decay width: appears as dissipative term

Example 2

- consider electromagnetic field A_μ
- field strength above Schwinger threshold: electron-positron pair production
- electron / positron field can be integrated out: dissipative term for electromagnetic field

Double time path formalism

- formalism for general, far-from-equilibrium situations: Schwinger-Keldysh double time path
- can be formulated with two fields $\Phi = \frac{1}{2}(\phi_+ + \phi_-)$, $\chi = \phi_+ - \phi_-$
- in principle for arbitrary initial density matrices, in praxis mainly Gaussian initial states
- allows to treat also dissipation
- useful also to treat initial state fluctuations or forced noise in classical statistical theories
- difficult to recover thermal equilibrium, in particular non-perturbatively
- formalism algebraically somewhat involved



Close-to-equilibrium situations

- Out-of-equilibrium situations
- Close-to-equilibrium: description by field expectation values and thermodynamic fields
- more complete description achieved by following more fields explicitly
- example: Viscous fluid dynamics plus additional fields
- usually discussed in terms of
  - phenomenological constitutive relations
  - as a limit of kinetic theory
  - in AdS/CFT
- want non-perturbative formulation in terms of QFT concepts
- Analytic continuation as an alternative to Schwinger-Keldysh
- direct generalization of equilibrium formalism

Local equilibrium states

- Dissipation: energy and momentum get transferred to a heat bath
- Even if one starts with pure state $T = 0$ initially, dissipation will generate nonzero temperature
- Close-to-equilibrium situations: dissipation is local
- Convenient to use general coordinates with metric

$$g_{\mu\nu}(x)$$

- Need approximate **local** equilibrium description with temperature $T(x)$ and fluid velocity $u^\mu(x)$, will appear in combination

$$\beta^\mu(x) = \frac{u^\mu(x)}{T(x)}$$

- **Global** thermal equilibrium corresponds to β^μ Killing vector

$$\nabla_\mu \beta_\nu(x) + \nabla_\nu \beta_\mu(x) = 0$$

Local equilibrium

- Use similarity between local density matrix and translation operator

$$e^{\beta^\mu(x)\mathscr{P}_\mu} \qquad \longleftrightarrow \qquad e^{i\Delta x^\mu \mathscr{P}_\mu}$$

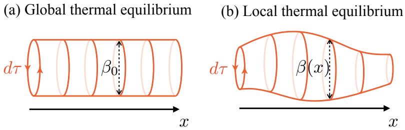
to represent partition function as functional integral with periodicity in imaginary direction such that

$$\phi(x^\mu - i\beta^\mu(x)) = \pm\phi(x^\mu)$$

- Partition function $Z[J]$, Schwinger functional $W[J]$ in Euclidean domain

$$Z[J] = e^{W_E[J]} = \int D\phi\, e^{-S_E[\phi]+\int_x J\phi}$$

- First defined on **Euclidean signature manifold** $\Sigma \times M$ at constant time
- Approximate local equilibrium at all times: Hypersurface Σ can be shifted



Quantum effective action (1 PI effective action)

- Defined in euclidean domain by Legendre transform

$$\Gamma_E[\Phi] = \int_x J_a(x)\Phi_a(x) - W_E[J]$$

with expectation values

$$\Phi_a(x) = \frac{1}{\sqrt{g}(x)} \frac{\delta}{\delta J_a(x)} W_E[J]$$

- Euclidean field equation

$$\frac{\delta}{\delta \Phi_a(x)} \Gamma_E[\Phi] = \sqrt{g}(x)\, J_a(x)$$

resembles classical equation of motion for $J = 0$.

- Need analytic continuation to obtain a viable equation of motion

Two-point functions

- Consider homogeneous background fields and global equilibrium

$$\beta^\mu = \left(\frac{1}{T}, 0, 0, 0\right)$$

- Propagator and inverse propagator

$$\frac{\delta^2}{\delta J_a(-p)\delta J_b(q)} W_E[J] = G_{ab}(i\omega_n, \mathbf{p})\ \delta(p-q)$$
$$\frac{\delta^2}{\delta \Phi_a(-p)\delta \Phi_b(q)} \Gamma_E[\Phi] = P_{ab}(i\omega_n, \mathbf{p})\ \delta(p-q)$$

