Ultracold quantum gases and functional renormalization

Stefan Flörchinger (Heidelberg)

Frankfurt, 25. Juni 2010

Why are Ultracold quantum gases interesting?

- Ultracold gases in the bulk are simple systems!

Why are Ultracold quantum gases interesting?

- Ultracold gases in the bulk are simple systems!
  - for example: Fermi surface is usually a sphere.

Why are Ultracold quantum gases interesting?

- Ultracold gases in the bulk are simple systems!
  - for example: Fermi surface is usually a sphere.
- Both fermions and bosons can be studied.

Why are Ultracold quantum gases interesting?

- Ultracold gases in the bulk are simple systems!
  - for example: Fermi surface is usually a sphere.
- Both fermions and bosons can be studied.
- Interactions can be tuned to arbitrary values.

Why are Ultracold quantum gases interesting?

- Ultracold gases in the bulk are simple systems!
    - for example: Fermi surface is usually a sphere.
- Both fermions and bosons can be studied.
- Interactions can be tuned to arbitrary values.
- Lower dimensional systems can be realized.

Why are Ultracold quantum gases interesting?

- Ultracold gases in the bulk are simple systems!
  - for example: Fermi surface is usually a sphere.
- Both fermions and bosons can be studied.
- Interactions can be tuned to arbitrary values.
- Lower dimensional systems can be realized.

Very nice model system to test methods of quantum and statistical *field theory*!

Quantum field theory

- Describes also electrons, atoms, quarks, gluons, protons,...
- Crucial object: quantum effective action

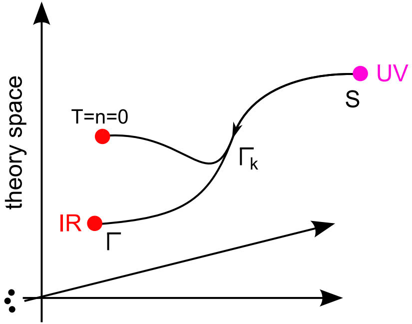
$$\Gamma[\phi] = S[\phi] + \frac{1}{2}\mathsf{Tr}\ln S^{(2)} + \dots$$
$$= \int dt \int d^d x \; U(\phi) + ...$$

- Quantum field equations from $\frac{\delta\Gamma}{\delta\phi} = 0$.
- Symmetries of Γ lead to conserved currents.
- All physical observables are easily obtained from Γ.
- Γ is generating functional of 1-PI Feynman diagrams and depends on external parameters like T, μ, or $\vec{B}$.

How do we obtain the quantum effective action $\Gamma[\phi]$?

Idea of functional renormalization: $\Gamma[\phi] \to \Gamma_k[\phi]$

- k is additional infrared cutoff parameter.
- $\Gamma_k[\phi] \to \Gamma[\phi]$ for $k \to 0$.
- $\Gamma_k[\phi] \to S[\phi]$ for $k \to \infty$.
- Dependence on T, μ or $\vec{B}$ trivial for $k \to \infty$.



$\Gamma[\phi]$ *and the grand canonical ensemble*

Functional integral representation of the partition function

$$Z = e^{-\beta\Omega_G} = \mathrm{Tr}\, e^{-\beta(H-\mu N)} = \int D\chi\, e^{-S[\chi]}.$$

Generalization with $J = \frac{\delta}{\delta\phi}\Gamma_k[\phi]$

$$e^{-\Gamma_k[\phi]} = \int D\chi\, e^{-S[\phi+\chi]+J\chi-\frac{1}{2}\chi\, R_k\, \chi}.$$

- R_k is an infrared cutoff function
  - suppresses all fluctuations $R_k \to \infty$ for $k \to \infty$.
  - is removed $R_k \to 0$ for $k \to 0$.
- $\Gamma_k[\phi]$ is the *average action* or *flowing action*.
- Grand canonical potential is obtained from $\beta\Omega_G = \Gamma_k[\phi]$ for $k = 0$ and $J = 0$.

How the flowing action flows

Simple and exact flow equation (Wetterich 1993)

$$\partial_k \Gamma_k[\phi] = \frac{1}{2}\mathsf{STr}\left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1} \partial_k R_k.$$

- Differential equation for a functional.
- For most cases not solvable exactly.
- Approximate solutions can be found from Truncations.
  - Ansatz for Γ_k with a finite number of parameters.
  - Derive ordinary differential equations for this parameters or couplings from the flow equation for Γ_k.
  - Solve these equations numerically.

