

A perturbative approach to hydrodynamics

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based on

- Fluctuations around Bjorken Flow and the onset of turbulent phenomena, [[JHEP 11, 100 \(2011\)](#), with U. A. Wiedemann]
- Mode-by-mode fluid dynamics for relativistic heavy ion collisions [[Phys. Lett. B, 728, 407 \(2014\)](#), with U. A. Wiedemann]
- Characterization of initial fluctuations for the hydrodynamical description of heavy ion collisions, [[Phys. Rev. C 88, 044906 \(2013\)](#), with U. A. Wiedemann]
- Kinetic freeze-out, particle spectra and harmonic flow coefficients from mode-by-mode hydrodynamics, [[Phys. Rev. C 89 \(2014\) 034914](#), with U. A. Wiedemann]
- How (non-) linear is the hydrodynamics of heavy ion collisions? [[arXiv:1312.5482](#), with U. A. Wiedemann, A. Beraudo, L. Del Zanna, G. Inghirami, V. Rolando]
- Mode-by-mode hydrodynamics: ideas and concepts [[arXiv:1401.2339](#)]
- Statistics of initial density perturbations in heavy ion collisions and their fluid dynamic response [[arXiv:1405.4393](#), with U. A. Wiedemann]

Introduction

What perturbations are interesting and why?

- **Initial fluid perturbations:** Event-by-event fluctuations around a background or average of fluid fields at time τ_0 :
 - energy density ϵ
 - fluid velocity u^μ
 - shear stress $\pi^{\mu\nu}$
 - more general also: baryon number density n_B , electric charge density, electromagnetic fields, ...
- governed by universal evolution equations
- can be used to constrain **thermodynamic and transport properties**
- contain interesting information from early times

A program to understand fluid perturbations

- 1 Characterize initial perturbations.
- 2 Propagated them through fluid dynamic regime.
- 3 Determine influence on particle spectra and harmonic flow coefficients.
- 4 Take also perturbations from non-hydro sources (jets) into account.
[see talk of K. Zapp]

Characterization of initial conditions

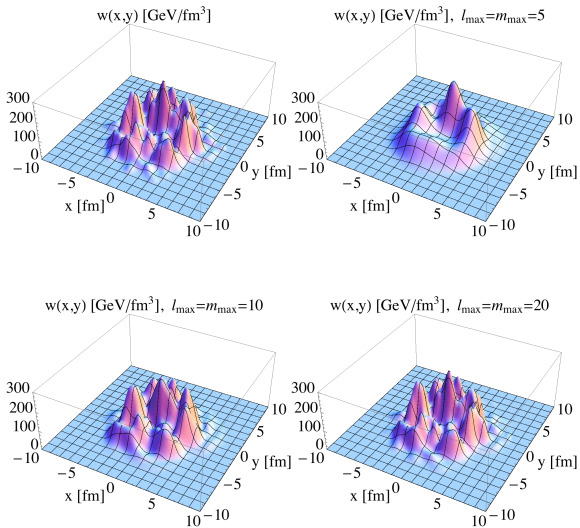
Transverse enthalpy density

Based on Bessel-Fourier expansion and background density
[Floerchinger & Wiedemann 2013, see also Coleman-Smith, Petersen & Wolpert 2012, Floerchinger & Wiedemann 2014]

$$w(r, \phi) = w_{\text{BG}}(r) + w_{\text{BG}}(r) \sum_{m,l} w_l^{(m)} e^{im\phi} J_m \left(z_l^{(m)} \rho(r) \right)$$

- azimuthal wavenumber m , radial wavenumber l
- $w_l^{(m)}$ dimensionless
- higher m and l correspond to finer spatial resolution
- coefficients $w_l^{(m)}$ can be related to eccentricities
- works similar for vectors (velocity) and tensors (shear stress)

Transverse density from Glauber model



Event ensembles

- Event-by-event probability distribution

$$p_{\tau_0}[w, u^\mu, \pi^{\mu\nu}, \dots]$$

- Moments / correlation functions

$$\left\langle w_{l_1}^{(m_1)} w_{l_2}^{(m_2)} \dots w_{l_n}^{(m_n)} \right\rangle$$

