On the Symmetries of Hamiltonian Systems

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Abstract

In this paper we show how the well-known local symmetries of Lagrangean systems, and in particular the diffeomorphism invariance, emerge in the Hamiltonian formulation. We show that only the constraints which are linear in the momenta generate transformations which correspond to symmetries of the corresponding Lagrangean system. The nonlinear constraints (which we have, for instance, in gravity, supergravity and string theory) rather generate the dynamics of the corresponding Lagrangean system. Only in a very special combination with "trivial" transformations proportional to the equations of motion do they lead to symmetry transformations. We reveal the importance of these special "trivial" transformations for the interconnection theorems which relate the symmetries of a system with its dynamics. We prove these theorems for general Hamiltonian systems. We apply the developed formalism to concrete physically relevant systems and in particular those which are diffeomorphism invariant. The connection between the parameters of the symmetry transformations in the Hamiltonian- and Lagrangean formalisms is found. The possible applications of our results are discussed.

Chapter 1

Introduction

Local symmetries play a very important role in all field theories being relevant in physics. The actions of such theories are invariant with respect to some group of local transformations. For example, for Yang-Mills theories these are the gauge transformations, for string theory and gravity diffeomorphisms and for supersymmetric theories coupled to gravity local supersymmetry transformations. These symmetries are quite transparent in the Lagrangean formulation and this is seen as one of the main virtues of this approach. Actually the Lagrangean of a theory is constructed such that it is invariant under gauge transformations and/or diffeomorphisms.

If we go from the Lagrangean to the first order Hamiltonian formalism then at first glance it seems that these symmetries are not manifest. This applies especially to diffeomorphism invariant theories and is of much relevance in general relativity [1, 2, 3, 4]. One of the purposes of this paper is to show that one can construct the symmetries of constrained Hamiltonian systems in an explicit manner.

It was found in [5, 6] that the first order action is invariant with respect to *infinitesimal* time-dependent transformations generated by the first class constraints if the Lagrangean multipliers are simultaneously transformed. However, we shall see that these transformations correspond to Lagrangean symmetries only if the constraints are *linear in momenta*. For instance, this is the case for Yang-Mills theories, where all gauge transformations (including time-dependent ones) can be recovered in such a manner in the Hamiltonian formalism.

For the constraints which are nonlinear in the momenta (as they exist in diffeomorphism invariant theories, e.g. gravity or string theory) this is not true anymore. The nonlinear constraints by themselves *do not* generate transformations which correspond to symmetries of the corresponding Lagrangean system. They are rather responsible for the dynamics of such systems.

Although the transformations generated by the nonlinear constraints are still symmetries of the Hamiltonian system (which cannot be identified with Lagrangean symmetries) it is not clear whether they are of any relevance, since only their infinitesimal form is known. It is not obvious whether for nontrivial theories they can be exponentiated, that is can be iterated to finite transformations.

The action in the Hamiltonian (and even Lagrangean) formalism is also invariant with respect to so-called *infinitesimal* 'trivial' transformations [7, 8, 9] which are proportional to antisymmetric combinations of the equations of motion and do not vanish off mass-shell. This huge class of additional transformations exists even in theories without local symmetries. It is clear that most of them (or sometimes even all) are irrelevant and can safely be ignored [7, 8, 9]. However, we shall see that not all of the "trivial" transformations are really unimportant for the systems with nonlinear constraints. Indeed, we shall demonstrate that all Lagrangean symmetries can be recovered in the Hamiltonian formalism only if we consider the transformations generated by the nonlinear constraints in a very special combination with *particular* "trivial" transformations. The combined transformations can be exponentiated since they correspond to known Lagrangean symmetries. Thus, *not all* of the trivial transformations are irrelevant for systems with nonlinear constraints, although they may be ignored for particular perturbative questions [9]. However, this is not alway the case. In particular we shall see later that it is impossible to get the theorems which relate the dynamics of a super-hamiltonian system with its symmetry properties (e.g. the interconnection theorems in general relativity) if we neglect the trivial transformations. Also, when one ignores them this can lead to wrong results in nonperturbative calculations. One last remark concerns the identification of transformations generated by the constraints with the Lagrangean symmetries on mass-shell. It seems that this identifictaion of *infinitesimal* transformations is meaningless, since on mass-shell any infinitesimal transformation can be viewed as "symmetry transformation" since solutions of the equations of motion are stationary points of the action.

