J. Phys. A: Math. Gen. 18 (1985) 2379-2384. Printed in Great Britain

Non-relativistic Yang–Mills particles in a spherically symmetric monopole field

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Received 30 October 1984, in final form 18 February 1985

Abstract. We discuss the equations of motion and the conservation laws for a non-relativistic isospin carrying particle in a spherically symmetric monopole field and for a vanishing Yukawa coupling. In the 't Hooft–Polyakov monopole field we find no classical counterpart of the Rubakov effect. In the Prasad–Sommerfeld limit we can solve the equations of motion analytically.

1. Introduction

Marciano and Muzinich solved the Dirac equation in an external Prasad–Sommerfeld monopole field for $J = 0$ analytically [1]. Their solution describes a charge exchanging and helicity conserving scattering process. Therefore, an 'up quark' is scattered into a 'down quark'.

It is one purpose of this paper to show that on the classical level the charge is conserved for $J = 0$.

In a similar way, as one obtains the Lorentz equation of motion for an electrically charged particle from the classical limit of the Dirac equation in an external electromagnetic field, one finds the equations of motion for an isospin carrying particle from the classical limit of the Dirac equation in an external Yang–Mills field [2].

We discuss these Wong equations for the motion of a non-relativistic Yang–Mills (YM) particle in a spherically symmetric (SS) monopole field. An SS monopole field is invariant under a rotation in space and a simultaneous gauge transformation [3–5]. Throughout this paper we assume a vanishing Yukawa coupling of the YM particle to the Higgs field of the monopole.

We denote the symmetry group by G and its Lie algebra by γ. The Dirac field which belongs to the YM particle transforms according to a representation V of G. Let V_* denote the induced representation of γ. We furthermore use the abbreviations $V(\mathrm{G})$ and $V_*(\gamma)$ for the sets $\{V(g)|g\in \mathrm{G}\}$ and $\{V_*(\alpha)|\alpha\in\gamma\}$.

2. Wong equations in a monopole field

The Wong equations generalise the Lorentz equation of motion for particles with non-Abelian charges [2, 9, 10]. The dynamical variables of a YM particle are its position $x\in\mathbb{R}^4$ and its isospin $I\in V_*(\gamma)$.

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Besides the generalised Lorentz law

$$\dot{p}^{\mu}=(mc)^{-1}\,\mathrm{Tr}\{I\cdot V_{*}(F^{\mu\nu})\}p_{\nu} \tag{1}$$

we have an equation for the time evolution of I. If D_Y is the covariant derivative in the direction of Y, then Wong's second equation reads

$$D_{\dot{x}}I=(\mathrm{d}/\mathrm{d}\tau)I-\mathrm{i}\,\dot{x}^{\mu}[V_{*}(A_{\mu}),I]=0 \tag{2}$$

i.e. the isospin is parallel transported along the particle path. From (2) one concludes that I performs a precessional motion

$$\mathrm{Tr}\,I^{2}=\text{constant}. \tag{3}$$

For a simple Lie group G and a faithful representation V the equations (1) and (2) are the Euler-Lagrange equations for the Lagrange function

$$L=-mc(g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu})^{1/2}-\mathrm{i}\,\mathrm{Tr}(Ks^{-1}D_{\dot{x}}s) \tag{4}$$

if $s(\tau)$ is a path in $V(\mathrm{G})$ such that

$$I(\tau)=s(\tau)Ks^{-1}(\tau). \tag{5}$$

We furthermore assume that $s(0)=e$ and hence $K=I(0)$.

A monopole field has vanishing electrical components F_{0i} and therefore the kinetic energy is conserved:

$$T=\gamma mc^{2}=\text{constant}. \tag{6}$$

With $B_i=\frac{1}{2}\varepsilon_{ijk}F_{jk}$ the non-relativistic Wong equations in the $A_0=0$ gauge are

$$m\ddot{\boldsymbol{x}}=\dot{\boldsymbol{x}}\wedge\mathrm{Tr}\,I\cdot V_{*}(\boldsymbol{B}) \tag{1a}$$

$$\dot{I}=\mathrm{i}[\dot{\boldsymbol{x}}\cdot V_{*}(\boldsymbol{A}),I]. \tag{1b}$$

For an Abelian gauge theory the isospin I is conserved and $(1a)$ reduces to the usual Lorentz law in a magnetic field.

