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Generalized Toda theories and W-algebras associated with integral gradings

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A general class of conformal Toda theories associated with integral gradings of the simple Lie algebras is investigated. These generalized Toda theories are obtained by reducing the Wess-Zumino-Novikov-Witten (WZNW) theory by first class constraints, and thus they inherite extended conformal symmetry algebras, generalized $\mathcal{W}$-algebras, and current dependent Kac-Moody (KM) symmetries from the WZNW theory, which are analysed in detail in a non-degenerate case. We uncover an $sl(2)$ structure underlying the generalized $\mathcal{W}$-algebras, which allows for identifying the primary fields, and give a simple algorithm for implementing the $\mathcal{W}$-symmetries by current dependent KM transformations, which can be used to compute the action of the $\mathcal{W}$-algebra on any quantity. We establish how the Lax pair of Toda theory arises in the WZNW framework, and show that a recent result of Mansfield and Spence, which interprets the $\mathcal{W}$-symmetry of the Toda theory by means of non-Abelian form preserving gauge transformations of the Lax pair, arises immediately as a consequence of the KM interpretation.

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I. Introduction

Conformally invariant and affine Toda type systems are important both in the theory of integrable non-linear equations [1-7] and in two-dimensional conformal field theory [8-17]. One of the key features of the conformal Toda theories is that they possess [3, 4, 10, 11, 13-17] interesting non-linear symmetry algebras, which are polynomial extensions of the Virasoro algebra by chiral conformal primary fields. The theory of such extended conformal algebras, called $\mathcal{W}$-algebras following A. B. Zamolodchikov who initiated their study in [18], is of great current interest, see [19] for a review, and, for example, refs. [20-23].

The traditional approach to Toda systems is the formalism of the Lax pair [1-8]. On the other hand, it has become clear recently [12-16, 24] that Toda theories are really nothing but reduced Wess-Zumino-Novikov-Witten (WZNW) theories.

In this paper we shall investigate a certain class of generalized conformal Toda theories, given by equation (2.2), which is associated with the integral gradings of the simple Lie algebras, see refs. [3, 4, 14-16] for earlier work on generalized Toda theories. We shall treat these theories by using the WZNW framework, which in our opinion is the natural setting for conformal Toda theories. In fact, the results of [13] demonstrate that the WZNW setting amounts to a linearization (in the sense that WZNW is a free theory) of the Toda theory which at the same time resolves the apparent singularities. Moreover, it is also clear from [13] that the WZNW setting is especially well suited for describing the $\mathcal{W}$-algebra symmetries of the conformal Toda theories.

It appears to us that the relationship between the WZNW and the Lax pair formalisms of Toda theory has not yet been properly elucidated, and the first purpose of the present paper is to clarify exactly this point. The second purpose of the paper is to make use of this relationship and the advantages of the WZNW approach for obtaining a description of the symmetries of the conformal Toda systems in both settings. Our results and methods using the WZNW formulation complement and generalize the ones given in [13], where the special case of the standard Toda theory, given by equation (2.5), was considered.

The outline of the paper is as follows. In Section II. we show how the Lax pair formalism arises in a natural way in the WZNW framework. We then analyse the $\mathcal{W}$-algebra of the Toda models in Section III. From this section on, we consider a certain non-degenerate case, defined by equation (3.3). The non-degeneracy condition ensures the existence of the Drinfeld-Sokolov gauges [5, 13], which are very convenient for analysing the $\mathcal{W}$-algebra. An important result we obtain is that, in this non-degenerate case, the $\mathcal{W}$-algebra of the generalized Toda theory is isomorphic to one from the list of $\mathcal{W}$-algebras which can be associated with the non-equivalent embeddings of $sl(2)$ into the simple Lie algebras [15, 25].

Section IV. is devoted to describing all the chiral, current dependent, local Kac-Moody (KM) type symmetry transformations of the WZNW theory surviving the reduction to Toda theory, and establishing an algorithm for implementing the $\mathcal{W}$-transformations by current dependent KM transformations. We also make clear that the $\mathcal{W}$-transformations are only a subset of the former transformations, namely the canonical ones.

In Section V. we relate our KM interpretation of the $\mathcal{W}$-symmetry to the one given recently in [17], where the authors interprete the $\mathcal{W}$-symmetry of the standard Toda theory in terms of the 'non-Abelian form preserving gauge transformations' of the Lax pair. We show that this interpretation arises immediately as a translation of the KM interpretation. This way we generalize the result of [17] from the standard case to our general situation, and also obtain a certain clarification. Namely, our results provide a general method for identifying the $\mathcal{W}$-transformations among the general 'form preserving gauge transformations' of the Lax pair.

We end the paper by giving our conclusions and commenting on some open problems.

II. WZNW versus Lax pair

First we define the Toda system we are going to investigate. To this we consider a real, maximally non-compact simple Lie algebra $\mathcal{G}$ together with an integral grading defined by some element H of a splitting (diagonalizable) Cartan subalgebra $\mathcal{H}$. This means that we have

$$\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_0 + \mathcal{G}_- \ , \qquad \mathcal{G}_\pm = \sum_{n=1}^{N} \mathcal{G}_{\pm n} \ , \tag{2.1}$$

where $\mathcal{G}_0$ and $\mathcal{G}_{\pm n}$ are eigenspaces of ad_H with eigenvalues 0 and $\pm n$, respectively. The generalized Toda equation we consider is an integrable non-linear equation for a field g_0 taking its values in G_0, the little group of H in the connected real Lie group G with Lie algebra $\mathcal{G}$. It is given as

$$\partial_-(\partial_+ g_0 \cdot g_0^{-1}) = [M_- \, , \, g_0 M_+ g_0^{-1}] \, , \tag{2.2}$$

where $M_\pm$ are some arbitrary but non-zero generators chosen from $\mathcal{G}_{\pm 1}$. It is easy to check that (2.2) is equivalent to the zero curvature condition

$$[\partial_+ - \mathcal{A}_+ \, , \, \partial_- - \mathcal{A}_-] = 0 \tag{2.3}$$

of the following 'Lax potential':

$$\mathcal{A}_+ = \partial_+ g_0 \cdot g_0^{-1} + M_- \, , \qquad\qquad \mathcal{A}_- = -g_0 M_+ g_0^{-1} \, . \tag{2.4}$$

A particular case of the above Toda system is obtained by considering an integral embedding of the Lie algebra $sl(2)$ into $\mathcal{G}$ and taking the standard generators of this $sl(2)$ subalgebra for H and $M_\pm$. This case has been investigated by using the Lax pair formalism in [3, 4], and by using the WZNW formulation in [15, 16]. It is well known that, by substituting $g_0 = \exp[\sum_i \varphi_i H_i]$, (2.2) reduces to the standard Toda equation

$$\partial_+\partial_-\varphi_i + \exp[\sum_{j=1}^{l} K_{ij}\varphi_j] = 0 \tag{2.5}$$

in the case of the so called principal $sl(2)$ subalgebra, which is characterized [26] by the condition that

$$[H \, , \, E_\alpha] = E_\alpha \tag{2.6}$$

for any root α from some system of simple positive roots of $\mathcal{G}$. Note that in equation (2.5) K_{ij} is the Cartan matrix of $\mathcal{G}$.