The questions which we address in this paper are the following. First we investigate how one recovers and generalizes the local Lagrangean symmetries in the first order Hamiltonian formalism. This question has also been raised recently in [10]. However, our approach is very different and can be viewed as complimentary to that in [10]. Also we explicitly reveal the connection between the parameters of the transformations in the Hamiltonian and Lagrangean formalisms in most physically important theories. Some of these results (but not all) can be found in the literature and our purpose here will be to clarify the confusing points which still exist. The other question concerns the difference between linear and nonlinear in momenta constraints. We will show that the transformations generated by the nonlinear constraints always take any trajectory which belongs to the subspace where the Lagrangean system lives [1] away from this subspace. Hence these transformations cannot correspond to Lagrangean symmetries. The role of the "trivial" transformations is to project the trajectory back to this subspace. The nonlinear constraints themselves rather generate the dynamics of the corresponding Lagrangean systems.

We will follow in detail how the closed Lie algebra belonging to the diffeomorphism group arises in a natural manner in the Hamiltonian formalism. We clarify the connection between the symmetry properties of the system and its dynamics and prove the so-called "interconnection" theorem [11] for general constrained Hamiltonian systems entirely in the Hamiltonian formalism. This theorem plays a crucial role in the Dirac quantization program and also in the Hamilton-Jacobi appraoach to classical general relativity. It will be shown that this theorem is nontrivial only for theories with an infinite number of degrees of freedom and only if there are nonlinear in momenta constraints. The special role played by the trivial transformations in proving it is emphasized. Most of our considerations are classical and we comment on the corresponding problems in the quantized theories at the end of the paper.

The paper is organized as follows. In the second section we describe the symmetries of general first order Hamiltonian systems. In the subsequent sections we apply the results to gauge theories, the relativistic particle, the locally supersymmetric relativistic particle, bosonic string and to general relativity. We show that the local symmetries of Hamiltonian systems coincide with the local symmetries of the corresponding Lagrangean systems by revealing the connection between the parameters of the corresponding groups for the Hamiltonian and Lagrangean systems. In the last chapter we discuss why from our point of view the Hamiltonian formalism is more 'fun-

[1] The subspace on which the momenta and velocities are related by the first half of Hamilton's equations

damental' than the Lagrangean one, in particular for the quantized theories, and describe the possible applications of the developed formalism.

Chapter 2

Symmetry Transformations

We shall consider a general first order Hamiltonian system with constraints, the action of which is

$$S = \int \Big(p^{\tilde{i}} \dot{q}_{\tilde{i}} - \mathcal{N}^{\tilde{\alpha}} C_{\tilde{\alpha}}(p,q) - \mathcal{H}(p,q) \Big) dt. \tag{2.1}$$

If the system contains fermions then some of the variables $p, q, \mathcal{N}$ will be of Grassmannian type. The first order action (2.1) describes both systems with a finite or infinite number of degrees of freedom since the following condensed notation [12] is assumed: the indices $\tilde{i}, \tilde{\alpha}$ are supposed to be composite ones, that is they may contain discrete and continuous variables. For systems with a finite number of degrees of freedom $\tilde{\alpha} = \alpha$ and $\tilde{i} = i$ are discrete. For field theories $\tilde{i} = \{i, x\}$ and $\tilde{\alpha} = \{\alpha, x\}$, where i and α are some discrete (internal) indices and in d space-time dimensions $x = \{x^1, \ldots, x^{d-1}\}$ are the d–1 space coordinates. For example, for a scalar field $q_{\tilde{i}}(t) = \phi_x(t) = \phi(x,t)$ and for a vector field $q_{\tilde{i}}(t) = A_{i,x}(t) = A_i(x,t)$, where in 4 spacetime dimensions $i = 1, 2, 3$. We adopt the Einstein convention for repeated indices in 'up' and 'down' positions, that is we assume summation over discrete repeated indices and integration over continuous ones, for example

$$\xi^x p^{i,x} \dot{q}_{i,x} = \sum_i \int dx \xi(x) p^i(x) \dot{q}_i(x), \tag{2.2}$$

but

$$\xi^x p^{i,x} q_i^x = \sum_i \xi(x) p^i(x) q_i(x), \qquad \text{no integration.} \tag{2.3}$$

Also, we shall not distinguish $q_{i,x}$ and q_i^x and use the position of the continuous index just to indicate when we should integrate. Sometimes it will be convenient to resolve the composite index $\tilde{i}$ (or $\tilde{\alpha}$) as i, x (or α, x). The dot always denotes derivative with respect to time t on which p, q and $\mathcal{N}$ may depend.