Lagrangians

We use a local field theory to describe the microscopic model.
Examples:

1. Bose gas with pointlike interaction

$$\mathcal{L} = \varphi^* \left(\partial_\tau - \vec{\nabla}^2 - \mu\right) \varphi + \frac{1}{2}\lambda \left(\varphi^* \varphi\right)^2 .$$

2. Fermions in the BCS-BEC-Crossover

$$\begin{aligned} \mathcal{L} &= \psi^\dagger(\partial_\tau - \vec{\nabla}^2 - \mu)\psi + \varphi^*(\partial_\tau - \frac{1}{2}\vec{\nabla}^2 - 2\mu + \nu)\varphi \\ &\quad -h(\varphi^*\psi_1\psi_2 + h.c.). \end{aligned}$$

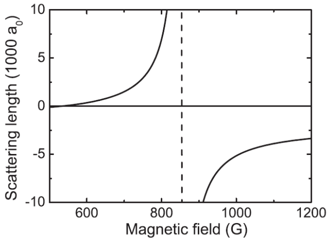
These are effective theories on the length scale of the Bohr radius or van-der-Waals length.

Symmetries of nonrelativistic field theories

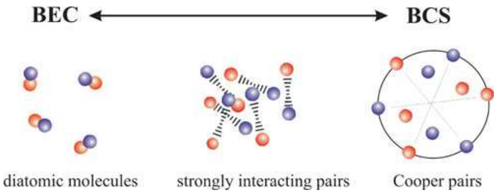
- U(1) for particle number conservation.
- Translations and Rotations.
- Galilean boost transformations.
- Possibly conformal symmetries.
- U(1) and Galilean invariance are broken spontaneously by a Bose-Einstein condensate.
- Galilean invariance is broken explicitely for $T > 0$.

Two component Fermi gas

- Two spin (or hyperfine-spin) components ψ_1 and ψ_2.
- For equal mass $M_{\psi_1} = M_{\psi_2}$, density $n_{\psi_1} = n_{\psi_2}$ etc. SU(2) spin symmetry
- s-wave interaction measured by scattering length a.
- Repulsive microscopic interaction: Landau Fermi liquid.
- Attractive interaction leads to many interesting effects!
- Scattering length can be tuned experimentally with Feshbach resonances.



BCS-BEC Crossover



- Small negative scattering length $a \to 0_-$
  - Formation of Cooper pairs in momentum space
  - BCS-theory valid
  - superfluid at small temperatures
  - order parameter $\varphi \sim \psi_1 \psi_2$
- Small positive scattering length $a \to 0_+$
  - Formation of dimers or molecules in position space
  - Bosonic mean field theory valid
  - superfluid at small temperatures
  - order parameter $\varphi \sim \psi_1 \psi_2$
- Between both limits: Continuous *BCS-BEC Crossover*
  - scattering length becomes large: strong interaction
  - superfluid, order parameter $\varphi \sim \psi_1 \psi_2$ at small T

Truncations

For many purposes *derivative expansions* are suitable approximations. For example we use for the BCS-BEC Crossover

$$\begin{aligned}\Gamma_k &= \int_{\tau,\vec{x}} \Big\{ \psi^\dagger (Z_\psi \partial_\tau - Z_\psi \vec{\nabla}^2 - \mu + \Delta m_\psi)\psi \\ &+ \varphi^* (Z_\varphi \partial_\tau - A_\varphi \frac{1}{2} \vec{\nabla}^2)\varphi \\ &- h(\varphi^* \psi_1 \psi_2 + h.c.) + \frac{1}{2}\lambda_\psi (\psi^\dagger \psi)^2 + U_k(\varphi^* \varphi, \mu) \Big\}\end{aligned}$$

Truncations

For many purposes *derivative expansions* are suitable approximations. For example we use for the BCS-BEC Crossover

$$\begin{aligned}\Gamma_k &= \int_{\tau,\vec{x}} \Big\{ \psi^\dagger (Z_\psi \partial_\tau - Z_\psi \vec{\nabla}^2 - \mu + \Delta m_\psi)\psi \\ &+\varphi^* (Z_\varphi \partial_\tau - A_\varphi \frac{1}{2}\vec{\nabla}^2)\varphi \\ &-h(\varphi^* \psi_1 \psi_2 + h.c.) + \frac{1}{2}\lambda_\psi (\psi^\dagger \psi)^2 + U_k(\varphi^*\varphi, \mu) \Big\}\end{aligned}$$

- The coefficients Z_φ, A_φ, λ_ψ, h, Z_ψ, Δm_ψ and the effective potential U_k are scale-dependent.

