Analytic continuation of functional renormalization group equations

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Short outline

- Quantum effective action and its analytic continuation
- Functional renormalization and exact flow equation
- Regulator function and analytic continuation
- Application to scalar $O(N)$ model

Quantum effective action and its analytic continuation

Partition and Schwinger functionals

$$Z[J] = e^{W[J]} = \int D\varphi \, e^{-S[\varphi] + \int_x J\varphi}$$

with

$$S[\varphi] = \int_0^{1/T} \int d^3x \, \mathcal{L}\left(\varphi(x), \partial_\mu \varphi(x)\right)$$

- generating functional of (connected) correlation functions
- contains contributions from all orders in perturbation theory

Quantum effective action

$$\Gamma[\phi] = \int J\phi - W[J]$$

with

$$\phi(x) = \frac{\delta}{\delta J(x)} W[J] = \langle \varphi(x) \rangle$$

- generating functional of one-particle irreducible diagrams
- exact on tree level
- contains renormalized parameters (masses, coupling constants, decay width,...)
- contains full inverse propagator $G^{-1}(p)$

$$\Gamma^{(2)}(p, p') = \frac{\delta}{\delta\phi(-p)} \frac{\delta}{\delta\phi(p')} \Gamma[\phi] = (2\pi)^d \, \delta^{(d)}(p - p') \;\; G^{-1}(p)$$

Källen-Lehmann spectral representation I

For standard quantum field theories

$$S[\varphi] = \int d^dx \, \mathcal{L}(\varphi, \partial_\mu \varphi)$$

one can derive from basic principles in operator picture

- space-time symmetry
- causality
- unitarity

$$G(p) = \int_0^\infty d\mu^2 \; \rho(\mu^2) \frac{1}{p^2 + \mu^2}$$

- holds for Euclidean space $p^2 = p_0^2 + \vec{p}^2$
- and Minkowski space $p^2 = -p_0^2 + \vec{p}^2$

Källen-Lehmann spectral representation II

$$G(p) = \int_0^\infty d\mu^2\ \rho(\mu^2)\ \frac{1}{2\sqrt{\vec{p}^2+\mu^2}} \times \left(\frac{1}{-p_0+\sqrt{\vec{p}^2+\mu^2}} - \frac{1}{-p_0-\sqrt{\vec{p}^2+\mu^2}} \right)$$

- all singularities and cuts are on the real frequency axis
- different variants of propagators correspond to different integration contours in the complex frequency plane
- close to the real frequency axis

$$G(p) = \underbrace{\int_0^\infty d\mu^2\ \rho(\mu^2)\ \mathcal{P}\,\frac{1}{-p_0^2+\vec{p}^2+\mu^2}}_{\in\mathbb{R}} + i\,\pi\ \underbrace{\underbrace{\mathsf{sign}(\mathsf{Re}\,p_0)\,\mathsf{sign}(\mathsf{Im}\,p_0)}_{=:s(p_0)}\ \rho(p_0^2-\vec{p}^2)}_{\in\mathbb{R}}$$

Inverse propagator

- close to real p_0 axis

$$P(p) = G^{-1}(p) = \underbrace{P_1(p_0^2 - \vec{p}^2)}_{\in \mathbb{R}} - is(p_0)\underbrace{P_2(p_0^2 - \vec{p}^2)}_{\geq 0}$$

- physical excitations correspond to singularities in $G(p)$ or minima in $P(p)$
- close to a point where $P_1(p)$ vanishes one can expand

$$P_1 = Z(-p_0^2 + \vec{p}^2 + m^2) + \dots$$
$$P_2 = Z\gamma^2 + \dots$$

- gives propagator of Breit-Wigner form

$$G(p) = \frac{1}{Z}\frac{-p_0^2 + \vec{p}^2 + m^2 + is(p_0)\gamma^2}{(-p_0^2 + \vec{p}^2 + m^2)^2 + \gamma^4}$$

Functional renormalization and exact flow equation

Modified partition and Schwinger functionals

in Euclidean space

$$Z_k[J] = e^{W_k[J]} = \int D\varphi \; e^{-S[\varphi] - \Delta S_k[\varphi] + \int J\varphi}$$

with infrared regulator term

$$\Delta S_k[\varphi] = \frac{1}{2} \int_p \varphi(-p) R_k(p) \varphi(p)$$

- R_k depends on $p_0^2 + \vec{p}^2$
- is real and positive
- decays for large $p_0^2 + \vec{p}^2$

Flowing action

$$\Gamma_k[\phi] = \underbrace{\int J\phi - W_k[J]}_{\text{Legendre transform}} - \Delta S_k[\phi]$$

has properties

$$\lim_{k\to\infty} \Gamma_k[\phi] = S[\phi] + \text{one loop}$$
$$\lim_{k\to 0} \Gamma_k[\phi] = \Gamma[\phi]$$

and fulfills exact flow equation (Wetterich 1993)