For first class Hamiltonian system [13] the contraints $C_{\tilde{\alpha}}$ and Hamiltonian $\mathcal{H}$ form a closed algebra with respect to the standard Poisson bracket $\{.,.\}$ (possibly extended to fermionic variables, in which case the algebra is graded [14]):

$$\{C_{\tilde{\alpha}}, C_{\tilde{\beta}}\} = t^{\tilde{\gamma}}_{\tilde{\alpha}\tilde{\beta}} C_{\tilde{\gamma}} \qquad \text{and} \qquad \{\mathcal{H}, C_{\tilde{\alpha}}\} = t^{\tilde{\beta}}_{\tilde{\alpha}} C_{\tilde{\beta}}, \tag{2.4}$$

where the t's are the structure coefficients [1]. These coefficients may depend on the canonical variables. For field theories $t^{\tilde{\gamma}}_{\tilde{\alpha}\tilde{\beta}} \equiv t^{\gamma z}_{\alpha x\, \beta y}$ and $t^{\tilde{\beta}}_{\tilde{\alpha}} \equiv t^{\beta y}_{\alpha x}$.

The equations of motion resulting from the variation of the action (2.1) with respect to q, p and the Lagrangean multipliers $\mathcal{N}$

$$\delta S = \int \Big(\delta p^{\tilde{i}} EM(q_{\tilde{i}}) - \delta q_{\tilde{i}} EM(p^{\tilde{i}}) - \delta \mathcal{N}^{\tilde{\alpha}} C_{\tilde{\alpha}}\Big) dt + \text{bound. terms} \tag{2.5}$$

are

$$\begin{aligned} EM(q_{\tilde{i}}) &\equiv \dot{q}_{\tilde{i}} - \{q_{\tilde{i}}, \mathcal{N}^{\tilde{\beta}} C_{\tilde{\beta}} + \mathcal{H}\} = 0, \\ EM(p^{\tilde{i}}) &\equiv \dot{p}^{\tilde{i}} - \{p^{\tilde{i}}, \mathcal{N}^{\tilde{\beta}} C_{\tilde{\beta}} + \mathcal{H}\} = 0, \\ C_{\tilde{\alpha}} &= 0. \end{aligned} \tag{2.6}$$

Below we shall often use these abbreviations $EM(q)$ and $EM(p)$ for the left hand sides in (2.6). Of course, on mass shell we have $EM = 0$, but off mass shell either $EM(q)$ or $EM(p)$ (or both) does not vanish.

To go from the Hamiltonian to the Lagrangean formalism we should express the momenta in terms of the velocities via the Hamiltonian equations $EM(q_{\tilde{i}}) = 0$. Thus not all off mass-shell trajectories of the Hamiltonian system can be considered in the Lagrangean formalism, but only those for which $EM(q) = 0$. Hence one can say that the Lagrangean system lives only in the subspace $\mathcal{M}$ of the 'extended phase space' defined by the conditions

$$\mathcal{M} : EM(q_{\tilde{i}}) \equiv \dot{q}_{\tilde{i}} - \{q_{\tilde{i}}, \mathcal{N}^{\tilde{\beta}} C_{\tilde{\beta}} + \mathcal{H}\} = 0 \tag{2.7}$$

Clearly the space of trajectories in phase space where the Hamiltonian system lives is much bigger than the space of Lagrangean trajectories.

[1] If some constraints depend explicitly on time, then we should add $\partial_t C_{\tilde{\alpha}}$ to the right hand side of the second relation.