- contain information from initial state physics / early dynamics
- universal (model independent) properties would be nice to have
- same information in cumulants (connected correlation functions)

Statistics of initial density perturbations

Independent point-sources model (IPSM)

$$w(\vec{x}) = \left[\frac{1}{\tau_0} \frac{dW_{\text{BG}}}{d\eta} \right] \frac{1}{N} \sum_{j=1}^N \delta^{(2)}(\vec{x} - \vec{x}_j)$$

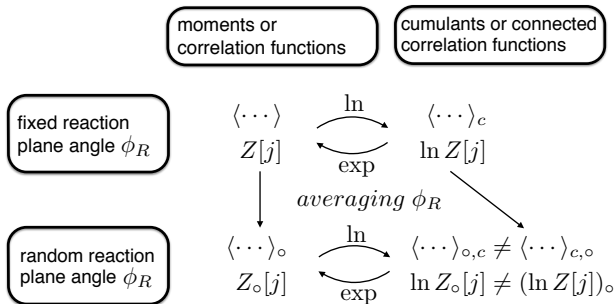
- random positions \vec{x}_j , independent and identically distributed
- probability distribution $p(\vec{x}_j)$ reflects collision geometry
- possible to determine correlation functions analytically for *central* and *non-central* collisions [Floerchinger & Wiedemann (2014)]
- Long-wavelength modes (small m and l) that don't resolve differences between point-like and extended sources have *universal statistics*.

Scaling with number of sources

- Connected correlation functions (cumulants) scale with N like [see also Ollitrault & Yan (2014), Bzdak & Skokov (2014)]

$$\langle w_{l_1}^{(m_1)} \dots w_{l_n}^{(m_n)} \rangle_c \sim \frac{1}{N^{n-1}}$$

- scaling broken for non-central collisions
- b -dependence of term that break scaling is known



Fluid dynamic response

Perturbative expansion

Write the hydrodynamic fields $h = (w, u^\mu, \pi^{\mu\nu}, \pi_{\text{Bulk}}, \dots)$

- at initial time τ_0 as

$$h = h_0 + \epsilon h_1$$

with background h_0 , fluctuation part ϵh_1

- at later time $\tau > \tau_0$ as

$$h = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \epsilon^3 h_3 + \dots$$

Solve for time evolution in this scheme

- h_0 is solution of full, non-linear hydro equations in symmetric situation: azimuthal rotation and Bjorken boost invariant
- h_1 is solution of linearized hydro equations around h_0 , can be solved mode-by-mode
- h_2 can be obtained by from interactions between modes etc.

Response to density perturbations

For a single event

$$\begin{aligned} V_m^* &= v_m e^{-i m \psi_m} \\ &= \sum_l S_{(m)l} w_l^{(m)} + \sum_{\substack{m_1, m_2, \\ l_1, l_2}} S_{(m_1, m_2)l_1, l_2} w_{l_1}^{(m_1)} w_{l_2}^{(m_2)} \delta_{m, m_1 + m_2} + \dots \end{aligned}$$

- $S_{(m)l}$ is linear dynamic response function
- $S_{(m_1, m_2)l_1, l_2}$ is quadratic dynamic response function etc.
- Symmetries imply conservation of azimuthal wavenumber
- Response functions depend on thermodynamic and transport properties, in particular viscosity.