Another, more general, subclass of Toda systems was investigated by using the WZNW picture in [14]. This class is obtained by assuming that (2.6) holds for a subset of the simple roots and that H commutes with the step operators associated to the rest of the simple roots.

Now we recall how the reduction of the WZNW theory, considered for the group G, to the Toda theory (2.2) comes about [12, 14], by using a method which brings out clearly the connection between the WZNW and Lax pair formalisms. The left and right KM currents, J and $\bar{J}$, are given* as

$$J = \partial_+ g \cdot g^{-1} \, , \qquad\qquad \bar{J} = -g^{-1}\partial_- g \, , \tag{2.7}$$

* The KM level k can be recovered by sustituting $\partial_\pm \longrightarrow \kappa\partial_\pm \, , \, \kappa = -\frac{k}{4\pi}$, everywhere below. We have chosen $\kappa = 1$ to simplify the notation.

and the field equation can be written equivalently as

$$\partial_- J = 0 \qquad \text{or} \qquad \partial_+ \bar{J} = 0 \ . \tag{2.8}$$

We identify the phase space of the WZNW theory with the space of solutions, given by the formula

$$g(x^+, x^-) = g_L(x^+) \cdot g_R(x^-) \ . \tag{2.9}$$

To describe the constraints of the WZNW $\longrightarrow$ Toda reduction we shall use the projection operators $\pi_{\pm,0}$ projecting $\mathcal{G}$ onto $\mathcal{G}_{\pm,0}$, respectively, and the connected subgroups $G_\pm$ of G obtained by exponentiating $\mathcal{G}_\pm$. The constraints are then given by

$$\pi_-(J) = M_- \qquad \text{and} \qquad \pi_+(\bar{J}) = -M_+ \ . \tag{2.10}$$

The gauge group generated by this first class system of constraints is the direct product of the left loop-group of G_+ and the right loop-group of G_-. Of course, this gauge group acts on the WZNW phase space according to

$$g(x^+, x^-) \longrightarrow A(x^+) \cdot g(x^+, x^-) \cdot C^{-1}(x^-) \ , \tag{2.11}$$

for any $A(x^+) \in G_+$, $C(x^-) \in G_-$, and the constraint surface (2.10) is left invariant under this action. To see what is the gauge invariant content of the constrained WZNW theory we consider the 'sector' where the generalized Gauss decomposition

$$g(x^+, x^-) = g_+(x^+, x^-) \cdot g_0(x^+, x^-) \cdot g_-(x^+, x^-) \ , \qquad g_{\pm,0} \in G_{\pm,0} \tag{2.12}$$

is valid. By substituting this Gauss decomposition into (2.7) it is easy to see that (2.10) is equivalent to

$$g_+^{-1} \cdot \partial_- g_+ = g_0 M_+ g_0^{-1} \ , \qquad \text{and} \qquad \partial_+ g_- \cdot g_-^{-1} = g_0^{-1} M_- g_0 \ , \tag{2.13}$$

that is the constrained currents can be written as

$$J = g_+[\partial_+ g_0 \cdot g_0^{-1} + M_-]g_+^{-1} + \partial_+ g_+ \cdot g_+^{-1} \tag{2.14a}$$

and

$$\bar{J} = -g_-^{-1}[g_0^{-1}\partial_- g_0 + M_+]g_- - g_-^{-1}\partial_- g_- \ . \tag{2.14b}$$

We see from (2.13) that the G_0 valued gauge invariant field g_0 represents the full gauge invariant content of the constrained WZNW field g, since $g_\pm$ can be determined from (2.13) in terms of g_0, up to gauge transformations

$$g_+(x^+, x^-) \longrightarrow A(x^+) \cdot g_+(x^+, x^-) \ , \qquad g_-(x^+, x^-) \longrightarrow g_-(x^+, x^-) \cdot C^{-1}(x^-) \ . \tag{2.15}$$

To obtain the gauge invariant dynamics, we have to project the WZNW field equation to the reduced theory. To this first we observe the obvious fact that the WZNW field equation is a zero curvature condition, namely

$$[\partial_+ - J\,,\,\partial_- - 0] = 0\ . \tag{2.16}$$

Conjugating this equation by the field $g_+^{-1}(x^+, x^-)$ in (2.12) and using the constraints expressed by (2.13) and (2.14), we see that (2.16) is equivalent to the zero curvature condition of the following gauge invariant Lax pair:

$$\begin{aligned}\mathcal{A}_+ &= g_+^{-1} J g_+ + \partial_+ g_+^{-1} \cdot g_+ = \partial_+ g_0 \cdot g_0^{-1} + M_-\ ,\\ \mathcal{A}_- &= \partial_- g_+^{-1} \cdot g_+ = -g_0 M_+ g_0^{-1}\ ,\end{aligned} \tag{2.17}$$

which is nothing but the usual Lax pair of Toda theory (2.4). Plainly, the Lax potential $\mathcal{A}_\pm$ is a pure gauge for solutions of the Toda field equation, $\mathcal{A}_\pm = \partial_\pm \hat{g} \cdot \hat{g}^{-1}$ for some G valued field $\hat{g}$. In terms of the WZNW variables we have

$$\hat{g}(x^+, x^-) = g_+^{-1}(x^+, x^-) \cdot g_L(x^+)\ . \tag{2.18}$$

Naturally, $\hat{g}$ is gauge invariant. For completeness we note that the above analysis could have been carried out by starting with the second equation in (2.8), the result would be an alternative form of the Toda Lax pair.

In summary, we have shown that the usual Lax pair of Toda theory is obtained by conjugation by a non-chiral, G_+-valued field from the trivial, chiral Lax 'pair' of the constrained WZNW theory. The Lax pair formalism provides a very convenient tool for investigating the Toda theory [1, 3]. Nevertheless, the WZNW formulation of Toda theory is the more fundamental one. The basic reason is that the Gauss decomposition (2.12) is valid only locally on G. This leads to the appearance of apparently singular but physically regular solutions in the traditional setting of Toda theory [12, 13]. Another reason is that in the WZNW framework the full power of the KM algebra becomes immediately available for describing the Toda theory.

