# *Turbulent fluctuations around Bjorken flow*

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work together with

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Quark Matter Annecy 2011

## *Why are fluctuations interesting?*

- "Standard model of heavy ion collisions" based on almost ideal hydrodynamics works rather well.
- This is also a puzzle:
  - Why is equilibration so fast?
  - Is there turbulence due to small viscosity?
- **Hydrodynamic fluctuations**: Local and event-by-event perturbations around the average of hydrodynamical fields:
  - energy density $\epsilon$
  - fluid velocity $u^\mu$
- Measure for deviations from equilibrium
- Contain interesting information from early times
- Might affect other phenomena, e.g. jet quenching

# *Theoretical framework*

- An ensemble average over many events with fixed impact parameter $b$ is described by smooth hydrodynamical fields

$$\bar{\epsilon} = \langle \epsilon \rangle$$
$$\bar{u}^{\mu} = \langle u^{\mu} \rangle$$

- Fluctuations are added on top

$$\epsilon = \bar{\epsilon} + \delta\epsilon$$
$$u^{\mu} = \bar{u}^{\mu} + \delta u^{\mu}$$

- Here we use Bjorkens model

$$\bar{\epsilon} = \bar{\epsilon}(\tau)$$
$$\bar{u}^{\mu} = (1, 0, 0, 0)$$
$$u^{\mu} = \bar{u}^{\mu} + \left(\delta u^{\tau}, u^{1}, u^{2}, u^{y}\right)$$

(in coordinates $\tau = \sqrt{(x^0)^2 - (x^3)^2}$, $x^1$, $x^2$, $y = \text{arctanh}(x^3/x^0)$)

# *Linear fluctuations*

- Consider only terms linear in $\delta\epsilon,\ (u^1, u^2, u^y)$
- We decompose velocity field into
  - gradient term, described by divergence

$$\vartheta = \partial_1 u^1 + \partial_2 u^2 + \partial_y u^y$$

  - rotation term, described by vorticity

$$\begin{aligned}
\omega_1 &= \tau\,\partial_2 u^y - \frac{1}{\tau}\partial_y u^2 \\
\omega_2 &= \frac{1}{\tau}\partial_y u^1 - \tau\,\partial_1 u^y \\
\omega_3 &= \partial_1 u^2 - \partial_2 u^1
\end{aligned}$$

- $\vartheta$ and $\delta\epsilon$ are coupled: sound waves
- Vorticity modes decouple from $\vartheta$ and $\delta\epsilon$
- Solution in Fourier space yields for ideal hydrodynamics

$$\omega_1, \omega_2 \sim \frac{1}{\tau^{2/3}}, \qquad\qquad \omega_3 \sim \tau^{1/3}.$$

# *Limits of linearized theory*

- Linear approximation only works for:
  - energy density
$$\frac{\delta\epsilon}{\bar{\epsilon}} \ll 1$$
  - velocity field
$$\mathrm{Re} = \frac{u_T\, l\,(\epsilon + p)}{\eta} = \frac{u_T\, l\,(Ts + \mu n)}{\eta} \ll 1$$
- Large Reynolds number $\mathrm{Re} \gg 1$ leads to turbulence!
- Typical numbers: $T = 0.3\,\mathrm{Gev}$, $l = 5\,\mathrm{fm}$, $u_T = 0.1c$, $\mu n = 0$
$$\Rightarrow \quad \mathrm{Re} \approx \frac{1}{\eta/s}$$

# *Mach number*

$$\mathsf{Ma} = \frac{\sqrt{u_1 u^1 + u_2 u^2 + u_y u^y}}{c_S}$$

- Turbulent motion can be described as "compression-less" for Ma $\ll 1$, which means one can take

$$\vartheta = \partial_1 u^1 + \partial_2 u^2 + \partial_y u^y = 0.$$

- We make a change of variables
  - kinematic viscosity

$$\nu_0 = \frac{\eta}{s\, T_{\mathsf{Bj}}(\tau_0)}$$

  - rescaled time / velocities

$$t = \frac{3}{4\tau_0^{1/3}}\, \tau^{4/3} \qquad v_j = \left(\frac{\tau_0}{\tau}\right)^{1/3} u_j$$