Truncations

For many purposes *derivative expansions* are suitable approximations. For example we use for the BCS-BEC Crossover

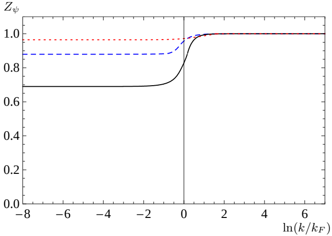
$$\begin{aligned}\Gamma_k &= \int_{\tau,\vec{x}} \Big\{ \psi^\dagger (Z_\psi \partial_\tau - Z_\psi \vec{\nabla}^2 - \mu + \Delta m_\psi)\psi \\ &\quad + \varphi^* (Z_\varphi \partial_\tau - A_\varphi \frac{1}{2} \vec{\nabla}^2)\varphi \\ &\quad - h(\varphi^* \psi_1 \psi_2 + h.c.) + \frac{1}{2}\lambda_\psi (\psi^\dagger \psi)^2 + U_k(\varphi^* \varphi, \mu) \Big\}\end{aligned}$$

- The coefficients Z_φ, A_φ, λ_ψ, h, Z_ψ, Δm_ψ and the effective potential U_k are scale-dependent.
- The effective potential U_k contains no derivatives - describes homogeneous fields.

Fermion self energy corrections

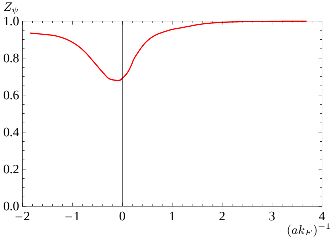
(Floerchinger, Scherer and Wetterich, PRA **81**, 063619 (2010))

- Flow of fermion wavefunction renormalization Z_ψ



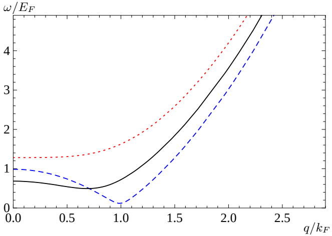
solid: $(ak_F)^{-1} = 0$,
long-dashed: $(ak_F)^{-1} = -1$,
short-dashed: $(ak_F)^{-1} = 1$

- At the macroscopic scale $k = 0$



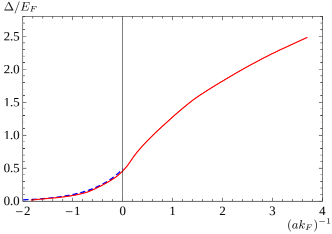
Dispersion relation and gap at zero temperature

- Dispersion relation $\omega = \pm\sqrt{\Delta^2 + (\vec{q}^2 - r_F^2)^2}$



solid: $(ak_F)^{-1} = 0$,
long-dashed: $(ak_F)^{-1} = -1$,
short-dashed: $(ak_F)^{-1} = 1$

- Single particle gap



Unitarity $(ak_F = \infty)$	μ/E_F	Δ/E_F
Carlson *et al.*	0.43	0.54
Perali *et al.*	0.46	0.53
Haussmann *et al.*	0.36	0.46
Diehl *et al.*	0.55	0.60
Bartosch *et al.*	0.32	0.61
present work	0.51	0.46

The effective potential

- We use a Taylor expansion around the minimum ρ_0

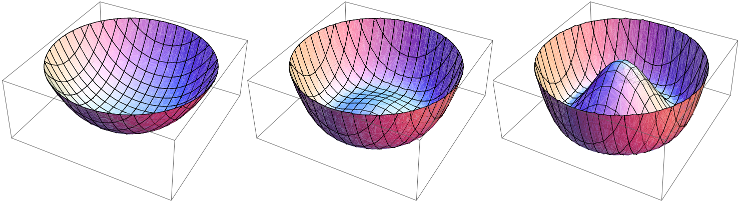
$$U_k(\varphi^*\varphi) = -p + m^2\left(\varphi^*\varphi - \rho_0\right) + \frac{1}{2}\lambda\left(\varphi^*\varphi - \rho_0\right)^2.$$

The effective potential

- We use a Taylor expansion around the minimum ρ_0

$$U_k(\varphi^*\varphi) = -p + m^2\,(\varphi^*\varphi - \rho_0) + \frac{1}{2}\lambda\,(\varphi^*\varphi - \rho_0)^2.$$