The action (2.1) is invariant (up to boundary terms) with respect to the infinitesimal transformations generated by the constraints if the Lagrangean multipliers are transformed simultaneously [5, 6]:

$$\begin{aligned} \delta_\lambda q_i^x &= \{q_i^x, \lambda^{\tilde\beta} C_{\tilde\beta}\}, \\ \delta_\lambda p^{ix} &= \{p^{ix}, \lambda^{\tilde\beta} C_{\tilde\beta}\}, \\ \delta_\lambda \mathcal{N}^{\tilde\alpha} &= \dot\lambda^{\tilde\alpha} - \lambda^{\tilde\beta} \mathcal{N}^{\tilde\gamma} t^{\tilde\alpha}_{\tilde\gamma\tilde\beta} - \lambda^{\tilde\beta} t^{\tilde\alpha}_{\tilde\beta}. \end{aligned} \tag{2.8}$$

The parameters $\lambda^{\tilde\alpha} = \lambda^\alpha(\mathcal{N}, x, t)$ in (2.8) are the parameters of the infinitesimal transformations. The order in which λ enters in (2.8) is important if some of the variables are of Grassmannian type. We shall only consider the case when the parameters λ depend explicitly on spacetime coordinates and Lagrangean multipliers, since this suffices to cover all known physically relevant theories[2]. Because of this $\mathcal{N}$-dependence we should keep λ inside the Poisson bracket even for purely bosonic theories since if we calculate the commutator of two subsequent infinitesimal transformations, then the parameter λ of the second transformation will depend on q, p if the structure coefficients depend on the canonical variables. It is not difficult to see that the variation of the action (2.1) under these transformations leads only to the boundary terms

$$\delta_\lambda S = \left(p^{\tilde i} \delta_\lambda\, q_{\tilde i} - \lambda^{\tilde\alpha} C_{\tilde\alpha}\right)\Big|_{t_i}^{t_f}. \tag{2.9}$$

This term can be removed even if the parameters λ do not vanish at the boundaries if we add to the action (2.1) the total derivative of some function $Q(p, q)$ which satisfies the equation

$$\frac{\delta Q}{\delta q_{\tilde i}} \delta_\lambda q_{\tilde i} + \frac{\delta Q}{\delta p^{\tilde i}} \delta_\lambda p^{\tilde i} = \lambda^{\tilde\alpha} C_{\tilde\alpha} - p^{\tilde i} \delta_\lambda q_{\tilde i}. \tag{2.10}$$

The question which naturally arise here is the following: do the symmetry transformations (2.8) correspond to Lagrangean symmetries, that is are they, for instance, the diffeomorphism transformations in general relativity and string theory?

[2]In principle, we could consider more general transformations for which the λ would also depend on the canonical variables q and p. One can show that in this case the action (2.1) is also invariant with respect to infinitesimal transformations generated by the constraints if $\mathcal{N}^{\tilde\alpha}$ are transformed as $\delta_\lambda \mathcal{N}^{\tilde\alpha} = \partial_t \lambda^{\tilde\alpha}(q, p, t) - \lambda^{\tilde\beta} \mathcal{N}^{\tilde\gamma} t^{\tilde\alpha}_{\tilde\gamma\tilde\beta} - \lambda^{\tilde\beta} t^{\tilde\alpha}_{\tilde\beta} - \mathcal{N}^{\tilde\beta}\{C_{\tilde\beta}, \lambda^{\tilde\alpha}\} - \{\mathcal{H}, \lambda^{\tilde\alpha}\}$.

As we shall see below the answer to this question crucially depends on the momenta dependence of the constraints. If the constraints are linear in the momenta, then the answer is yes. However, it is not the case if some of the constraints are nonlinear. The reason is that the transformation (2.8) generated by a nonlinear constraint take a trajectory in $\mathcal{M}$ (see (2.7)) away from it and the transformed trajectory can not be viewed as a trajectory of the Lagrangean system, since $EM(q)=0$ does not hold anymore. To proceed in this case we should consider extra compensating symmetry transformations of the Hamiltonian system.

Actually the set of infinitesimal off mass-shell transformations which leave the action (2.1) invariant is much bigger than (2.8). Any infinitesimal transformation $(\delta q, \delta p, \delta \mathcal{N})$ orthogonal to the (functional) gradient $\nabla S = (-EM(p), EM(q), -C)$ leaves the action invariant, as can be easily seen from (2.5). Hence we could add to the transformations generated by the constraints for example any transformation of the form

$$\begin{aligned} \delta q_{\tilde{i}} &= EM(q_{\tilde{j}})\xi^{\tilde{j}}{}_{\tilde{i}} + EM(p^{\tilde{j}})\eta_{\tilde{j}\tilde{i}}, \\ \delta p^{\tilde{i}} &= EM(p^{\tilde{j}})\xi^{\tilde{i}}{}_{\tilde{j}} + EM(q_{\tilde{j}})\zeta^{\tilde{j}\tilde{i}}, \\ \delta \mathcal{N}^{\tilde{\alpha}} &= 0, \end{aligned} \tag{2.11}$$

where $\xi^{\tilde{j}}{}_{\tilde{i}}$ are arbitrary 'matrices' (kernels) and the $\eta_{\tilde{j}\tilde{i}}$, $\zeta^{\tilde{j}\tilde{i}}$ are antisymmetric. Generically such transformations are nonlocal, and they exist for all systems even for those without any symmetries.