3. Spherically symmetric monopole fields

For the introduction of ss monopole fields we need an embedding

$$g:\mathrm{SU}(2)\to\mathrm{G}$$
$$a\to g(a)$$

of the spin group into the gauge group. If $R:\mathrm{SU}(2)\to\mathrm{SO}(3)$ is a covering map, then a spherical rotation is a simultaneous rotation in space with $R^{-1}(a)$ and a gauge transformation with $g(a)$. Hence a ss field $\boldsymbol{A}$ fulfils

$$\boldsymbol{A}(\boldsymbol{x})=(\Gamma(a)\boldsymbol{A})(\boldsymbol{x})=g(a)R(a)\boldsymbol{A}(R^{-1}(a)\boldsymbol{x})g^{-1}(a) \tag{7}$$

i.e. is invariant with respect to spherical rotations. If $s_p:=\sigma_p/2$, $t_p:=g_{*}(s_p)$ and $\boldsymbol{L}$ is the orbital angular momentum, then the infinitesimal form of (7) is

$$(\boldsymbol{w}\cdot\boldsymbol{J})\boldsymbol{A}:=\{\boldsymbol{w}\cdot\boldsymbol{L}+\mathrm{Ad}(\boldsymbol{w}\cdot\boldsymbol{t})\}\boldsymbol{A}=-\mathrm{i}\boldsymbol{w}\wedge\boldsymbol{A}=:-\mathrm{i}(\boldsymbol{w}\cdot\boldsymbol{\Omega})\boldsymbol{A} \tag{8}$$

i.e. $\boldsymbol{A}$ is a $\boldsymbol{J}$ vector field.

Because of (7) an ss field $\boldsymbol{A}(\boldsymbol{x})$ is determined by its values on the positive 3-axis $\boldsymbol{A}(r)=\boldsymbol{A}(r\boldsymbol{e}_3)$.

In the spherical gauge $\boldsymbol{x}\cdot\boldsymbol{A}(\boldsymbol{x})=0$ [4] the construction rules for $\boldsymbol{A}(r)$ are simple [5]:

(i) Compute the subspace $\tilde{\gamma}=\text{kernel}\,\{\text{Ad}(t_3)\}$ of γ.

(ii) Let $b_1,\ldots,b_q$ be a base of $\tilde{\gamma}$. Then

$$\begin{aligned} A_1(r)&=\frac{\mathrm{i}}{r}\sum_{j=1}^{q} g_j(r)[b_j,t_1]\\ A_2(r)&=\frac{\mathrm{i}}{r}\sum_{j=1}^{q} g_j(r)[b_j,t_2]\\ A_3(r)&=0. \end{aligned} \tag{9}$$

Now let us define the $\boldsymbol{J}$ vector field $\boldsymbol{M}$ by

$$\begin{aligned} r\cdot V_*(\boldsymbol{A})&=(\boldsymbol{M}(\boldsymbol{x})-\boldsymbol{T})\wedge\hat{\boldsymbol{x}}\\ [M_1(r),M_2(r)]&=\mathrm{i}M_3(r) \end{aligned} \tag{10}$$

where $T_i=V_*(t_i)$, and the $\boldsymbol{J}$ vector field $\boldsymbol{N}$ is given by its values on the positive 3-axis

$$\boldsymbol{N}(r)=\boldsymbol{M}'(r). \tag{11}$$

With these definitions the $\boldsymbol{B}$ field reads

$$r^2\boldsymbol{B}=(\boldsymbol{M}-\boldsymbol{T})_{\|}+(r\boldsymbol{N})_{\perp} \tag{12}$$

where $\boldsymbol{C}_{\|}=(\boldsymbol{C}\cdot\hat{\boldsymbol{x}})\hat{\boldsymbol{x}}$ and $\boldsymbol{C}_{\perp}=(\hat{\boldsymbol{x}}\wedge\boldsymbol{C})\wedge\hat{\boldsymbol{x}}$ denote the components of the vector $\boldsymbol{C}$ which are parallel and orthogonal to $\hat{\boldsymbol{x}}$.

