Theory of particle number correlations

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Equilibrium: grand canonical ensemble

- describes volume V with temperature T and chemical potentials μ_B, μ_C and μ_S associated with conserved baryon, charge and strangeness numbers
- exchange of energy and particles with heat bath
- can be simulated with Lattice QCD
- all thermodynamic properties follow from

$$p(T, \mu_B, \mu_C, \mu_S)$$

- For example mean value of net baryon number

$$M_q = \langle N_B \rangle = V \frac{\partial}{\partial \mu_B} p(T, \mu_B, \mu_C, \mu_S)$$

variance

$$\sigma_B^2 = \langle \delta N_B^2 \rangle = TV \frac{\partial^2}{\partial \mu_B^2} p(T, \mu_B, \mu_C, \mu_S)$$

skewness

$$S_B = \frac{\langle \delta N_q^3 \rangle}{\sigma_B^3} = \frac{1}{\sigma_B^3} T^2 V \frac{\partial^3}{\partial \mu_B^3} p(T, \mu_B, \mu_C, \mu_S)$$

kurtosis

$$\kappa_B = \frac{\langle \delta N_B^4 \rangle - 3\langle \delta N_B^2 \rangle^2}{\sigma_B^4} = \frac{1}{\sigma_B^4} T^3 V \frac{\partial^4}{\partial \mu_B^4} p(T, \mu_B, \mu_C, \mu_S)$$

Hadron resonance gas

- Pressure for free hadrons and resonances with vacuum masses

$$p = \frac{T^2}{\pi^2} \sum_i d_i m_i^2 K_2 \left(\frac{m_i}{T} \right) \cosh \left(\frac{B_i \mu_B + Q_i \mu_Q + S_i \mu_S}{T} \right)$$

- Implies relations like

$$\kappa_B \sigma_B^2 = \frac{T^2 \frac{\partial^4}{\partial \mu_B^4} p}{\frac{\partial^2}{\partial \mu_B^2} p} = \frac{\langle B_i^4 \rangle}{\langle B_i^2 \rangle} = 1$$

when only baryons with $B_i = \pm 1$ contribute.

- Similarly,

$$\kappa_B M_B = S_B \sigma_B$$

- And for $\mu_S = \mu_Q = 0$ one has relations like

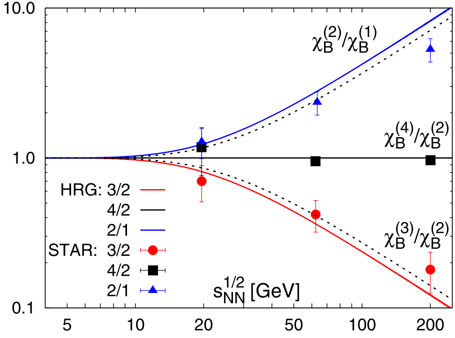
$$S_B \sigma_B = \frac{T \frac{\partial^3}{\partial \mu_B^3} p}{\frac{\partial^2}{\partial \mu_B^2} p} = \tanh \left(\frac{\mu_B}{T} \right)$$

Hadron resonance gas versus experiment

- Ratios of cumulants are independent of volume V and less sensitive to kinematic cuts

$$\frac{\chi_B^{(2)}}{\chi_B^{(1)}} = \frac{\sigma_q^2}{M_q}, \qquad \frac{\chi_B^{(3)}}{\chi_B^{(2)}} = S_q\sigma_q, \qquad \frac{\chi_B^{(4)}}{\chi_B^{(2)}} = \kappa_q\sigma_q^2$$

- Particularly well suited to compare to experiment

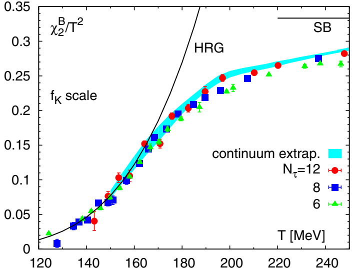


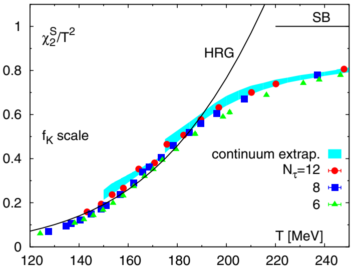
(Data: STAR, Lines: HRG, [F. Karsch, K. Redlich, PLB 695, 136 (2011)])

