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WKB Properties of the Time-Dependent Schrödinger System

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It is shown that the time-dependent WKB expansion highlights some of the hidden properties of the Schrödinger equation and forms a natural bridge between that equation and the functional integral formulation of quantum mechanics. In particular it is shown that the leading (zero- and first-order in $\hbar$) terms in the WKB expansion are essentially classical, and the relationship of this result to the classical nature of the WKB partition function, and of the anomalies in quantum field theory, is discussed.

1. INTRODUCTION

Although the Schrödinger equation has been in existence for 60 years and its applications pervade most of modern physics and chemistry, the equation itself continues to be of great intrinsic interest and to yield new insights.

In the present paper we wish to show that some such insights can be gained by writing the time-dependent Schrödinger equation in its (nonlinear) real form. In this form one sees that the Schrödinger system depends basically on $\hbar^2$ rather than $\hbar$, and is ripe for development in $\hbar^2$. In particular one sees that the term linear in $\hbar$ (the time-dependent WKB$^{(1)}$ term) is essentially classical and is, in fact, generated by the purely classical Hamilton–Jacobi equation. As such, this term should have a classical meaning, and the classical meaning is elucidated.

Apart from its intrinsic interest, the time-dependent WKB term and the subsequent WKB expansion of the (time-dependent) Schrödinger

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system is of interest because it forms a bridge between the Schrödinger formulation and the functional integral formulation of quantum mechanics.[2] Indeed, the most popular first approximation to the functional integral is the classical-path approximation, and the subsequent development of the functional integral in powers of $\hbar$ (loops) corresponds exactly to the WKB development just mentioned.[3] In particular, the term of order $\hbar$ relative to the classical approximation (one-loop term) in the functional integral is just the leading WKB term discussed above. Since in the functional integral formalism this term always appears as a determinant (coming from the Gaussian distribution around the classical path), we shall call it the WKB determinant. The derivations of this term from the wave equation and from the functional integral formalism (which we shall give) are very different, and give different insights into its nature. In each case, however, both the derivation and the end result is essentially classical.

The classical nature of the WKB term becomes even more evident when one considers the quantum-mechanical partition[4] function $Z(t)$, because in the WKB approximation to $Z(t)$, the Planck constant $\hbar$ simply does not appear. An interesting consequence of this result is that it helps to explain the apparent paradox that the so-called global "anomalies" of quantum field theory,[5] which play such a central role in grand unification and string theories, are quantum mechanical from the point of view of Feynman graphs[6] but are classical from the point of view of index theorems.[7] The point is that, as we shall show, the anomalies are WKB (one-loop) contributions and can be expressed as quantum-mechanical partition functions. The relationship between anomalies and essentially classical WKB partition functions is illustrated in detail for one simple model.

For quadratic Lagrangians the WKB approximation becomes exact, and hence we thought it worthwhile to write down the wave function, or functional integral, for a wide class of quadratic Lagrangians called abelian (Section 8). It turns out that all abelian quadratic Lagrangians can be reduced to one which describes a particle in the presence of a harmonic oscillator potential and a potential which corresponds to a constant magnetic field. Furthermore, there exists a transformation which eliminates the magnetic field, so that the computations can be reduced to those for a harmonic oscillator and the general result recovered by using the inverse transformation.

Finally it should be stated that most of the results presented here are not new, though some of them are relatively recent. What we hope to have done is to have shown that a number of previous results can be simplified and woven into a coherent whole, and to emphasize that the WKB parts of these results are essentially classical.

2. REAL FORM AND HIDDEN PROPERTIES OF THE SCHRÖDINGER EQUATION

By writing the Schrödinger wave function Ψ in the form

$$\Psi = \rho^{1/2} e^{\frac{i}{\hbar} S}, \qquad \text{where} \quad \rho \text{ and } S \text{ are real} \tag{1}$$

one may decompose the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V(x)\, \Psi \tag{2}$$

into the two real equations

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + V(x) = \frac{\hbar^2}{2m} \left(\frac{\Delta \rho^{1/2}}{\rho^{1/2}} \right) \tag{3}$$

and

$$\hbar \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} \right) = 0, \qquad \text{where} \quad \mathbf{j} = \frac{1}{m} \rho (\nabla S) \tag{4}$$

and these equations exhibit the following rather remarkable properties of the Schrödinger system which are otherwise not so evident:

(1) As might be inferred from the manner in which it is written, the second equation (4) is just the usual Schrödinger continuity equation. Indeed, one sees by inspection that

$$\mathbf{j} = \frac{1}{m} \rho (\nabla S) = \frac{\hbar}{2im} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \tag{5}$$

Thus the decomposition of the Schrödinger equation into real parts is simultaneously a decomposition into the continuity equation and the equation (3), which may then be regarded as the "kernel" of the Schrödinger equation.

(2) Once the overall factor $\hbar$ has been eliminated from the continuity equation (4), this equation is *independent* of $\hbar$. Thus the continuity equation gives a purely classical relationship between ρ and S, the quantum mechanical ($\hbar$) appearing only in the "kernel" equation (3).

(3) Even in the kernel equation (3) there appears only the *square* of $\hbar$. This shows that, once an $\hbar$ has been absorbed in the definition of S in (1), the Schrödinger system actually depends only on $\hbar^2$. Furthermore, one sees from (3) that $\hbar^2$ appears in a manner that leaves the equation ripe for

an iteration in $\hbar^2$. (Of course, whether the iteration converges depends on the potential).

(4) If, in particular, one considers the WKB approximation (neglect of terms of order $\hbar^2$), one sees that one obtains not only the usual classical Hamilton–Jacobi equation but *two* classical equations, namely,

$$\frac{\partial S_c}{\partial t}+\frac{1}{2m}(\nabla S_c)^2+V(x)=0 \quad \textit{and} \quad \frac{\partial \rho_c}{\partial t}+\frac{1}{m}\nabla(\rho_c \nabla S_c)=0 \tag{6}$$

the second of which may be regarded as coming from the first-order term in $\hbar$ (but not containing $\hbar$!). In this sense the (time-dependent) WKB approximation may be regarded as essentially classical. As will be seen later, this corresponds to the fact that the one-loop approximation to the partition function (and to the Schwinger function in QFT) is essentially classical.

(5) As will be shown in the next section, a solution to the second equation in (6) can always be found in terms of the Hamilton–Jacobi function S_c, namely,

$$\rho_{sc}=\det\left(-\frac{\partial^2 S_c(x, a, t)}{\partial x_i \partial a_j}\right) \tag{7}$$

where $a_i=x_i(0)$ is the initial value of $x_i(t)$. In fact, the derivations[8][9] and analysis of the solution (7) will constitute a substantial part of this paper.

