Functional renormalization group equations and analytic continuation

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Introduction

The quantum effective action

Schwinger functional from functional integral

$$e^{W[J]} = \int D\varphi \, e^{-S[\varphi] + \int_x J\varphi}$$

- Generates *connected* correlation functions, e.g.

$$\frac{\delta^2}{\delta J(x)\delta J(y)} W[J] = \langle \varphi(x)\varphi(y) \rangle_c = \langle \varphi(x)\varphi(y) \rangle - \langle \varphi(x) \rangle \langle \varphi(y) \rangle$$

Quantum effective action by Legendre transform

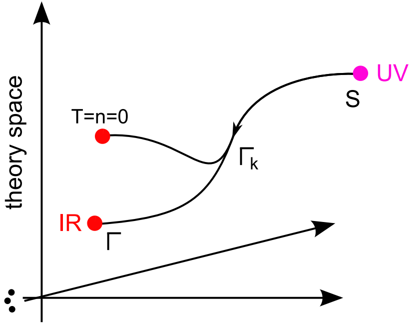
$$\Gamma[\phi] = \int_x J\phi - W[J], \qquad \text{with} \qquad \phi = \frac{\delta W[J]}{\delta J}.$$

- Generates *one-particle irreducible* correlation functions: Propagators and vertices obtained by functional derivatives of $\Gamma[\phi]$ can be used in tree-level expressions to yield *exact* correlation functions.
- Contains renormalized masses, coupling constants etc.
- Thermodynamic and transport properties follow from $\Gamma[\phi]$, as well.

How do we obtain the quantum effective action $\Gamma[\phi]$*?*

Idea of functional renormalization: $\Gamma[\phi] \to \Gamma_k[\phi]$

- k is additional infrared cutoff parameter.
- $\Gamma_k[\phi] \to S[\phi]$ for $k \to \infty$.
- $\Gamma_k[\phi] \to \Gamma[\phi]$ for $k \to 0$.



Flowing action

Modified Schwinger functional

$$e^{W_k[J]} = \int D\varphi \, e^{-S[\varphi] - \frac{1}{2}\int_p \varphi(-p)\, R_k(p)\, \varphi(p) + \int_x J\varphi}$$

- $R_k(p)$ is infrared cutoff function,

$$\begin{aligned} \lim_{p^2\to 0} R_k(p) &\approx k^2, \qquad \lim_{p^2\to\infty} R_k(p) = 0, \\ \lim_{k\to 0} R_k(p) &= 0, \qquad \lim_{k\to\infty} R_k(p) = \infty. \end{aligned}$$

Flowing action defined as

$$\Gamma_k[\phi] = \int_x J\phi - W_k[J] - \frac{1}{2}\int_p \phi(-p)\, R_k(p)\, \phi(p)$$

- Interpolates between classical action and quantum effective action

$$\begin{aligned} \lim_{k\to\infty} \Gamma_k[\phi] &= S[\phi], \\ \lim_{k\to 0} \Gamma_k[\phi] &= \Gamma[\phi]. \end{aligned}$$

How the flowing action flows

Simple and exact flow equation (Wetterich 1993)

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \mathsf{STr} \left(\Gamma_k^{(2)}[\phi] + R_k \right)^{-1} \partial_k R_k .$$

- Differential equation for a functional.
- For most cases not solvable exactly.
- Approximate solutions can be found from Truncations.
  - Ansatz for Γ_k with a finite number of parameters.
  - Derive ordinary differential equations for this parameters or couplings from the flow equation for Γ_k.
  - Solve these equations numerically.

Truncations

Main truncation schemes usually employed are

- Derivative expansion

$$\Gamma_k = \int_x \left\{ U_k(\varphi^*\varphi) + Z_k\, \varphi^*(-\partial_\mu \partial^\mu)\varphi + \ldots \right\}$$

- Vertex expansion

$$\Gamma_k = \int_q \varphi^*(q) P_k(q) \varphi(q) + \int_{q_1..q_4} A_k(q_1, .., q_4)\, \varphi^*(q_1)\, \varphi(q_2)\, \varphi^*(q_3)\, \varphi(q_4) + \ldots$$

- Momentum dependence of vertices is crucial but this gets quickly complicated...
- One needs a way to take the most important structures into account.