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr}(\Gamma_k^{(2)} + R_k)^{-1} \partial_k R_k$$

(equivalent to infinity hierarchy of flow equations for n-point functions)

Derivative expansion

- flow is "local" in momentum space due to $\partial_k R_k(p)$
- acts as effective UV regulator if $R_k(p)$ decays sufficiently fast for large $p_0^2 + \vec{p}^2$
- derivative expansion works often well since only IR modes contribute to flow

$$\Gamma_k = \int d^d x \left\{ U_k(\rho) + \frac{1}{2} Z_k \vec{\nabla}\phi \vec{\nabla}\phi + \frac{1}{4} Y_k \vec{\nabla}\rho \vec{\nabla}\rho + \dots \right\}$$

Regulator function and analytic continuation

Why analytic continuation?

- Functional renormalization works well in Euclidean/Matsubara space but many physical quantities are easier to access in Minkowski space
- Examples:
  - spectral density and particle decay width
  - transport properties such as conductivities, viscosities etc.
- Different strategies to access real time properties
  - extend formalism to Minkowski space functional integral
  - keep on working with Matsubara space functional integral, use analytic continuation at $k = 0$
  - keep on working with Matsubara space functional integral, use analytic continuation of flow equations

Strategy 1: Extend formalism to Minkowski space

- some technical problems
  - factors i appear at various places
  - $-p_0^2 + \vec{p}^2$ is not positive definite: what is IR and what is UV?
  - not obvious how to choose $R_k(p)$ such that

$$\lim_{k\to\infty} \Gamma_k[\phi] = S[\phi]$$

- needs Schwinger-Keldysh closed time contour
  - technically involved formalism
  - averaging over initial density matrix sometimes difficult
- can be used also in far-from-equilibrium situations

Strategy 2: Work with functional integral in Matsubara space and use analytic continuation at $k = 0$

- can be done with numerical techniques: Padé approximants
- numerical effort rather large
- knowledge about spectral properties does not improve RG running
- only linear response properties accessible

Strategy 3: Work with functional integral in Matsubara space and use analytic continuation of flow equations

- no numerical methods needed for analytical continuation
- truncations with only a few parameters that parameterize efficiently the quasi-particle properties can be used
- flow equations for real-time properties
- space-time symmetries can be preserved
- only linear response properties accessible

Follow this strategy here!

Choosing a regulator function 1

- following discussion for truncations where inverse propagator is of the form

$$P_k = Z\left(-p_0^2 + \vec{p}^2 + m^2 - is(p_0)\gamma^2\right)$$

- choose regulator function such that flow equations can be analytically continued
- requirements on R_k in Minkowski space a priori not clear
- here: choose cutoff that has all desired properties for Euclidean arguments and use analytic continuation
- not possible for all regulator functions: $R_k(p)$ that is smooth for Euclidean argument can have singularities, poles, cuts etc. in the complex plane!

Choosing a regulator function 2

- choose here a class of rather simple regulator functions

$$R_k(p) = \frac{Z\ k^2}{1 + c_1 \left(\frac{-p_0^2 + \vec{p}^2}{k^2}\right) + c_2 \left(\frac{-p_0^2 + \vec{p}^2}{k^2}\right)^2 + \dots}$$

  with $c_i \geq 0$. Simplest non-trivial choice is $c_1 = c \geq 0$, $c_2 = c_3 = \cdots = 0$.

- allows for convenient decomposition

$$(P_k + R_k)^{-1} = \frac{1}{Z} \left(\frac{\beta_1}{p^2 + \alpha_1 k^2} + \frac{\beta_2}{p^2 + \alpha_2 k^2} + \dots \right)$$

  with complex coefficients β_i, α_i that depend also on $s(p_0)$.

Application to $O(N)$ model

Model & truncation

Microscopic action for $O(N)$ model

$$S = \int_0^{iT} dt \int d^3x \left\{ \sum_{m=1}^{N} \frac{1}{2}\phi_m(\partial_\tau^2 - \vec{\nabla}^2)\phi_m + \frac{1}{2}\lambda\,\rho^2 \right\}$$

Truncation

$$\Gamma_k = \int \Bigg\{ \sum_{m=1}^{N} \frac{1}{2}\phi_m\, P_\phi\left(i\partial_t, -i\vec{\nabla}\right)\, \phi_m + \frac{1}{4}\rho\, P_\rho\left(i\partial_t, -i\vec{\nabla}\right)\, \rho + U_k(\rho) \Bigg\}$$

with $\rho = \frac{1}{2}\sum_{m=1}^{N} \phi_m^2$.