(6) Given the solution (7), the general solution ρ_c is evidently given by $\rho_c=\eta\rho_{sc}$, where η is any solution of the simpler residual equation

$$\frac{\partial \eta}{\partial t}+\frac{1}{m}(\nabla\eta)\cdot(\nabla S_c)=0 \tag{8}$$

Geometrically, the solutions η are those functions which are constant along the lines with tangents parallel to $(m, \nabla S_c)$ in (t, x)-space. From this, and the fact that the initial value $\rho_{sc}(x, 0)$ of $\rho_{sc}(x, t)$ is completely determined by the initial value of $S_c(x, t)$ from (7), one sees that one of the main roles of η is to free the initial value of $\rho_c(x, t)$ from that of $S_c(x, t)$.

Although Eqs. (3) and (4) were derived from the Schrödinger equation in order to have a familiar starting point, the properties we have just described are actually more transparent if one starts from the Lagrangian density for the Schrödinger equation, namely,

$$\mathscr{L}(x)=\frac{i\hbar}{2}(\Psi^*\partial_t\Psi-\Psi\partial_t\Psi^*)-\frac{\hbar^2}{2m}|\nabla\Psi|^2-V|\Psi|^2 \tag{9}$$

Indeed, if one writes the wave function Ψ in the form (1), this Lagrangian reduces to

$$\mathscr{L}(x) = -\rho^2\left(\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 + V\right) - \frac{\hbar^2}{2m}(\nabla\rho)^2 \tag{10}$$

from which Eqs. (3) and (4) and points (1)–(4) above follow by inspection. One also sees that ρ may be regarded as a Lagrange multiplier for the Hamilton–Jacobi equation in the classical part of $\mathscr{L}$, and that the current conservation equation (4) may be regarded either as a field equation, or as the Noether-current conservation law[10] for the symmetry $S(x, t) \to S(x, t) + c$ of $\mathscr{L}$, where $c = \text{const}$. Indeed, the usual Noether rule for fields S that appear in Lagrangian densities only through their derivatives yields the current

$$j_\mu = \frac{\partial\mathscr{L}}{\partial S_{,\mu}}\frac{\partial S}{\partial c} = \frac{\partial\mathscr{L}}{\partial S_{,\mu}} \tag{11}$$

whose conservation law is just the field equation, and in the present instance the current j_μ is evidently $\rho(1, \nabla S/m)$.

3. DERIVATION OF THE SPECIAL SOLUTION

In this section we wish to derive the special solution (7) of the continuity equation in (6) from the Hamilton–Jacobi equation

$$\frac{\partial S_c}{\partial t} + \frac{1}{m}(\nabla S_c)^2 + V = 0 \tag{12}$$

For this purpose one differentiates (12) with respect to $x_i(t)$ and $x_i(0) = a_i$. Since $V(x)$ does not depend on a_i, it drops out and one obtains

$$\partial_t\left(\frac{\partial^2 S_c}{\partial x_i\,\partial a_j}\right) + \frac{1}{m}\left(\frac{\partial^2 S_c}{\partial a_j\,\partial x_k}\right)\left(\frac{\partial^2 S_c}{\partial x_k\,\partial x_i}\right) + \frac{1}{m}\left(\frac{\partial S_c}{\partial x_k}\right)\left(\frac{\partial^3 S_c}{\partial x_k\,\partial a_j\,\partial x_i}\right) = 0 \tag{13}$$

or, on writing the second derivatives in matrix form,

$$\partial_t M_{ij} + \frac{1}{m}M_{ik}N_{kj} + \frac{1}{m}(\nabla S_c)\cdot(\nabla M_{ij}) = 0 \tag{14}$$

where M and N are the matrices indicated. On multiplying (14) to the left by M^{-1}, taking the trace of the result, and using the general formula

$$\mathrm{tr}(M^{-1}\,\delta M) = \delta(\ln \det M) \tag{15}$$

for the first variation of any nonsingular matrix M, one obtains

$$\partial_t(\det M) + \frac{1}{m}(\det M)\,\mathrm{tr}\,N + \frac{1}{m}(\nabla S_c)\cdot(\nabla \det M) = 0 \tag{16}$$

But since tr N is just ΔS_c, the last two terms in (16) can be combined, and one obtains

$$\partial_t(\det M) + \frac{1}{m}\nabla((\det M)\,\nabla S_c) = 0 \tag{17}$$

The result then follows by making the obvious identification

$$\rho_{sc} = \det(-M) = \det\left(-\frac{\partial^2 S}{\partial x_i\,\partial a_j}\right) \tag{18}$$

the minus sign being chosen because $\partial^2 S/\partial a\,\partial x$ is generically negative.[3]

Note that the derivation hinges on the fact that $V(x)$ is independent of a_i. Hence the result (17) does not generalize to the case when the $\hbar^2$ term in (3) is not neglected. Indeed in that case it is easy to see that (17) picks up the term

$$\frac{\hbar^2}{2}(\det M)\,\mathrm{tr}(M^{-1}K), \qquad \text{where} \quad K_{ij} = \frac{\partial}{\partial a_i}\frac{\partial}{\partial x_j}\left(\frac{\Delta\rho^{1/2}}{\rho^{1/2}}\right) \tag{19}$$

Clearly K does not vanish unless $\partial^4 S/\partial x^2\,\partial a^2$ is zero, and while this may happen in exceptional cases (see Sections 8 and 9, for example), it does not happen in general.

4. CLASSICAL MEANING OF THE SPECIAL SOLUTION

The fact that the special solution ρ_{sc} can be expressed in terms of the Hamilton–Jacobi function S_c means that even a purely classical theory must have hidden in it an analogue of the Schrödinger density $\Psi^*\Psi$ and its accompanying conservation law! One may then ask: What is the physical meaning of this classical "$\Psi^*\Psi$" and its conservation law?

One meaning of the conservation law is obtained, of course, from Noether's theorem, because just as the quantum-mechanical conservation

[3] We shall assume throughout this paper that $\rho_{sc} \neq 0$. The zeros of ρ_{sc} are, in fact, of great physical interest in themselves (see, for example, Ref. 8), but they shall not concern us here.

law follows from the invariance of the Schrödinger equation under the phase transformation $\Psi(x) \to e^{ic}\Psi$, where c is constant, the classical conservation law follows from the invariance of the Hamilton–Jacobi equation under the corresponding transformation $S_c \to S_c + c$.