Flow correlations from initial density correlations

Moments of flow coefficients

$$\langle V_{m_1}^* \cdots V_{m_n}^* \rangle = S_{(m_1)l_1} \cdots S_{(m_n)l_n} \langle w_{l_1}^{(m_1)} \cdots w_{l_n}^{(m_n)} \rangle$$

+ non-linear terms

- combination of **dynamical response coefficients** and **correlation functions** of initial density perturbations
- linear, quadratic and higher-order terms
- For N independent sources and central collisions

$$v_m \{n\}^n \sim \frac{1}{N^{n-1}}$$

- holds also for **extended sources**
- holds also for **non-linear response contributions**
- can be extended to other conn. correlation functions, e. g. $\langle V_2 V_3 V_5^* \rangle$
- gets broken for non-central collisions
- impact parameter dependence of corrections is known

Scaling with system size

- Large (PbPb) and small systems (pPb) have different number of independent sources N .
- For linear dynamics one has parametrically

$$v_m\{n\} \sim S_{(m)l} \frac{N^{1/n}}{N}.$$

- To have $v_m\{n\}|_{\text{PbPb}} = v_m\{n\}|_{\text{pPb}}$ one needs

$$\frac{S_{(m)l}|_{\text{pPb}}}{S_{(m)l}|_{\text{PbPb}}} = \left(\frac{N_{\text{pPb}}}{N_{\text{PbPb}}} \right)^{1 - \frac{1}{n}} < 1.$$

- Response function $S_{(m)l}$ must be **smaller** for pPb than for PbPb.
- $S_{(m)l}$ is independent of n , above equation can be true only for one n .
- $S_{(m)l}$ depends on system size only via initial background $w_{\text{BG}}(r)$.
Precise dependence can be investigated more closely.

*How response functions can be
determined:*

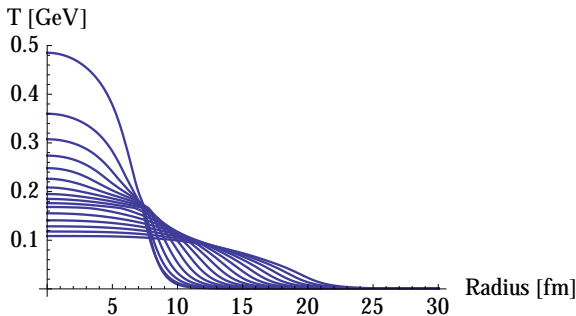
- 1. Hydrodynamic evolution*

Background evolution

System of coupled 1 + 1 dimensional non-linear partial differential equations for

- enthalpy density $w(\tau, r)$ (or temperature $T(\tau, r)$)
- fluid velocity $u^\tau(\tau, r), u^r(\tau, r)$
- two independent components of shear stress $\pi^{\mu\nu}(\tau, r)$

Can be easily solved numerically

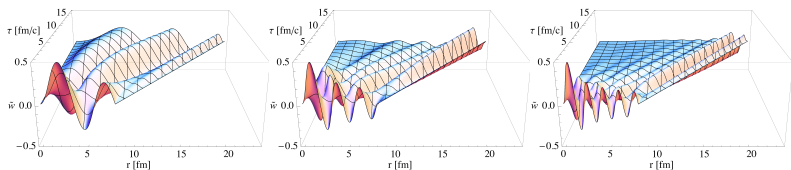


Evolving perturbation modes

- Linearized hydro equations: set of coupled 3 + 1 dimensional, linear, partial differential equations.
- Use Fourier expansion

$$h_j(\tau, r, \phi, \eta) = \sum_m \int \frac{dk_\eta}{2\pi} h_j^{(m)}(\tau, r, k_\eta) e^{i(m\phi + k_\eta \eta)}.$$

- Reduces to 1 + 1 dimensions.
- Can be solved numerically for each initial Bessel-Fourier mode.



Mode interactions

- Non-linear terms in the evolution equations lead to mode interactions.
- Quadratic and higher order in initial perturbations.
- Can be determined from iterative solution but has not been fully worked out yet.
- Convergence can be tested with numerical solution of full hydro equations.