- Symmetry breaking:



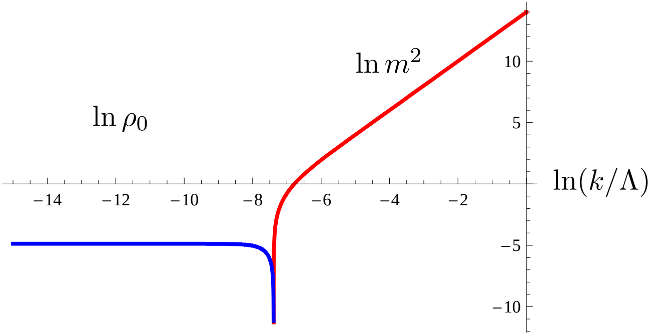
$T > T_c$ $\qquad$ $T < T_c$ $\qquad$ $T \ll T_c$

The effective potential

- We use a Taylor expansion around the minimum ρ_0

$$U_k(\varphi^*\varphi) = -p + m^2\left(\varphi^*\varphi - \rho_0\right) + \frac{1}{2}\lambda\left(\varphi^*\varphi - \rho_0\right)^2.$$

- Typical flow:



Solving the flow equation - Phase diagram

- Information on phase diagram is contained in form of the effective potential $U(\rho, \mu, T)$ at macroscopic scale.

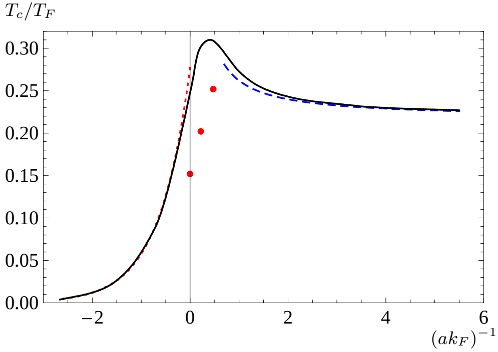
Solving the flow equation - Phase diagram

- Information on phase diagram is contained in form of the effective potential $U(\rho, \mu, T)$ at macroscopic scale.
- Very nice generalization of Landau's theory!

Solving the flow equation - Phase diagram

- Information on phase diagram is contained in form of the effective potential $U(\rho, \mu, T)$ at macroscopic scale.
- Very nice generalization of Landau's theory!
- Example: BCS-BEC Crossover

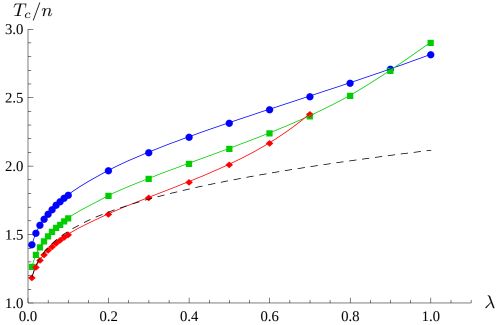
(Floerchinger, Scherer and Wetterich, PRA **81**, 063619 (2010).)



Solving the flow equation - Phase diagram

- Information on phase diagram is contained in form of the effective potential $U(\rho, \mu, T)$ at macroscopic scale.
- Very nice generalization of Landau's theory!
- Example: Superfluid Bose gas in $d = 2$

  (Floerchinger and Wetterich, PRA 79, 013601 (2009)).

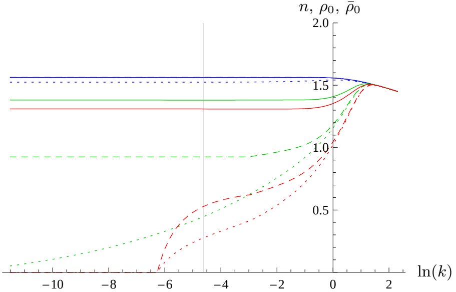


Superfluid order in two dimensions

- Bose gas with truncation

$$\Gamma_k = \int_{\tau,\vec{x}} \left\{ \varphi^*(Z\partial_\tau - V\partial_\tau^2 - A\vec{\nabla}^2)\varphi + U(\varphi^*\varphi) \right\}$$

- Mermin-Wagner theorem: No true long range order at $T > 0$ in $d = 2$.
- This implies: $n_c = \bar{\rho}_0 \to 0$ for $k \to 0$.