Lattice QCD results on cumulants

- Cumulants of conserved quantum numbers can be calculated in lattice QCD. e. g. [HotQCD, PRD 86, 034509 (2012)]
- Results for

$$\frac{\chi_B^{(2)}}{T^2} = \frac{\sigma_B^2}{VT^5} = \frac{\langle \delta N_B^2 \rangle}{VT^5} \qquad \frac{\chi_S^{(2)}}{T^2} = \frac{\sigma_S^2}{VT^5} = \frac{\langle \delta N_S^2 \rangle}{VT^5}$$

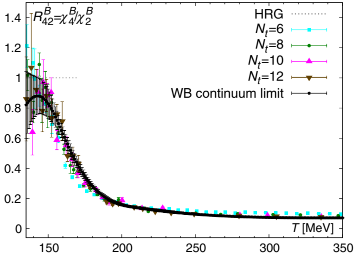


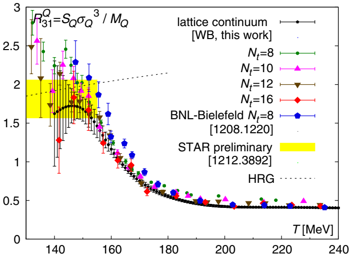


Ratios of cumulants from Lattice QCD

- From Wuppertal-Budapest collaboration [PRL 111, 062005 (2013)]

$$R^B_{42} = \frac{\chi^B_4}{\chi^B_2} = \frac{\langle \delta N^4_B \rangle}{\langle \delta N^2_B \rangle} \qquad R^Q_{31} = \frac{\chi^Q_3}{\chi^Q_1} = \frac{\langle \delta N^3_Q \rangle}{\langle \delta N_Q \rangle}$$





More realistic description of fireball

- Ratios of cumulants of conserved quantum numbers N_B, N_C and N_S have been proposed as particularly nice observables
- Problem 1: What is optimal range of acceptance?
  - Full coverage for ^{208}Pb - ^{208}Pb: No fluctuations at all

$$N_B = 2 \times 208 = 416, \qquad N_C = 2 \times 82 = 164, \qquad N_S = 0.$$

  - Too small coverage: Poisson statistics
- Problem 2: Fireball is not in equilibrium
  - At best approx. local equilibrium (viscous fluid dynamics) for some time
  - Freeze-out corresponds to dropping out of equilibrium
  - Need more differential description including dependence on rapidity and p_T

Correlation functions as generalized cumulants

- Correlation functions for conserved quantum numbers can also be studied locally
- Baryon number density fluctuations

$$C_2^{(B,B)}(t,\vec{x};t',\vec{x}') = \langle n_B(t,\vec{x})\, n_B(t',x')\rangle - \langle n_B(t,\vec{x})\rangle\langle n_B(t',\vec{x}')\,\rangle$$

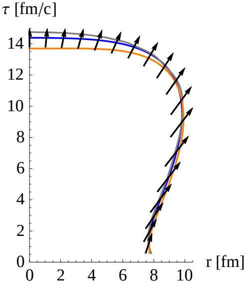
- Integral over equal time correlation determines variance

$$\sigma_B^2 = \langle \delta N_B^2 \rangle = \int_V d^3x \int_V d^3x'\; C_2^{(B,B)}(t,\vec{x};t,\vec{x}')$$

- Similar for higher order correlation functions
- Correlation functions for different thermodynamic variables can be translated

$$(\epsilon, n_B, n_C, n_S) \quad \leftrightarrow \quad (T, \mu_B, \mu_C, \mu_S)$$

Cooper-Frye freeze-out



- Single particle distribution (Cooper & Frye 1974)

$$E\frac{dN_i}{d^3p} = -p_\mu \int_{\Sigma_f} \frac{d\Sigma^\mu}{(2\pi)^3} f_i(p;x)$$

with close-to equilibrium distribution

$$f_i(p;x) = f_i(p;T(x),\mu_i(x),u^\mu(x),\pi^{\mu\nu}(x),\varphi(x),\ldots)$$

- Precise position of freeze-out surface is not known. Usual assumption

$$\langle T(x)\rangle = T_{\text{fo}} = \text{const}$$

Particle correlations from freeze-out

- Can be used for expectation values...

$$\left\langle E\frac{dN_i}{d^3p}\right\rangle = \left\langle -p_\mu \int_{\Sigma_f} \frac{d\Sigma^\mu}{(2\pi)^3}\, f_i(p;x)\right\rangle$$

- ... but also for correlation functions

$$\left\langle E\frac{dN_i}{d^3p} E'\frac{dN_j}{d^3p'}\right\rangle = p_\mu p'_\nu \int_{\Sigma_f} \frac{d\Sigma^\mu}{(2\pi)^3}\frac{d\Sigma'^\nu}{(2\pi)^3} \left\langle f_i(p;x)\, f_j(p';x')\right\rangle$$

- The right hand side involves correlation functions

$$\left\langle f_i(p;x)\, f_j(p';x')\right\rangle$$

between different points x and x' on the freeze-out surface.

