

Low Energy Descriptions of Pure Gauge Theories: Effective Actions and Lattice Ward Identities

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Heidelberg, 06.June, 2002

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OUTLINE:

- CHO-connection and Abelian Projections
- Low Energy Approximations
- Toward calculating S_{eff} by Inverse MC

Cho connection and Abelian Projections

$SU(2)$: given $n(x) \in S^2$, $n \cdot n = 1$, $n = n_a \sigma_a$

$$\hat{A}_\mu = (n, A_\mu)n + i[n, \partial_\mu n] = \hat{A}_\mu(A, n)$$

PROPERTIES:

- $\hat{D}_\mu n = 0$
- $(A_\mu, n) = (\hat{A}_\mu, n)$
- n^* constant: $\hat{A}_\mu = (A_\mu, n^*)n^*$

Conversely:

$$D_\mu n = 0 \implies A_\mu = C_\mu n + i[n, \partial_\mu n] \quad \text{reducible}$$

GAUGE TRANSFORMATIONS:

$$\begin{aligned} A_\mu &\longrightarrow {}^V A_\mu = V(A_\mu + i\partial_\mu)V^{-1} \\ n &\longrightarrow {}^V n = VnV^{-1} \end{aligned}$$

$$\Rightarrow \hat{A}_\mu({}^V A, {}^V n) = {}^V \hat{A}_\mu(A, n).$$

restricted and reducible gauge connection!

GENERALIZATION FOR $SU(N)$ (SIMPLY LACED):

n_i : r commuting orthogonal generators. Demand

$$\begin{aligned} \hat{A}_\mu({}^V A, {}^V n_i) &= {}^V \hat{A}_\mu(A, n_i) \quad , \quad \hat{A}_\mu(n_i^*, A) = P_{n_i^*}(A) \\ \longrightarrow \hat{A}_\mu &= P_n(A_\mu) + i \sum [n_i, \partial_\mu n_i] \end{aligned}$$

reducible with holonomy group $\supset U^r(1)$.

Abelian Gauges/Projections for $SU(2)$

$\longrightarrow n(A)$. Decompose

$$A_\mu = \hat{A}_\mu(A, n) + X_\mu$$

A_μ, \hat{A}_μ gauge potentials $\rightarrow X_\mu$ in adjoint reps.

MAG: A_μ given

$$\text{minimize } F_{\text{MAG}}[n] = \|A_\mu - \hat{A}_\mu(A, n)\|$$

$$\begin{aligned} \min_n F[n] &= \min_V \|A_\mu - \hat{A}_\mu(A, \overbrace{V^{-1}n^*V}^n)\| \\ &= \min_V \|{}^V A_\mu - \hat{A}_\mu({}^V A, n^*)\| = \min_V \|({}^V A_\mu)^\perp\|. \end{aligned}$$

PAG: A_0 given

$$\text{minimize } F_{\text{Pol}}[n] = \|A_0 - \hat{A}_0\|$$

Singularities of n : position of magnetic monopoles^a

relation $n \longrightarrow$ instantons, monopoles

Abelian projections:

- Abelian *gauge fixing*: $\min F[n]$, residual $U(1)$
- *Truncation*: throw away X_μ in 'measured' object

^aH. Reinhardt et.al; F. Lenz et.al; A.W. et.al

Low Energy Approximations

- $n \rightarrow$ topological defects
- good variable for QCD in infrared?

Would like: $A_\mu = \hat{A}_\mu + X_\mu$

$$\begin{aligned}\int \mathcal{D}A e^{-S} O(A) &= \int \mathcal{D}\hat{A} \mathcal{D}X e^{-S[\hat{A}, X]} O(\hat{A}, X) \\ &\stackrel{\text{AG}}{\approx} \int \mathcal{D}C \mathcal{D}n \mathcal{D}X \text{ (J FP)} e^{-S[C, n, X]} O(C, n) \\ &= \int \mathcal{D}C \mathcal{D}n e^{-S_{\text{eff}}[C, n]} O(C, n)\end{aligned}$$

One step further

$$e^{-S_{\text{eff}}[n]} \stackrel{\text{AG}}{=} \int \mathcal{D}C \mathcal{D}X \text{ (J FP)} e^{-S[C, n, X]}$$