Most of them are actually irrelevant for the physically interesting transformations [7, 8, 9]. However, if some of the constraints are nonlinear then, as we will see, the particular "trivial" transformations from (2.11) play an important role for recovering the Lagrangean symmetries in the Hamiltonian formalism.

We will show that in all theories containing only one nonlinear constraint (e.g. gravity and string theory) we need only very special transformations from (2.11), namely

$$\delta_\xi q_i^x = EM(q_i^x)\xi^x \quad \text{and} \quad \delta_\xi p^{ix} = EM(p^{ix})\xi^x \tag{2.12}$$

to recover all Lagrangean symmetries.

Only for theories with several nonlinear constraints do we need extra transformations from (2.11) in addition to (2.8) to recover all off mass-shell Lagrangean symmetries in the Hamiltonian approach. For a system with only one nonlinear constraint we consider the combined transformations

$$\hat{I}_{\xi,\lambda}\, F(q,p,\mathcal{N}) = F(\hat{I}_{\xi,\lambda}\, q, \hat{I}_{\xi,\lambda}\, p, \hat{I}_{\xi,\lambda}\, \mathcal{N}), \qquad \hat{I}_{\xi,\lambda} = \hat{1} + \delta_{\xi,\lambda} + \cdots, \tag{2.13}$$

where

$$\begin{aligned}
\delta_{\xi,\lambda}\, q_i^x &= EM(q_i^x)\xi^x + \{q_i^x, \lambda^{\tilde{\beta}} C_{\tilde{\beta}}\}, \\
\delta_{\xi,\lambda}\, p^{ix} &= EM(p^{ix})\xi^x + \{p^{ix}, \lambda^{\tilde{\beta}} C_{\tilde{\beta}}\}, \\
\delta_{\xi,\lambda}\, \mathcal{N}^{\tilde{\alpha}} &= \dot{\lambda}^{\tilde{\alpha}} - \lambda^{\tilde{\beta}} \mathcal{N}^{\tilde{\gamma}} t^{\tilde{\alpha}}_{\tilde{\gamma}\tilde{\beta}} - \lambda^{\tilde{\beta}} t^{\tilde{\alpha}}_{\tilde{\beta}}.
\end{aligned} \tag{2.14}$$

The number of functions (ξ, λ^α) which appear here is equal to the number of constraints (per point of space) plus one. This seems strange since for all constrained theories the number of parameters in the Lagrangean symmetry transformations is equal to the number of constraints. Thus not all of the parameters in (2.14) should be independent for these transformations to be symmetries of the corresponding Lagrangean systems. To understand why we need the "trivial" transformations (2.12) in additions to (2.8) and to reveal the connections between the parameters ξ and λ^α we derive the conditions under which the transformations (2.14) can be viewed as Lagrangean symmetries.

For that the transformations (2.14) should at least leave the subspace $\mathcal{M}$ (see (2.7)) in which the Lagrangean system lives, invariant. That is, they should leave any trajectory which belongs to the subspace $\mathcal{M}$ in this subspace. The necessary conditions for that can be gotten by varying (2.7) as follows

$$\begin{aligned}
\frac{d}{dt}(\delta_{\xi,\lambda} q_{\tilde{i}}) &= \frac{\delta^2(\mathcal{H} + \mathcal{N}^{\tilde{\sigma}} C_{\tilde{\sigma}})}{\delta p^{\tilde{i}} \delta q_{\tilde{j}}}\, \delta_{\xi,\lambda} q_{\tilde{j}} \\
&+ \frac{\delta^2(\mathcal{H} + \mathcal{N}^{\tilde{\sigma}} C_{\tilde{\sigma}})}{\delta p^{\tilde{i}} \delta p^{\tilde{j}}}\, \delta_{\xi,\lambda} p^{\tilde{j}} + \{q_{\tilde{i}}, \delta_{\xi,\lambda} \mathcal{N}^{\tilde{\sigma}} C_{\tilde{\sigma}}\}.
\end{aligned} \tag{2.15}$$