What we really seek, however, is a direct physical meaning for ρ_{sc}. This is obtained by noting that, according to standard Hamilton–Jacobi theory, the quantity $(-\partial S_c/\partial a_i)$ is just the initial momentum $p_i(0)$. Indeed, from the general Lagrangian variational principle, one has

$$\delta S_c = \delta \int_a^x L\, dt + [\mathbf{p} \cdot \delta \mathbf{x}]_a^x, \qquad p_k = \frac{\partial \mathscr{L}}{\partial \dot{x}_k} = m\dot{x}_k + g_k(x) \tag{20}$$

where p_k is the canonical momentum, and g_k is a function of x only. And since for classical paths the first term is zero, one has, in particular,

$$\frac{\partial S_c}{\partial a_k} = -p_k(0) \tag{21}$$

Hence the density ρ_{sc} may also be written in the form

$$\rho_{sc} = \det\left(\frac{\partial p_i(0)}{\partial x_j}\right) \tag{22}$$

Equation (22) shows that ρ_{sc} is just the Jacobian for the transformation from initial momenta to final positions (keeping the initial positions fixed). Thus ρ_{sc} converts any small distribution $\Delta^3 p(0)$ in initial momentum into a distribution in final position according to

$$\Delta^3 x = (\rho_{sc})^{-1}\, \Delta^3 p(0) \tag{23}$$

So finally one sees that the physical meaning of ρ_{sc} is that it determines the spread in the final position (e.g., area where a projectile may land) in terms of the spread in the initial momentum (e.g., spread in the angle of projection). Thus, for artillery computations at least, ρ_{sc} is an important quantity! Of course, the fact that $\int d^3x\, \rho_{sc}(x)$ is conserved simply means that (if ρ_{sc} is not singular) a moving particle must arrive *somewhere.*

5. FREE CASE, GENERAL INITIAL CONDITIONS, AND THE WKB GREEN'S FUNCTION

Although the free ($V = 0$) Schrödinger wave equation and the free classical action are each fairly trivial, it is nevertheless worthwhile to

investigate the relationship between them. It is well known that the free classical action is

$$S_{cf}(x, a, t) = \frac{m}{2t}(\mathbf{x} - \mathbf{a})^2 \tag{24}$$

and from this one sees at once that

$$-\frac{\partial^2 S_{cf}}{\partial a_i \partial x_j} = \frac{m}{t}\delta_{ij} \quad \text{and} \quad \rho_{scf}^{1/2} = \left(\frac{m}{t}\right)^{3/2} \tag{25}$$

where ρ_{scf} is the special solution of (4). Furthermore, since $\Delta\rho_{scf}$ is zero, one sees that the pair (S_{cf}, ρ_{scf}) in this case actually form an exact solution to the Schrödinger system, and hence

$$\Psi_f(x, a, t) = \left(\frac{m}{t}\right)^{3/2} \exp\left[\frac{i}{\hbar} \cdot \frac{m(\mathbf{x} - \mathbf{a})^2}{2t}\right] \tag{26}$$

is an exact solution of the free Schrödinger equation. Furthermore, since for any smooth test-function $g(x)$, one has

$$\lim_{t=0} \int \Psi_f(x, a, t)\, g(x) = (2\pi\hbar)^{3/2}\, g(a) \tag{27}$$

one sees that Ψ_f is the solution that tends to $(2\pi\hbar)^{3/2}\,\delta^3(x-a)$ as $t \to 0$. Thus the combination of the special solution of (4) (i.e., the solution with $\eta = 0$) and the free classical action corresponds to the wave function that starts out as a delta function. Alternatively, since

$$\Phi(x, t) = (2\pi\hbar)^{-3/2} \int d^3x\, \Psi_f(x, a, t)\, \Phi(a) \tag{28}$$

then satisfies the free Schrödinger equation with initial value $\Phi(a)$, one may say that $(2\pi\hbar)^{-3/2}\,\Psi_f(x, a, t)$ is the Green's function for the free Schrödinger operator.

Perhaps the most interesting feature of this result is that it generalizes to the nonfree case $V \neq 0$ in the WKB approximation. The point is that for any (smooth, etc.) potential V it is consistent to let $S_c(x, a, t)$ be of the form

$$S_c(x, a, t) = \frac{m}{2t}[(\mathbf{x} - \mathbf{a})^2 + O(t)] \tag{29}$$

for small t, because in the Hamilton–Jacobi equation this $S_c(x, a, t)$ will be of order t^{-2}, and so will dominate the potential term for small t. On

choosing this initial condition the special solution ρ_{sc} will still take the form $(M/t)^3$ for small t, and the WKB Schrödinger wave function

$$\Psi(x, a, t) = (2\pi\hbar)^{-3/2} \rho_{sc}^{1/2}(x, a, t) \exp\left[\frac{i}{\hbar} S_c(x, a, t)\right] \tag{30}$$

will still tend to $\delta^3(x-a)$ as $t \to 0$. In other words, the effect of the potential will only be felt at later times. Thus the quantity (30) which satisfies the Schrödinger equation in the WKB approximation and tends to $\delta^3(x-a)$ as $t \to 0$ must be the WKB approximation to the Green's function for the Schrödinger operator for $V \neq 0$.

Note incidentally that the "spread in the wave packet" of a free particle comes from the exponential (classical) part of the wave function in (26), the WKB part $(m/t)^{-3/2}$ acting only as a normalizing factor. Thus the classical spread $\rho_{sc}^{-1}(x, a, t) = \det(\partial x_i/\partial p_j(0))$ discussed in the previous section is not the spread of the wave packet but the additional spread introduced by the existence of the potential.

6. FUNCTIONAL INTEGRAL APPROACH

In the following sections we wish to compare our previous results with those obtained using the functional integral approach, partly because the functional integral approach is interesting in its own right and gives an alternative derivation of the central formula (7), and partly because it will relate the previous results directly to the partition function. In particular, it will help to bring out clearly the essentially classical nature of the WKB (one-loop) approximation to the partition function.

We start from the standard Dirac–Feynman–Kac functional integral expression[2] for the wave function, namely,

$$\Psi(x, a, t, V) = N^{-1/2} \int_a^x d[y] \exp\left[\frac{i}{\hbar} S(y, t)\right] \tag{31}$$

where N is a universal (V-independent) normalizing factor, $d[y]$ denotes integration over all paths, and $S(y, t)$ denotes the classical action function

$$S(y, t) = \int_0^t \left\{\frac{m}{2}\left(\frac{\partial y}{\partial \tau}, \frac{\partial y}{\partial \tau}\right) - V(y)\right\} d\tau, \qquad \left(\frac{\partial y}{\partial \tau}, \frac{\partial y}{\partial \tau}\right) = \sum_{i=1}^{3} \left(\frac{\partial y_i}{\partial \tau}\right)^2 \tag{32}$$