- From definition of effective action

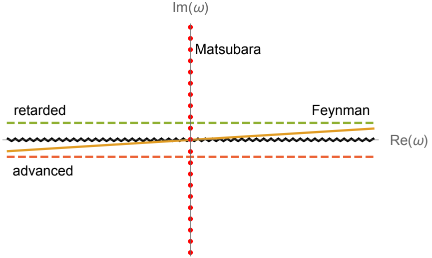
$$\sum_b G_{ab}(p) P_{bc}(p) = \delta_{ac}$$

Spectral representation

- Källen-Lehmann spectral representation

$$G_{ab}\left(\omega,\mathbf{p}\right)=\int_{-\infty}^{\infty}dz\;\frac{\rho_{ab}(z^2-\mathbf{p}^2,z)}{z-\omega}$$

  with $\rho_{ab}\in\mathbb{R}$
- correlation functions can be analytically continued in $\omega=-u^\mu p_\mu$
- branch cut or poles on real frequency axis $\omega\in\mathbb{R}$ but nowhere else
- different propagators follow by evaluation of G_{ab} in different regions



$$\Delta_{ab}^M(p)=G_{ab}\left(i\omega_n,\mathbf{p}\right)$$
$$\Delta_{ab}^R(p)=G_{ab}\left(p^0+i\epsilon,\mathbf{p}\right)$$
$$\Delta_{ab}^A(p)=G_{ab}\left(p^0-i\epsilon,\mathbf{p}\right)$$
$$\Delta_{ab}^F(p)=G_{ab}\left(p^0+i\epsilon\,\text{sign}\left(p^0\right),\mathbf{p}\right)$$

Inverse propagator

- spectral representation for G_{ab} implies that *inverse* propagator $P_{ab}(\omega, \mathbf{p})$
  - can have zero-crossings for $\omega = p^0 \in \mathbb{R}$
  - has in general branch-cut for $\omega = p^0 \in \mathbb{R}$
- so far reference frame with $u^\mu = (1, 0, 0, 0)$
- more general: analytic continuation with respect to

$$\omega = -u^\mu p_\mu$$

- use **decomposition**

$$P_{ab}(p) = P_{1,ab}(p) - i s_\mathsf{I}(-u^\mu p_\mu)\, P_{2,ab}(p)$$

with **sign function**

$$s_\mathsf{I}(\omega) = \mathsf{sign}(\mathsf{Im}\ \omega)$$

- both functions $P_{1,ab}(p)$ and $P_{2,ab}(p)$ are regular (no discontinuities)

Sign operator in position space

- In position space, **sign function** becomes **operator**

$$s_{\mathsf{I}}\left(-u^{\mu}p_{\mu}\right) = \mathsf{sign}\left(\mathsf{Im}(-u^{\mu}p_{\mu})\right)$$
$$\rightarrow \mathsf{sign}\left(\mathsf{Im}\left(iu^{\mu}\frac{\partial}{\partial x^{\mu}}\right)\right) = \mathsf{sign}\left(\mathsf{Re}\left(u^{\mu}\frac{\partial}{\partial x^{\mu}}\right)\right) = s_{\mathsf{R}}\left(u^{\mu}\frac{\partial}{\partial x^{\mu}}\right)$$

- Geometric representation in terms of Lie derivative

$$s_{\mathsf{R}}(\mathcal{L}_u) \qquad \text{or} \qquad s_{\mathsf{R}}(\mathcal{L}_\beta)$$

- **Sign operator** appears also in analytically continued quantum effective action $\Gamma[\Phi]$

Analytically continued 1 PI effective action

- Analytically continued quantum effective action defined by analytic continuation of correlation functions
- Quadratic part

$$\Gamma_2[\Phi] = \frac{1}{2}\int_{x,y} \Phi_a(x)\left[P_{1,ab}(x-y) + P_{2,ab}(x-y)s_\mathsf{R}\left(u^\mu \frac{\partial}{\partial y^\mu}\right)\right]\Phi_b(y)$$

- Higher orders correlation functions less understood: no spectral representation
- Use inverse Hubbard-Stratonovich trick: terms quadratic in auxiliary field can be integrated out
- Allows to understand analytic structures of higher order terms [Floerchinger, JHEP09 (2016) 099]