- Works similar for higher order correlation functions.
- Thermal fluctuations and initial state fluctuations contribute to correlation functions

Particle correlations from field correlation functions

- One can decompose

$$T(x) = \bar{T}(x) + \delta T(x), \qquad \mu(x) = \bar{\mu}(x) + \delta\mu(x)$$

  and expand the distribution functions

$$\begin{aligned} f_i(p;x) =& f_i(p;\bar{T}(x),\bar{\mu}_i(x),\ldots) \\ &+ \delta T(x)\frac{\partial}{\partial T} f_i(p;\bar{T}(x),\bar{\mu}(x),\ldots) \\ &+ \delta\mu(x)\frac{\partial}{\partial \mu} f_i(p;\bar{T}(x),\bar{\mu}(x),\ldots) + \ldots \end{aligned}$$

- Two-particle correlation function governed by integral over

$$\begin{aligned} \langle f_i(p;x)\, f_j(p';x')\rangle =& f_i(p;\bar{T}(x),\ldots)\, f_j(p';\bar{T}(x'),\ldots) \\ &+ \langle \delta T(x)\delta T(x')\rangle \frac{\partial}{\partial T} f_i(p;\bar{T}(x),\ldots)\frac{\partial}{\partial T} f_j(p;\bar{T}(x'),\ldots) \\ &+ \langle \delta\mu(x)\delta\mu(x')\rangle \frac{\partial}{\partial \mu} f_i(p;\bar{T}(x),\ldots)\frac{\partial}{\partial \mu} f_j(p;\bar{T}(x'),\ldots) \\ &+ \langle \delta\varphi(x)\delta\varphi(x')\rangle \frac{\partial}{\partial \varphi} f_i(p;\bar{T}(x),\ldots)\frac{\partial}{\partial \varphi} f_j(p;\bar{T}(x'),\ldots) \\ &+ \ldots \end{aligned}$$

Critical physics

- Critical physics shows up in correlation functions
- In homogeneous space

$$\langle \varphi(\vec{x})\varphi(\vec{x}+\vec{r})\rangle \sim \frac{1}{r^{d-2+\eta}} \exp\left(\frac{r}{\xi}\right)$$

  with correlation length

$$\xi \sim \frac{1}{|T-T_c|^{\nu}}$$

- Critical slowing down leads drop out of equilibrium (talk by R. Venugopalan)
- Could lead to (partial) freeze-out
- Correlation functions of net baryons, net charge etc could be sensitive to this.

Mode-by-mode fluid dynamic description

- Correlation functions in a fireball can be expanded in Bessel-Fourier basis
- Chemical and kinetic freeze-out can be studied in that basis, as well
  [S. Floerchinger, U. A. Wiedemann, PRC 89, 034914 (2014)]
- Leads to differential correlation functions

$$C_2(\eta_1, \phi_1, p_{T1}; \eta_2, \phi_2, p_{T2}) = \left\langle \frac{dN}{d\eta_1 d\phi_1 dp_{T1}} \frac{dN}{d\eta_2 d\phi_2 dp_{T2}} \right\rangle_c$$

  and similar for higher orders C_3, C_4 etc.
- Integration over ranges of η, ϕ and p_T gives cumulants of particle numbers
- Should help to improve theory-experiment comparison.
- Besides thermal fluctuations, also initial state fluctuations contribute.
- Propagation of initial state fluctuations in net baryon number is sensitive to baryon diffusion i. e. heat conductivity
  [S. Floerchinger, M. Martinez, PRC 92, 064906 (2015)]

Effective action

- Equations of motion for expectation values and correlation functions can be obtained from one-particle irreducible effective action

$$\Gamma[\Phi, T, \mu] = \int d^d x \, \{Z \partial_\mu \Phi^* \partial^\mu \Phi + U(T, \mu, \Phi) + \ldots\}$$

with effective potential U related to pressure

$$p(T, \mu) = -U(T, \mu, \Phi)$$

- Also viscous fluid dynamics can be described in this way [S. Floerchinger, 1603.07148]
- Correlation functions can be obtained from functional derivatives of Γ.
- Well understood for order parameter fields - less understood for thermodynamic / fluid dynamic fields.