If $S(y, t)$ is expanded around its classical value $S_c(y, t)$, where $y(t)$ is the

classical path from a to x in the potential $V(x)$, the terms linear in $(y-y_c)$ drop out because y_c is extremal and one obtains

$$S(y,t)=S_c(y,t)+\frac{m\hbar}{2}\int_0^t[(\delta\dot{y},\delta\dot{y})-(\delta y,U(\tau)\,\delta y)]\,d\tau+O(\hbar^{3/2}) \tag{33}$$

where $y-y_c=\hbar^{1/2}\,\delta y$ and $U(\tau)$ is the matrix

$$U(\tau)=U_{jk}(\tau)=\frac{1}{m}\left(\frac{\partial^2 V}{\partial y_j\,\partial y_k}\right)_{y=y_c} \tag{34}$$

From this one sees that the WKB approximation to $\Psi(x,t)$ is

$$\Psi_{\mathrm{WKB}}(x,t)=[ND(S_c)]^{-1/2}\exp\left[\frac{i}{\hbar}S_c(x,t,V)\right] \tag{35}$$

where the exponential term is the zero-order (classical) approximation and

$$\begin{aligned}D^{-1/2}(S_c)&=\int_0^0 d[y]\exp\left(\frac{im}{2}\right)\int_0^t(\delta y,+\partial_\tau^2+U(\tau)\,\delta y)\,d\tau\\&=[\det(+\partial_\tau^2+U(\tau))]^{-1/2}\end{aligned} \tag{36}$$

is the first-order correction. Note that the paths in (36) are closed loops (from 0 to 0) and that a partial integration has been used in order to express the kinetic term in operator form. The quantity $D(S_c)$, which we shall call the WKB determinant, is a closed expression for the WKB correction for any potential $V(x)$, and, as we shall see, it reduces to the expression ρ_{sc} of the previous sections. It has the nice property that since the time interval $[0,t]$ is finite, and the boundary conditions are $\delta y(0)=\delta y(t)=0$, the spectrum of the operator $-\partial^2-U(\tau)$ is discrete. If, in addition, one assumes that the potential is such that there is a one-to-one correspondence between the eigenvalues ε_r of this operator and the eigenvalues $\mathring{\varepsilon}_r=\pi^2r^2/t^2$ of $-\partial^2$, and that the indicated products converge, we may write the ratio of the corresponding WKB determinants as

$$D(S_c)/D(\mathring{S}_c)=\prod(\varepsilon_r)\Big/\prod(\mathring{\varepsilon}_r)=\prod(\varepsilon_r/\mathring{\varepsilon}_r) \tag{37}$$

For example, for the harmonic oscillator in one dimension,

$$V(x)=\omega^2x^2/2,\qquad U(\tau)=\omega^2/m,\qquad \varepsilon_r=\mathring{\varepsilon}_r-\omega^2/m \tag{38}$$

and hence

$$\frac{D(S_c)}{D(\mathring{S}_c)}=\prod_{r=1}^{\infty}\left(1-\frac{\omega^2t^2}{m\pi^2r^2}\right)=\frac{\sin(\omega t/\sqrt{m})}{(\omega t/\sqrt{m})} \tag{39}$$

For a general potential, it is not to be expected, of course, that the entire spectrum ε_r can be computed explicitly, but since the WKB determinant $D(S_c)$ depends only on a single combination of the ε_r (namely their product), one might expect to be able to compute $D(S_c)$ even when the individual ε_r are not computable. One method[11] of computing $D(S_c)$ is the following: Consider the related differential equation

$$\left(\frac{\partial^2}{\partial\tau^2}+U(\tau)\right)_{ij} f_j(\tau, z)=zf_i(\tau, z) \tag{40}$$

with boundary conditions

$$f_i(0, z)=0, \qquad \left(\frac{\partial}{\partial\tau}f_i(\tau, z)\right)_{\tau=0}=c \tag{41}$$

where z is any complex number and c_i is any real nonzero vector. One sees by inspection that the $f_i(\tau, z)$ are entire functions of z. Furthermore, one sees that the $f_i(\tau, z)$ are eigenvectors of the operator $-\partial_\tau^2-U(\tau)$ whenever z is such that $f_i(t, z)=0$, since then both boundary conditions $f_i(0, z)=0$ and $f_i(t, z)=0$ are satisfied. Since the converse is also true, it follows that if $f_i^{(k)}(\tau, z)$, $k=1\cdots n$ denote the n linearly independent solutions of (40), (41) then the $n\times n$ determinant $\det f_i^{(k)}(t, z)$ must be of the form

$$\det f_i^{(k)}(t, z)=g(t, z)\prod(\varepsilon_r-z) \tag{42}$$

where $g(t, z)$ is an entire function *without zeros*. Letting subscript zero denote the corresponding quantities for $U(\tau)=0$, one sees that

$$G(t, z)=\frac{\mathring{g}(t, z)}{g(t, z)}=\frac{\det \mathring{f}_i^{(k)}(t, z)}{\det f_i^{(k)}(t, z)}\cdot\prod\left(\frac{\varepsilon_r-z}{\mathring{\varepsilon}_r-z}\right) \tag{43}$$

is an entire function. On the other hand, for any smooth (even bounded) potential $U(\tau)$ in the interval $[0, t]$, Eq. (40) becomes independent of $U(\tau)$ for large $|z|$. Hence, if the *same* matrix $c_i^{(k)}$ is used in the boundary condition for $f_i^{(k)}$ and $\mathring{f}_i^{(k)}$, these functions must approach each other asymptotically, and the ratio of the determinants in (43) must tend to unity, as $|z|\to\infty$. Similarly, if one assumes that the spectrum ε_r is such that the order of the limits in forming the product $\prod$ and letting $|z|\to\infty$ can be interchanged, the $\prod$ term in (43) tends to unity for large $|z|$. Under these circumstances, one sees that $G(z)\to 1$ as $|z|\to\infty$ and hence by Liouville's theorem $G(z)=1$ for all z. Thus

$$\prod\left(\frac{\varepsilon_r-z}{\mathring{\varepsilon}_r-z}\right)=\frac{\det f(t, z)}{\det \mathring{f}(t, z)} \tag{44}$$

Setting $z=0$ one then sees that, finally, the WKB determinant (37) is given by

$$\frac{D(S_c)}{D(\mathring{S}_c)}=\prod\left(\frac{\varepsilon_r}{\mathring{\varepsilon}_r}\right)=\frac{\det f_i^{(k)}(t,0)}{\det \mathring{f}_i^{(k)}(t,0)} \tag{45}$$

where $f_i^{(j)}(\tau,0)$ are the n linearly independent solutions of the zero-mode equation

$$(\partial_\tau^2+U(\tau))_{ij}f_j^{(k)}(\tau)=0, \qquad f_j^{(k)}(0)=0, \qquad \left(\frac{d}{d\tau}f_j^{(k)}\right)(0)=c_j^{(k)} \tag{46}$$

and $\mathring{f}_j^{(k)}=c_j^{(k)}\tau$ are the corresponding solutions for $U(\tau)=0$. Thus from (45) and (46) one sees that in order to compute D in (37) one need only solve the system (46), which is actually simpler than any of the eigenvalue equations for the individual ε_r, and is the simplest case of (40). In fact, as we shall see in the next section, the solutions of (46) are provided by the classical equations of motion!