# *Compression-less flow*

This leads us to

$$\partial_t v_j + \sum_{m=1}^{2} v_m \partial_m v_j + \frac{1}{\tau^2} v_y \partial_y v_j + \partial_j d$$
$$-\nu_0 \left( \partial_1^2 + \partial_2^2 + \frac{1}{\tau^2} \partial_y^2 \right) v_j = 0.$$

- $d$ is related to temperature fluctuations
- solenoidal constraint

$$\partial_1 v_1 + \partial_2 v_2 + \frac{1}{\tau^2} \partial_y v_y = 0$$

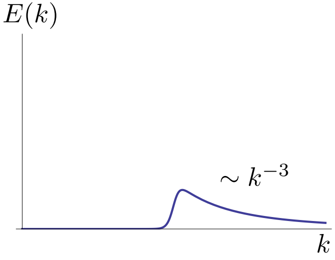
- for large times $\tau$ effectively *two-dimensional*

# *Turbulence in $d = 2$*

KRAICHNAN (1967):

- inverse cascade of energy to small wave numbers
- cascade of vorticity to large wave numbers

$$E(k) \sim k^{-3}$$




- qualitatively different to $d = 3$, emerges here dynamically

BATCHELOR (1969):

- scaling theory of decaying turbulence in $d = 2$

$$E(t,k) = \lambda^3\, t\; f(k\,\lambda\,t) \qquad \text{with} \qquad \lambda^2 = \langle \vec{v}^2 \rangle = \text{const.}$$

- turbulent motion goes to smaller and smaller wave numbers

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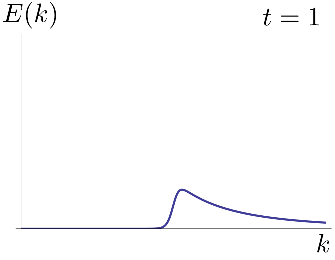


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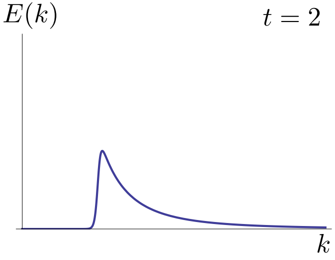


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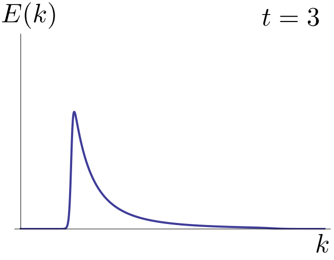


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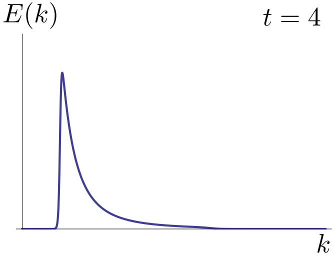


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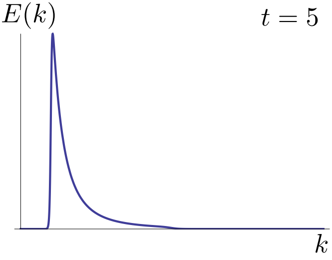


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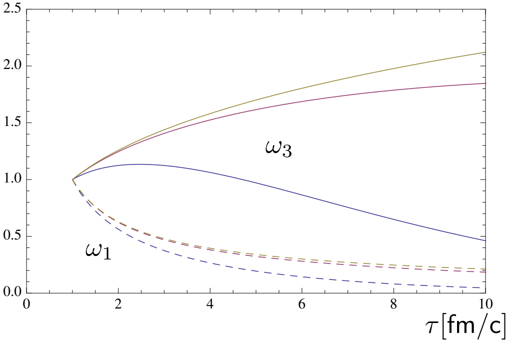
# *Vorticity with viscosity*

$$\omega_3 = \partial_1 u^2 - \partial_2 u^1$$
$$\omega_1 = \tau \partial_2 u^y - \frac{1}{\tau}\partial_y u^2$$