Effective propagators

Expand around background field

$$\phi_1 = \phi_0 + \delta\phi_1, \quad \phi_1 = \delta\phi_2, \ldots$$

and keep only quadratic part

$$\Gamma_{k,2} = \int_p \left\{ \frac{1}{2}\delta\phi_1 \left[P_\phi + \rho P_\rho + U' + 2\rho U''\right] \delta\phi_1 + \sum_{m=2}^{N} \frac{1}{2}\delta\phi_m \left[P_\phi + U'\right] \delta\phi_m \right\}$$

use also the decomposition

$$P_\phi(p_0, \vec{p}) = Z_\phi(-p_0^2 + \vec{p}^2) \left[-p_0^2 + \vec{p}^2\right] - is(p_0)\gamma_\phi^2(-p_0^2 + \vec{p}^2)$$

and similar for P_ρ.

Goldstone propagator

- at minimum $U' = 0$ and Goldstone mode is massless
- use expansion point $-p_0^2 + \vec{p}^2 = 0$
- imaginary part vanishes $\gamma_\phi(0) = 0$
- Goldstone propagator:

$$\frac{1}{Z_\phi(0)(-p_0^2 + \vec{p}^2) + U'}$$

- flow can be characterized in terms of the anomalous dimension

$$\eta_\phi = -\frac{1}{Z_\phi(0)} k\partial_k Z_\phi(0)$$

Radial mode (or Higgs mode) propagator

- radial mode is massive
- use expansion point

$$p_0 = \sqrt{\vec{p}^2 + m^2}$$

- and write there

$$P_\phi + \rho P_\rho \approx Z_1 \left[-p_0^2 + \vec{p}^2 \right] - is(p_0)\gamma_1^2$$

- decay width is due to possible decay of radial excitation into two Goldstone excitations
- characterized by flow equations for Z_1, γ_1 and U_k

Flow of the effective potential

$$\partial_k U_k = \frac{1}{2}\int_{p_0=i\omega_n}\int_{\vec{p}}\Bigg\{\frac{1}{Z_1(\vec{p}^2-p_0^2)-is(p_0)\gamma_1^2+U'+2\rho U''+\frac{1}{Z}R_k} + \frac{1}{\vec{p}^2-p_0^2+U'+\frac{1}{Z}R_k}\Bigg\}\frac{1}{Z}\partial_k R_k$$

- frequencies are summed over the discrete Matsubara values $p_0 = i\omega_n = i2\pi T n$
- one should not take the above expression literally for Matsubara frequencies: $s(p_0)$ only defined close to real axis
- use contour integration methods to perform the Matsubara summation: boils down to evaluating residues and integrals along branch cuts close to the real frequency axis!

Numerical results

- only first application of formalism for illustration
- for $O(2)$ model in $3+1$ dimensions at $T=0$
- for expansion of effective potential around minimum

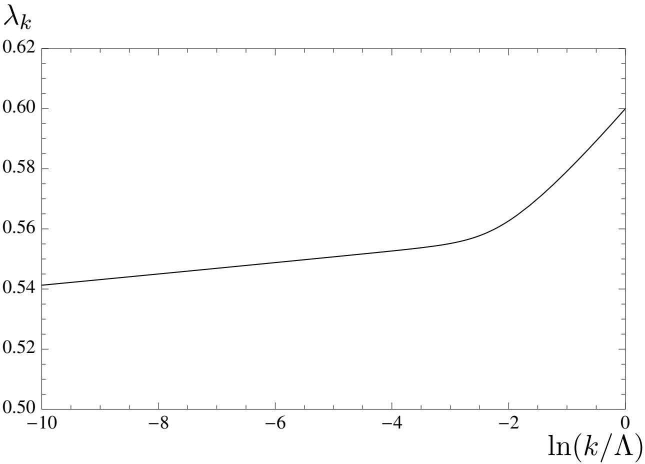
$$U_k(\rho) = U_k(\rho_0) + \frac{1}{2}\lambda_k \left(\rho - \rho_{0,k}\right)^2$$