III. The $\mathcal{W}$-algebra

The fundamental ingredient of the WZNW model is its KM symmetry, therefore it is natural to ask what part of this symmetry survives the reduction leading to Toda theory. This way we are led to searching for those chiral quantities formed out of the KM currents which are constant along the gauge orbits on the constraint surface (2.10). On general grounds, locally there should exist $\dim\mathcal{G} - 2\dim\mathcal{G}_+ = \dim\mathcal{G}_0$ independent gauge invariant chiral quantities. The special feature of our system is that here one can find a complete set of gauge invariant objects which are globally well defined, finite polynomials in the currents and their derivatives. The gauge invariant differential polynomials form a closed algebra under the KM Poisson bracket. This polynomial Poisson bracket algebra can be called a classical $\mathcal{W}$-algebra since it contains the Virasoro subalgebra generated by the gauge invariant polynomial

$$L_H = \frac{1}{2}\mathrm{Tr}(J^2) - \mathrm{Tr}\,(H \cdot \partial_+ J)\ , \tag{3.1}$$

which can be used to define a conformal automorphism of the Toda system (2.2).

Concentrating on the left moving sector, now we analyse the reduced chiral algebra. The gauge transformations act on the constrained current, J, according to

$$J \longrightarrow \mathrm{Adj}_A\,(J) \equiv AJA^{-1} + \partial_+ A \cdot A^{-1}\ , \quad A(x^+) \in G_+\ , \tag{3.2}$$

and we are looking for differential polynomials $W(J)$ invariant under this action. The existence of a complete set of such gauge invariant quantities can be deduced from the existence of the so called Drinfeld-Sokolov (DS) gauges [5, 13]. The existence of the DS gauges is ensured by the non-degeneracy condition

$$\mathrm{Ker}(\mathrm{ad}_{M_-}) \cap \mathcal{G}_+ = \{0\}\ , \tag{3.3}$$

which we assume to be satisfied from now on.

One can always choose a system of simple roots in such a way that the corresponding step operators have *non-negative* integral grades. If in this basis there occurs a simple root β such that $[H\,,\,E_\beta] = n_\beta E_\beta$ with $n_\beta \geq 2$ then a non-degenerate M_- cannot exist. In fact, it is easily seen that in this case $[E_\beta\,,\,\mathcal{G}_{-1}] = 0$. On the other hand, we conjecture that if the grades of the step operators corresponding to some system of simple roots are all from the set $\{0\,,\,1\}$ then there exists at least one non-degenerate M_-. This was true in all the examples we considered so far [14], and, moreover, in these examples the non-degenerate M_- was found to be unique up to conjugation by the little group of H.

The construction of the DS gauges proceeds as follows. First we choose a direct sum decomposition

$$\mathcal{G}_i = \mathcal{I}_i + \mathcal{V}_i \qquad \text{for} \qquad i = 0, 1, \ldots, N \ , \tag{3.4}$$

where

$$\mathcal{I}_i = [M_- \, , \, \mathcal{G}_{i+1}] \tag{3.5}$$

and $\mathcal{V}_i$ is some complementary subspace. Note that $d_i = \dim(\mathcal{V}_i)$ can be zero for some i, and $\sum_{i=0}^{N} d_i = \dim(\mathcal{G}_0)$. By considering the direct sums

$$\mathcal{I} = \sum_{i=0}^{N} \mathcal{I}_i = [M_- \, , \, \mathcal{G}_+] \ , \qquad \mathcal{V} = \sum_{i=0}^{N} \mathcal{V}_i \ , \tag{3.6}$$

we have

$$\mathcal{G} = \mathcal{G}_- + \mathcal{I} + \mathcal{V} \ . \tag{3.7}$$

Thus an arbitrary KM current J can be written as

$$J = \pi_-(J) + \pi_{\mathcal{I}}(J) + \pi_{\mathcal{V}}(J) \ , \tag{3.8}$$

where π_-, $\pi_{\mathcal{I}}$ and $\pi_{\mathcal{V}}$ are the projection operators corresponding to (3.7). We define the Drinfeld-Sokolov gauge corresponding to the subspace $\mathcal{V}$ by supplementing the first class constraints $\pi_-(J) = M_-$ with the gauge condition

$$\pi_{\mathcal{I}}(J) = 0 \ . \tag{3.9}$$

One has to investigate whether the general constrained current

$$J = M_- + \sum_{i=0}^{N} J_i \ , \qquad J_i \in \mathcal{G}_i \tag{3.10}$$

can be brought to the DS form

$$J^{\mathrm{DS}} = M_- + \sum_{i=0}^{N} J_i^{\mathrm{DS}} \ , \qquad J_i^{\mathrm{DS}} \in \mathcal{V}_i \tag{3.11}$$

by a gauge transformation. To answer this question one inserts the above expressions and

$$A = e^{a_1} \cdot e^{a_2} \cdots e^{a_N} \ , \qquad a_k(x^+) \in \mathcal{G}_k \ , \qquad k = 1, 2, \ldots, N \tag{3.12}$$

into the equation

$$J^{\mathrm{DS}} = \mathrm{Adj}_A\,(J)\ , \tag{3.13}$$

and considers this equation grade by grade, starting from grade 0, in terms of the decomposition (3.4). At every grade one tries to gauge away the $\mathcal{I}_i$ component of J by choosing a_{i+1} appropriately. One sees that this is indeed possible as a consequence of the nondegeneracy condition (3.3), which implies that ad_{M_-} maps $\mathcal{G}_{i+1}$ onto $\mathcal{I}_i$ in a one-to-one manner, for any $i \geq 0$. Moreover, one also sees from (3.13) that the components of the gauge representative J^{DS} and those of the a_k are *uniquely determined differential polynomials* in terms of the components of J. This property of the DS gauges guarantees the polynomial character of the reduced chiral algebra.

Indeed, let us introduce some basis F_{i,n_i} in $\mathcal{V}_i$ and write the unique intersection point of the DS gauge section with the gauge orbit passing through J as

$$\mathrm{Adj}_{A(J)}\,(J) = M_- + \sum_{i,n_i} W^{i,n_i}(J)\,F_{i,n_i}\ . \tag{3.14}$$

It immediately follows from the above that the $W^{i,n_i}(J)$ are gauge invariant differential polynomials, which can be used as coordinates in the chiral sector of the reduced WZNW theory. In particular, the $W^{i,n_i}(J)$ corresponding to a DS gauge in the above manner always form a basis of the $\mathcal{W}$-algebra. Note that, of course, any unique gauge fixing can be used to define gauge invariant quantities, but they are in general not polynomial, not even local in J.

Plainly, a differential polynomial $W(J)$ reduces to a differential polynomial of the components of the gauge fixed current in an arbitrary gauge defined by putting a linear gauge fixing condition on the components of J. The Poisson brackets of the W's can then be computed in that gauge by using the Dirac brackets of the current components surviving the gauge fixing.