Effective action

- The effective potential or pressure $p(T, \mu)$ contains information about equation of state and can be determined by Lattice QCD.
- Other elements of Γ can be calculated by
  - analytically continued LQCD results
  - perturbation theory
  - functional renormalization
  - classical simulations
  - universal critical physics
  - model calculations, ...
- Effective actions can be very helpful to think about physics.
- For QCD, the microscopic theory is very well known and heavy ions collisions provide a chance to investigate resulting effective action Γ
- Challenge for theory will be to calculate effective action Γ for QCD
- Challenge for phenomenology is to constrain Γ from observables.

Initial state fluctuations in baryon number and their evolution

Evolution of baryon number in fluid dynamics

- Small perturbation in static medium with $u^\mu = (1,0,0,0)$

$$\frac{\partial}{\partial t}\delta n(t,\vec{x}) = D\vec{\nabla}^2 \delta n(t,\vec{x})$$

- Baryon number diffusion constant

$$D = \kappa \left[\frac{nT}{\epsilon + p}\right]^2 \left(\frac{\partial(\mu/T)}{\partial n}\right)_\epsilon$$

- Heat capacity κ appears here because

baryon diffusion in Landau frame	$\hat{=}$	heat conduction in Eckart frame

- Is D finite for $n \to 0$?

Heat conductivity

- Heat conductivity of QCD rather poorly understood theoretically so far.
- From perturbation theory [Danielewicz & Gyulassy, PRD 31, 53 (1985)]

$$\kappa \sim \frac{T^4}{\mu^2 \alpha_s^2 \ln \alpha_s} \qquad (\mu \ll T)$$

- From AdS/CFT [Son & Starinets, JHEP 0603 (2006)]

$$\kappa = 8\pi^2 \frac{T}{\mu^2} \eta = 2\pi \frac{sT}{\mu^2} \qquad (\mu \ll T)$$

- Baryon diffusion constant D finite for $\mu \to 0$!

Relativistic fluid dynamics

- Evolution of baryon number density from conservation law

$$u^\mu \partial_\mu n + n \nabla_\mu u^\mu + \nabla_\mu \nu^\mu = 0$$

- Diffusion current ν^α determined by heat conductivity κ

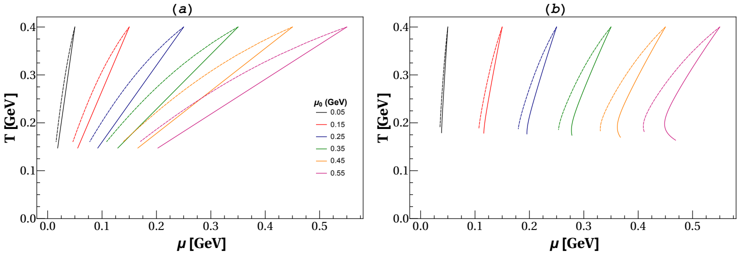
$$\nu^\alpha = -\kappa \left[\frac{nT}{\epsilon + p} \right]^2 \Delta^{\alpha\beta} \partial_\beta \left(\frac{\mu}{T} \right)$$

Bjorken expansion

- Consider Bjorken type expansion

$$\partial_\tau \epsilon + (\epsilon + p)\frac{1}{\tau} - \left(\tfrac{4}{3}\eta + \zeta\right)\frac{1}{\tau^2} = 0$$
$$\partial_\tau n + n\frac{1}{\tau} = 0$$

- Heat conductivity κ does not enter by symmetry argument
- Compare ideal gas to lattice QCD equation of state
  [Borsanyi et al., JHEP 08 (2012) 053]



Perturbations around Bjorken expansion

- Consider situation with $\langle n(x) \rangle = \langle \mu(x) \rangle = 0$
- Local event-by-event fluctuation $\delta n \neq 0$
- Concentrate now on Bjorken flow profile for u^μ
- Consider perturbation δn

$$\partial_\tau \delta n + \frac{1}{\tau} \delta n - D(\tau) \left(\partial_x^2 + \partial_y^2 + \frac{1}{\tau^2} \partial_\eta^2 \right) \delta n = 0$$

- Structures in transverse and rapidity directions are "flattened out" by heat conductive dissipation