Equations of motion

- Can one obtain **causal** and **real** renormalized equations of motion from the 1 PI effective action?
- naively: time-ordered action / Feynman $i\epsilon$ prescription:

$$\frac{\delta}{\delta\Phi_a(x)}\Gamma_{\text{time ordered}}[\Phi] = \sqrt{g}\, J_a(x)$$

- This does not lead to causal and real equations of motion !
  [e.g. Calzetta & Hu: *Non-equilibrium Quantum Field Theory* (2008)]

Retarded functional derivative

[Floerchinger, JHEP09 (2016) 099]

- **Real** and **causal dissipative field equations** follow from analytically continued effective action

$$\left.\frac{\delta\Gamma[\Phi]}{\delta\Phi_a(x)}\right|_{\text{ret}} = \sqrt{g}J(x)$$

- to calculate retarded variational derivative determine

$$\delta\Gamma[\Phi]$$

  by varying the fields $\delta\Phi(x)$ including dissipative terms

- set signs according to

$$s_{\mathsf{R}}(u^\mu\partial_\mu)\;\delta\Phi(x) \to -\delta\Phi(x), \qquad\qquad \delta\Phi(x)\;s_{\mathsf{R}}(u^\mu\partial_\mu) \to +\delta\Phi(x)$$

- proceed as usual
- opposite choice of sign: field equations for backward time evolution

Causality

- consider derivative of field equation (in flat space with $\sqrt{g} = 1$)

$$\frac{\delta}{\delta \Phi_b(y)} \frac{\delta \Gamma}{\delta \Phi_a(x)} \bigg|_{\text{ret}} = \frac{\delta}{\delta \Phi_b(y)} J_a(x)$$

- inverting this equation gives retarded Green's function

$$\frac{\delta}{\delta J_b(y)} \Phi_a(x) = \Delta^R_{ab}(x, y)$$

- only non-zero for x future or null to y
- **Causality**: Field expectation value $\Phi_a(x)$ can only be influenced by the source $J_b(y)$ in or on the past light cone ✓

Damped harmonic oscillator 1

- Equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0$$

or

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0$$

with $\omega_0 = \sqrt{k/m}$ and $\zeta = c/\sqrt{4mk}$

- What is action for damped oscillator? This does *not* work:

$$\int \frac{d\omega}{2\pi}\, \frac{m}{2} x^*(\omega) \left[\omega^2 + 2i\,\omega\,\zeta\omega_0 - \omega_0^2\right] x(\omega)$$

- Consider inverse propagator

$$\omega^2 + 2i\, s_\mathsf{I}(\omega)\, \omega\, \zeta\omega_0 - \omega_0^2$$

with

$$s_\mathsf{I}(\omega) = \mathsf{sign}\left(\mathsf{Im}\,\omega\right)$$

zero crossings (poles in the eff. propagator) are broadened to branch cut

Damped harmonic oscillator 2

- Take for effective action

$$\Gamma[x] = \int \frac{d\omega}{2\pi}\ \frac{m}{2} x^*(\omega)\left[-\omega^2 - 2i\, s_{\mathsf{I}}(\omega)\,\omega\,\zeta\omega_0 + \omega_0^2\right] x(\omega)$$
$$= \int dt \left\{-\frac{1}{2} m\dot{x}^2 + \frac{1}{2} c\, x\, s_{\mathsf{R}}(\partial_t)\dot{x} + \frac{1}{2} k x^2\right\}$$

where the second line uses

$$s_{\mathsf{I}}(\omega) = \mathsf{sign}(\mathsf{Im}\,\omega) \to \mathsf{sign}(\mathsf{Im}\, i\partial_t) = \mathsf{sign}(\mathsf{Re}\,\partial_t) = s_{\mathsf{R}}(\partial_t)$$

- Variation gives up to boundary terms

$$\delta\Gamma = \int dt \left\{m\ddot{x}\,\delta x + \frac{1}{2} c\,\delta x\, s_{\mathsf{R}}(\partial_t)\dot{x} - \frac{1}{2} c\,\dot{x}\, s_{\mathsf{R}}(\partial_t)\delta x + kx\,\delta x\right\}$$

Set now $s_{\mathsf{R}}(\partial_t)\delta x \to -\delta x$ and $\delta x\, s_{\mathsf{R}}(\partial_t) \to \delta x$. Defines $\frac{\delta\Gamma}{\delta x}\big|_{\mathsf{ret}}$.