7. DERIVATION OF THE WKB DETERMINANT FROM THE CLASSICAL EQUATION OF MOTION

In this section we wish to derive the solutions of the system (46) from the classical equations of motion

$$m\frac{d^2}{d\tau^2}x_i^c(\tau)=-\frac{\partial V(x^c)}{\partial x_i^c} \tag{47}$$

The reason that this can be done is that Eq. (46) is just the equation that one obtains on making an infinitesimal variation of $x^c(t)$ in (47). That is, if one assumes that both x_c and $x_c+\delta x_c$ satisfy (47), then

$$m\frac{d^2}{d\tau^2}\delta x_i^c(\tau)=-\frac{\partial^2 V(x^c)}{\partial x_i^c\,\partial x_j^c}\delta x_j^c \tag{48}$$

which is just (46). However, since both x^c and $x^c+\delta x^c$ satisfy the classical equations of motion, they can only differ in their initial conditions, and thus

$$\delta x_i^c(\tau)=\frac{\partial x_i^c(\tau)}{\partial x_j(0)}\delta x_j(0)+\frac{\partial x_i^c(\tau)}{\partial \dot{x}_j(0)}\delta \dot{x}_j(0) \tag{49}$$

But since we wish to identify the solutions $f_j(t)$ with $\delta x_i^c(t)$, and $\delta x_i^c(t)$ satisfies the boundary condition $\delta x_i^c(0)=f_i(0)=0$ only if the first term on

the right-hand side of (49) is zero, one must choose $\delta x_j(0)=0$. Thus the solutions with the required boundary conditions are

$$f_i^{(k)}(\tau)=\frac{\partial x_i^c(\tau)}{\partial \dot{x}_j^c(0)}\, c_j^{(k)}=m\,\frac{\partial x_i^c(\tau)}{\partial p_j^c(0)}\, c_j^{(k)} \tag{50}$$

where in the last equation we have used the second equation in (20).

[Another interpretation of (49) may be obtained by noting that the classical Hamiltonian

$$H=\frac{1}{2m}\left(\frac{\partial x}{\partial \tau}, \frac{\partial x}{\partial \tau}\right)+V(x_c) \tag{51}$$

for (47) is invariant with respect to variations of x^c which respect to the equations of motion, that any particular solution $x^c(t)$ spontaneously breaks this symmetry, and that the $\delta x_i^c(t)$ in (49) are just the Goldstone[12] modes for this breaking!]

In any case, Eq. (50) provides the required solutions of the system (46), and thus the WKB determinant $D(S_c)$ is given by

$$\frac{D(S_c)}{D(\mathring{S}_c)}=\left(\frac{m}{t}\right)^3 \det\left(\frac{\partial x_i^c}{\partial p_j(0)}\right)=\left(\frac{m}{t}\right)^3 \det\left(-\frac{\partial^2 S_c}{\partial x_i\, \partial a_j}\right) \tag{52}$$

Hence if one chooses the universal constant N to be $(mD(\mathring{S}_c)/t)$, one sees that finally the WKB approximation to the wave function (31) is just

$$\Psi_{\mathrm{WKB}}(x, t)=\left\{\det\left(-\frac{\partial^2 S_c}{\partial x_i\, \partial a_j}\right)\right\}^{1/2} e^{\frac{i}{\hbar} S_c(x,t,V)} \tag{53}$$

in agreement with the result obtained directly from the Schrödinger equation in Section 3.

8. QUADRATIC LAGRANGIANS AND THEIR REDUCTION

Since for quadratic Lagrangians the WKB approximation becomes exact, and quadratic Lagrangians are physically important, it may be worthwhile to consider them in detail. The most general quadratic Lagrangian in three dimensions is

$$\mathscr{L}(Y)=\tfrac{1}{2}(\dot{Y}, \dot{Y})-\tfrac{1}{2}(Y, \omega^2 Y)-(Y, \sigma \dot{Y})-(\lambda, Y) \tag{54}$$

where the variable Y and the parameter λ are 3-vectors, (,) denotes the three-dimensional Euclidean inner product, and ω^2 and σ are real sym-

metric and real antisymmetric matrices respectively. Note that an antisymmetric part of ω^2 would automatically drop out, and a symmetric part of σ and a term linear in $\dot{Y}$ would be total time derivatives. This is why such terms have been omitted. We shall call Lagrangians for which ω^2 and σ commute "abelian" quadratic Lagrangians and shall show that all such Lagrangians can be reduced to an "effective" harmonic oscillator Lagrangian.

The first stage in the reduction is to eliminate the linear term (λ, Y). If ω^2 is not singular, this is easily achieved by making the linear transformation $Y \to y = Y + \omega^{-2}\lambda$ and using y as the new variable. However, ω^2 (and σ) may be singular, so the general transformation needed to eliminate the linear term is more complicated, namely,

$$Y \to y = Y - l(t), \qquad \text{where} \quad l(t) = \underline{\omega}^{-2}\lambda_\omega + \frac{t}{2}\underline{\sigma}^{-1}\lambda_\sigma + \frac{t^2}{2}\lambda_0 \tag{55}$$

the vectors λ_0 and λ_σ are those parts of λ which are annihilated by (both ω^2 and σ) and by (ω^2 but not σ) respectively, and $\underline{\omega}^2$ and $\underline{\sigma}$ are the restrictions of ω^2 and σ to the subspaces on which they are not singular. In other words,

$$\lambda = \lambda_\omega + \lambda_\sigma + \lambda_0, \qquad \text{where} \quad \omega\lambda_\sigma = 0,\ \sigma\lambda_\sigma \neq 0,\ \omega\lambda_0 = \sigma\lambda_0 = 0 \tag{56}$$

On making the transformation (55), the Lagrangian (54) becomes

$$\mathscr{L}(y) = L(y) - L(l(t)) - \frac{d}{dt}(y, \dot{l} + \sigma l) \tag{57}$$

where $L(y)$ is the Lagrangian

$$L(y) = \tfrac{1}{2}(\dot{y}, \dot{y}) - \tfrac{1}{2}(y, \omega^2 y) - (y, \sigma\dot{y}) \tag{58}$$

i.e., the original Lagrangian (54) without the linear term. It is easy to see that $L(l)$ depends only on λ, ω, σ, and t, i.e., does not depend on the initial and final values of y, and indeed is just

$$-L(l(t)) = \frac{1}{2}(\underline{\omega}^{-1}\lambda_\omega)^2 - \frac{1}{8}(\sigma^{-1}\lambda_\sigma)^2 - \frac{t^2}{2}\lambda_0^2 \tag{59}$$

Thus this term and the total time derivative can be discarded and we can concentrate on the purely quadratic Lagrangian (58). In this Lagrangian the ω^2 term describes the three-dimensional harmonic oscillator and the σ term (which can be strictly nonzero only on two-dimensional spaces) describes the interaction of the particle with a constant magnetic field.