Thus the transformations δq, δp and $\delta\mathcal{N}$ should satisfy this equation in the subspace $\mathcal{M}$. If this is not the case, then the trajectories for which

$$EM(q) = 0 \Longleftrightarrow p^{\tilde{i}} = f^{\tilde{i}}(\dot{q}^{\tilde{j}}, q^{\tilde{j}}, \mathcal{N}^{\tilde{\alpha}}) \tag{2.16}$$

are transformed into trajectories for which this equality fails and they cannot be viewed as trajectories of the corresponding Lagrangean system. The transformation of p which would follow from (2.16) as

$$\delta p^{\tilde{i}} = \frac{\partial f^{\tilde{i}}}{\partial q_{\tilde{j}}}\delta q_{\tilde{j}} + \frac{\partial f^{\tilde{i}}}{\partial \dot{q}_{\tilde{j}}}\delta(\dot{q}_{\tilde{j}}) + \frac{\partial f^{\tilde{i}}}{\partial \mathcal{N}^{\tilde{\alpha}}}\delta\mathcal{N}^{\tilde{\alpha}} \tag{2.17}$$

would be different from the transformation (2.14) for p. Thus (2.14) will not be a Lagrangean symmetry for which (2.17) must hold. Substituting (2.14) into (2.15) this condition simplifies to

$$\frac{\delta^2(\mathcal{H} + \mathcal{N}^{\tilde{\sigma}}C_{\tilde{\sigma}})}{\delta p^{ix}\delta p^{jy}} EM(p^{jy})\xi^y = \frac{\delta^2 C_{\tilde{\sigma}}}{\delta p^{ix}\delta p^{jy}} EM(p^{jy})\lambda^{\tilde{\sigma}}. \tag{2.18}$$

The point is that this equation relates ξ and λ^α and if it holds then the phase space transformations (2.14) can be interpreted as Lagrangean symmetries. At the same time the number of free functions becomes equal to the number of constraints as it should be.

Let us note that the "trivial" transformations (2.12) alone do not satisfy (2.18) for off mass-shell trajectories if the Hamiltonian $\mathcal{H}$ and/or $C_{\tilde{\alpha}}$ are nonlinear in momenta. Hence they cannot be identified with the Lagrangean symmetries.

If some constraints $C_{\tilde{\alpha}}$ are nonlinear, then the transformations (2.8) generated by them alone also cannot satisfy (2.18) for off mass-shell trajectories. Hence they cannot be viewed as Lagrangean symmetries either. Only when the transformations generated by the nonlinear constratins are taken in a very special combination with the "trivial" transformations (2.12) one can satisfy the condition (2.18). The reason why the nonlinear constraints alone do not generate the Lagrangean symmetries is simple. They always take off mass-shell trajectories away form the subspace $\mathcal{M}$, where the Lagrangean system lives. The "trivial" transformations (2.12) return the trajectories back in $\mathcal{M}$, if for the given λ^α in (2.14) we take the appropriate $\xi(\lambda^\alpha)$ to satisfy (2.18). They play the role of compensating transformations. As we shall see later, the nonlinear constraints themselves generate the dynamics for Lagrangean systems in the subspace $\mathcal{M}$.

Now we would like to consider two important examples:

Gauge Invariance. If the constraints $C_{\tilde{\alpha}}$ are linear and $\mathcal{H}$ at least quadratic in the momenta then only for $\xi^z = 0$ can equation (2.18) be satisfied [3]. So,

[3] If $\mathcal{H}$ and all constraints are linear in momenta then the Hamiltonian system is strongly degenerate.

in this case the transformations which are generated by the constraints will also be symmetry transformations for the corresponding Lagrangean system. We shall call them *gauge transformations*:

$$\hat{G}_\lambda = \hat{I}_{\xi=0,\lambda} \Longrightarrow \begin{cases} \delta_\lambda q_{\tilde{i}} = \{q_{\tilde{i}}, \lambda^{\tilde{\beta}} C_{\tilde{\beta}}\}, \quad \delta_\lambda p^{\tilde{i}} = \{p^{\tilde{i}}, \lambda^{\tilde{\beta}} C_{\tilde{\beta}}\}, \\ \delta_\lambda \mathcal{N}^{\tilde{\alpha}} = \dot{\lambda}^{\tilde{\alpha}} - \lambda^{\tilde{\beta}} \mathcal{N}^{\tilde{\gamma}} t^{\tilde{\alpha}}_{\tilde{\gamma}\tilde{\beta}} - \lambda^{\tilde{\beta}} t^{\tilde{\alpha}}_{\tilde{\beta}}. \end{cases} \tag{2.19}$$

For example, in Yang-Mills theories all constraints are linear in the momenta and (as we shall see in the next section) the finite gauge transformations can be recovered as transformations generated only by the constraints ($\xi = 0$) in the Hamiltonian formalism [4]. Thus the extra transformations (2.12) are irrelevan t in this case.