- Equation of motion for forward time evolution

$$\left.\frac{\delta\Gamma}{\delta x}\right|_{\mathsf{ret}} = m\ddot{x} + c\dot{x} + kx = 0$$

Scalar field with $O(N)$ symmetry

- Consider effective action (with $\rho = \frac{1}{2}\varphi_j\varphi_j$)

$$\Gamma[\varphi, g_{\mu\nu}, \beta^\mu] = \int d^dx\sqrt{g}\Big\{\frac{1}{2}Z(\rho,T)g^{\mu\nu}\partial_\mu\varphi_j\partial_\nu\varphi_j + U(\rho,T)$$
$$+\frac{1}{2}C(\rho,T)\ [\varphi_j, s_{\mathsf{R}}(u^\mu\partial_\mu)]\ \beta^\nu\partial_\nu\varphi_j\Big\}$$

- Variation at fixed metric $g_{\mu\nu}$ and β^μ gives

$$\delta\Gamma = \int d^dx\sqrt{g}\Big\{Z(\rho,T)g^{\mu\nu}\partial_\mu\delta\varphi_j\partial_\nu\varphi_j + \frac{1}{2}Z'(\rho,T)\varphi_m\delta\varphi_m\ g^{\mu\nu}\partial_\mu\varphi_j\partial_\nu\varphi_j$$
$$+U'(\rho,T)\varphi_m\delta\varphi_m$$
$$+\frac{1}{2}C(\rho,T)\ [\delta\varphi_j, s_{\mathsf{R}}(u^\mu\partial_\mu)]\,\beta^\nu\partial_\nu\varphi_j$$
$$+\frac{1}{2}C(\rho,T)\ [\varphi_j, s_{\mathsf{R}}(u^\mu\partial_\mu)]\,\beta^\nu\partial_\nu\delta\varphi_j$$
$$+\frac{1}{2}C'(\rho,T)\varphi_m\delta\varphi_m\ [\varphi_j, s_{\mathsf{R}}(u^\mu\partial_\mu)]\,\beta^\nu\partial_\nu\varphi_j\Big\}$$

- set now $\delta\varphi_j\ s_{\mathsf{R}}(u^\mu\partial_\mu) \to \delta\varphi_j$ and $s_{\mathsf{R}}(u^\mu\partial_\mu)\,\delta\varphi_j \to -\delta\varphi_j$

Scalar field with $O(N)$ symmetry

- Field equation becomes

$$-\nabla_\mu \left[Z(\rho,T)\partial^\mu \varphi_j\right] + \frac{1}{2}Z'(\rho,T)\varphi_j\partial_\mu\varphi_m\partial^\mu\varphi_m$$
$$+U'(\rho,T)\varphi_j + C(\rho,T)\beta^\mu\partial_\mu\varphi_j = 0$$

- Generalized Klein-Gordon equation with additional damping term

Where do energy & momentum go?

- Modified variational principle leads to equations of motion with dissipation.
- But what happens to the dissipated energy and momentum?
- And other conserved quantum numbers?
- What about entropy production?

Energy-momentum tensor expectation value

- Analogous to field equation, obtain by retarded variation

$$\left.\frac{\delta\Gamma[\Phi, g_{\mu\nu}, \beta^\mu]}{\delta g_{\mu\nu}(x)}\right|_{\mathsf{ret}} = -\frac{1}{2}\sqrt{g}\,\langle T^{\mu\nu}(x)\rangle$$

- Leads to Einstein's field equation when $\Gamma[\Phi, g_{\mu\nu}, \beta^\mu]$ contains Einstein-Hilbert term
- Useful to decompose

$$\Gamma[\Phi, g_{\mu\nu}, \beta^\mu] = \Gamma_R[\Phi, g_{\mu\nu}, \beta^\mu] + \Gamma_D[\Phi, g_{\mu\nu}, \beta^\mu]$$

  where reduced action Γ_R contains no dissipative / discontinuous terms and Γ_D only dissipative terms
- Energy-momentum tensor has two parts

$$\langle T^{\mu\nu}\rangle = (\bar{T}_R)^{\mu\nu} + (\bar{T}_D)^{\mu\nu}$$

General covariance

- Infinitesimal general coordinate transformations as a "gauge transformation" of the metric

$$\delta g^G_{\mu\nu}(x) = g_{\mu\lambda}(x)\frac{\partial\epsilon^\lambda(x)}{\partial x^\nu} + g_{\nu\lambda}(x)\frac{\partial\epsilon^\lambda(x)}{\partial x^\mu} + \frac{\partial g_{\mu\nu}(x)}{\partial x^\lambda}\epsilon^\lambda(x)$$