From this point of view, the main advantage of the DS gauge is that in this gauge the W^{i,n_i} become *linear* functions of the current. In fact, writing

$$\pi_{\mathcal{V}}(J) = \sum_{i,n_i} U^{i,n_i}\ F_{i,n_i}\ , \tag{3.15}$$

we have

$$U^{i,n_i}{}_{|\mathrm{DS}} = W^{i,n_i}{}_{|\mathrm{DS}}\ . \tag{3.16}$$

We call the current components U^{i,n_i} DS currents. Their Dirac bracket algebra represents the basic Poisson brackets in the DS gauge. As a consequence of

(3.16) and the gauge invariance of the W^{i,n_i}, the $\mathcal{W}$-algebra can be interpreted as the Dirac bracket algebra of the DS currents:

$$\{U^{i,n_i}\,,\,U^{k,n_k}\}^*{}_{|\mathrm{DS}} = \{W^{i,n_i}\,,\,W^{k,n_k}\}_{|\mathrm{DS}}\ . \tag{3.17}$$

We shall use this interpretation to show that our $\mathcal{W}$-algebra is isomorphic to one from the set of $\mathcal{W}$-algebras associated to the inequivalent embeddings of $sl(2)$ into the simple Lie algebras [15, 25].

Let $(I_-\,, I_0\,, I_+)$ be the standard generators of an $sl(2)$ subalgebra, $\mathcal{S}$, of $\mathcal{G}$. One can put a set of *second class* constraints on the KM phase space by requiring the constrained current to be of the form

$$J^{\mathcal{S}}(x) \equiv I_- + j^{\mathcal{S}}(x)\ , \qquad j^{\mathcal{S}}(x) \in \mathrm{Ker}\,(\mathrm{ad}_{I_+})\ , \tag{3.18}$$

that is $j^{\mathcal{S}}(x)$ is a linear combination of the highest weight states of $\mathcal{S}$ in the adjoint representation of $\mathcal{G}$. One can prove (see [15, 25]) that the components of $j^{\mathcal{S}}(x)$ form a $\mathcal{W}$-algebra, denoted as $\mathcal{W}^{\mathcal{G}}_{\mathcal{S}}$, under Dirac bracket. The canonical Virasoro subalgebra is generated by the I_+-component of $j^{\mathcal{S}}$. The components corresponding to the other highest weight states in $\mathrm{Ker}\,(\mathrm{ad}_{I_+})$ are conformal primary fields with respect to this Virasoro algebra, their conformal weight is $(1+l)$, where l is the $sl(2)$ spin. This construction is motivated by our earlier result [13] that in the usual Toda theory one finds the primary fields by going to the highest weight DS gauge of the principal $sl(2)$ subalgebra of $\mathcal{G}$.

By the method described previously, we constructed a $\mathcal{W}$-algebra by starting with the data $(H\,,\,M_-)$, where H is an integral grading operator of $\mathcal{G}$ and M_- is a non-degenerate generator of grade -1. On the other hand, we have the following mathematical result.

Lemma: Let $\mathcal{G} = \mathcal{G}_0 + \sum_{n=1}^{N} \mathcal{G}_{\pm n}$ be the decomposition of $\mathcal{G}$ defined by the integral grading operator H, and let M_- be an element of $\mathcal{G}_{-1}$, which is non-degenerate with respect to this grading in the sense of (3.3). Then there exists an $sl(2)$ subalgebra $\mathcal{S}$ of $\mathcal{G}$ with standard generators $(I_-\,,\,I_0\,,\,I_+)$ satisfying

$$I_- = M_-\ , \qquad I_0 \in \mathcal{G}_0\ , \qquad I_+ \in \mathcal{G}_{+1}\ . \tag{3.19}$$

The generator I_+ here is always non-degenerate, that is $\mathrm{Ker}\,(\mathrm{ad}_{I_+}) \cap \mathcal{G}_- = \{0\}$. The conjugacy class of the $sl(2)$ subalgebra depends only on the conjugacy class of M_- in $\mathcal{G}$.

This result is an easy consequence of some powerful theorems by Morozov, Jacobson and Kostant on $sl(2)$ embeddings [27], as explained in detail in [25].

It follows from the lemma that we can construct a highest weight DS gauge by choosing the complement of $\mathcal{I}_i$ in (3.4) according to

$$\mathcal{V}_i \equiv \mathcal{G}_i \cap \mathrm{Ker}\,(\mathrm{ad}_{I_+})\ . \tag{3.20}$$

Let us remember that every DS gauge defines a basis of the $\mathcal{W}$-algebra, and that by using this basis the $\mathcal{W}$-algebra can be identified with the Dirac bracket algebra of the DS currents. This fact and the existence of the highest weight gauge imply that our $\mathcal{W}$-algebra is isomorphic to $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ with $\mathcal{S}$ provided by the lemma. We also see that the equivalence class of the $\mathcal{W}$-algebra depends only on the conjugacy class of the nilpotent element M_- in $\mathcal{G}$.

The Virasoro generator associated to the I_+-component of the highest weight gauge can be identified with the gauge invariant differential polynomial

$$L_{I_0} = \frac{1}{2}\mathrm{Tr}(J^2) - \mathrm{Tr}\,(I_0 \cdot \partial_+ J)\ . \tag{3.21}$$

The generators of the $\mathcal{W}$-algebra corresponding to the other highest weight components are primary fields with respect to the conformal action generated by this Virasoro density. It should be noted that the spectrum of I_0, and thus the spectrum of conformal weights, is in general half-integral. We also remark that in general there is no basis of the $\mathcal{W}$-algebra consisting of the Virasoro density L_H in (3.1) and primary fields with respect to the conformal action generated by L_H [14].

Besides the highest weight gauge one has another particularly important gauge, namely the 'diagonal gauge' for which the gauge fixed current is given as

$$J^{\mathrm{diag}} = M_- + j_0\ , \qquad\qquad j_0 \in \mathcal{G}_0\ . \tag{3.22}$$

The advantage of this gauge is that the Dirac brackets of the components of j_0 coincide with their original Poisson brackets, given by the $\mathcal{G}_0$ KM algebra. We denote the differential polynomial representing an element, $W(J)$, of the $\mathcal{W}$-algebra in the diagonal gauge as W_0, $W_0(j_0) \equiv W(J^{\mathrm{diag}})$. In the case of the usual Toda theory $\mathcal{G}_0$ is the Cartan subalgebra and W_0 is the Miura transform of W.