Problems with momentum dependence

Numerical schemes to resolve the momentum dependence face various problems

- Symmetries / Ward identities
- Numerical effort
- Singularities
- Spontaneous symmetry breaking
- Analytic continuation to real frequencies
- Unitarity and Causality
- Physical interpretation

Idea followed here: Concentrate on physical important singular structures.

Bound states

Motivation

- Formation of bound states was one of the first problems discussed in quantum mechanics

1926. № 6.

ANNALEN DER PHYSIK.

VIERTE FOLGE. BAND 79.

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1. *Quantisierung als Eigenwertproblem;*
von E. Schrödinger.

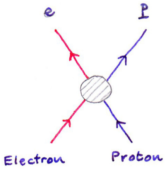
- Bound state formation is much more difficult to treat in Quantum field theory.
- Bethe-Salpeter equation can be used to sum Ladder diagrams but it is difficult to go beyond.
- Good alternatives needed!

Flow equations and Bound states

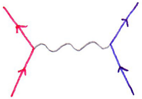
- Wetterich's flow equation was used by Ellwanger to study bound states in the Wick-Cutkosky model.
  (U. Ellwanger, Z. Phys. C **62**, 503 (1994).)
- Wegner's flow equation for Hamiltonians was used to investigate bound states in two dimensions
  (S. D. Glazek and K. G. Wilson, PRD **57**, 3558 (1998).)
- Partial bosonization and k-dependent, non-linear field transformations were used for the NJL-model
  (H. Gies and C. Wetterich, PRD **65**, 065001 (2002).)

Four point function in QED

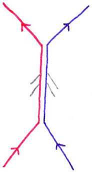
- Exact four point function in QED



- Two very different contributions
  - Photon exchange



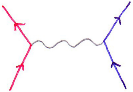
  - Bound state formation



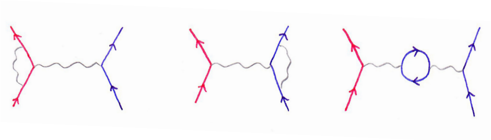
- Different physics with different description but both included in exact four-point function.

Perturbative QED point of view

- basic process

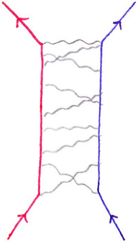


- gets renormalized by



- leads for example to

$$\frac{g-2}{2} \approx \frac{\alpha}{2\pi} \approx 0.0011614$$



- Bound state formation is non-perturbative
- Bethe-Salpeter equation allows to resum parts of this

Quantum mechanics point of view 1

- Integrate photon out, take non-relativistic limit

$$\frac{-e^2}{(\vec{p}-\vec{p}')^2} \sim \frac{-e^2}{4\pi|\vec{x}_1-\vec{x}_2|}$$

- Schrödinger equation

$$H\psi = E\psi$$

- Hamiltonian

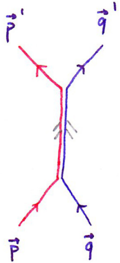
$$H = \frac{1}{2(m_e+m_P)}(\vec{p}_e+\vec{p}_P)^2 + \frac{1}{2\mu}\vec{p}_r^{\,2} + V$$

- Solution gives series of bound states

$$H\psi_{nlm} = E_n\psi_{nlm}$$
$$\psi_{nlm} = R_{nl}(r)Y_{lm}(\Omega_{\vec{r}})$$

Quantum mechanics point of view 2

- Four point function



$$\sum_{nlm} \frac{g_{nlm}(m_P\vec{p}\,' - m_e\vec{q}\,')\; g^*_{nlm}(m_P\vec{p} - m_e\vec{q})}{q_0 + p_0 - \frac{1}{2(m_e+m_P)}(\vec{p}+\vec{q})^2 - E_n}.$$