- Temperature / fluid velocity field transforms as vector

$$\delta\beta^\mu_G(x) = -\beta^\nu(x)\frac{\partial\epsilon^\mu(x)}{\partial x^\nu} + \frac{\partial\beta^\mu(x)}{\partial x^\nu}\epsilon^\nu(x)$$

- Also fields Φ_a transform in some representation, e. g. as scalars

$$\delta\Phi^G_a(x) = \epsilon^\lambda(x)\frac{\partial}{\partial x^\lambda}\Phi_a(x)$$

- Reduced action is invariant

$$\Gamma_R[\Phi + \delta\Phi^G, g_{\mu\nu} + \delta g^G_{\mu\nu}, \beta^\mu + \beta^\mu_G] = \Gamma_R[\Phi, g_{\mu\nu}, \beta^\mu]$$

Situation without dissipation

- Consider first situation **without dissipation** $\Gamma[\Phi, g_{\mu\nu}, \beta^\mu] = \Gamma_R[\Phi, g_{\mu\nu}]$
- Field equation implies (for $J = 0$)

$$\frac{\delta}{\delta\Phi_a(x)}\Gamma_R[\Phi, g_{\mu\nu}] = 0$$

- Gauge variation of the metric

$$\delta\Gamma_R = \int d^dx\sqrt{g}\,\epsilon^\lambda(x)\nabla_\mu\langle T^\mu{}_\lambda(x)\rangle$$

- General covariance $\delta\Gamma_R = 0$ and field equations imply covariant energy-momentum conservation

$$\nabla_\mu\,\langle T^\mu{}_\lambda(x)\rangle = 0$$

Situation with dissipation

- Consider now situation **with dissipation**. General covariance of Γ_R:

$$\delta\Gamma_R = \int d^d x \left\{ \frac{\delta\Gamma_R}{\delta\Phi_a}\delta\Phi_a^G + \sqrt{g}\,\epsilon^\lambda \nabla_\mu (\bar{T}_R)^\mu{}_\lambda + \frac{\delta\Gamma_R}{\delta\beta^\mu}\delta\beta^\mu_G \right\} = 0$$

- Reduced action **not stationary** with respect to field variations

$$\frac{\delta\Gamma_R}{\delta\Phi_a(x)} = -\frac{\delta\Gamma_D}{\delta\Phi_a(x)}\bigg|_{\text{ret}} =: -\sqrt{g}(x)\,M_a(x)$$

- Reduced energy-momentum tensor **not conserved**

$$\nabla_\mu(\bar{T}_R)^\mu{}_\lambda(x) = -\nabla_\mu(\bar{T}_D)^\mu{}_\lambda(x)$$

- Dependence on $\beta^\mu(x)$ **cannot be dropped**

$$\frac{\delta\Gamma_R}{\delta\beta^\mu(x)} =: \sqrt{g}(x)\,K_\mu(x)$$

- General covariance implies **four additional differential equations** that determine β^μ

$$M_a \partial_\lambda \Phi_a + \nabla_\mu(\bar{T}_D)^\mu{}_\lambda = \nabla_\mu\left[\beta^\mu K_\lambda\right] + K_\mu \nabla_\lambda \beta^\mu$$

Entropy production

- Contraction of previous equation with β^λ gives

$$M_a \beta^\lambda \partial_\lambda \Phi_a + \beta^\lambda \nabla_\mu (\bar{T}_D)^\mu{}_\lambda = \nabla_\mu \left[\beta^\mu \beta^\lambda K_\lambda\right]$$

- Consider special case

$$\sqrt{g}\, K_\mu(x) = \frac{\delta \Gamma_R}{\delta \beta^\mu(x)} = \frac{\delta}{\delta \beta^\mu(x)} \int d^d x \sqrt{g}\, U(T)$$

with grand canonical potential density $U(T) = -p(T)$ and temperature

$$T = \frac{1}{\sqrt{-g_{\mu\nu}\beta^\mu \beta^\nu}}$$

- Using $s = \partial p / \partial T$ gives entropy current

$$\beta^\mu \beta^\lambda K_\lambda = s^\mu = s u^\mu$$

- Local form of **second law of thermodynamics**

$$\nabla_\mu s^\mu = M_a \beta^\lambda \partial_\lambda \Phi_a + \beta^\lambda \nabla_\mu (\bar{T}_D)^\mu{}_\lambda \geq 0$$

