Analytic continuation of functional renormalization group equations

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based on

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What is functional renormalization?

- Gives a formulation of quantum and statistical field theories.
- Tool to solve difficult non-perturbative problems and answer questions such as
  - What are the critical exponents at classical phase transitions?
  - What are the phases of the Hubbard model?
  - Is Gravity asymptotically safe?
- But is it only a reformulation and a tool for special purposes or is it more?
- Here I want to argue: Functional RG is much more and can be used to solve one of the biggest problems in modern physics!

The complexity problem

Arises in many ways in modern physics (and other sciences...):

- Many degrees of freedom.
- Fundamental or microscopic laws are known.
- Consequences of the fundamental laws for the macroscopic or collective behavior are not known.
- Calculations are simply getting too complex.

What we aim for

- Simple but precise macroscopic laws.
- They should be derived from microscopic laws including values for all relevant coupling constants.
- Real theoretical understanding of complex phenomena and not only numerical simulations.
- A formalism that is sufficiently general to be used for a large class of problems and is not based on specific *a priori* knowledge from other approaches or experiments.

How to reduce the complex to the essential?

- We have to loose information. But which one?
- RG theory can provide information on this: Think about classification of coupling constants into relevant, marginal and irrelevant close to a Gaussian fixed point.
- But: Exact functional RG equation alone does not yet solve the complexity problem!
- We need: Simple and efficient approximate solutions.
- From experience: Quantum field theories at a particular scale often well described in terms of some sort of quasi-particles:
  - May be composite particles or collective fields.
  - Different scales can be dominated by different collective fields.
  - Transition regions are more complicated.
  - A formalism that uses this could be rather helpful.
- How to find the right composite fields?
- How to describe them efficiently?

Singular structures matter

- Physical propagating degrees of freedom are characterized by a pole or cut in the correlation function.
- A pole in the propagator corresponds to a stable particle, a cut corresponds to a resonance.
- Many technical methods e.g. to perform Matsubara summations use the analytic structures and at the end one needs the residue at a pole or the integral along a cut.
- Idea: Concentrate on the singular structures and describe them by as few parameters as possible.
- Singular structures in vertex functions can be described efficiently using scale-dependent Hubbard-Stratonovich transformations.

Physics takes place in Minkowski space

- Many singular structures can only be properly seen in Minkowski space. (In Euclidean space there are some at $\vec{p} = 0$ for massless particles or at Fermi surfaces.)
- Numerical approaches have difficulties with singularities and try to avoid them as far as possible (and therefore usually work in Euclidean space).
- But: Singularities in correlation functions are physical and very important. We should not be afraid of them!
- Functional renormalization as a semi-analytic method has the potential to cope well with singularities but is mainly used in Euclidean space so far.
- Idea followed here: Derive flow equations directly for real time properties by using analytic continuation.

Analytic structure of the effective action

Consider the Quantum effective action

$$\Gamma[\phi] = \int_x J\phi - W[J].$$

The propagator

$$\Gamma^{(2)}(p, p') = (2\pi)^d \delta^{(d)}(p - p')\; G^{-1}(p)$$

has the Källen-Lehmann spectral representation

$$G(p) = \int_0^\infty d\mu^2\; \rho(\mu^2) \frac{1}{p^2 + \mu^2}.$$

This holds both for

- Euclidean space: $p^2 = \vec{p}^2 + p_4^2$
- Minkowski space: $p^2 = -p_0^2 + \vec{p}^2$

Propagator in Minkowski space

Consider $p_0 \in \mathbb{C}$ as complex. Close to real p_0 axis one has

- From spectral representation

$$P(p) = G(p)^{-1} = P_1(p_0^2 - \vec{p}^2) - i\, s(p_0)\, P_2(p_0^2 - \vec{p}^2)$$

  with

$$s(p_0) = \mathrm{sign}(\mathrm{Re}\, p_0)\, \mathrm{sign}(\mathrm{Im}\, p_0)$$

  and real functions P_1 and P_2.

- Nonzero P_2 leads to a branch cut in the propagator: The imaginary part of $P(p)$ jumps at the real p_0 axis.
- Physical implication of non-zero P_2 is non-zero decay width of quasi-particles (finite life-time).

Analytic continuation setup

- Keep on working with Euclidean space functional integral.
- Definition of Γ_k and flow equation remains unchanged,

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}(\Gamma_k^{(2)}[\phi] + R_k)^{-1} \partial_k R_k.$$

- Choose cutoff function R_k with correct properties for Euclidean argument $p^2 \geq 0$
  - $R_k(p^2) \to \infty$ for $k \to \infty$ (implies $\Gamma_k[\phi] \to S[\phi]$)
  - $R_k(p^2) \to 0$ for $k \to 0$ (implies $\Gamma_k[\phi] \to \Gamma[\phi]$)
  - $R_k(p^2) \geq 0$, $R_k(p^2) \to 0$ for $p^2 \gg k^2$
- Flow equations for n-point functions

$$\Gamma_k^{(n)}(p_1, ..., p_n)$$

  are analytically continued towards the real frequency axis.
- Truncation uses expansion around real p_0 (Minkowski space).