Reparametrization invariance. Very often the reparametrization invariance of a Lagrangean system, if it exists, is identified with the gauge invariance (2.19) in the Hamiltonian formalism. As we shall see they are actually very different.

If some of the constraints in (2.1) are nonlinear then it is obvious that the transformations generated by the constraints only ($\xi = 0$) do not satisfy (2.18). However, in all known theories with nonlinear constraints $\mathcal{H} = 0$ and the condition (2.18) can be satisfied if we impose some functional dependence between λ and ξ in (2.14) so that $\xi \neq 0$ for such theories. Thus the nonlinear constraints generate the Lagrangean symmetries only in very special combination with 'trivial' transformations from (2.12). More explicitly taking $\lambda^{\tilde{\sigma}}$ to be $\lambda^{\sigma z} = \mathcal{N}^{\sigma z} \xi^z$ in (2.18) we reduce this equation to

$$\mathcal{N}^{\sigma z} (\xi^y - \xi^z) \frac{\delta^2 C_{\sigma z}}{\delta p^{ix} \delta p^{jy}} EM(p^{jy}) = 0. \tag{2.20}$$

One sees at once that if

$$\frac{\delta^2 C_{\sigma z}}{\delta p^{ix} \delta p^{jy}} \sim \delta(z - y) \tag{2.21}$$

then even for constraints nonlinear in the momenta the equation (2.18) is satisfied off mass shell ($EM(p) \neq 0$). From that it follows immediately that the transformations (2.14) with $\lambda^{\sigma z} = \mathcal{N}^{\sigma z} \xi^z$ are symmetry transformations

[4] Another interesting class of theories where all constraints are linear in momenta are the constrained Wess-Zumino-Novikov-Witten theories [15].

for the corresponding Lagrangean system if $\mathcal{H} = 0$. We shall call this symmetry *reparametrization invariance*: $\hat{R}_\xi = \hat{I}_{\xi,\lambda^{\sigma z}=\mathcal{N}^{\sigma z}\xi^z}$. The explicit form of the reparametrization transformations generalized to field theories is

$$\begin{aligned} \delta_\xi q_i^x &= \dot{q}_i^x \xi^x + (\xi^y - \xi^x)\{q_i^x, \mathcal{N}^{\beta y} C_{\beta y}\} \\ \delta_\xi p^{ix} &= \dot{p}^{ix} \xi^x + (\xi^y - \xi^x)\{p^{ix}, \mathcal{N}^{\beta y} C_{\beta y}\} \\ \delta_\xi \mathcal{N}^{\alpha x} &= (\mathcal{N}^{\alpha x}\xi^x)^{\cdot} - \xi^y \mathcal{N}^{\beta y} \mathcal{N}^{\gamma z} t^{\alpha x}_{\gamma z, \beta y}. \end{aligned} \tag{2.22}$$

We would like to remind that according to our notation we assume here integration over y and z but no integration over x. For systems with a finite number of degrees of freedom the second terms or the right hand sides are absent and (2.22) has a familiar form.

If several constraints are nonlinear in momenta then there are extra reparametrisation transformations in addition to (2.22). They can be obtained by combining the transformations generated by the constraints with 'trivial' transformations (2.11) in such a manner that (2.18) is fulfilled (see section 5).

In some of the theories we shall study (string, gravity) there are both linear and nonlinear constraints. For such theories the symmetry transformations which correspond to Lagrangean symmetries are combinations of gauge transformations (generated by the linear constraints) and reparametrization transformations.