Now let us discuss the motion of a YM particle in a ss monopole field. Since the Lagrange function

$$L=\tfrac{1}{2}m\dot{x}^2-\mathrm{i}\,\mathrm{Tr}\,Ks^{-1}D_{\dot{x}}s$$

for a non-relativistic isospin carrying particle in an ss monopole field is invariant under the transformations

$$\begin{aligned} \boldsymbol{x}&\to R(a)\boldsymbol{x}\\ s&\to V(g(a))s \end{aligned}$$

angular momentum is conserved. A straightforward calculation gives

$$\boldsymbol{J}=m\boldsymbol{x}\wedge\dot{\boldsymbol{x}}-\boldsymbol{x}\wedge\mathrm{Tr}(I\cdot V_*(\boldsymbol{A})\}-\mathrm{Tr}\{I\cdot\boldsymbol{T}\} \tag{13}$$

and therefore we obtain

$$(\boldsymbol{J}+\mathrm{Tr}\,I\cdot\boldsymbol{T})\cdot\hat{\boldsymbol{x}}=0. \tag{14}$$

Using (1) and (11)-(13) we end up with the following equations:

$$r^2\dot{I}=\mathrm{i}(\boldsymbol{x}\wedge\dot{\boldsymbol{x}})[\boldsymbol{M}-\boldsymbol{T},I] \tag{15}$$

$$mr^2\ddot{\boldsymbol{x}}=\dot{\boldsymbol{x}}\wedge\{\mathrm{Tr}\,I(\boldsymbol{M}-\boldsymbol{T})_{\|}+r\,\mathrm{Tr}\,I\boldsymbol{N}_{\perp}\} \tag{16}$$

$$\boldsymbol{J}=m\boldsymbol{x}\wedge\dot{\boldsymbol{x}}-\mathrm{Tr}\,I\boldsymbol{M}_{\perp}+\mathrm{Tr}\,I\boldsymbol{T}_{\|}. \tag{17}$$

Now we restrict ourselves to the case $J=0$. With (17) we obtain for a vanishing angular momentum

$$m\boldsymbol{x}\wedge\dot{\boldsymbol{x}}=\operatorname{Tr} I\boldsymbol{M}_{\perp} \tag{18}$$

$$\operatorname{Tr} I\boldsymbol{T}_{\parallel}=0. \tag{18a}$$

By using these identities in (15), we find

$$mr^2\dot{I}=\mathrm{i}\operatorname{Tr} I\boldsymbol{M}_{\perp}\cdot[\boldsymbol{M}-\boldsymbol{T}, I]. \tag{19}$$

Now we specialise further and discuss these equations for the 't Hooft-Polyakov monopole solution.

4. A Yang–Mills particle in a 't Hooft–Polyakov monopole field

For a 't Hooft-Polyakov monopole solution [6, 7] the symmetry group is $G=SU(2)$ and hence

$$M_1(r)=K(r)S_1 \qquad M_2(r)=K(r)S_2 \qquad M_3(r)=K^2(r)S_3. \tag{20}$$

We furthermore assume that the YM particle is a SU(2) doublet. Thus we may expand $(T\equiv S)$

$$I=\boldsymbol{\gamma}(t)\cdot S.$$

Because of (18a) we have $\hat{\boldsymbol{x}}\cdot\boldsymbol{\gamma}=0$. Using this, together with (20), yields the stated conservation of the isospin

$$\dot{I}=0 \tag{21}$$

or equivalently

$$\boldsymbol{\gamma}=\text{constant}. \tag{21a}$$

With (14) we conclude that

$$(\hat{\boldsymbol{x}}\cdot\boldsymbol{\gamma})=0$$

i.e. the YM particle moves on a plane which is perpendicular to $\boldsymbol{\gamma}$. The equation of motion (16) simplifies to

$$m\ddot{\boldsymbol{x}}=(K'(r)/r)\boldsymbol{\gamma}\wedge\dot{\boldsymbol{x}}. \tag{22}$$

In a Prasad-Sommerfeld monopole field $K=Dr/\sinh(Dr)$, we can solve equation (22) analytically. For that we introduce polar coordinates (ρ, ϕ) in the plane of the particle path. With the abbreviation $A^2=D^2/2mT$ we obtain

$$\rho(t)=(4\pi/M)\cosh^{-1}[(1+A^2)^{1/2}\cosh(2ATt)]. \tag{23}$$

Here $M=4\pi D$ is the mass of the monopole. One should compare (23) with the solution in the field of a Dirac pole:

$$\rho^2(t)=(2T/m)t^2.$$

As one expects, $\rho(t)$, $\phi(t)$ approach the Dirac solution for $M\to\infty$.