In the WZNW framework, we identified the left moving chiral algebra of the Toda system as the algebra of gauge invariant differential polynomials in the current J. On the other hand, in the 'Gauss decomposable sector' of the constrained WZNW theory a complete set of gauge invariant quantities is provided by the Toda field g_0. In particular, J can be expressed in terms of g_0 up to gauge transformations. It follows that for any $W(J)$ there exists a unique function $W_{\rm Toda}(g_0)$ such that

$$W_{\rm Toda}(g_0) = W(J) \ . \tag{3.23}$$

To find the explicit form of the function $W_{\rm Toda}$ let us first point out that, by construction, $W(J)$ is a differential polynomial whose form is invariant under any change of variables of the form

$$J \longrightarrow {\rm Adj}_{\,\alpha}(J) \ , \qquad\qquad \alpha \in G_+ \ . \tag{3.24}$$

The point is that here α can depend on the dynamical variables in an arbitrary way and is not even restricted to be chiral. (Of course, (3.24) then does not necessarily represent a gauge transformation in the sense of (3.2).) Equation (2.17) tells us that J and the Lax potential $\mathcal{A}_+$ are related by a transformation of the form (3.24), with $\alpha = g_+^{-1}(x^+, x^-)$. Therefore we see that $W_{\rm Toda}$ depends on g_0 only through $\mathcal{A}_+$ and that $W_{\rm Toda}(\mathcal{A}_+)$ is obtained by simply substituting $\mathcal{A}_+$ for the argument of the differential polynomial W. In other words, taking also into account that the form of $\mathcal{A}_+$ is the same as that of $J^{\rm diag}$, we have

$$W_{\rm Toda}(g_0) = W(\mathcal{A}_+) = W_0(j_0 \rightarrow \partial_+ g_0 \cdot g_0^{-1}) \ . \tag{3.25}$$

Thus the chiral W's depend on the non-chiral 'Toda current' $\partial_+ g_0 \cdot g_0^{-1}$ in the same way as on the chiral variable j_0. We note that in their Lax pair approach to Toda theory Leznov and Savaliev [3, 4] constructed the conserved currents by directly solving the 'characteristic equation'

$$\partial_- W_{\rm Toda}(g_0) = 0 \tag{3.26}$$

for $W_{\rm Toda}$. The above arguments, essentially due to Palla [28] who observed (3.25) in the case of the usual Toda theory, provide the translation between the Lax pair [4, 11, 17] and the constrained KM descriptions of the $\mathcal{W}$-algebra of the Toda system (2.2), for non-degenerate $M_\pm$.

IV. Local symmetry transformations of KM type and $\mathcal{W}$-transformations

In this section we set up a scheme for describing all the local KM type symmetry transformations of the WZNW theory surviving the reduction to Toda theory. In particular, we shall give a natural generalization of the results of [13] about the KM implementation of the $\mathcal{W}$-symmetry.

In the WZNW theory, we call a transformation of the form

$$\delta_K g = K \cdot g \tag{4.1}$$

a (left, current dependent, local) KM transformation if the $\mathcal{G}$-valued function K is a differential polynomial in the components of J and their derivatives. (For simplicity, we shall often refer to these transformations simply as KM transformations. This is an abuse of terminology since the standard KM transformations are current independent.) The KM transformations are symmetries, i.e. they act on the space of solutions of the theory. Their action on the left moving field $g_L(x^+)$ and on the current $J(x^+)$ is given as

$$\delta_K g_L = K(J) \cdot g_L \tag{4.2}$$

and

$$\delta_K J = \mathrm{D}_J K = \partial_+ K + [K\ ,\ J]\ . \tag{4.3}$$

A KM type symmetry is an infinitesimal canonical transformation if and only if it can be written as

$$K = \frac{\delta Q}{\delta J}\ , \qquad\qquad Q = \int_0^{2\pi} dx^+\, q(J, J', ...)\ , \tag{4.4}$$

where q is some differential polynomial in the components of J, possibly depending on some test functions too. Indeed, as follows easily from the form of the KM Poisson bracket, we have $\delta_Q = \delta_K$ for K in (4.4), where, by definition, δ_Q acts on any quantity via Poisson bracket, $\delta_Q(\,\cdot\,) \equiv -\{Q\ ,\ \cdot\,\}$.

We are looking for those KM transformations which preserve the constraint surface and project to transformations on the space of gauge orbits in a well defined way. Taking into account the form of the constrained current, denoted from now on as J^c,

$$J^c = M_- + j\ , \qquad\qquad j \in (\mathcal{G}_0 + \mathcal{G}_+)\ , \tag{4.5}$$

the first condition is equivalent to

$$\pi_-(\delta_K J^c) = 0\ . \tag{4.6}$$

The second condition requires δ_K to be invariant under gauge transformations, up to infinitesimal gauge transformations, which projects to zero when going to the space of gauge orbits. To see the meaning of this condition, let us observe that under a gauge transformation, $A(x^+) \in G_+$, K transforms according to the rule

$$K \longrightarrow K^A , \qquad\qquad K^A(J^c) \equiv A \cdot K(\mathrm{Adj}_{A^{-1}}(J^c)) \cdot A^{-1} , \tag{4.7}$$

since $\delta_{K(J^c)}$ is a vector field on the constraint surface. Thus $\delta_{K(J^c)}$ projects to a well defined vector field on the space of gauge orbits if and only if

$$\pi_{\leq 0}(K^A - K) = 0 . \tag{4.8}$$

Here we introduced the notation $\pi_{\leq 0}$ for $(\pi_- + \pi_0)$, operating according to the decomposition (2.1).

Combining our previous equations, we conclude that the 'residual' KM type symmetries, i.e. the ones surviving the reduction, can be determined as the local solutions $K(J^c)$ of the following two conditions:

$$\pi_-(\partial_+ K + [K\, ,\, M_-] + [K\, ,\, j]) = 0 , \tag{4.9}$$

and

$$\pi_{\leq 0}(K(J^c)) = \pi_{\leq 0}(A \cdot K(\mathrm{Adj}_{A^{-1}}(J^c)) \cdot A^{-1}) , \qquad\qquad \forall A(x^+) \in G_+ . \tag{4.10}$$

It is easy to see that K solves these equations if and only if its 'lower triangular part' $\pi_{\leq 0}(K)$ does, and that K and $\pi_{\leq 0}(K)$ give rise to the same transformation on the space of gauge orbits. For this reason, without loss of generality, it is enough to consider those solutions which are lower triangular, $\pi_+(K) = 0$, for which (4.10) becomes

$$K(J^c) = \pi_{\leq 0}(A \cdot K(\mathrm{Adj}_{A^{-1}}(J^c)) \cdot A^{-1}) , \qquad\qquad \forall A(x^+) \in G_+ . \tag{4.11}$$

The canonical transformations generated by the elements of the $\mathcal{W}$-algebra are implemented by those residual current dependent KM transformations which are given by means of equation (4.4), where q is some extension of a gauge invariant differential polynomial from the constraint surface to the KM phase space. Note that if one takes the trivial extension for which q depends only on $\pi_{\geq 0}(J)$ then K is lower triangular. It should be noted that not every residual KM transformation is a $\mathcal{W}$-transformation, for the same reason that not every current dependent KM transformation is canonical in WZNW theory.

Namely, a current dependent KM transformation is canonical in WZNW theory provided K can be written as a gradient, eq. (4.4).

Every residual KM transformation K induces a transformation on the space of gauge orbits. Representing the orbits by a gauge section, the induced transformations become gauge preserving KM transformations. From this point of view, as we shall see below, the advantage of the DS gauge is that there is a one-to-one correspondence between *local* KM transformations preserving the DS gauge and *local* KM transformations preserving the constraint surface and satisfying (4.11). Moreover, we shall give an effective algorithm which allows for finding all the KM transformations preserving the DS gauge, and for identifying the subset of induced $\mathcal{W}$-transformations.