Energy-momentum tensor for scalar field

- Analytic action

$$\Gamma[\varphi, g_{\mu\nu}, \beta^\mu] = \int d^dx\sqrt{g}\Big\{\frac{1}{2}Z(\rho,T)g^{\mu\nu}\partial_\mu\varphi_j\partial_\nu\varphi_j + U(\rho,T) + \frac{1}{2}C(\rho,T)\,[\varphi_j, s_{\mathsf{R}}(u^\mu\partial_\mu)]\;\beta^\nu\partial_\nu\varphi_j\Big\}$$

- Energy-momentum tensor

$$\langle T^{\mu\nu}(x)\rangle = Z(\rho,T)\partial^\mu\varphi_j\partial^\nu\varphi_j - \left(g^{\mu\nu} + u^\mu u^\nu T\frac{\partial}{\partial T}\right)\left\{\frac{1}{2}Z(\rho,T)g^{\mu\nu}\partial_\mu\varphi_j\partial_\nu\varphi_j + U(\rho,T)\right\}$$

- Generalizes $T^{\mu\nu}$ for scalar field and $T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + g^{\mu\nu}p$ for ideal fluid with pressure $p = -U$ and enthalpy density $\epsilon + p = sT = -T\frac{\partial}{\partial T}U$.
- General covariance and covariant conservation law imply

$$\nabla_\mu\langle T^{\mu\nu}(x)\rangle = 0 \quad \Longrightarrow \quad \text{Differential eqs. for } \beta^\mu(x)$$

Entropy production for scalar field

- Entropy current

$$s^\mu = \beta^\mu \beta^\lambda K_\lambda = -\beta^\mu T \frac{\partial}{\partial T} \left\{ \frac{1}{2} Z(\rho, T) g^{\alpha\beta} \partial_\alpha \varphi_j \partial_\beta \varphi_j + U(\rho, T) \right\}$$

- Generalized entropy density

$$s_G = -\frac{\partial}{\partial T} \left\{ \frac{1}{2} Z(\rho, T) g^{\alpha\beta} \partial_\alpha \varphi_j \partial_\beta \varphi_j + U(\rho, T) \right\}$$

- Entropy generation positive semi-definite for $C(\rho, T) \geq 0$

$$\nabla_\mu s^\mu = C(\rho, T) \left(\beta^\mu \partial_\mu \varphi_j\right) \left(\beta^\nu \partial_\nu \varphi_j\right) \geq 0$$

- For fluid at rest $u^\mu = (1, 0, 0, 0)$

$$\nabla_\mu s^\mu = \dot{s}_G = \frac{C(\rho, T)}{T^2} \dot{\varphi}_j \dot{\varphi}_j$$

entropy increases when φ_j oscillates. For example reheating after inflation.

Ideal fluid

- Consider effective action

$$\Gamma[g_{\mu\nu}, \beta^\mu] = \Gamma_R[g_{\mu\nu}, \beta^\mu] = \int d^dx \sqrt{g}\, U(T)$$

with effective potential $U(T) = -p(T)$ and temperature

$$T = \frac{1}{\sqrt{-g_{\mu\nu}\beta^\mu \beta^\nu}}$$

- Variation of $g_{\mu\nu}$ at fixed β^μ leads to

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu}$$

where $\epsilon + p = Ts = T\frac{\partial}{\partial T} p$ is the enthalpy density
- Describes ideal fluid. General covariance of covariant conservation $\nabla_\mu T^{\mu\nu} = 0$ leads to ideal fluid equations

$$u^\mu \partial_\mu \epsilon + (\epsilon + p)\nabla_\mu u^\mu = 0 \qquad\qquad (\epsilon + p) u^\mu \nabla_\mu u^\nu + \Delta^{\nu\mu}\partial_\mu p = 0$$