What is effective action $S_{\text{eff}}[n] = S_{\text{eff}}[\mathbf{n}]?$

Faddeev–Niemi Proposal

$$\begin{aligned}\mathcal{L}_{\text{SFN}} &= \frac{m^2}{2}(\partial_\mu \mathbf{n})^2 - \frac{1}{4e^2}H_{\mu\nu}^2 \\ &\equiv \text{'sigma-model'} + \text{'Skyrme term'}\end{aligned}$$

- 'field strength' $H_{\mu\nu} \equiv (\mathbf{n}, \partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})$
- $\tilde{A}_\mu = \frac{1}{2i}[n, \partial_\mu n] \Rightarrow F_{\mu\nu}(\tilde{A}) = 8H_{\mu\nu}$ classical
- *dynamically* generated mass scale m !

FN: “*unique local and Lorentz invariant action for the unit vector \mathbf{n} which is at most quadratic (!) in time derivatives ... and involves all such terms that are either relevant or marginal in the infrared*”

S_{SFN} supports stable knot-like *solitons*

FADDEEV–NIEMI CONJECTURE:

- S_{SFN} low-energy effective action for YM-theory
- knot solitons \simeq glue-balls

- Bound on energy of static configurations: rescale x

$$\frac{2e}{m}E = \tilde{E} = \int d^3x \left((\nabla \mathbf{n})^2 + \frac{1}{2} H_{ij} \right)$$

configuration space consists of topological sectors

$$\mathbf{n}(|\vec{x}| \rightarrow \infty) = \mathbf{n}_0 \quad \mathbf{n} : S^3 \rightarrow S^2$$

$$\pi_2(S^3) = \mathbb{Z} \quad \text{classified by } Q$$

$\mathbf{n}(C_i) = \mathbf{n}_i \Rightarrow Q = \text{linking number of loops } C_1, C_2$

Hopf invariant $Q \sim \text{instanton number}$

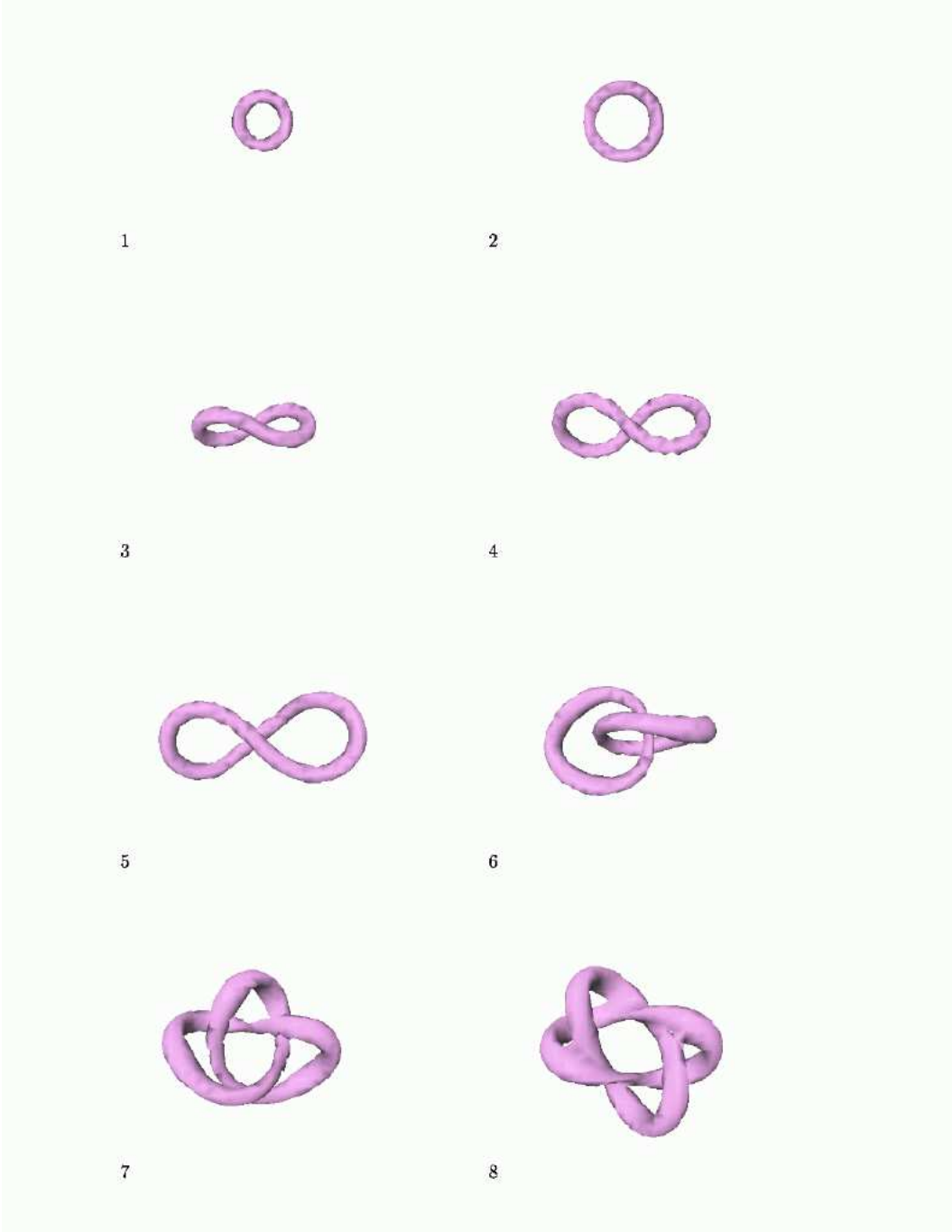
VALENKO AND KAPITANSKY (1979):