- Limits are
  - Only instantaneous interactions
  - No radiation corrections
  - Not Lorentz invariant

Unified treatment

Should describe both

- Perturbative QED (High energies / momenta)
- Bound states (Small energy / momenta)

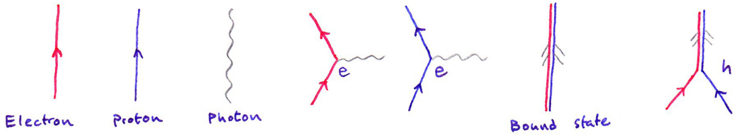
Basic ideas

- Introduce auxiliary fields for the orbitals
  - simple description of bound states
  - efficient treatment of singular momentum structure
- Keep photon exchange picture for interaction
  - retardation effects
  - radiation corrections
  - simple scattering theory for large energies
- On large scale only photon exchange
  - introduce orbitals gradually during flow

Can be done with flowing bosonization.

Flowing bosonization

- Start with QED + auxiliary fields for bound states



- Auxiliary fields decouple at the microscopic scale $h_\Lambda = 0$.
- Need one auxiliary field for every orbital $j = (n, l, m)$.
- For instantaneous photon ($c \to \infty$):
  - Yukawa vertex depends on relative velocity of electron and proton

$$h_j = h_j \left(\vec{p}/m_e - \vec{q}/m_P \right)$$

.

  - Propagator matrix depends on center of mass momentum

$$G_{jj'} = G_{jj'}(p + q).$$

Flowing bosonization with exact flow equation 1

- Exact flow equation

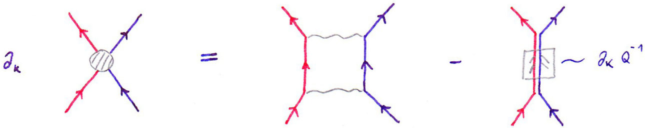
$$\partial_k \Gamma_k = \frac{1}{2}\mathsf{STr}(\Gamma_k^{(2)} + R_k)^{-1}(\partial_k R_k - R_k(\partial_k Q^{-1})R_k) - \frac{1}{2}\Gamma_k^{(1)}(\partial_k Q^{-1})\Gamma_k^{(1)}.$$

(S. Floerchinger and C. Wetterich, PLB **680**, 371 (2009).)

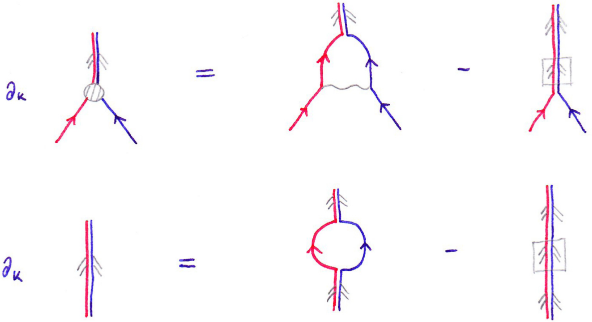
- Derived from k-dependent Hubbard-Stratonovich transformation.
- $\Gamma_k^{(1)}$ is functional derivative with respect to the composite field.
- $\partial_k Q^{-1}$ can be chosen arbitrary.

Flowing bosonization with exact flow equation 2

- Flow of four point function can be absorbed by convenient choice of $\partial_k Q^{-1}$.