Although the ω^2 and σ terms have different physical meanings, it is

convenient for computations to convert the σ term into an effective ω^2 term, and this conversion constitutes the second stage of the reduction. In fact, if one makes the transformation

$$y \to q = e^{\sigma t} y \tag{60}$$

which is actually orthogonal since σ is real antisymmetric, one finds that (58) reduces to

$$L(y) = \Lambda(q) = \tfrac{1}{2}(\dot{q}, \dot{q}) - \tfrac{1}{2}(q, \Omega^2 q), \qquad \text{where} \quad \Omega^2 = \omega^2 - \sigma^2 \tag{61}$$

(and we note that $-\sigma^2$ is positive). The Lagrangian $\Lambda(q)$ is the "effective" harmonic oscillator Lagrangian referred to earlier. Note incidentally that the inverse conversion of an ω^2 term into a σ term is possible only if the spectrum of ω^2 is proportional to (0, 1, 1).

9. COMPUTATION OF THE CLASSICAL ACTION AND FUNCTIONAL INTEGRAL FOR QUADRATIC LAGRANGIANS

In this section we carry out the computation of the functional integral for all quadratic Lagrangians by computing the classical action for the "effective" harmonic oscillator Lagrangian (61) and recovering the result for the general quadratic Lagrangian (58) by making the inverse transformation to (60).

For the "effective" harmonic oscillator Lagrangian the classical equations of motion and their solutions are evidently

$$\ddot{q} = (-\Omega^2)\, q \quad \text{and} \quad q(\tau) = \left(\frac{\sin \Omega\tau}{\sin \Omega t}\right) x + \left(\frac{\sin \Omega(t-\tau)}{\sin \Omega t}\right) a \tag{62}$$

where the vectors a and x are the initial and final positions and the 3×3 matrix Ω^2 acts on these vectors. One then sees by inspection that the Lagrangian $\Lambda(q)$ is given by

$$\begin{aligned} 2\Lambda(q) &= (\dot{q}, \dot{q}) - (q, \Omega^2 q) \\ &= \left(x, \frac{\Omega^2 \cos 2\Omega\tau}{\sin^2 \Omega t}\, x\right) + \left(a, \frac{\Omega^2 \cos 2\Omega(t-\tau)}{\sin^2 \Omega t}\, a\right) \\ &\quad - 2\left(x, \frac{\Omega^2 \cos \Omega(t-2\tau)}{\sin^2 \Omega t}\, a\right) \end{aligned} \tag{63}$$

and hence, after a simple integration, that the classical action $S_c(x, a, t)$ is given by

$$S_c(x, a, t) = \int_0^t \Lambda(q)\, d\tau = \frac{1}{2}\left(x, \frac{\Omega \cos \Omega t}{\sin \Omega t} x\right) + \frac{1}{2}\left(a, \frac{\Omega \cos \Omega t}{\sin \Omega t} a\right) - \left(x, \frac{\Omega}{\sin \Omega t} a\right) \tag{64}$$

This is the usual expression for the harmonic oscillator, but expressed in matrix form.

Having computed the classical action for the "effective" harmonic oscillator in this simple manner, one may now obtain the classical action for the general quadratic Lagrangian (58) by making the inverse transformation to (60). However, it is only necessary to apply this transformation to the boundary values a and x of q, for which it reduces to

$$a \to a, \qquad x \to (e^{-\sigma t})\, x \tag{65}$$

On account of the orthogonality of σ and the fact that it commutes with Ω^2, only the last term on the right-hand side of (64) changes under (65), and one obtains

$$S_c(x, a, t, L) = \frac{1}{2}\left(x, \frac{\Omega \cos \Omega t}{\sin \Omega t} x\right) + \frac{1}{2}\left(a, \frac{\Omega \cos \Omega t}{\sin \Omega t} a\right) - \left(x, \frac{\Omega e^{\sigma t}}{\sin \Omega t} a\right) \tag{66}$$

Note that this expression can also be written as

$$S_c(x, a, t, L) = \frac{1}{2}\left(x - a, \frac{\Omega \cos \Omega t}{\sin \Omega t} x - a\right) - \left(x, \frac{\Omega}{\sin \Omega t} (e^{\sigma t} - \cos \Omega t)\, a\right) \tag{67}$$

Having computed the classical action, we are now ready to compute the functional integral, since for quadratic Lagrangians the WKB approximation becomes exact, and for the WKB approximation we need, in addition to S_c, only the WKB determinant

$$D(S_c) = \det\left\{-\frac{\partial^2 S}{\partial a\, \partial x}\right\} \tag{68}$$

From (66) we see that the required second derivative of S_c is simply

$$-\frac{\partial^2 S_c}{\partial a\,\partial x}=\left(\frac{\Omega}{\sin \Omega t}\right)e^{-\sigma t} \tag{69}$$

and is independent of x and a. Furthermore, since σ is orthogonal, $\det(\exp \sigma t)$ is unity, so actually,

$$D(S_c)=\det\left(\frac{\Omega}{\sin \Omega t}\right) \tag{70}$$

It follows that the final expression for the functional integral, or wave function, of the purely quadratic Lagrangian (58) is

$$\Psi_Q(x,t)=\left\{\det\left(\frac{\Omega}{\sin \Omega t}\right)\right\}^{1/2}\exp\left(\frac{i}{2\hbar}\right)\left\{(x-a,(\Omega\cot \Omega t)\,x-a)-\left(x,\frac{\Omega(e^{\sigma t}-\cos \Omega t)}{\sin \Omega t}\,a\right)\right\} \tag{71}$$

where, of course, $\Omega^2=\omega^2-\sigma^2$ (and if one or more of the eigenvalues ω_i of Ω is zero, one takes the limit $\omega_i/\sin \omega_i t\to 1/t$). In particular, for the pure harmonic oscillator ($\sigma=0$), one has

$$\Psi_\omega(x,t)=\left\{\det\left(\frac{\omega}{\sin \omega t}\right)\right\}^{1/2}\exp\left(\frac{i}{2\hbar}\right)\left\{(x-a,(\omega\cot \omega t)\,x-a)-\left(x,\left(\omega\tan\frac{\omega t}{2}\right)a\right)\right\} \tag{72}$$

and for the pure magnetic field ($\omega=0$), restricted to two of the three dimensions,

$$\Psi_\sigma(x,t)=\left\{\det\frac{\sigma}{\sinh \sigma t}\right\}^{1/2}\times\exp\left(\frac{i}{2\hbar}\right)\{(x-a,(\sigma\coth \sigma t)\,x-a)+(x,\sigma a)\} \tag{73}$$

Note that $\sigma/\sinh \sigma t$ and $\coth \sigma t$ are even in σ, so $(a,\sigma x)$ is the only odd term. Note also that this term, and hence the whole classical action, vanishes for $x=a$, corresponding to the fact that, according to the Lorentz equation, the classical path from a to a in a magnetic field is the rest position $q=a$.