Viscous fluid

- Analytic action

$$\Gamma[g_{\mu\nu}, \beta^\mu] = \int_x \left\{ U(T) + \frac{1}{4} \left[g_{\mu\nu}, s_{\mathsf{R}}(\mathcal{L}_u)\right] \left(2\eta(T)\sigma^{\mu\nu} + \zeta(T)\Delta^{\mu\nu}\nabla_\rho u^\rho\right) \right\}$$

with projector

$$\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$$

and

$$\sigma^{\mu\nu} = \left(\frac{1}{2}\Delta^{\mu\alpha}\Delta^{\mu\beta} + \frac{1}{2}\Delta^{\mu\beta}\Delta^{\mu\alpha} - \frac{1}{d-1}\Delta^{\mu\nu}\Delta^{\alpha\beta}\right)\nabla_\alpha u_\beta$$

leads to

$$\langle T^{\mu\nu}\rangle = -\tfrac{2}{\sqrt{g}}\tfrac{\delta\Gamma[g_{\mu\nu},\beta^\mu]}{\delta g_{\mu\nu}}\Big|_{\mathsf{ret}} = (\epsilon + p)u^\mu u^\nu + p g^{\mu\nu} - 2\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\nabla_\rho u^\rho$$

- Describes viscous fluid with shear viscosity $\eta(T)$ and bulk viscosity $\zeta(T)$
- Entropy production

$$\nabla_\mu s^\mu = \frac{1}{T}\left[2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta(\nabla_\rho u^\rho)^2\right]$$

Conclusions

- A variational principle for theories with dissipation can be based on analytic continuation.
- Needs a local equilibrium setup: Generalized Gibbs ensemble with $T(x)$ and $u^\mu(x)$.
- Works at least for close-to-equilibrium situations, e. g. fluid dynamics coupled to additional fields.
- General covariance and energy-momentum conservation lead to equations for fluid velocity and entropy production.
- Local form of second law of thermodynamics is implemented on the level of the effective action $\Gamma[\Phi]$.
- Fluid dynamics equations of motion can follow from quantum effective action $\Gamma[\phi]$.
- Higher order functional derivatives contain information about hydrodynamic fluctuations.
- Different methods of QFT can be used to determine $\Gamma[\phi]$ from more microscopic calculations.

Backup

Equations of motion from the Feynman action ?

- Consider damped harmonic oscillator as example. Time-ordered or Feynman action is obtained from analytic action by replacing $s_\mathsf{I}(\omega) \to \mathsf{sign}(\omega)$

$$\Gamma_{\text{time ordered}}[x] = \int \frac{d\omega}{2\pi}\, \frac{m}{2} x^*(\omega)\left[-\omega^2 - 2i|\omega|\,\zeta\omega_0 + \omega_0^2\right] x(\omega)$$

- Field equation $\frac{\delta}{\delta x(t)}\Gamma_{\text{time ordered}}[x] = J(t)$ would give

$$\left[-\omega^2 - 2i|\omega|\,\zeta\omega_0 + \omega_0^2\right] x(\omega) = J(\omega)$$

- Violates reality constraint $x^*(\omega) = x(-\omega)$ for $J^*(\omega) = J(-\omega)$
- Solution not causal

$$x(t) = \int_{t'} \Delta_F(t-t')J(t')$$

  because Feynman propagator $\Delta_F(t-t')$ not causal.
- In contrast, retarded variation of analytically continued action leads to real and causal equation of motion

Tree-like structures

- Discontinuous terms in analytic action could be of the form

$$\Gamma_{\mathsf{Disc}}[\Phi] = \int d^d x \sqrt{g}\ \left\{ f[\Phi](x)\ s_{\mathsf{R}}\big(u^\mu(x)\tfrac{\partial}{\partial x^\mu}\big)\ g[\Phi](x) \right\}$$

- More general, tree-like structure are possible such as

$$\Gamma_{\mathsf{Disc}}[\Phi] = \int_{x,y} \Big\{ f[\Phi](x)\ s_{\mathsf{R}}\big(u^\mu(x)\tfrac{\partial}{\partial x^\mu}\big)\ g[\Phi](x,y)\ s_{\mathsf{R}}\big(u^\mu(y)\tfrac{\partial}{\partial y^\mu}\big)\ h[\Phi](y) \Big\}$$

or

$$\Gamma_{\mathsf{Disc}}[\Phi] = \int_{x,y,z} \Big\{ f[\Phi](x)\ s_{\mathsf{R}}\big(u^\mu(x)\tfrac{\partial}{\partial x^\mu}\big)\ g[\Phi](x,y,z)\ s_{\mathsf{R}}\big(u^\mu(y)\tfrac{\partial}{\partial y^\mu}\big)\ h[\Phi](y) \times s_{\mathsf{R}}\big(u^\mu(z)\tfrac{\partial}{\partial z^\mu}\big)\ j[\Phi](z) \Big\}$$