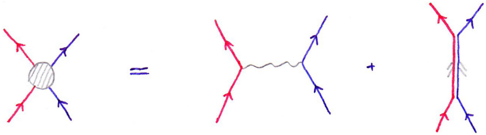


- This modifies flow of coupling h and bound state propagator



Flowing bosonization with exact flow equation 3

- For non-relativistic particles with instantaneous interaction one can solve the flow equations. Equivalence to Schrödinger equation can be shown (S. Floerchinger, Eur. Phys. J. C **69**, 119 (2010)).
- For $k = 0$ the effective four-point function has two main contributions



- Fundamental fields and composite fields are treated equal.
- This allows to treat
  - Interactions between composite fields
  - Spontaneous symmetry breaking
  - Bound states of composite fields

Analytic continuation

Why analytic continuation

- Physical propagating degrees of freedom are characterized by a pole or cut in the correlation function.
- A pole in the propagator corresponds to a stable particle, a cut corresponds to a resonance.
- Many technical methods e.g. to perform Matsubara summations use the analytic structures and at the end one needs the residue at a pole or the integral along a cut.
- Idea: Concentrate on the singular structures and describe them by as few parameters as possible.

Physics takes place in Minkowski space

- Many singular structures can only be properly seen in Minkowski space. (In Euclidean space there are some at $\vec{p} = 0$ for massless particles or at Fermi surfaces.)
- Numerical approaches have difficulties with singularities and try to avoid them as far as possible (and therefore usually work in Euclidean space).
- But: Singularities in correlation functions are physical and very important. We should not be afraid of them!
- Functional renormalization as a semi-analytic method has the potential to cope well with singularities but is mainly used in Euclidean space so far.
- Idea followed here: Derive flow equations directly for real time properties by using analytic continuation.

Different strategies for analytic continuation

- 1. Extend formalism to Minkowski space functional integral
- 2. Keep on working with Matsubara space functional integral, use analytic continuation at $k = 0$.
- 3. Keep on working with Matsubara space functional integral, use analytic continuation of flow equations.

Strategy 1: Extend formalism to Minkowski space

- some technical problems
  - factors i appear at various places
  - $-p_0^2 + \vec{p}^2$ is not positive definite: what is IR and what is UV?
  - not obvious how to choose $R_k(p)$ such that

$$\lim_{k \to \infty} \Gamma_k[\phi] = S[\phi]$$

- needs Schwinger-Keldysh closed time contour
  - technically involved formalism
  - averaging over initial density matrix sometimes difficult
- can be used also in far-from-equilibrium situations

Strategy 2: Work with functional integral in Matsubara space and use analytic continuation at $k = 0$

- can be done with numerical techniques: Padé approximants or maximal entropy methods
- numerical effort rather large
- knowledge about spectral properties does not improve RG running
- only linear response properties accessible
- some results already available:
  - N. Dupuis, PRA 80, 043627 (2009).
  - A. Sinner, N. Hasselmann, P. Kopietz, PRL 102, 120601 (2009).
  - R. Schmidt, T. Enss, Phys. Rev. A 83, 063620 (2011).
  - M. Haas, L. Fister, J. M. Pawlowski, arXiv:1308.4960

Strategy 3: Work with functional integral in Matsubara space and use analytic continuation of flow equations

- no numerical methods needed for analytical continuation
- truncations with only a few parameters that parameterize efficiently the quasi-particle properties can be used
- flow equations for real-time properties
- space-time symmetries can be preserved
- only linear response properties accessible

Follow this strategy here!

Analytic structure of the effective action

Consider the Quantum effective action

$$\Gamma[\phi] = \int_x J\phi - W[J].$$

The propagator

$$\Gamma^{(2)}(p, p') = (2\pi)^d \delta^{(d)}(p - p')\, G^{-1}(p)$$

has the Källen-Lehmann spectral representation

$$G(p) = \int_0^\infty d\mu^2\ \rho(\mu^2) \frac{1}{p^2 + \mu^2}.$$

This holds both for

- Euclidean space: $p^2 = \vec{p}^2 + p_4^2$
- Minkowski space: $p^2 = -p_0^2 + \vec{p}^2$