density n (solid),

superfluid density ρ_0 (dashed),

condensate density $\bar{\rho}_0$ (dotted),

Solving the flow equation - Thermodynamic observables

From grand canonical potential

$$dU = -dp = -s\,dT - n\,d\mu$$

take derivatives e. g. for Bose gas in $d = 3$

(Floerchinger and Wetterich, PRA 79, 063602 (2009))

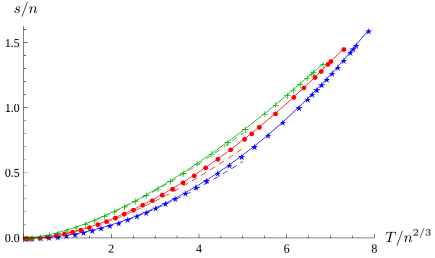
Solving the flow equation - Thermodynamic observables

From grand canonical potential

$$dU = -dp = -s\,dT - n\,d\mu$$

take derivatives e. g. for Bose gas in $d = 3$

(Floerchinger and Wetterich, PRA 79, 063602 (2009))



- entropy density $s = -\frac{\partial U}{\partial T}$,

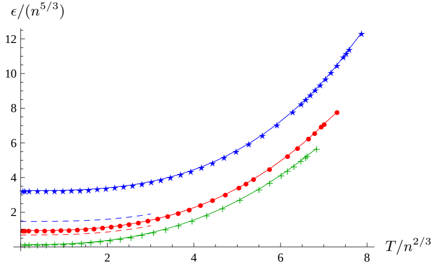
Solving the flow equation - Thermodynamic observables

From grand canonical potential

$$dU = -dp = -s\,dT - n\,d\mu$$

take derivatives e. g. for Bose gas in $d = 3$

(Floerchinger and Wetterich, PRA 79, 063602 (2009))



- entropy density $s = -\frac{\partial U}{\partial T}$,
- energy density $\epsilon = -p + Ts + \mu n$,

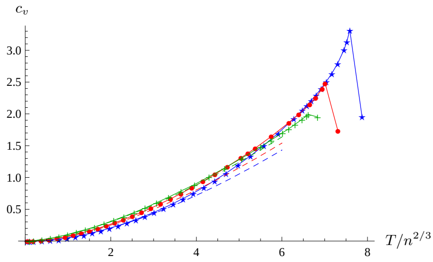
Solving the flow equation - Thermodynamic observables

From grand canonical potential

$$dU = -dp = -s\,dT - n\,d\mu$$

take derivatives e. g. for Bose gas in $d = 3$

(Floerchinger and Wetterich, PRA 79, 063602 (2009))



- entropy density $s = -\frac{\partial U}{\partial T}$,
- energy density $\epsilon = -p + Ts + \mu n$,
- specific heat c_v,

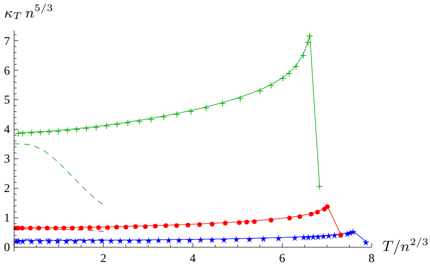
Solving the flow equation - Thermodynamic observables

From grand canonical potential

$$dU = -dp = -s\,dT - n\,d\mu$$

take derivatives e. g. for Bose gas in $d = 3$

(Floerchinger and Wetterich, PRA 79, 063602 (2009))



- entropy density $s = -\frac{\partial U}{\partial T}$,
- energy density $\epsilon = -p + Ts + \mu n$,
- specific heat c_v,
- isoth. compressibility κ_T,

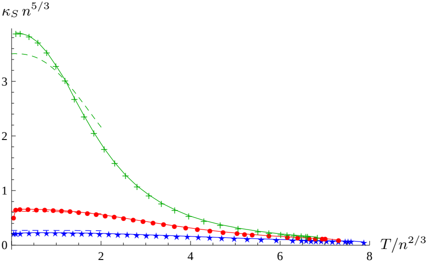
Solving the flow equation - Thermodynamic observables

From grand canonical potential

$$dU = -dp = -s\,dT - n\,d\mu$$

take derivatives e. g. for Bose gas in $d = 3$

(Floerchinger and Wetterich, PRA 79, 063602 (2009))



- entropy density $s = -\frac{\partial U}{\partial T}$,
- energy density
  $\epsilon = -p + Ts + \mu n$,
- specific heat c_v,
- isoth. compressibility κ_T,
- adiab. compressibility κ_S,