$$\tilde{E} \geq C|Q|^{3/4}, \quad C = 16\pi^2 3^{3/8} \sim 238$$

Can be improved. Is this a BPS-bound?

- Extensive numerical studies

Battye/Sutcliffe '98:



Problems

- relation to Yang Mills? (cf. Gies 2001)
- $H^2 = (\mathbf{n} \square \mathbf{n})^2 - (\mathbf{n} \partial_\mu \partial_\nu \mathbf{n})^2$; why same coupling?
- $S_{\text{SFN}} \rightarrow \text{SSB}: \text{SO}(3) \rightarrow \text{SO}(2)$

2 Goldstone bosons, no mass gap???

where are these Goldstones?

Reformulations: (PLB 515 (01) 181)

- CP_1 formulation: $n_a = z^\dagger \tau_a z$, $z \in \mathbb{C}^2$, $z^\dagger z = 1$

$$\mathcal{L}_{\text{SFN}} = \frac{m^2}{2} (D_\mu z)^\dagger D^\mu z - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}$$
$$Q = \frac{1}{4\pi^2} \int d^3x A \wedge F, \quad \text{CS-term}$$

gauge potential = composite:

$$A = -iz^\dagger dz, \quad D_\mu = \partial_\mu - iA_\mu, \quad F = dA$$

- The $SU(2)/U(1)$ formulation

$$n_a(x) = \frac{1}{2} \text{tr} \left(\tau_3 g^\dagger(x) \tau_a g(x) \right).$$

current $J = g^\dagger dg \sim \text{flat potential}$

$$A \wedge F = -\frac{1}{2} \text{tr} \left(J \wedge dJ + \frac{2}{3} J \wedge J \wedge J \right)$$

$$E \sim \int d^3x \left\{ (J_i^1 J_i^1 + J_i^2 J_i^2) + \frac{1}{2} (J_i^1 J_j^1 - J_j^2 J_i^2)^2 \right\}$$

- $Q =$ gauge field winding number
- first term of E : defines MAG
- $E \sim$ gauge fixing functional for a non-linear MAG

minima of E (Q fixed) \sim gauge fixed pure gauge potentials in a sector with winding number Q .

pure gauge theory vacua \longleftrightarrow knots

Toward calculating S_{eff}

non-local S_{eff} should vary slowly with $x \rightarrow$

- systematic derivative expansion

$$S_{\text{eff}}[\mathbf{n}] = \sum_i \lambda_i S_i[\mathbf{n}] + \sum_i \lambda'_i S'_i[\mathbf{n}] = S + S'$$

- list of symmetric operators S_i :

$\partial \mathbf{n} \cdot \partial \mathbf{n}$	$\square \mathbf{n} \cdot \square \mathbf{n}$	$(\mathbf{n} \cdot \square \mathbf{n})^2$	$(\mathbf{n} \cdot \partial_\mu \partial_\nu \mathbf{n})^2$
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- list of non-symmetric operators S'_i :