Propagator in Minkowski space

Consider $p_0 \in \mathbb{C}$ as complex. Close to real p_0 axis one has

- From spectral representation

$$P(p) = G(p)^{-1} = P_1(p_0^2 - \vec{p}^2) - i\, s(p_0)\, P_2(p_0^2 - \vec{p}^2)$$

  with

$$s(p_0) = \mathsf{sign}(\mathsf{Re}\, p_0)\, \mathsf{sign}(\mathsf{Im}\, p_0)$$

  and real functions P_1 and P_2.
- Nonzero P_2 leads to a branch cut in the propagator:
  The imaginary part of $P(p)$ jumps at the real p_0 axis.
- Physical implication of non-zero P_2 is non-zero decay width of quasi-particles (finite life-time).

Analytic continuation setup

- Keep on working with Euclidean space functional integral.
- Definition of Γ_k and flow equation remains unchanged,

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}(\Gamma_k^{(2)}[\phi] + R_k)^{-1} \partial_k R_k.$$

- Choose cutoff function R_k with correct properties for Euclidean argument $p^2 \geq 0$
  - $R_k(p^2) \to \infty$ for $k \to \infty$ (implies $\Gamma_k[\phi] \to S[\phi]$)
  - $R_k(p^2) \to 0$ for $k \to 0$ (implies $\Gamma_k[\phi] \to \Gamma[\phi]$)
  - $R_k(p^2) \geq 0$, $R_k(p^2) \to 0$ for $p^2 \gg k^2$
- Flow equations for n-point functions

$$\Gamma_k^{(n)}(p_1, ..., p_n)$$

  are analytically continued towards the real frequency axis.
- Truncation uses expansion around real p_0 (Minkowski space).

Derivative expansion in Minkowski space

- Consider a point $p_0^2 - \vec{p}^2 = m^2$ where $P_1(m^2) = 0$.
- One can expand around this point

$$P_1 = Z(-p_0^2 + \vec{p}^2 + m^2) + \cdots$$
$$P_2 = Z\gamma^2 + \cdots$$

- Leads to Breit-Wigner form of propagator (with $\gamma^2 = m\Gamma$)

$$G(p) = \frac{1}{Z}\frac{-p_0^2 + \vec{p}^2 + m^2 + i\, s(p_0)\, m\Gamma}{(-p_0^2 + \vec{p}^2 + m^2)^2 + m^2\Gamma^2}.$$

- A few flowing parameters describe efficiently the singular structure of the propagator.

Choosing a regulator

- The analytic properties of correlation functions at $k > 0$ depend on the choice of $R_k(p)$.
- One would like to perform loop integrations analytically as far as possible to facilitate analytic continuation.
- Useful are the following choices

$$R_k(p_0, \vec{p}) = Zk^2 \frac{1}{1 + c_1 \left(\frac{-p_0^2 + \vec{p}^2}{k^2}\right) + c_2 \left(\frac{-p_0^2 + \vec{p}^2}{k^2}\right)^2 + \ldots}.$$

- Allows to do the Matsubara summations analytically for truncation based on derivative expansion.

Truncation for relativistic scalar $O(N)$ theory

$$\Gamma_k = \int_{t,\vec{x}} \left\{ \sum_{j=1}^{N} \frac{1}{2} \bar{\phi}_j \, \bar{P}_\phi(i\partial_t, -i\vec{\nabla}) \, \bar{\phi}_j + \frac{1}{4} \bar{\rho} \, \bar{P}_\rho(i\partial_t, -i\vec{\nabla}) \, \bar{\rho} + \bar{U}_k(\bar{\rho}) \right\}$$

with $\bar{\rho} = \frac{1}{2} \sum_{j=1}^{N} \bar{\phi}_j^2$.