Scaling tests

- Start with single enthalpy density mode ($m = 2, l = 1$) on top of background

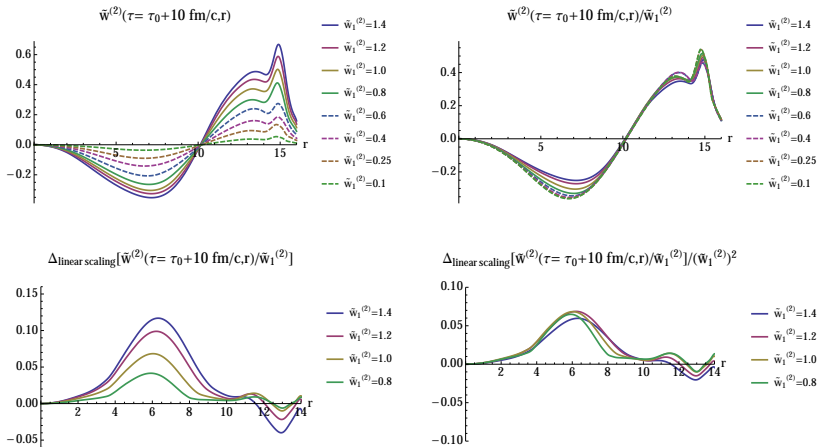
$$w(\tau_0, r, \phi) = w_{\text{BG}}(\tau_0, r) \left[1 + 2 \tilde{w}_1^{(2)} J_2(k_1^{(2)} r) \cos(2\phi) \right].$$

- Evolve this with hydro solver ECHO-QGP
[Del Zanna *et al.*, EPJC 73, 2524 (2013), see also following talk]
- Determine Fourier components

$$\tilde{w}^{(m)}(\tau, r) = \frac{1}{w_{\text{BG}}(r)} \frac{1}{2\pi} \int d\phi e^{-im\phi} w(\tau, r, \phi)$$

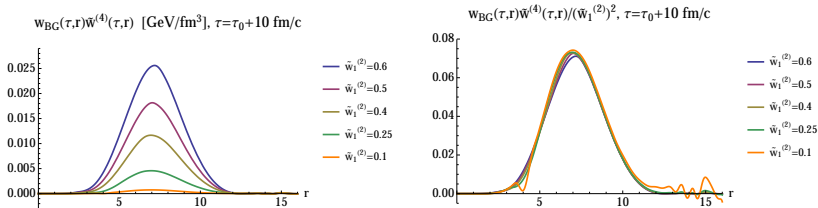
Scaling tests at first order

Compare enthalpy $\tilde{w}^{(2)}(\tau, r)$ at fixed τ for different initial weights



Scaling tests at second order

From symmetry considerations one expects that modes with $m = 0$ and $m = 4$ receive mainly quadratic contributions $\sim (\tilde{w}_1^{(2)})^2$.



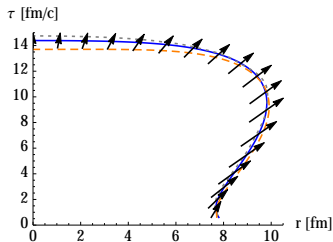
- Hydrodynamic response to initial enthalpy density fluctuations is perturbative.
- Non-linearities can be understood order-by-order and lead to characteristic “overtones”.
- Results motivate more thorough development of fluid dynamic perturbation theory.

*How response functions can be
determined:*

2. Kinetic freeze-out

Freeze-out surface

- Perturbative expansion can be used also at freeze-out.
[Floerchinger, Wiedemann 2013]
- Freeze-out surface is azimuthally symmetric as background.
- Generalization to kinetic hadronic scattering and decay phase possible.



(solid: $\eta/s = 0.08$, dotted: $\eta/s = 0$, dashed: $\eta/s = 0.3$)

Particle distribution

for single event

$$\ln \left(\frac{dN^{\text{single event}}}{p_T dp_T d\phi dy} \right) = \underbrace{\ln S_0(p_T)}_{\text{from background}} + \underbrace{\sum_{m,l} w_l^{(m)} e^{im\phi} \theta_l^{(m)}(p_T)}_{\text{from fluctuations}}$$