$\mathbf{n} \cdot \mathbf{h}$	$(\mathbf{n} \cdot \mathbf{h})^2$	$(\partial \mathbf{n} \cdot \partial \mathbf{n})(\mathbf{n} \cdot \mathbf{h})$
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Explicit calculations for $SU(2)$:

- generate lattice configurations $\sim e^{-S_{\text{YM}}[A]}$
- gauge fixings + projections: $A_\mu \longrightarrow \mathbf{n} (??)$

- \mathbf{n} distributed according

$$e^{-S_{\text{YM}}[A]} \longrightarrow e^{-S_{\text{eff}}[\mathbf{n}]}.$$

- gradient expansion for S_{eff}

calculate couplings via inverse MC

Schwinger Dyson and Ward identities

$$\text{SO(3)-invariant} \quad \mathcal{D}\mathbf{n} = \prod_x d\mathbf{n}_x \delta(\mathbf{n}^2 - 1)$$

- Isometries of $S^2 \rightarrow i\mathbf{L}_x = \mathbf{n} \wedge \frac{\partial}{\partial \mathbf{n}_x}$

$$\int \mathcal{D}\mathbf{n} \mathbf{L}_x \left\{ G[\mathbf{n}] \exp(-S_{\text{eff}}[\mathbf{n}, \mathbf{h}]) \right\} = 0$$

$$\implies \langle \mathbf{L}_x G[\mathbf{n}] - G[\mathbf{n}] \mathbf{L}_x S_{\text{eff}} \rangle = 0.$$

- SD-equations: field monomials $G_i \rightarrow$

$$\sum_j \langle G_i \mathbf{L}_x S_j \rangle \lambda_j + \sum_j \langle G_i \mathbf{L}_x S'_j \rangle \lambda'_j = \langle \mathbf{L}_x G_i \rangle.$$

exact $\langle \dots \rangle$, linear system $\rightarrow \lambda_i, \lambda'_i$ (least squares)

- Ward-Identities (consistency checks)

$$\mathbf{L} = \sum_x \mathbf{L}_x \implies \mathbf{L} S_i = 0, \quad \mathbf{L} S_{\text{eff}} = \mathbf{L} S'$$

$$\langle \mathbf{L} G[\mathbf{n}] - G[\mathbf{n}] \mathbf{L} S'[\mathbf{n}, \mathbf{h}] \rangle = 0.$$

Minimal ansatz

$$S_{\text{eff}} = \lambda_1 \sum_x (\partial_\mu \mathbf{n})^2 + \sum_x \mathbf{n}_x \cdot \mathbf{h}$$

Ward identity

$$G = 1 \rightarrow \mathfrak{M} \wedge \mathbf{h} = 0, \quad \mathfrak{M} = \langle \mathbf{n}_x \rangle$$

$$G = n_y \cdot m^* \longrightarrow \mathfrak{M} = -\chi^\perp \mathbf{h}$$

Technicalities:

- fix to lattice Landau gauge by maximizing

$$F_{\text{LLG}}[{}^V U] = \sum_{x,\mu} \text{tr} \left({}^V U_\mu(x) + \text{h.c.} \right)$$

→ residual global $SU(2)$

- gauge transform to MAG by maximizing

$$F_{\text{MAG}} = \sum_{x,\mu} \text{tr} \left(\tau_3 {}^V U_\mu(x) \tau_3 {}^V U_\mu^\dagger(x) \right)$$