Solution by Bessel-Fourier expansion

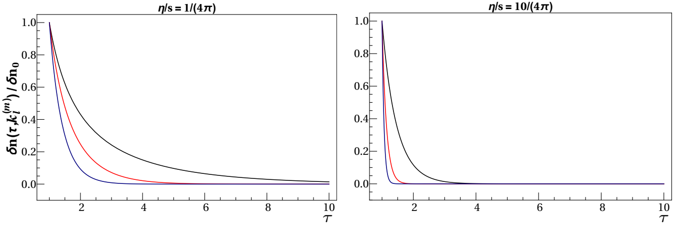
- Expand perturbations like

$$\delta n(\tau, r, \phi, \eta) = \int_0^\infty dk\, k \sum_{m=-\infty}^{\infty} \int \frac{dq}{2\pi}\, \delta n(\tau, k, m, q)\, e^{i(m\phi + q\eta)}\, J_m(kr)$$

- Leads to ODE

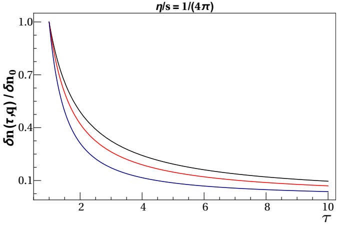
$$\partial_\tau \delta n + \frac{1}{\tau}\delta n + D(\tau)\left(k^2 + \frac{q^2}{\tau^2}\right)\delta n = 0.$$

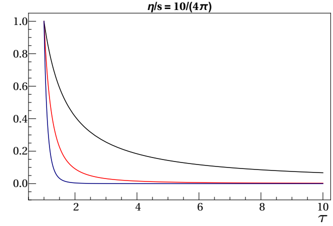
- For $q = 0$ and different $k \approx 1/\text{fm}$, AdS/CFT value $\kappa = 8\pi^2 \frac{T}{\mu^2}\eta$



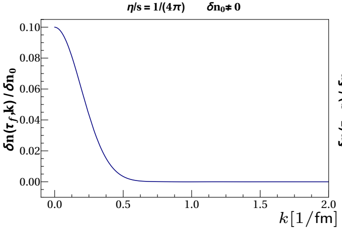
Evolution of perturbations

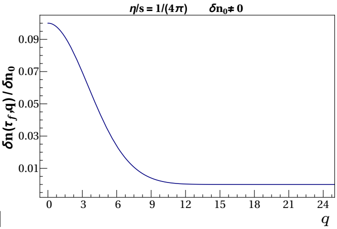
- For $k = 0$ and different $q = 1, 3, 5$, AdS/CFT value $\kappa = 8\pi^2 \frac{T}{\mu^2}\eta$





- At $\tau = 10\mathrm{fm/c}$





- Only long-range fluctuations survive diffusive damping.

Fluctuations at freeze-out

- Background-perturbation splitting can also be used at freeze-out
- Interesting observable is net baryon number

$$n(\phi, \eta) = (B - \bar{B})(\phi, \eta)$$

- Correlation functions and distributions contain information about baryon number fluctuations
- Two-particle correlation function of baryons minus anti-baryons

$$C_{\text{Baryon}}(\phi_1 - \phi_2, \eta_1 - \eta_2) = \langle n(\phi_1, \eta_1)\, n(\phi_2, \eta_2) \rangle_c$$

Baryon number correlation function

- In Fourier representation

$$C_{\text{Baryon}}(\Delta\phi, \Delta\eta) = \sum_{m=-\infty}^{\infty} \int \frac{dq}{2\pi}\, \tilde{C}_{\text{Baryon}}(m, q)\, e^{im\Delta\phi + iq\Delta\eta}$$

heat conductivity leads to exponential suppression

$$\tilde{C}_{\text{Baryon}}(m, q) = e^{-m^2 I_1 - q^2 I_2}\; \tilde{C}_{\text{Baryon}}(m, q)\Big|_{\kappa=0}$$

- I_1 and I_2 can be approximated as

$$I_1 \approx \int_{\tau_0}^{\tau_f} d\tau\, \frac{2}{R^2}\, \kappa \left[\frac{nT}{\epsilon + p}\right]^2 \left(\frac{\partial(\mu/T)}{\partial n}\right)_\epsilon$$

$$I_2 \approx \int_{\tau_0}^{\tau_f} d\tau\, \frac{2}{\tau^2}\, \kappa \left[\frac{nT}{\epsilon + p}\right]^2 \left(\frac{\partial(\mu/T)}{\partial n}\right)_\epsilon$$

- $I_2 \gg I_1$ would lead to long-range correlations in rapidity direction ("baryon number ridge")