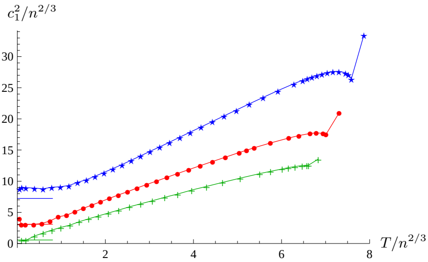
Solving the flow equation - Thermodynamic observables

From grand canonical potential

$$dU = -dp = -s\, dT - n\, d\mu$$

take derivatives e. g. for Bose gas in $d = 3$

(Floerchinger and Wetterich, PRA 79, 063602 (2009))



- entropy density $s = -\frac{\partial U}{\partial T}$,
- energy density $\epsilon = -p + Ts + \mu n$,
- specific heat c_v,
- isoth. compressibility κ_T,
- adiab. compressibility κ_S,
- velocity of sound I,

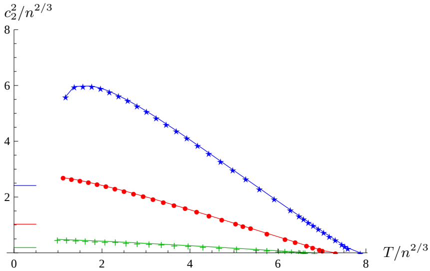
Solving the flow equation - Thermodynamic observables

From grand canonical potential

$$dU = -dp = -s\,dT - n\,d\mu$$

take derivatives e. g. for Bose gas in $d = 3$

(Floerchinger and Wetterich, PRA 79, 063602 (2009))



- entropy density $s = -\frac{\partial U}{\partial T}$,
- energy density $\epsilon = -p + Ts + \mu n$,
- specific heat c_v,
- isoth. compressibility κ_T,
- adiab. compressibility κ_S,
- velocity of sound I,
- velocity of sound II,

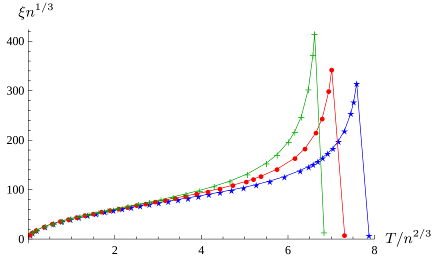
Solving the flow equation - Thermodynamic observables

From grand canonical potential

$$dU = -dp = -s\,dT - n\,d\mu$$

take derivatives e. g. for Bose gas in $d = 3$

(Floerchinger and Wetterich, PRA 79, 063602 (2009))



- entropy density $s = -\frac{\partial U}{\partial T}$,
- energy density $\epsilon = -p + Ts + \mu n$,
- specific heat c_v,
- isoth. compressibility κ_T,
- adiab. compressibility κ_S,
- velocity of sound I,
- velocity of sound II,
- correlation length.

Fermi gases with different physics

- 1 component Fermi gas - no s-wave interaction
- 2 component Fermi gas - BCS-BEC crossover
- 3 component Fermi gas - ??

Fermi gases with different physics

- 1 component Fermi gas - no s-wave interaction
- 2 component Fermi gas - BCS-BEC crossover
- 3 component Fermi gas - ??
  - Three-body problem: Efimov effect

    (Efimov, Phys. Lett. **33B**, 563 (1970),
    Review: Braaten and Hammer, Phys. Rep. **428**, 259 (2006))

Fermi gases with different physics

- 1 component Fermi gas - no s-wave interaction
- 2 component Fermi gas - BCS-BEC crossover
- 3 component Fermi gas - ??
  - Three-body problem: Efimov effect

    (Efimov, Phys. Lett. **33B**, 563 (1970),
    Review: Braaten and Hammer, Phys. Rep. **428**, 259 (2006))
  - On the lattice: Trion formation

    (Rapp, Zarand, Honerkamp, and Hofstetter, PRL **98**, 160405 (2007),
    Rapp, Hofstetter and Zarand, PRB **77**, 144520 (2008).)