Derivative expansion in Minkowski space

- Consider a point $p_0^2 - \vec{p}^2 = m^2$ where $P_1(m^2) = 0$.
- One can expand around this point

$$P_1 = Z(-p_0^2 + \vec{p}^2 + m^2) + \cdots$$
$$P_2 = Z\gamma^2 + \cdots$$

- Leads to Breit-Wigner form of propagator (with $\gamma^2 = m\Gamma$)

$$G(p) = \frac{1}{Z}\frac{-p_0^2 + \vec{p}^2 + m^2 + i\,s(p_0)\,m\Gamma}{(-p_0^2 + \vec{p}^2 + m^2)^2 + m^2\Gamma^2}.$$

- A few flowing parameters describe efficiently the singular structure of the propagator.

Choosing a regulator

- The analytic properties of correlation functions at $k > 0$ depend on the choice of $R_k(p)$.
- One would like to perform loop integrations analytically as far as possible to facilitate analytic continuation.
- Useful are the following choices

$$R_k(p_0, \vec{p}) = Zk^2 \frac{1}{1 + c_1 \left(\frac{-p_0^2 + \vec{p}^2}{k^2}\right) + c_2 \left(\frac{-p_0^2 + \vec{p}^2}{k^2}\right)^2 + \dots}.$$

- Allows to do the Matsubara summations analytically for truncation based on derivative expansion.

Truncation for relativistic scalar $O(N)$ theory

$$\Gamma_k = \int_{t,\vec{x}} \left\{ \sum_{j=1}^{N} \frac{1}{2}\bar{\phi}_j \, \bar{P}_\phi(i\partial_t, -i\vec{\nabla}) \, \bar{\phi}_j + \frac{1}{4}\bar{\rho} \, \bar{P}_\rho(i\partial_t, -i\vec{\nabla}) \, \bar{\rho} + \bar{U}_k(\bar{\rho}) \right\}$$

with $\bar{\rho} = \frac{1}{2}\sum_{j=1}^{N} \bar{\phi}_j^2$.

- Goldstone propagator massless, expanded around $p_0 - \vec{p}^2 = 0$

$$\bar{P}_\phi(p_0, \vec{p}) \approx \bar{Z}_\phi \, (-p_0^2 + \vec{p}^2)$$

- Radial mode is massive, expanded around $p_0^2 - \vec{p}^2 = m_1^2$

$$\bar{P}_\phi(p_0, \vec{p}) + \bar{\rho}_0 \bar{P}_\rho(p_0, \vec{p}) + \bar{U}_k' + 2\bar{\rho}\bar{U}_k'' \approx \bar{Z}_\phi Z_1 \left[(-p_0^2 + \vec{p}^2 + m_1^2) - is(p_0) \, \gamma_1^2 \right]$$

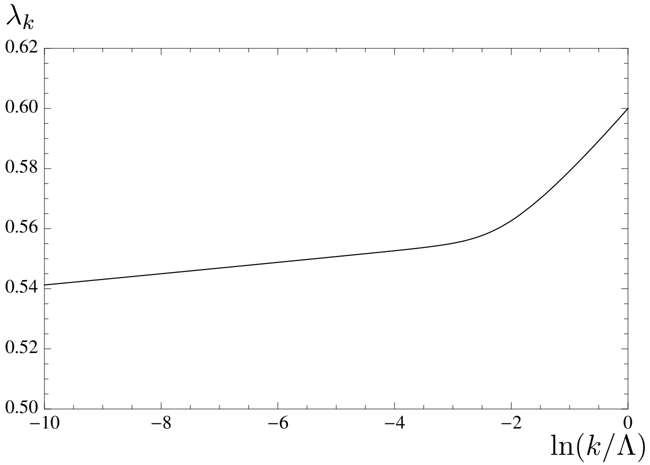
Flow of the effective potential

$$\partial_t U_k(\rho)\big|_{\bar\rho} = \frac{1}{2}\int_{p_0=i\omega_n,\vec p}\left\{\frac{(N-1)}{\vec p^2 - p_0^2 + U' + \frac{1}{\bar Z_\phi}R_k}\right.$$
$$\left. + \frac{1}{Z_1\left[(\vec p^2 - p_0^2) - i\, s(p_0)\gamma_1^2\right] + U' + 2\rho U'' + \frac{1}{\bar Z_\phi}R_k}\right\}\frac{1}{\bar Z_\phi}\partial_t R_k.$$