- truncation with the parameters

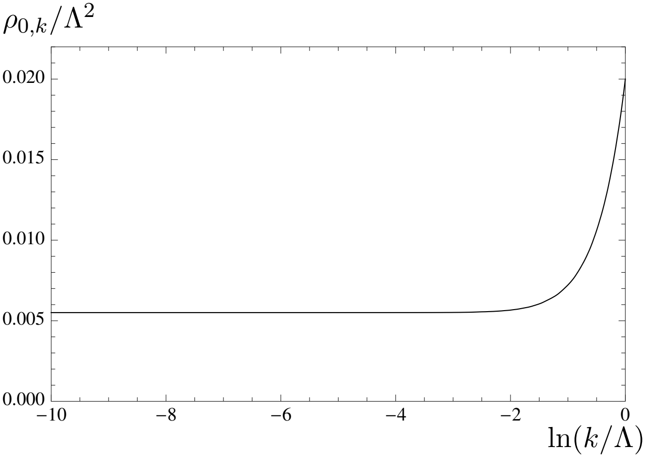
$$\lambda_k, \quad \rho_0, \quad Z_1, \quad \gamma_1$$

supplemented by the anomalous dimension

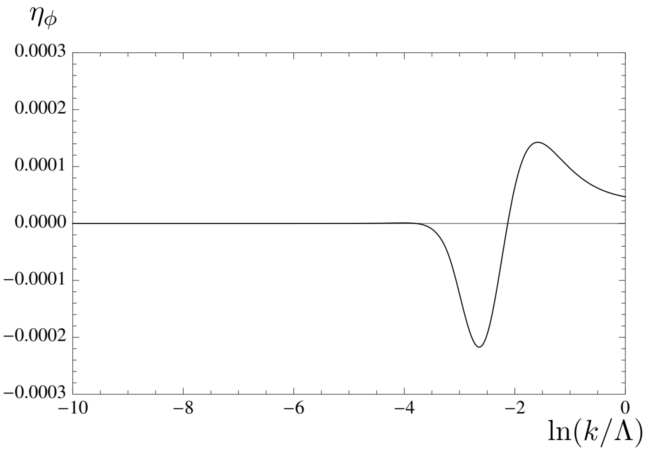
$$\eta_\phi = -\frac{1}{Z_\phi} k\partial_k Z_\phi$$



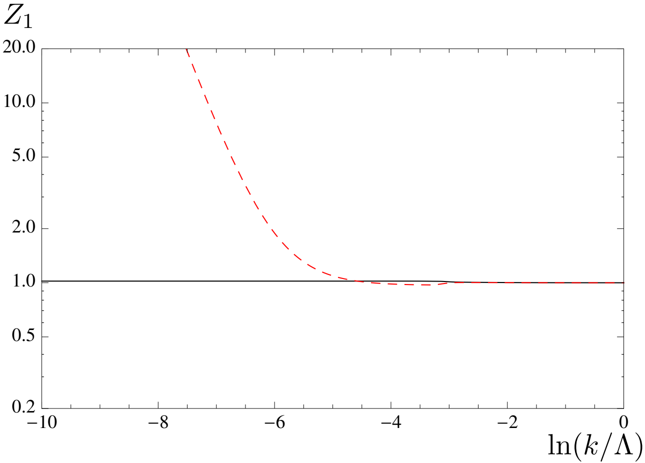
Flow of the interaction strength λ_k.



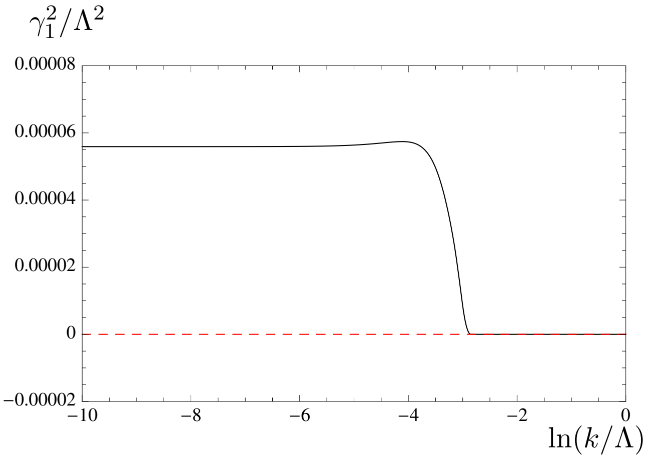
Flow of the minimum of the effective potential $\rho_{0,k}$.



Anomalous dimension η_ϕ.



Flow of the coefficient Z_1 (solid line). We also show the resulting behavior if the flow equation is evaluated at $q_0 = 0$ instead (dashed line). Interestingly, one finds $Z_1 \to \infty$ for $k \to 0$ in the latter case whereas the result is completely regular if the flow equation is evaluated on-shell.



Flow of the discontinuity coefficient γ_1^2 (solid line). We also show the resulting behavior if the flow equation is evaluated at $q_0 = 0$ instead (dashed line). As expected, the discontinuity γ_1^2 is non-zero on-shell whereas it vanishes for $q_0 = 0$.

Conclusions

- Analytic continuation of flow equations works in praxis
- Improved derivative expansion in Minkowski space possible
- Many dynamical and linear response properties can now be calculated from this formalism