Three component Fermi gas

- For equal masses, densities etc. global SU(3) symmetry

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} \to u \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}, \quad u \in \text{SU(3)}.$$

  Similar to flavor symmetry in the Standard model!
- For small scattering length $|a| \to 0$
  - BCS $(a < 0)$ or BEC $(a > 0)$ superfluidity at small T.
  - order parameter is conjugate triplet $\bar{\mathbf{3}}$ under SU(3)

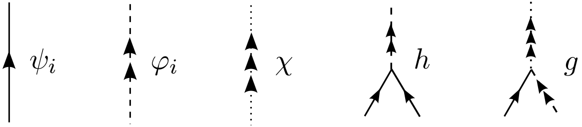
$$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} \sim \begin{pmatrix}\psi_2\psi_3\\ \psi_3\psi_1\\ \psi_1\psi_2\end{pmatrix}.$$

  - SU(3) symmetry is broken spontaneously for $\varphi \neq 0$.
- What happens for large $|a|$?

Simple truncation for fermions with three components

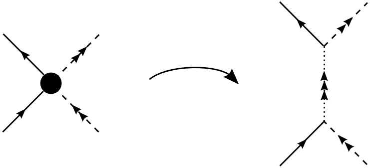
$$\Gamma_k = \int_x \psi^\dagger(\partial_\tau - \vec{\nabla}^2 - \mu)\psi + \varphi^\dagger(\partial_\tau - \frac{1}{2}\vec{\nabla}^2 + m_\varphi^2)\varphi$$
$$+\chi^*(\partial_\tau - \frac{1}{3}\vec{\nabla}^2 + m_\chi^2)\chi$$
$$+h\epsilon_{ijk}(\varphi_i^*\psi_j\psi_k + h.c.) \quad + g(\varphi_i\psi_i^*\chi + h.c.).$$

- Units are such that $\hbar = k_B = 2M = 1$
- Wavefunction renormalization for ψ, φ and χ is implicit.
- Γ_k contains terms for
  - fermion field $\psi = (\psi_1, \psi_2, \psi_3)$
  - bosonic field $\varphi = (\varphi_1, \varphi_2, \varphi_3) \sim (\psi_2\psi_3, \psi_3\psi_1, \psi_1\psi_2)$
  - trion field $\chi \sim \psi_1\psi_2\psi_3$

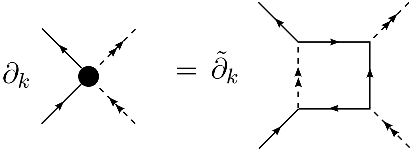


"Refermionization"

- Trion field is introduced via a generalized Hubbard-Stratonovich transformation



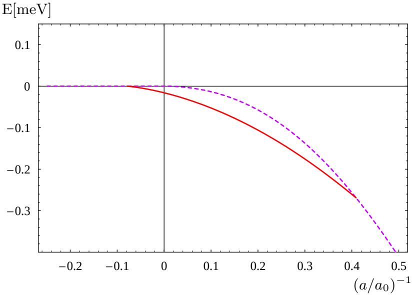
- Fermion-boson coupling is regenerated by the flow

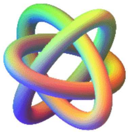


- Express this again by trion exchange
  (Gies and Wetterich, PRD **65**, 065001 (2002), Floerchinger and Wetterich, PLB **680**, 371 (2009).)

Binding energies

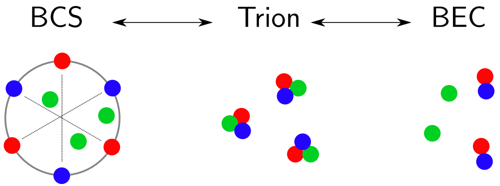
- Vacuum limit $T \to 0, \quad n \to 0.$





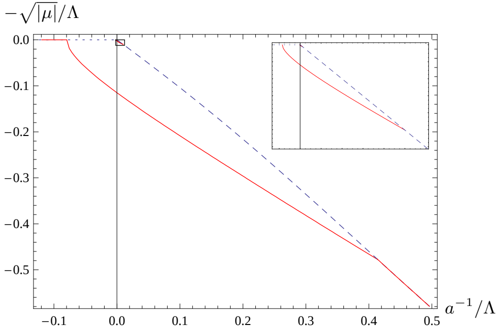
- Binding energy per atom for
  - molecule or dimer φ (dashed line)
  - trion or trimer χ (solid line)
- For large scattering length a trion is energetically favorable!
- Three-body bound state even for $a < 0$.