- For retarded variation calculate $\delta\Gamma$ and set $s_{\mathsf{R}}(u^\mu\partial_\mu) \to -1$ if derivative operator points towards node that is varied and $s_{\mathsf{R}}(u^\mu\partial_\mu) \to 1$ if derivative operator points in opposite direction

Analytic continuation of FRG equations

[Floerchinger, JHEP 1205 (2012) 021]

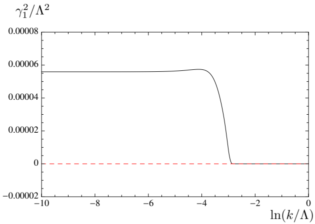
- Consider a point $p_0^2 - \vec{p}^2 = m^2$ where $P_1(m^2) = 0$.
- One can expand around this point

$$P_1 = Z(-p_0^2 + \vec{p}^2 + m^2) + \cdots$$
$$P_2 = Z\gamma^2 + \cdots$$

- Leads to Breit-Wigner form of propagator (with $\gamma^2 = m\Gamma$)

$$G(p) = \frac{1}{Z}\,\frac{-p_0^2 + \vec{p}^2 + m^2 + i\,s(p_0)\,m\Gamma}{(-p_0^2 + \vec{p}^2 + m^2)^2 + m^2\Gamma^2}.$$

- A few flowing parameters describe efficiently the singular structure of the propagator.



Truncation for relativistic scalar O(N) theory

$$\Gamma_k = \int_{t,\vec{x}} \left\{ \sum_{j=1}^{N} \frac{1}{2} \bar{\phi}_j \, \bar{P}_\phi(i\partial_t, -i\vec{\nabla}) \, \bar{\phi}_j + \frac{1}{4} \bar{\rho} \, \bar{P}_\rho(i\partial_t, -i\vec{\nabla}) \, \bar{\rho} + \bar{U}_k(\bar{\rho}) \right\}$$

with $\bar{\rho} = \frac{1}{2} \sum_{j=1}^{N} \bar{\phi}_j^2$.

- Goldstone propagator massless, expanded around $p_0 - \vec{p}^2 = 0$

$$\bar{P}_\phi(p_0, \vec{p}) \approx \bar{Z}_\phi \, (-p_0^2 + \vec{p}^2)$$

- Radial mode is massive, expanded around $p_0^2 - \vec{p}^2 = m_1^2$

$$\bar{P}_\phi(p_0, \vec{p}) + \bar{\rho}_0 \bar{P}_\rho(p_0, \vec{p}) + \bar{U}_k' + 2\bar{\rho}\bar{U}_k'' \approx \bar{Z}_\phi Z_1 \left[(-p_0^2 + \vec{p}^2 + m_1^2) - is(p_0) \, \gamma_1^2 \right]$$

Flow of the effective potential

$$\partial_t U_k(\rho)\big|_{\bar{\rho}} = \frac{1}{2}\int_{p_0=i\omega_n,\vec{p}} \left\{ \frac{(N-1)}{\vec{p}^2 - p_0^2 + U' + \frac{1}{\bar{Z}_\phi}R_k} \right.$$

$$\left. + \frac{1}{Z_1\left[(\vec{p}^2 - p_0^2) - i\,s(p_0)\gamma_1^2\right] + U' + 2\rho U'' + \frac{1}{\bar{Z}_\phi}R_k} \right\} \frac{1}{\bar{Z}_\phi}\partial_t R_k.$$

- Summation over Matsubara frequencies $p_0 = i2\pi T n$ can be done using contour integrals.
- Radial mode has non-zero decay width since it can decay into Goldstone excitations.
- Use Taylor expansion for numerical calculations

$$U_k(\rho) = U_k(\rho_{0,k}) + m_k^2(\rho - \rho_{0,k}) + \frac{1}{2}\lambda_k(\rho - \rho_{0,k})^2$$