- each mode comes with an angle, $w_l^{(m)} = |w_l^{(m)}| e^{im\psi_l^{(m)}}$
- each mode has different p_T -dependence, $\theta_l^{(m)}(p_T)$
- quadratic order correction

$$\sum_{m_1, m_2, l_1, l_2} w_{l_1}^{(m_1)} w_{l_2}^{(m_2)} e^{i(m_1+m_2)\phi} \kappa_{l_1, l_2}^{(m_1, m_2)}(p_T)$$

- non-linearities from hydro evolution and freeze-out

Harmonic flow coefficients

Double differential harmonic flow coefficient (to lowest order)

$$v_m\{2\}^2(p_T^a, p_T^b) = \sum_{l_1, l_2} \theta_{l_1}^{(m)}(p_T^a) \theta_{l_2}^{(m)}(p_T^b) \langle w_{l_1}^{(m)} w_{l_2}^{(m)*} \rangle$$

- intuitive matrix expression
- in general no factorization
- can be generalized to higher order flow cumulants

Summary and Conclusions

Conclusions

- Systematic expansion in initial fluid perturbations seems possible (good convergence properties) and very useful.
- Formalism works in praxis (see backup slides for results of “proof of principle” study).
- Initial density perturbations have some universal properties that can help to better constrain thermodynamic and transport properties.
- Fluid dynamic response allows to access correlation functions of initial perturbations.

Backup

Characterization of transverse density via eccentricities

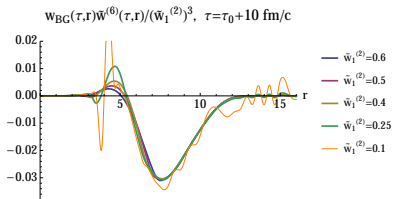
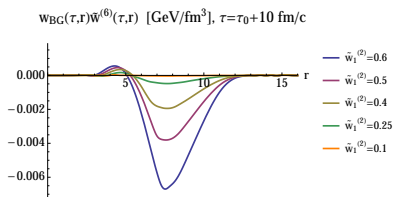
Fluctuations in initial transverse enthalpy density $w(r, \phi)$ can be characterized in terms of eccentricities $\epsilon_{n,m}$ and angles $\psi_{n,m}$
[Ollitrault, Teaney, Yan, Luzum, and others]

$$\epsilon_{n,m} e^{im\psi_{n,m}} = \frac{\int dr \int_0^{2\pi} d\phi r^{n+1} e^{im\phi} w(r, \phi)}{\int dr \int_0^{2\pi} d\phi r^{n+1} w(r, \phi)}$$

- $w(r, \phi)$ completely determined by set of all $\epsilon_{n,m}$ and $\psi_{n,m}$
- closely related method is based on cumulants [Teaney, Yan]
- no positive transverse density can be associated to small set of cumulants (beyond Gaussian order) such that higher order cumulants vanish
- generalization to velocity and shear fluctuations not known

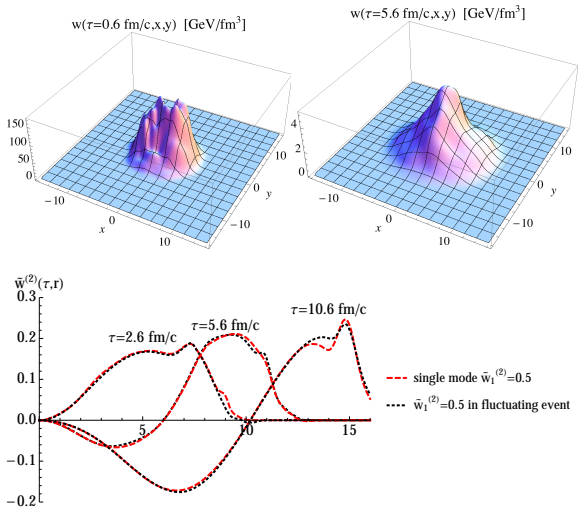
Scaling tests at third order

From symmetry considerations one expects that modes $m = 6$ receive mainly cubic contributions $\sim (\tilde{w}_1^{(2)})^3$.