In some fixed DS gauge, a gauge preserving KM transformation $K_{\rm DS}$ is a local solution of the equation

$$\delta J^{\rm DS} = \partial_+ K_{\rm DS} + [K_{\rm DS}\,,\,M_-] + [K_{\rm DS}\,,\,j^{\rm DS}]\ . \tag{4.12}$$

Here the condition is that this variation preserves the form of $J^{\rm DS} = M_- + j^{\rm DS}$, that is one must have $\delta J^{\rm DS} \in \mathcal{V}$, where $\mathcal{V}$ is the complementary space in (3.7) defining the DS gauge in question. Any residual KM transformation $K(J^c)$ gives rise to a gauge preserving variation defined by

$$K_{\rm DS}(J^{\rm DS}) = K(J^{\rm DS}) + k(J^{\rm DS})\ , \qquad k \in \mathcal{G}_+\ , \tag{4.13}$$

where k is a uniquely determined local infinitesimal gauge transformation compensating for the fact that δ_K does not necessarily preserve the DS gauge. Conversely, any differential polynomial solution of (4.12) can be uniquely extended to a differential polynomial $K(J^c)$ defining a residual KM transformation on the full constraint surface. To this it is enough to take (4.13) as the definition of $K(J^{\rm DS})$ and then use (4.11).

We note that, of course, there is a natural one-to-one relationship between gauge preserving KM transformations and solutions of (4.9), (4.11) for any unique gauge fixing. The special feature of the DS gauge is that $K(J^c)$ is local in J^c if and only if the corresponding $K_{\rm DS}(J^{\rm DS})$ is local in $J^{\rm DS}$. This follows from (4.11) and (4.13) and the fact that J^c can be brought to the DS gauge by a gauge transformation which is local in J^c.

Next we give a technical result about equations (4.9) and (4.12), which plays a crucial role in our analysis. These linear equations determine the allowed set of KM transformations $K(J^c)$ and $K_{\rm DS}(J^{\rm DS})$, which define form preserving variations of J^c and $J^{\rm DS}$, respectively. Here we shall establish the structure of their general solution. We start by observing that the non-degeneracy condition (3.3) is equivalent to the fact that the map

$$\mathrm{ad}_{M_-} : \mathcal{G}_{-i} \longrightarrow \mathcal{G}_{-i-1} \ , \qquad\qquad i = 0, 1, \ldots, N \ , \tag{4.14}$$

is always *onto*. This is easily proven by using the invariance of the Cartan-Killing form and the fact that under this scalar product the dual space to $\mathcal{G}_i$ is $\mathcal{G}_{-i}$. Thus, by using the non-degeneracy assumption, we can choose a direct sum decomposition

$$\mathcal{G}_{-i} = \mathcal{P}_{-i} + \mathcal{Z}_{-i} \ , \qquad\qquad i = 0, 1, \ldots, N \ , \tag{4.15}$$

in such a way that ad_{M_-} maps $\mathcal{Z}_{-i}$ onto $\mathcal{G}_{-i-1}$ in a one-to-one manner. For example, a possible, in some sense canonical, choice is to take $\mathcal{P} \equiv \sum_{i=0}^{N} \mathcal{P}_{-i}$ to be $\mathrm{Ker}(\mathrm{ad}_{M_-})$, and $\mathcal{Z} \equiv \sum_{i=0}^{N} \mathcal{Z}_{-i}$ to be some complement to $\mathcal{P}$ in $(\mathcal{G}_- + \mathcal{G}_0)$. We remark that one always has $\mathcal{Z}_{-N} = \{0\}$ for the lowest grade $-N$. It turns out to be useful to decompose any lower triangular solution of (4.9) and, respectively, any solution of (4.12) in the following manner:

$$K(J^c) = p(J^c) + z(J^c) \ , \qquad p \in \mathcal{P}, \quad z \in \mathcal{Z} \ , \tag{4.16}$$

and

$$K_{\mathrm{DS}}(J^{\mathrm{DS}}) = p(J^{\mathrm{DS}}) + z(J^{\mathrm{DS}}) + k(J^{\mathrm{DS}}), \qquad p \in \mathcal{P}, \quad z \in \mathcal{Z}, \quad k \in \mathcal{G}_+ \, . \tag{4.17}$$

Substituting (4.16) into (4.9), one can prove that the components of $p(J^c)$ can be arbitrarily given and then the components of z are *uniquely determined* in terms of p and J^c by this equation. Moreover, one sees by inspection that the components of z are *differential polynomials in* J^c and *linear differential polynomials in the components of* p. Similarly, equation (4.12) determines z and k as *unique differential polynomials linear in the arbitrarily given components of* p *and in general non-linear in* J^{DS}.

To prove that the general solutions of (4.9) and (4.12) are parametrized by arbitrary $\mathcal{P}$-valued functions in the above manner one has to consider these equations grade by grade, starting from the lowest grade, and at every grade use the non-degeneracy condition and the decompositions (4.15), and also the decomposition (3.4) when considering (4.12). The crucial property of these decompositions to be used in this analysis is that ad_{M_-} maps $\mathcal{Z}_{-i}$ onto $\mathcal{G}_{-i-1}$ and $\mathcal{G}_{i+1}$ onto $\mathcal{I}_i$ in a one-to-one manner, for any $i = 0, 1, \ldots, N$. The iterative procedure of solving (4.9) and (4.12) grade by grade provides one with an algorithm for computing K and K_{DS} in terms of the parameters $p(J^c)$ and $p(J^{\mathrm{DS}})$.

This algorithm is very convenient, e.g. since one obtains $\delta_K J^c$ (resp. $\delta_{K_{\rm DS}} J^{\rm DS}$) essentially by means of the same computation which produces K (resp. $K_{\rm DS}$).

A particular consequence of the above result is that $K(J^c)$ and $K_{\rm DS}(J^{\rm DS})$ are local functions if and only if the parameter functions $p(J^c)$ and $p(J^{\rm DS})$ are local. Equation (4.11) imposes a further condition on $p(J^c)$ which is hard to handle practically. On the other hand, any local $p(J^{\rm DS})$ defines a gauge preserving KM transformation implementing a residual KM symmetry. In conclusion, we see that the set of residual, local, current dependent KM symmetry transformations is parametrized, in a one-to-one manner, by $\dim(\mathcal{P}) = \dim(\mathcal{G}_0)$ arbitrary but local functions of $J^{\rm DS}$. (Note that the equality $\dim(\mathcal{P}) = \dim(\mathcal{G}_0)$ is a consequence of the non-degeneracy condition of M_-, eq. (3.3).)