residual local $U(1)$, 3-direction distinguished

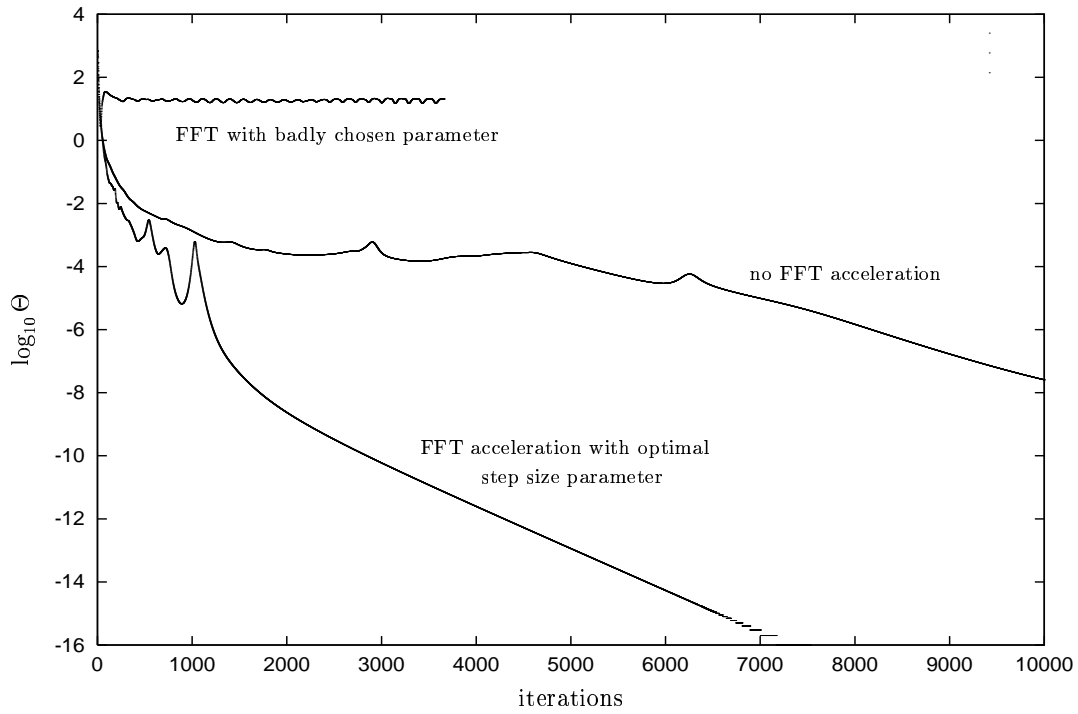
- chose g_x , defined by

$$U_{\text{LLG}} \xrightarrow{g} U_{\text{MAG}}$$

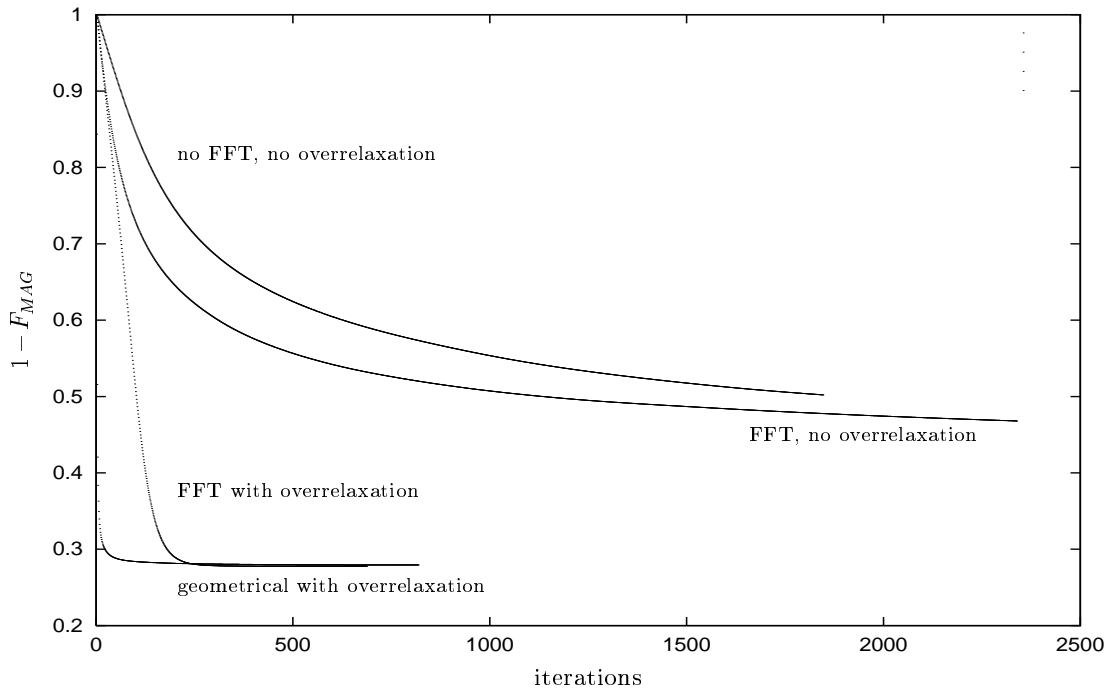
to define (almost gauge invariant) $n_x \equiv g_x^\dagger \tau_3 g_x$
 $\{n_x\}$ distributed $\sim \exp(-S_{\text{YM}})$ in LLG