Scaling tests embedded in realistic event

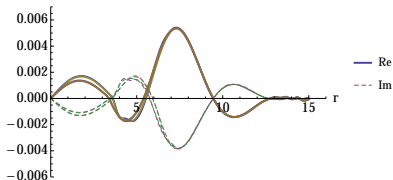
Embed mode ($m = 2, l = 1$) into realistic fluctuating event and compare to embedding into pure background.



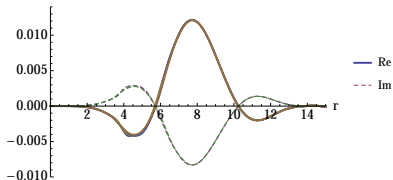
Scaling tests with several initial modes

Start with linear combination of $(m = 2, l = 2)$ and $(m = 3, l = 1)$ modes and test scaling for $m = 1$ and $m = 5$ response.

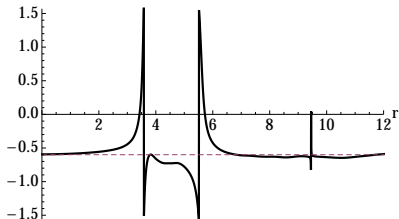
$$w_{BG}(\tau, r) \bar{w}^{(1)}(\tau, r) / |\bar{w}_2^{(2)} \bar{w}_1^{(3)}| \text{ [GeV/fm}^3\text{]}$$



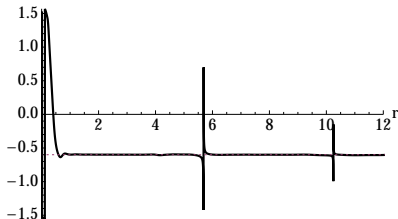
$$w_{BG}(\tau, r) \bar{w}^{(5)}(\tau, r) / |\bar{w}_2^{(2)} \bar{w}_1^{(3)}| \text{ [GeV/fm}^3\text{]}$$



$$\text{Arg}[\bar{w}^{(1)}(\tau, r)] \text{ for } \psi^{(2)}=0.0, \psi^{(3)}=-0.2$$



$$\text{Arg}[\bar{w}^{(5)}(\tau, r)] \text{ for } \psi^{(2)}=0.0, \psi^{(3)}=-0.2$$



Generalized Glauber model

- Fluctuations due to nucleon positions: used so far

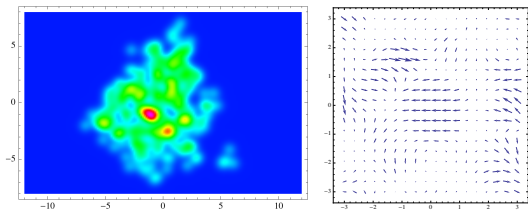
$$\epsilon(\tau, \mathbf{x}, y) = \sum_{i=1}^{N_{\text{part}}} \epsilon_w(\tau, \mathbf{x} - \mathbf{x}_i, y), \quad u^\mu = (1, 0, 0, 0)$$

- can be generalized to include also velocity fluctuations

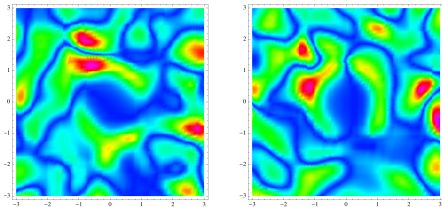
$$T^{\mu\nu}(\tau, \mathbf{x}, y) = \sum_{i=1}^{N_{\text{part}}} T_w^{\mu\nu}(\tau, \mathbf{x} - \mathbf{x}_i, y)$$

- More generally describe primordial fluid fields by
 - expectation values $\langle \epsilon(\tau_0, \mathbf{x}, y) \rangle$, $\langle u^\mu(\tau_0, \mathbf{x}, y) \rangle$, $\langle n_B(\tau_0, \mathbf{x}, y) \rangle$
 - correlation functions $\langle \epsilon(\tau_0, \mathbf{x}, y) \epsilon(\tau_0, \mathbf{x}', y') \rangle$, etc.
- Origin of this fluctuations is initial state physics and early-time, non-equilibrium dynamics.