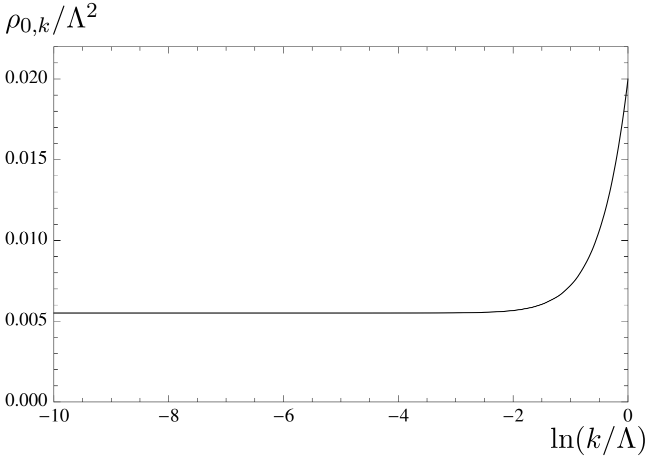
- Summation over Matsubara frequencies $p_0 = i2\pi T n$ can be done using contour integrals.
- Radial mode has non-zero decay width since it can decay into Goldstone excitations.
- Use Taylor expansion for numerical calculations

$$U_k(\rho) = U_k(\rho_{0,k}) + m_k^2(\rho - \rho_{0,k}) + \frac{1}{2}\lambda_k(\rho - \rho_{0,k})^2$$

Flow of the interaction strength λ_k



Flow of the minimum of the effective potential $\rho_{0,k}$



Flow of the propagator

- Goldstone mode propagator characterized by anomalous dimension

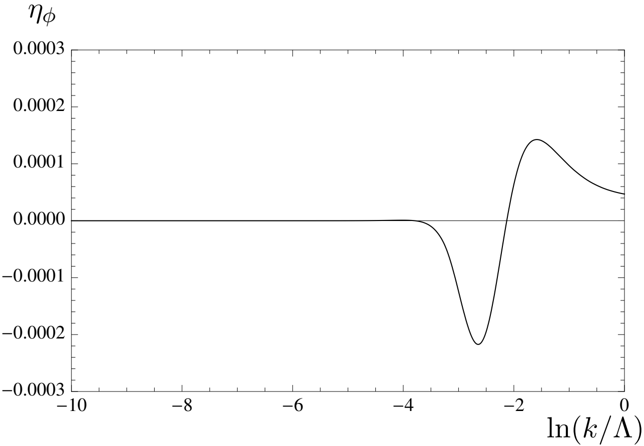
$$\eta_\phi = -\frac{1}{\bar{Z}_\phi} k \partial_k \bar{Z}_\phi$$

- Radial mode propagator

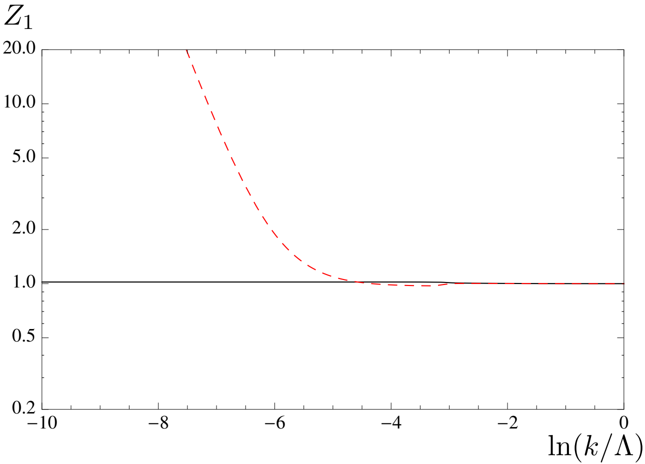
$$G_1 = \frac{1}{Z_1 \left[(-p_0^2 + \vec{p}^2) - i s(p_0) \gamma_1^2 \right] + 2\lambda_k \rho_0^2}$$

  - flow equation for Z_1 is evaluated in the standard way
  - flow equation for γ_1^2 is evaluated from discontinuity at $p_0 = m_1 \pm i\epsilon$

Anomalous dimension η_ϕ

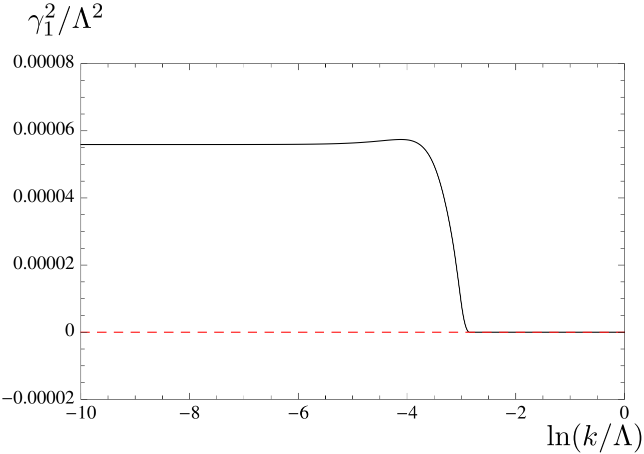


Flow of the coefficient Z_1



- black solid line: evaluation at $p_0 = m_1$
- red dashed line: evaluation at $p_0 = 0$

Flow of the discontinuity coefficient γ_1^2



- black solid line: evaluation at $p_0 = m_1$
- red dashed line: evaluation at $p_0 = 0$

Conclusions

- Analytic continuation of flow equations is now possible.
- An improved derivative expansion in Minkowski space was developed.
- Many dynamical and linear response properties can now be calculated from functional renormalization.
- Together with k-dependent Hubbard-Stratonovich transformation this will allow for efficient truncations with few parameteres taking all singular structures into account.
- Usefulness of formalism must be proven in applications.