Now we establish the KM implementation of the induced $\mathcal{W}$-transformation

$$\delta^*_Q J^{\rm DS}(x^+) = -\{Q\,,\, J^{\rm DS}(x^+)\}^*\ , \qquad Q = \int_0^{2\pi} dx^+\, q(W^{i,n_i})\ , \tag{4.18}$$

where q is an arbitrary element of the $\mathcal{W}$-algebra, that is an arbitrary differential polynomial in the $\mathcal{W}$-basis W^{i,n_i} associated with the DS gauge. We note that under the Dirac bracket one can substitute the DS currents U^{i,n_i} for W^{i,n_i}, and that we allow q to depend on some test functions as well. Our purpose is to find the function $K_{\rm DS}(J^{\rm DS})$ for which

$$\delta^*_Q J^{\rm DS} = \delta_{K_{\rm DS}(J^{\rm DS})} J^{\rm DS}\ . \tag{4.19}$$

By the meaning of the Dirac bracket, δ^*_Q is nothing but the gauge preserving KM transformation induced by the $\mathcal{W}$-transformation

$$\delta_Q J(x^+) = -\{Q\,,\, J(x^+)\}\ , \tag{4.20a}$$

for which

$$\delta_Q J = \delta_{K(J)} J \tag{4.20b}$$

with $K(J)$ given by (4.4).

We have seen that any gauge preserving KM transformation can be computed from (4.12) if its components in $\mathcal{P}$ are known, so our problem boils down to establishing how the 'parameter function' $p(J^{\rm DS})$ of $K_{\rm DS}$ in (4.19) depends on Q. To formulate the solution of this problem, first we note that, as a consequence of their definition, the space of parameters $\mathcal{P} \subset (\mathcal{G}_- + \mathcal{G}_0)$ is necessarily dual with respect to the Cartan-Killing form to the space $\mathcal{V} \subset (\mathcal{G}_0 + \mathcal{G}_+)$ defining the DS gauge. (This is trivial to see in the special case of the highest weight gauge and $\mathcal{P} = \mathrm{Ker}(\mathrm{ad}_{M_-})$. The general case is conveniently treated as a modification of this situation.) This allows us to introduce a basis $\hat{F}^{m,n_m}$ in $\mathcal{P}$ in such a way that

$$< \hat{F}^{m,n_m} \, , \, F_{i,n_i} >= \delta_i^m \delta_{n_i}^{n_m} \ , \tag{4.21}$$

where F_{i,n_i} is the basis of $\mathcal{V}$, which we used to define the W^{i,n_i}, and $< \ , \ >$ is the Cartan-Killing form. Furthermore, we choose the space $\mathcal{Z} \subset (\mathcal{G}_- + \mathcal{G}_0)$ to be the *annihilator* of $\mathcal{V}$ with respect to the scalar product, that is $\mathcal{Z}$ consists of all the elements $z \in (\mathcal{G}_- + \mathcal{G}_0)$ satisfying

$$< z \, , \, v >= 0 \ , \qquad\qquad \forall v \in \mathcal{V} \ . \tag{4.22}$$

We can now write the general solution of (4.12) as

$$K_{\mathrm{DS}}(J^{\mathrm{DS}}) = \sum_{i,n_i} p_{i,n_i} \hat{F}^{i,n_i} + z(J^{\mathrm{DS}}) + k(J^{\mathrm{DS}}), \tag{4.23}$$

where $z(J^{\mathrm{DS}}) \in \mathcal{Z}$ and $k(J^{\mathrm{DS}}) \in \mathcal{G}_+$ are uniquely determined by the parameter functions p_{i,n_i}. Then we have the following

Theorem: The parameters of $K_{\mathrm{DS}}(J^{\mathrm{DS}})$ satisfying (4.19) are the functional derivatives of Q with respect to the W^{i,n_i}, that is

$$p_{i,n_i}(x^+) = \frac{\delta Q}{\delta W^{i,n_i}(x^+)}(J^{\mathrm{DS}}) \ . \tag{4.24}$$

This is one of our main results. Before explaining how to prove this theorem, we mention some of its consequences. An important special case is obtained by taking Q to be a moment of the W^{i,n_i}, that is by considering

$$Q_a = \int_0^{2\pi} dx^+ \sum_{i,n_i} a_{i,n_i}(x^+) W^{i,n_i}(x^+) \ , \tag{4.25}$$

for arbitrary *test functions* $a_{i,n_i}(x^+)$. It follows from (4.24) that in this particular case the parameters are just the test functions themselves. Specializing further, we denote by $K_{y^+}^{k,n_k}$ the solution of equation (4.12) belonging to the following choice of parameters:

$$p_{i,n_i}(x^+) \equiv \delta_{i,k}\delta_{n_i,n_k}\delta(x^+ - y^+) \ , \tag{4.26}$$

for any fixed k, n_k and y^+. By combining our previous results, it then follows that

$$\{U^{k,n_k}(y^+) \, , \, J^{\mathrm{DS}}(x^+)\}^* = -\frac{\partial}{\partial x^+} K_{y^+}^{k,n_k}(x^+) - [K_{y^+}^{k,n_k}(x^+) \, , \, J^{\mathrm{DS}}(x^+)] \ , \tag{4.27}$$

where U^{k,n_k} is the DS current component introduced in (3.15). This gives us an algorithm for computing the $\mathcal{W}$-algebra itself by solving equation (4.12), which is a simple linear problem. This is a direct generalization of the algorithm given in [13]. This algorithm provides one with an effective tool for working out non-trivial examples [13], and it is also useful for studying the qualitative features, e.g. the sub-algebra structure [15], of the $\mathcal{W}$-algebras.

To sketch the proof of the above theorem we first point out that it can be reduced to (4.27) by using the Leibniz rule. On the other hand, (4.27) can be obtained by considering the problem for Q_a with test functions chosen as $a_{i,n_i} = p_{i,n_i}$ in (4.26). The point then is that for Q_a, with arbitrary test functions, (4.24) follows from (4.4) and (4.13) by taking into account that W^{i,n_i} becomes the current component U^{i,n_i} in the DS gauge. Alternatively, the proofs given in [13] for the special case of (4.25) in the context of the usual Toda theory can also be easily adapted to our general situation.

V. On a Lax pair interpretation of the $\mathcal{W}$-symmetry

In a recent paper Mansfield and Spence [17] proposed a new interpretation of the $\mathcal{W}$-symmetry of the Toda system (2.2). Their interpretation, developed in [17] for the case of the standard Toda theory given by (2.5), is that the $\mathcal{W}$-symmetry corresponds to 'non-Abelian gauge transformations' preserving the form of the Lax pair (2.4). More exactly, they consider variations of $\mathcal{A}_\pm$ of the type

$$\delta_{\mathcal{K}} \mathcal{A}_\pm = \partial_\pm \mathcal{K} + [\mathcal{K}\,,\, \mathcal{A}_\pm]\ , \tag{5.1}$$

where $\mathcal{K}$ is a $\mathcal{G}$-valued function. They find that, upon requiring the above variation to respect the form of $\mathcal{A}_\pm$, this equation determines the allowed set of $\mathcal{K}$'s in terms of $\dim(\mathcal{G}_0)$ independent chiral parameter functions. They are then able to interpret these form preserving variations as the ones underlying the $\mathcal{W}$-symmetry, and also find some nice explicit formulae.