Velocity fluctuations



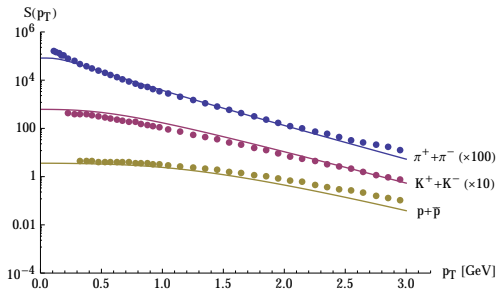
- also the velocity field will fluctuate at the initialization time τ_0
- take here transverse velocity for every participant to be Gaussian distributed with width $0.1c$
- vorticity $|\partial_1 u^2 - \partial_2 u^1|$ and divergence $|\partial_1 u^1 + \partial_2 u^2|$



“Proof of principle” study: One-particle spectrum

Initial conditions from Glauber Monte Carlo Model

$$S(p_T) = dN/(2\pi p_T dp_T d\eta d\phi)$$

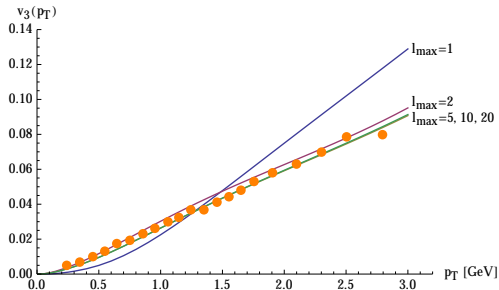


Points: 5% most central collisions, ALICE [[PRL 109, 252301 \(2012\)](#)]

Curves: Our calculation, no hadron rescattering and decays after freeze-out.

Harmonic flow coefficients for central collisions

Triangular flow for charged particles

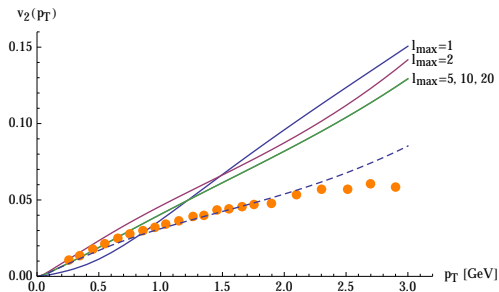


Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]

Curves: Different maximal resolution l_{\max}

Harmonic flow coefficients for central collisions

Elliptic flow for charged particles



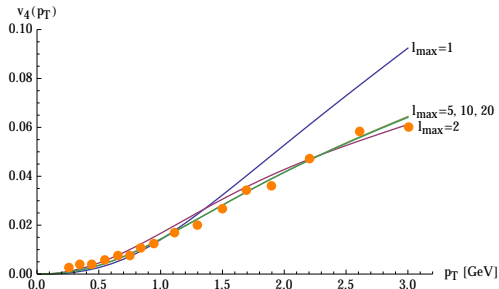
Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]

Solid curves: Different maximal resolution l_{\max}

Dashed curve: Mode ($m = 2, l = 1$) suppressed by factor 0.7

Harmonic flow coefficients for central collisions

Flow coefficient v_4 for charged particles

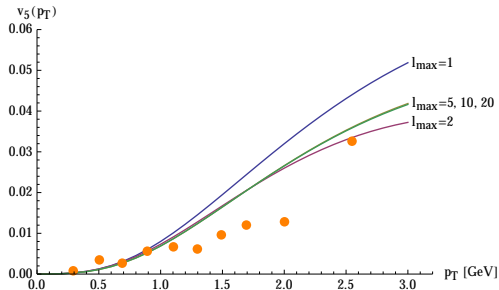


Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]

Curves: Different maximal resolution l_{\max}

Harmonic flow coefficients for central collisions

Flow coefficient v_5 for charged particles



Points: 2% most central collisions, ALICE [PRL 107, 032301 (2011)]

Curves: Different maximal resolution l_{\max}

Harmonic flow coefficients, central, particle identified