- Goldstone propagator massless, expanded around $p_0 - \vec{p}^2 = 0$

$$\bar{P}_\phi(p_0, \vec{p}) \approx \bar{Z}_\phi \, (-p_0^2 + \vec{p}^2)$$

- Radial mode is massive, expanded around $p_0^2 - \vec{p}^2 = m_1^2$

$$\bar{P}_\phi(p_0, \vec{p}) + \bar{\rho}_0 \bar{P}_\rho(p_0, \vec{p}) + \bar{U}_k' + 2\bar{\rho}\bar{U}_k'' \approx \bar{Z}_\phi Z_1 \left[(-p_0^2 + \vec{p}^2 + m_1^2) - is(p_0) \, \gamma_1^2 \right]$$

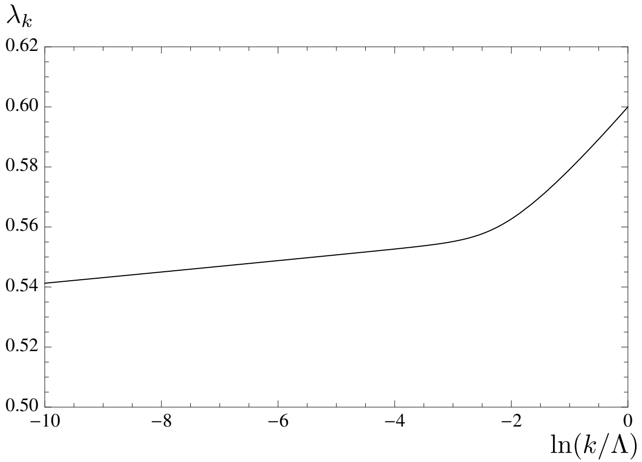
Flow of the effective potential

$$\partial_t U_k(\rho)\big|_{\bar\rho} = \frac{1}{2}\int_{p_0=i\omega_n,\vec{p}} \left\{ \frac{(N-1)}{\vec{p}^2 - p_0^2 + U' + \frac{1}{\bar{Z}_\phi}R_k} + \frac{1}{Z_1\left[(\vec{p}^2 - p_0^2) - i\,s(p_0)\gamma_1^2\right] + U' + 2\rho U'' + \frac{1}{\bar{Z}_\phi}R_k} \right\} \frac{1}{\bar{Z}_\phi}\partial_t R_k .$$

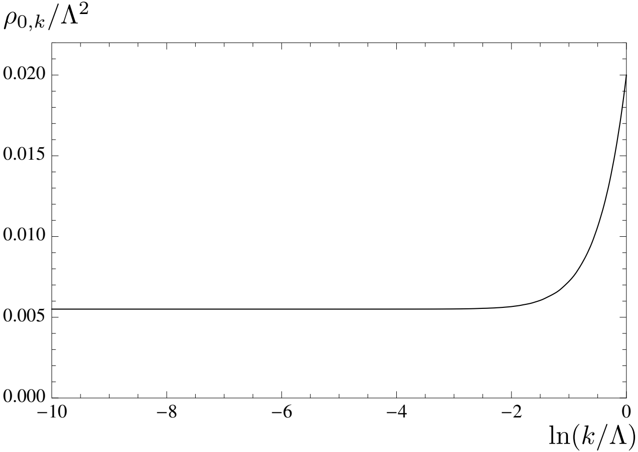
- Summation over Matsubara frequencies $p_0 = i2\pi T n$ can be done using contour integrals.
- Radial mode has non-zero decay width since it can decay into Goldstone excitations.
- Use Taylor expansion for numerical calculations

$$U_k(\rho) = U_k(\rho_{0,k}) + m_k^2(\rho - \rho_{0,k}) + \frac{1}{2}\lambda_k(\rho - \rho_{0,k})^2$$