global $SU(2)$ broken *explicitly* down to $U(1)$

- lattice: $V = 16^4$, $\beta = 2.35$, $a \simeq 0.13$ fermi
- LLG: Fourier acc. steepest descent, 6000 iterations
- MAG: two independent algorithms
 - MAGI: 'geometrical' iterations + over-relaxation
 - MAGII: as LLG + over-relaxation



LLG: with and without FFT



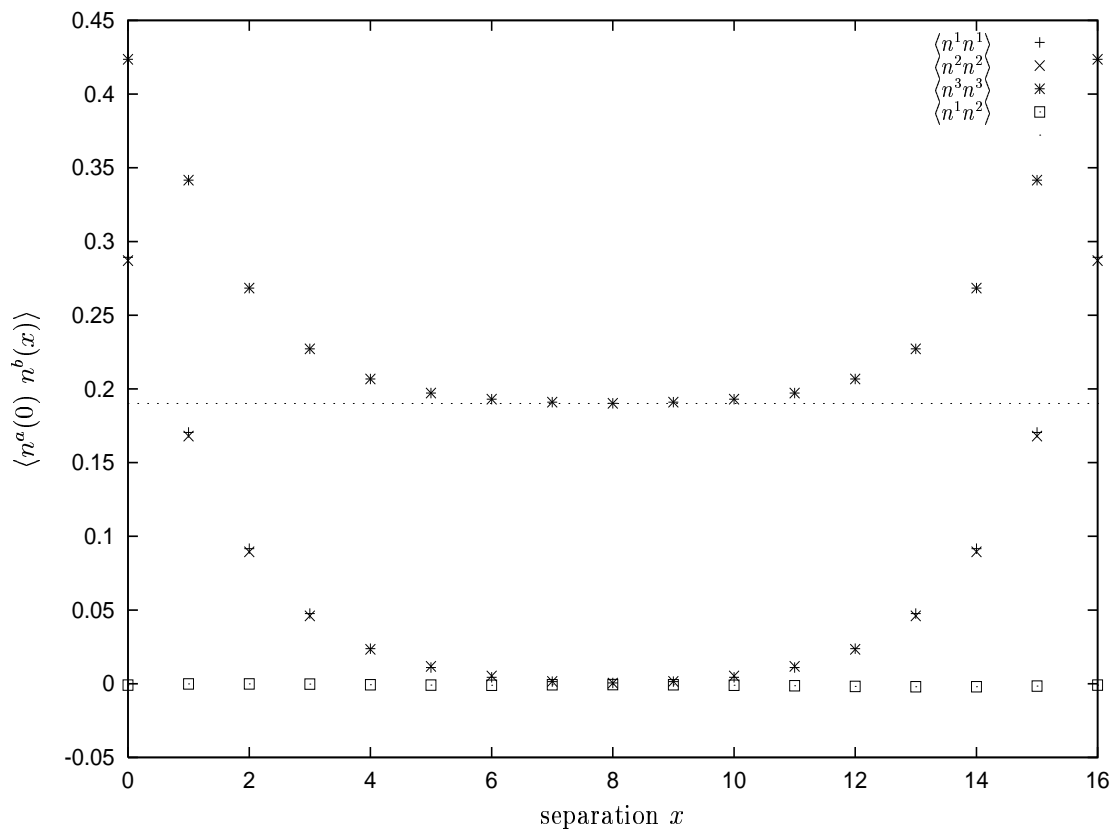
MAG: different algorithms

exhibits explicit SB $SO(3) \rightarrow SO(2)$

$$\langle n^1 \rangle = \langle n^2 \rangle = 0 \quad , \quad \langle n^3 \rangle = 0.44 \quad (0.35)$$