Quantum phase diagram



- BCS-Trion-BEC transition
  (Floerchinger, Schmidt, Moroz and Wetterich, PRA **79**, 013603 (2009)).
  - $a \to 0_-$: Cooper pairs, $SU(3) \times U(1) \to SU(2) \times U(1)$.
  - $a \to 0_+$: BEC of molecules, $SU(3) \times U(1) \to SU(2) \times U(1)$.
  - $a \to \pm\infty$: Trion phase, $SU(3)$ unbroken.
- Quantum phase transitions
  - from BCS to Trion phase
  - from Trion to BEC phase.

Efimov effect



- Self-similarity in energy spectrum.
- Efimov trimers become more and more shallow. At $a = \infty$

$$E_{n+1} = e^{-2\pi/s_0}\, E_n.$$

- Simple truncation: $s_0 \approx 0.82$.
- Advanced truncation: $s_0 \approx 1.006$ (exact result)

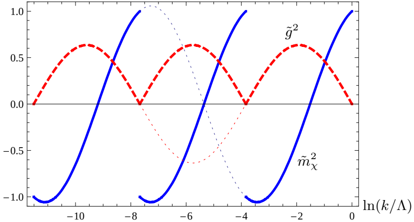
  (Moroz, Floerchinger, Schmidt and Wetterich, PRA **79**, 042705 (2009).)

Renormalization group limit cycle

- For $\mu = 0$ and $a^{-1} = 0$ flow equations for rescaled couplings

$$k\frac{\partial}{\partial k}\begin{pmatrix}\tilde{g}^2 \\ \tilde{m}_\chi^2\end{pmatrix} = \begin{pmatrix} 7/25 & -13/25 \\ 36/25 & 7/25 \end{pmatrix}\begin{pmatrix}\tilde{g}^2 \\ \tilde{m}_\chi^2\end{pmatrix}.$$

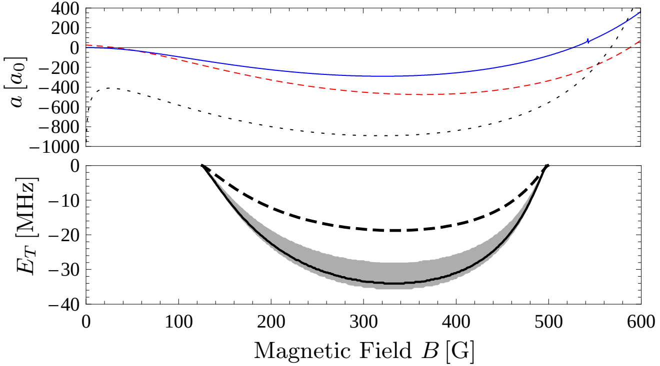
- Solution is log-periodic in scale.



- Every zero-crossing of $\tilde{m}_\chi^2$ corresponds to a new bound state.
- For $\mu \neq 0$ or $a^{-1} \neq 0$ limit cycle scaling stops at some scale k. Only finite number of Efimov trimers.

Contact to experiments

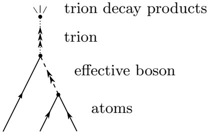
- Model can be generalized to case without SU(3) symmetry (Floerchinger, Schmidt and Wetterich, PRA A **79**, 053633 (2009)).
- Hyperfine states of ^{6}Li have large scattering lengths.

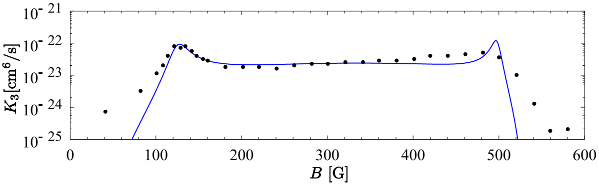


- Binding energies might be measured using RF-spectroscopy.
- Lifetime is quite short ~ 10ns.

Three-body loss rate

- Three-body loss rate measured experimentally (Ottenstein et al., PRL **101**, 203202 (2008); Huckans et al., PRL **102**, 165302 (2009))





- Trion may decay into deeper bound molecule states
- Calculate B-field dependence of loss process above.
- Left resonance (position and width) fixes model parameters.
- Form of curve for large B is prediction.
- Similar results obtained by other methods (Braaten, Hammer, Kang and Platter, PRL **103**, 073202 (2009); Naidon and Ueda, PRL **103**, 073203 (2009).)

Conclusions

- Functional renormalization is a useful method to describe ultracold quantum gases.
- Quantitative precision seems reachable.
- Unified description of
  - Bosons and Fermions,
  - Weak and strong coupling,
  - Few-Body and Many-Body physics.