Our aim now is to understand how the above interpretation of the $\mathcal{W}$-symmetry relates to the KM interpretation, given in [13] for the case of the standard Toda theory and generalized in this paper. In fact, originally this question provided our motivation for investigating the relationship between the WZNW and Lax pair descriptions of Toda theory.

Let us consider a residual KM type symmetry transformation δ_K of the constrained WZNW theory, given by some local solution $K(J^c)$ of equations (4.9) and (4.11). We know that δ_K gives a well defined variation of any gauge invariant quantity, because of (4.11). We would like to compute the variation of the Lax pair. To this first we recall that a KM transformation acts on the WZNW field $g(x^+,x^-)$ and on its chiral part $g_L(x^+)$ according to (4.1) and (4.2). Second, supposing that we are in the Gauss decomposable sector, the action of δ_K on the upper triangular field $g_+(x^+,x^-)$ in (2.12) is also fixed in principle by (4.1), since g_+ is a unique function of g. Combining these, we get that the variation of the gauge invariant field $\hat{g}$ defined by (2.18) reads as

$$\delta_K \hat{g} = \mathcal{K}\cdot\hat{g}\ , \tag{5.2}$$

where, from (4.2),

$$\mathcal{K} = g_+^{-1}\cdot K(J^c)\cdot g_+ - g_+^{-1}\cdot\delta_K g_+\ . \tag{5.3}$$

It then follows from $\mathcal{A}_\pm = \partial_\pm\hat{g}\cdot\hat{g}_+^{-1}$ that the variation of the Lax pair under the residual KM transformation is given by equation (5.1) with $\mathcal{K}$ determined by formula (5.3). Since δ_K preserves the constraints reducing the WZNW theory to Toda theory, it is obvious that any variation arising in the above manner preserves the form of the Lax pair. Moreover, since we have seen that the $\mathcal{W}$-transformations are implemented by certain residual KM transformations, namely by those which are local in J^c and canonical, it follows that the $\mathcal{W}$-algebra indeed acts on the Lax pair by form preserving transformations of the type (5.1). On the other hand, it is also clear that not every form preserving variation of $\mathcal{A}_\pm$ is a $\mathcal{W}$-transformation, simply because not all of them are even canonical transformations. This is an obvious consequence of the fact that for $\mathcal{W}$-transformations one has a non-trivial gradient condition corresponding to (4.24).

The function $\mathcal{K}$ given by (5.3) describes the transformation of the gauge invariant objects $\hat{g}$ and $\mathcal{A}_\pm$ under the residual KM symmetry, and therefore it can be expressed as some function of the Toda field $g_0(x^+,x^-)$, $\mathcal{K}=\mathcal{K}_{\mathrm{Toda}}(g_0)$. Below we establish the functional form of $\mathcal{K}_{\mathrm{Toda}}$.

To this first we introduce the notation K_0 for the restriction of $K(J^c)$ to the diagonal gauge (3.22), $K_0(j_0)\equiv K(J^{\mathrm{diag}})$. We assumed that K is lower triangular and from this it follows that δ_K preserves the diagonal gauge. Indeed, $\delta_K J^{\mathrm{diag}}$ has only grade zero components because of (4.3) and (4.9). This means that the lower triangular matrix K_0 implements the residual KM symmetry in the diagonal gauge.

The result we prove is that $\mathcal{K}_{\rm Toda}$ depends on g_0 only through $\mathcal{A}_+$ and that $\mathcal{K}_{\rm Toda}(\mathcal{A}_+)$ can be obtained by simply substituting $\mathcal{A}_+$ for the argument of K. In other words, analogously to (3.25), we have

$$\mathcal{K}_{\rm Toda}(g_0) = K(\mathcal{A}_+) = K_0(j_0 \to \partial_+ g_0 \cdot g_0^{-1}) \ . \tag{5.4}$$

For $\mathcal{W}$-transformations this follows from (3.25) by taking into account that $J^{\rm diag}$ and $\mathcal{A}_+$ not only have the same form, but the form of the Poisson bracket relations of their components is the same as well. This then implies (5.4) for general residual KM transformations too, simply because (5.3) is an algebraic formula.

In summary, we see that the Lax pair interpretation of the $\mathcal{W}$-symmetry proposed in [17] arises immediately as a translation of our KM interpretation. This way we not only generalized the result of [17] to the case of the general Toda system (2.2), but also obtained a certain clarification, namely we have a general method allowing us to single out the $\mathcal{W}$-transformations amongst the general 'form preserving gauge transformations' of the Lax pair.

VI. Conclusions

In this paper we analysed the structure of the generalized conformal Toda theories associated with the integral gradings of the simple Lie algebras. Our main results are the following.

First, we established the relationship between the constrained WZNW and the Lax pair descriptions of the Toda theories. Second, in the non-degenerate case, we set up a general framework for analysing the extended conformal symmetry algebras and the current dependent, residual KM type symmetries present in these models as a consequence of their WZNW origin. In particular, by exhibiting the highest weight gauge, we uncovered the $sl(2)$ structure underlying the $\mathcal{W}$-algebras considered and found their conformal primary field basis. Furthermore, we have given an algorithm for finding the KM implementation of the symmetry transformations generated by the $\mathcal{W}$-algebra, which can be used, for example, to compute the $\mathcal{W}$-algebra itself. Our results on the $\mathcal{W}$-algebra generalize and complement the ones given in [13], where the special case of the standard Toda theory was considered. Here we have also shown how to express these results in terms of the Lax pair framework. In particular, we recovered the interpretation of the $\mathcal{W}$-symmetry given in [17] as a consequence of our KM interpretation.

This paper is a continuation of the series [12-14, 29], and most of the other results obtained in those papers can be generalized to the case considered here. This is also true for the case of the conformal Toda theories associated with the half-integral embeddings of $sl(2)$ into the simple Lie algebras [15, 25].

In addition to the outstanding problem of finding the quantum analogues of our $\mathcal{W}$-algebras, the following 'classical' problems would deserve further investigation. The conformal reductions of the KM phase space leading to chiral algebras of polynomial nature should be classified. We think that the $\mathcal{W}_{\mathcal{S}}^{\mathcal{G}}$ algebras mentioned in Section III. constitute an important class of $\mathcal{W}$-algebras and their structure should be analysed in detail. Furthermore, it would be important to explore the KdV like hierarchies of integrable equations which should correspond to the generalized $\mathcal{W}$-algebras (see also [15, 16, 30]). Finally, the (affine) WZNW framework [24] of describing affine Toda theories should also be further developed. We hope to be able to report on some of these issues in a future publication.

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