$$G^{\parallel}(x) = \langle n^3(x)n^3(0) \rangle$$

$$G^{\perp}(x) = \frac{1}{2} \langle n^a(x)n^a(0) \rangle, \quad a = 1, 2$$



Two-point functions with MAG I; clustering

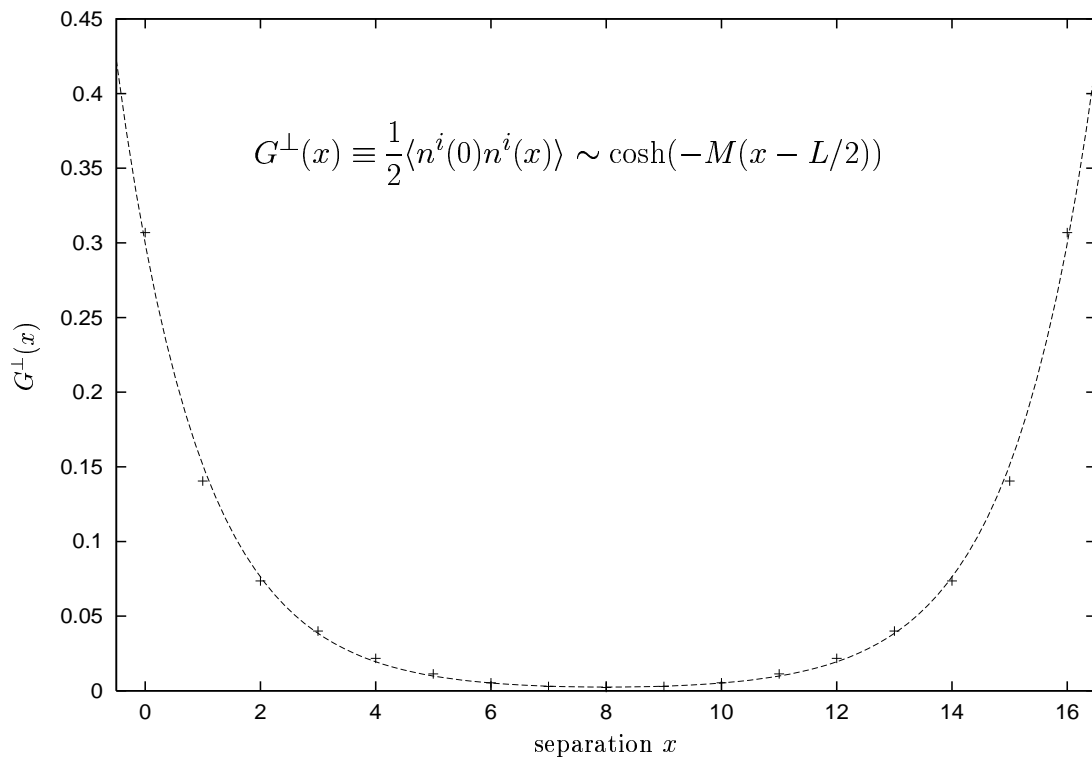
Magnetization $\langle n^a \rangle = \mathfrak{M} \delta^{a3}$

susceptibility $\chi^\perp \equiv \sum_x G^\perp(x)$

- Numerical values:

MAGI: $\mathfrak{M} = 0.436 \quad \chi^\perp = 0.636 \quad M = 0.95$

MAGII: $\mathfrak{M} = 0.352 \quad \chi^\perp = 0.596 \quad M = 1.01$



Cosh-fit to $G^\perp \rightarrow M = 1.0 \text{ GeV}$

COMPARISON WITH MINIMAL ANSATZ:

- Ward identity: $M \simeq (\mathfrak{M}/\chi^\perp)^{1/2} \simeq 1.2 \text{ GeV}$
- cosh-fit to $G^\perp(x)$: $M \sim 1.0 \text{ GeV} \rightarrow$

gradient expansion of S_{eff} reasonable

CHECKED:

- explicit (not spontaneous) symmetry breaking
- $\langle \text{tr} \mathcal{P} \rangle(\beta)$ ($N_s = 20, N_t = 4$), Binder cum: $\beta_c = 2.325$

FUTURE:

- λ_i from DS-equations via inverse MC
- inclusion of higher derivative terms
- how sensitive to choice of n