10. THE CLASSICAL NATURE OF THE WKB PARTITION FUNCTION

As is well known, the quantum-mechanical partition function $Z(t) = \operatorname{tr} \exp(-Ht/\hbar)$, where H is the Hamiltonian, may be expressed in terms of the Green's function for the Schrödinger operator as follows: Let ϕ_n denote any orthonormal basis. Then

$$\begin{aligned} Z(t) = \operatorname{tr}(e^{-\frac{Ht}{\hbar}}) &= \sum_n (\phi_n, e^{-(Ht/\hbar)}\phi_n) \\ &= \sum_n \int d^3x\, \phi_n^*(x)\, e^{-(Ht/\hbar)}\phi_n(x) \\ &= \sum_n \int d^3x\, d^3a\, \phi_n^*(x)\, G(x, a, -it)\, \phi_n(a) \\ &= \int d^3x\, d^3a\, \delta^3(x-a)\, G(x, a, -it) \\ &= \int d^3a\, G(a, a, -it) \end{aligned} \tag{74}$$

Furthermore, if for $G(x, a, t)$ one uses the Schrödinger wave function with appropriate initial conditions as discussed in Section 5, then

$$Z(t) = (2\pi\hbar)^{-3/2} \int d^3a\, \Psi(a, a, -it) \tag{75}$$

In the WKB approximation the partition function (75) evidently becomes

$$Z_{\mathrm{WKB}}(t) = (2\pi\hbar)^{-3/2} \int d^3a \left\{ \det\left(-\frac{\partial^2 S_c}{\partial a\, \partial x} \right)_{x=a} \right\}^{1/2} \exp\left(\frac{i}{\hbar}\right) S_c(a, a, -it) \tag{76}$$

where $S_c(x, a, t)$ is the classical action.

For quadratic Lagrangians the WKB approximation becomes exact, the determinant becomes independent of a, and the exponential term becomes a Gaussian. In fact, from (67) one sees that $S_c(a, a, -it)$ reduces to

$$S_c(a, a, -it) = \frac{i}{2}(a, N(t)\, a), \qquad \text{where} \quad N(t) = \left(\frac{\Omega}{\sinh \Omega t}\right)(\cosh \Omega t - \cos \sigma t) \tag{77}$$

Hence by letting $\mathbf{b} = N^{1/2}\mathbf{a}$ and using the identity

$$(2\pi\hbar)^{-3/2} \int d^3b \, e^{\frac{i}{2\hbar}(b,b)} = 1 \tag{78}$$

one obtains [for $N(t) \neq 0$]

$$\begin{aligned} Z_a(t) &= \left\{ \det \frac{\Omega}{\sinh \Omega t} \right\}^{3/2} \{\det N(t)\}^{-3/2} \\ &= \{\det(\cosh \Omega t - \cos \sigma t)\}^{-3/2} \end{aligned} \tag{79}$$

In particular, in the case of a pure harmonic oscillator, one obtains

$$Z_\omega(t) = \left\{ \det \left(\sqrt{2} \sinh \frac{\omega t}{2} \right) \right\}^{-3} \tag{80}$$

In the special case $N(t) = 0$, which actually corresponds to the case of a magnetic field ($\omega = 0$, $\Omega^2 = -\sigma^2$), one obtains instead the product

$$Z_\sigma(t) = Z_f^{(1)}(t)\, Z_\sigma^{(2)}(t), \qquad \text{where} \quad Z_f^{(1)}(t) = \left(\frac{L}{2\pi\hbar t} \right)^{-1/2} \tag{81}$$

and

$$Z_\sigma^{(2)}(t) = \left\{ \det \frac{\sigma}{\sin \sigma t} \right\}^{1/2} \left(\frac{V}{2\pi\hbar} \right) = \frac{B/2}{\sinh Bt/2} \left(\frac{V}{2\pi\hbar} \right)$$

$Z_f^{(1)}$ being the (free) partition function for the one-dimensional space L parallel to the magnetic field $\mathbf{B}$, and $Z_\sigma^{(2)}(t)$ the partition function for the two-dimensional space V orthogonal to $\mathbf{B}$. Thus the pure magnetic field case is really two-dimensional (or, more generally, even dimensional). The magnitude B of $\mathbf{B}$ is only half the magnitude of σ because the Lorentz force equation corresponding to (58) is $\ddot{y} = -2\sigma y$. An independent, direct, derivation of the pure magnetic field case is given in the next section.

Note that the Planck constant $\hbar$ does not appear in any of the above expressions [except (81), where it appears only as a factor for the volume] and that the WKB contribution $\det(\Omega/\sinh \Omega t)$ (which is nominally of order one) is cancelled by part of the classical contribution $[\det N(t)]^{-1/2}$ (which is nominally of order $\hbar^{-1}$), a result which shows that both contributions must be of the same order and therefore be classical. A closer examination, of course, shows that what has actually happened is that the nominal factor $\hbar^{-1}$ for the classical action is absorbed by the volume

integration [see (78)], thus reducing the classical contribution to order one. Since $Z(t)$ can also be written as

$$Z(t) = \mathrm{tr}(e^{-\frac{Ht}{\hbar}}) = \sum_n e^{-\frac{E_n t}{\hbar}} \tag{82}$$

one sees that $\hbar$ is not completely lost, but reappears when $Z(t)$ is expanded in terms of the energy levels.

11. DIRECT DERIVATION FOR THE CASE OF A PURE MAGNETIC FIELD

In view of the importance of the pure magnetic case for anomalies, it is perhaps worthwhile to give a simple, direct, derivation of the formula (81) for that case.