The linearized equations can be solved in Fourier space.
For $k_1 = 1\,\mathrm{fm}^{-1}$, $k_2 = k_y = 0$ and different viscosities



# *Phenomenological consequences*

Effect of hydrodynamical fluctuations can be calculated for Blast-wave model and Cooper-Frye freeze-out

- correction to one-particle spectrum is sensitive to the *numbers*

$$\langle (u^1)^2 + (u^2)^2 \rangle, \qquad \langle (u^y)^2 \rangle, \qquad \langle T^2 \rangle - \langle T \rangle^2$$

- effect qualitatively similar to the one of viscosity
- two-particle spectrum is sensitive to the correlation *functions* of hydrodynamical fluctuations
- allows to compare to predictions of Kraichnan and Batchelor for Re $\to \infty$

Also, macroscopic flow can be directly influenced by turbulent fluctuations.

## *Summary*

We have shown that

- Transverse vorticity mode grows!
- Hydrodynamical fluctuations on expanding medium can become turbulent
- Evolution laws can be mapped to two-dimensional Navier-Stokes equation for late times
- Turbulence has interesting effects on the two-particle spectrum

More details will be published soon.

# Backup

# *Little Bang vs. Big Bang*

**Heavy Ions**

Bjorken model

$$x^\mu = (\tau, x^1, x^2, y)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \tau^2 \end{pmatrix}$$

$$\epsilon_0(\tau), \quad u_0^\mu = (1,0,0,0)$$

$+$ hydrodyn. fluctuations

$$\epsilon = \epsilon_0(\tau) + \epsilon_1(\tau, x^1, x^2, y)$$

$$u^\mu = u_0^\mu + u_1^\mu(\tau, x^1, x^2, y)$$

**Cosmology**

Friedmann-Robertson-Walker

$$x^\mu = (t, x^1, x^2, x^3)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a(t) & & \\ & & a(t) & \\ & & & a(t) \end{pmatrix}$$

$$\epsilon_0(t), \quad u_0^\mu = (1,0,0,0)$$

$+$ hydrodyn. fluctuations

$$\epsilon = \epsilon_0(t) + \epsilon_1(t, x^1, x^2, x^3)$$

$$u^\mu = u_0^\mu + u_1^\mu(t, x^1, x^2, x^3)$$

$+$ gravity fluctuations

# *Turbulence in $d = 3$*

fully developed turbulence

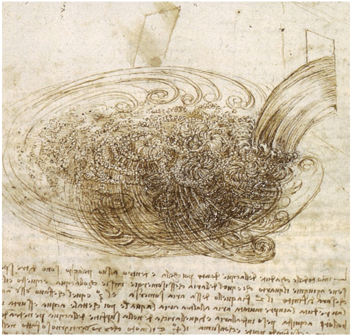
$$\mathrm{Re} = \frac{u\, l}{\nu_0} \to \infty$$

dissipated energy per unit time

$$\frac{d}{dt}\langle \vec{v}^2 \rangle = -\nu_0 \left\langle (\vec{\nabla} \times \vec{v})^2 \right\rangle = -\varepsilon$$

RICHARDSON (1922):

*Big whorls have little whorls,*
*Which feed on their velocity;*
*And little whorls have lesser whorls,*
*And so on to viscosity.*



L. DA VINCI (CA. 1500)

KOLMOGOROV (1941):

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$

with

$$\tfrac{1}{2}\langle \vec{v}^2 \rangle = \int_0^\infty dk\; E(k)$$

# *Effects on macroscopic motion of fluid*

- Turbulent fluctuations might affect macroscopic motion
  - modified equation of state
  - modified transport properties
- Anomalous, turbulent or eddy viscosity
  - proposed by Asakawa, Bass, Müller (2006) for plasma turbulence and Romatschke (2007) for fluid turbulence
  - could become negative in $d = 2$ (Kraichnan (1976))
  - depends on detailed state of turbulence – not universal
  - gradient expansion needs separation of scales
- More work needed