5. The asymptotic solutions

The vector fields $\boldsymbol{M}(x)$ and $\boldsymbol{N}(x)$ in (15)-(17) decay exponentially for $|\boldsymbol{x}|\to\infty$ [4]. With (17) we obtain

$$\boldsymbol{J} \sim m\boldsymbol{x} \wedge \dot{\boldsymbol{x}} + \mathrm{Tr}\, I\boldsymbol{T}_{\parallel}. \tag{24}$$

Therefore $\boldsymbol{J}\boldsymbol{x} \sim r\,\mathrm{Tr}\, I(\boldsymbol{T}\hat{\boldsymbol{x}})$ and $\boldsymbol{J}\dot{\boldsymbol{x}} = \dot{r}\,\mathrm{Tr}\, I(\boldsymbol{T}\hat{\boldsymbol{x}})$. We conclude that

$$\mathrm{Tr}\, I(\boldsymbol{T}\hat{\boldsymbol{x}}) \sim \text{constant} \tag{25}$$

and

$$\boldsymbol{J}\cdot\hat{\boldsymbol{x}} \sim \text{constant} \tag{26}$$

i.e. the particle moves asymptotically on a cone. The asymptotic form of (16)

$$m\ddot{\boldsymbol{x}} \sim -(\dot{\boldsymbol{x}} \wedge \boldsymbol{x})\mathrm{Tr}\, I(\boldsymbol{T}\hat{\boldsymbol{x}})/r^3 = -\dot{\boldsymbol{x}} \wedge \boldsymbol{B}_{\mathrm{D}}(\boldsymbol{x}) \tag{27}$$

is the equation of motion of an electron in the field of a Dirac pole with a magnetic charge $g = \mathrm{Tr}\, I(\boldsymbol{T}\hat{\boldsymbol{x}})/e$.

6. Additional remarks

Throughout this paper we confined ourselves to the case of a vanishing Yukawa coupling of the YM particle to the Higgs field. If the Dirac field which corresponds to the YM particle also couples to the scalar field, i.e.

$$L = \bar{\psi}\mathrm{i}\not{D}\psi + \bar{\psi}(\Gamma\phi)\psi$$

then, using Wong's arguments, one finds instead of (1) and (2) the equations

$$\dot{p}^{\mu} = (mc)^{-1}\mathrm{Tr}\{I\cdot V_{*}(F^{\mu\nu})\}p_{\nu} - \mathrm{Tr}\, I\cdot D^{\mu}(\Gamma\phi)$$

$$\mathrm{d}I/\mathrm{d}\tau = \mathrm{i}\dot{x}^{\mu}[V_{*}(A_{\mu}), I] + \mathrm{i}[\Gamma\phi, I]$$

where $D_{\mu}(\Gamma\phi) = \partial_{\mu}(\Gamma\phi) - \mathrm{i}[V_{*}(A_{\mu}), \Gamma\phi]$. From the second of these modified Wong equations, using $\dot{x}^{\mu}\dot{x}_{\mu} = 1$, we especially obtain a time-dependent mass

$$\dot{m} = -\mathrm{Tr}\, ID_{\nu}(\Gamma\phi)\dot{x}^{\nu}.$$

The Lagrangian (4) gets an additional term $\mathrm{Tr}\, Ks^{-1}(\Gamma\phi)s$. For an ss field configuration (ϕ, A) the angular momentum (13) remains the same and our discussion of the equations of motion is only slightly modified. But the qualitative behaviour of the particle may be dramatically affected by the introduction of this additional Yukawa coupling.

7. Conclusions

Using the most general ansatz for a spherically symmetric monopole field (with arbitrary gauge group G) we derived with (15)-(17) the equations of motion and the conserved angular momentum for a YM particle in such a field. Then we discuss these equations in various circumstances, e.g. for a vanishing angular momentum and in the asymptotic region $|x|\to\infty$.

With (21) we proved that for $G = SU(2)$ an isospin doublet suffers no charge exchange for $J = 0$. Hence there exists no non-relativistic 'classical Rubakov effect'.

This is, of course, not very surprising, since the $J = 0$ sector is in the 'anti-classical regime'. But nevertheless our results tells us something about the (qualitative) limits of Wong's equations.

After solving the Wong equations for a YM particle in a Prasad–Sommerfield monopole field we dsicussed the asymptotic form of these equations in an *arbitrary* SS monopole field.

We find that the YM particle behaves asymptotically like an electron in an Dirac pole field with magnetic charge $g = \mathrm{Tr}\, I(T\hat{x})/e$.

Acknowledgments

The author thanks P Horvathy for valuable discussions. This work was supported by the Swiss National Science Foundation and the Max-Planck-Gesellschaft.

Note added. After this work was completed we learned of the article by Fehér [8]. This paper takes a possible Yukawa coupling into account but restricts it to the PS limit. We point out that our equations (28)–(30) differ from these obtained by Fehér. We thank P Horvathy for bringing this paper to our attention.

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