For a pure magnetic field B in two dimensions, the classical Lagrangian is

$$L = \tfrac{1}{2}(\dot{y}, \dot{y}) - (y, \sigma\dot{y}), \qquad \text{where} \quad \sigma = \frac{B}{2}, \qquad \eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{83}$$

This leads to the classical Hamiltonian

$$H_c = \tfrac{1}{2}(p - \sigma y)^2, \qquad \text{where} \quad p = \dot{y} + \sigma y \tag{84}$$

with corresponding quantum mechanical Hamiltonian

$$H_q = \frac{1}{2}\left(\frac{\hbar}{i}\partial_y - \sigma y\right)^2 \tag{85}$$

Now from (83) the classical field equations are evidently

$$\ddot{y}_c = -2\sigma\dot{y}_c = -B\eta\dot{y} \tag{86}$$

which integrates to

$$\dot{y}_c(t) = e^{-\eta B t}\dot{y}_c(0) \tag{87}$$

and thence to

$$y_c(t) = \frac{\eta}{B}(e^{-\eta B t} - 1)\,\dot{y}_c(0) + y_c(0) \tag{88}$$

Now it is clear from either (83) or (87) that the classical path from $y_c(0) = a$ to a itself is the trivial path $y_c(t) = a$ $(\dot{y}_c(0) = 0)$ and thus the

classical action $S_c(a, a, t)$ is zero. On the other hand, for classical paths with neighboring momenta, one has from (84) and (88),

$$\frac{\partial y_c(t)}{\partial p_c(0)} = \frac{\partial y_c(t)}{\partial \dot{y}_c(0)} = \frac{\eta}{B}(e^{-\eta Bt} - 1) \tag{89}$$

and thus

$$\left\{\det\left(\frac{\partial y_c(t)}{\partial p_c(0)}\right)\right\} = \frac{1}{B^2}\det(e^{-\eta Bt} - 1) = \left(\frac{\sin Bt/2}{B/2}\right)^2 \tag{90}$$

Hence finally, for the WKB partition function, one has

$$\begin{aligned}(2\pi\hbar)^{-1}\int d^2a \left\{\det\left(\frac{\partial y(t)}{\partial \pi(0)}\right)_{-it}\right\}^{-1/2} \exp\left[\frac{i}{\hbar}S_c(a, a, -it)\right] \\ = (2\pi\hbar)^{-1}\frac{B/2}{\sinh Bt/2}\int d^2a \\ = \frac{B/2}{\sinh Bt/2}\left(\frac{V}{2\pi\hbar}\right)\end{aligned} \tag{91}$$

which agrees with the result (81) obtained earlier.

12. CLASSICAL NATURE OF ANOMALIES

As mentioned in the Introduction, the result that the WKB approximation to the quantum-mechanical partition function is essential classical ($\hbar$-independent) is related to the paradox that "anomalies" in quantum field theory seem to be at the same time quantum-mechanical and classical. In this section we wish to make the relationship precise by showing how QFT anomalies may be expressed as QM partition functions, and to resolve the paradox for the simplest anomaly.

The starting point for our considerations is the Schwinger action[13] functional for the Dirac operator without sources (in any number of dimensions), namely,

$$\begin{aligned}e^W &= N^{-1}\int d(\psi\bar{\psi})\, e^{\bar{\psi}(\not{D} + im)\psi} \\ \not{D} &= \frac{\hbar}{i}\not{\partial} + \not{A}\end{aligned} \tag{92}$$

where m is a small mass-term, N is a normalization constant, and $\mathbf{A}$ is a Yang–Mills potential. By the usual QFT arguments,

$$W = \ln\det(\not{D} + im) - \ln N \tag{93}$$

and the anomaly A, which is defined to be the variation of W under the transformation $m \to m + i\gamma_5 \delta m$ in the limit $m \to 0$, is

$$A = \lim_{m=0} (im) \operatorname{tr}[(\not{D} + im)^{-1} \gamma_5] = \lim_{m=0} \operatorname{tr}\left[\frac{m^2}{m^2 + \not{D}^2} \gamma_5\right] \tag{94}$$

It is well known[14] that if the expression (94) is expanded in powers of $\not{A}$, then only a single one-loop term survives, and this term represents the QFT version of the anomaly. When the spectrum of $\not{D}^2$ is discrete, A is evidently also the index of $\not{D}$.

On the other hand, one may also write the expression (94) in the form

$$A = \lim_{m=0} m^2 \operatorname{tr} \gamma_5 \int_0^\infty e^{-(\not{D}^2 + m^2)t}\, dt = \lim_{m=0} m^2 \int_0^\infty e^{-m^2 t}\, Z_5(2\hbar t) = Z_5(0) \tag{95}$$

where $Z_5(t)$ is the γ_5-weighted partition function

$$Z_5(t) = \operatorname{tr}(e^{-\not{D}^2 t/2\hbar} \gamma_5) = Z_+(t) - Z_-(t) \equiv \operatorname{tr}[(e^{-\not{D}_+^2 t/2\hbar}) - (e^{-\not{D}_-^2 t/2\hbar})] \tag{96}$$

and $\not{D}_\pm^2$ are the chiral projections of $\not{D}^2$, i.e., $(1 \pm \gamma_5)\, \not{D}^2/2$. Thus $Z_5(t)$, and hence the anomaly A, is seen to be the difference of two ordinary QM partition functions, namely those for the Hamiltonians $\not{D}_+^2/2$ and $\not{D}_-^2/2$. This is the required relationship between the anomalies and the partition functions.

In order to illustrate how this relationship exhibits the essentially classical nature of anomalies, let us consider the simplest possible anomaly, namely that arising from the electromagnetic potential $\mathbf{A} = \frac{1}{2}B(y, -x)$ for a constant magnetic field B in two dimensions. In this simple case

$$-\not{D}_\pm^2 = -D^2 \pm \hbar B \tag{97}$$

and the partition functions $Z_\pm(t)$ are then easily seen to be

$$Z_\pm(2t) = \operatorname{tr}(e^{-D^2 t/\hbar \pm Bt}) = e^{\pm Bt} \operatorname{tr}(e^{-D^2 t/\hbar}) = e^{\pm Bt} Z_\sigma(2t) \tag{98}$$

where $Z_\sigma(t)$ is the partition function for the pure magnetic field discussed in Section 10. It follows from (96) that

$$\begin{aligned} Z_5(2t) &= (e^{Bt} - e^{-Bt})\, Z_\sigma(2t) \\ &= (2 \sinh Bt) \left(\frac{B/2}{\sinh Bt}\right)\left(\frac{V}{2\pi\hbar}\right) = \frac{BV}{2\pi\hbar} = \Phi \end{aligned} \tag{99}$$

where Φ is the total flux [normalized so that it is an integer (the index of $\not{D}$) whenever $\not{D}^2$ is discrete and Φ quantized]. Thus, in particular, the anomaly $A = Z_5(0)$ is equal to the total flux.

Of course, it is well known[15] that for the two-dimensional magnetic field the anomaly is equal to the flux, but the point to be noticed here is how the result is obtained via the quantum-mechanical partition function and thus by essentially classical arguments. It should also be emphasized that, although this example is very simple, it is nevertheless typical, in that anomalies are always determined by one-loop graphs, or equivalently by WKB determinants, and are thus always related to WKB partition functions, which are essentially classical.

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