Flow of the interaction strength λ_k



Flow of the minimum of the effective potential $\rho_{0,k}$



Flow of the propagator

- Goldstone mode propagator characterized by anomalous dimension

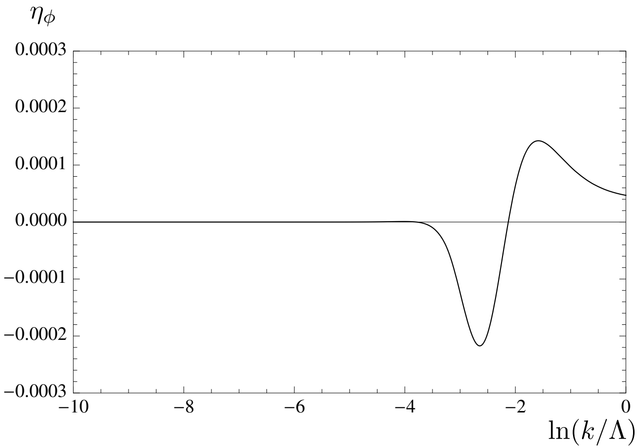
$$\eta_\phi = -\frac{1}{\bar{Z}_\phi} k\partial_k \bar{Z}_\phi$$

- Radial mode propagator

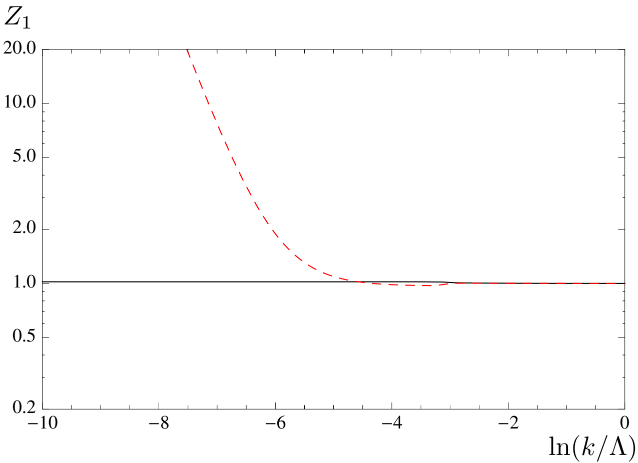
$$G_1 = \frac{1}{Z_1\left[(-p_0^2 + \vec{p}^2) - is(p_0)\gamma_1^2\right] + 2\lambda_k \rho_0^2}$$

  - flow equation for Z_1 is evaluated in the standard way
  - flow equation for γ_1^2 is evaluated from discontinuity at $p_0 = m_1 \pm i\epsilon$

Anomalous dimension η_ϕ

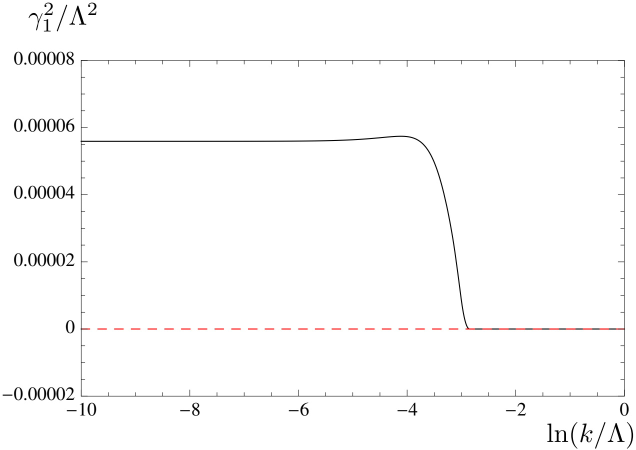


Flow of the coefficient Z_1



- black solid line: evaluation at $p_0 = m_1$
- red dashed line: evaluation at $p_0 = 0$

Flow of the discontinuity coefficient γ_1^2



- black solid line: evaluation at $p_0 = m_1$
- red dashed line: evaluation at $p_0 = 0$

Conclusions

Conclusions

- Functional renormalization is powerful method for non-perturbative QFT studies.
- Analytic continuation allows to access directly physical information in real time.
- Together with k-dependent Hubbard-Stratonovich transformation this will allow for efficient truncations with few parameters taking all singular structures into account.
- Bound states can be treated as well.
- Allows unified treatment of fundamental and composite fields.
- Would be interesting to make connection to RG in light cone coordinates (DGLAP).