Algebra of transformations To construct the finite transformations we need to apply the infinitesimal transformations many times. To be successful in this 'exponentiation' it is clear that the following *necessary* condition should be fulfilled: The algebra of infinitesimal transformations should be closed, that is the commutator of two subsequent transformations should be a transformation of the same type. To check the algebra of transformations let us calculate the result for the commutator of two subsequent infinitesimal transformations (2.14) with parameters ξ_1, λ_1 and ξ_2, λ_2 respectively. For an arbitrary algebraic function $F(q,p)$ of the canonical variables (for example $F=q$ or $F=p$) a rather lengthy but straightforward calculation yields the commutator

$$\begin{aligned} [\hat{I}_{\xi_2\lambda_2}, \hat{I}_{\xi_1\lambda_1}]\, F^x(q,p) &= \Big(\frac{\delta F^x}{\delta q_i^z} EM(q_i^z) + (q \to p)\Big)(\dot{\xi}_1^z \xi_2^z - \xi_1^z \dot{\xi}_2^z) \\ &+ \Big((\xi_2^x - \xi_2^y)\lambda_1^{\tilde{\gamma}} - \mathcal{N}^{\tilde{\gamma}} \xi_2^x \xi_1^y - (1 \leftrightarrow 2)\Big)\Big(\{F^x, \frac{\delta C_{\tilde{\gamma}}}{\delta q_j^y}\} EM(q_j^y) + (q \to p)\Big) \end{aligned}$$

$$-(\xi_2^x\xi_1^y-\xi_1^x\xi_2^y)\Big(\{F^x,\frac{\delta\mathcal{H}}{\delta q_j^y}\}EM(q_j^y)+(q\to p)\Big)+\{F^x,\bar\lambda^{\tilde\gamma}C_{\tilde\gamma}\} \quad (2.23)$$

and correspondingly for the Lagrangean multipliers one has

$$\begin{aligned}[\hat I_{\xi_2\lambda_2},\hat I_{\xi_1\lambda_1}]\mathcal{N}^{\tilde\alpha} &= (\hat I_{\bar\lambda}-1)\mathcal{N}^{\tilde\alpha}+\lambda_2^{\tilde\delta}\lambda_1^{\tilde\gamma}\Big(\dot t^{\tilde\alpha}_{\tilde\gamma\tilde\delta}-\{t^{\tilde\alpha}_{\tilde\gamma\tilde\delta},\mathcal{N}^{\tilde\sigma}C_{\tilde\sigma}+\mathcal{H}\}\Big)\\ &-(\lambda_2^{\tilde\gamma}\xi_1^x-\lambda_1^{\tilde\gamma}\xi_2^x)\Big(\frac{\delta}{\delta q_i^x}(\mathcal{N}^{\tilde\beta}t^{\tilde\alpha}_{\tilde\beta\tilde\gamma}+t^{\tilde\alpha}_{\tilde\gamma})EM(q_i^x)+(q\to p)\Big),\end{aligned} \quad (2.24)$$

where we have introduced

$$\bar\lambda^{\tilde\alpha}=\lambda_1^{\tilde\sigma}\lambda_2^{\tilde\beta}t^{\tilde\alpha}_{\tilde\beta\tilde\sigma}+\frac{\delta\lambda_2^{\tilde\alpha}}{\delta\mathcal{N}^{\tilde\beta}}\delta_{\lambda_1}\mathcal{N}^{\tilde\beta}-\frac{\delta\lambda_1^{\tilde\alpha}}{\delta\mathcal{N}^{\tilde\beta}}\delta_{\lambda_2}\mathcal{N}^{\tilde\beta}. \quad (2.25)$$

In deriving (2.23) and (2.24) we used the identities

$$(\lambda_1^{\tilde\gamma}\lambda_2^{\tilde\delta}-\lambda_2^{\tilde\gamma}\lambda_1^{\tilde\delta})(\{t^{\tilde\alpha}_{\tilde\sigma\tilde\gamma},C_{\tilde\delta}\}+t^{\tilde\beta}_{\tilde\sigma\tilde\gamma}t^{\tilde\alpha}_{\tilde\beta\tilde\delta})=\lambda_1^{\tilde\gamma}\lambda_2^{\tilde\delta}(t^{\tilde\beta}_{\tilde\gamma\tilde\delta}t^{\tilde\alpha}_{\tilde\sigma\tilde\beta}-\{t^{\tilde\alpha}_{\tilde\gamma\tilde\delta},C_{\tilde\sigma}\}), \quad (2.26)$$

and

$$(\lambda_1^{\tilde\gamma}\lambda_2^{\tilde\delta}-\lambda_2^{\tilde\gamma}\lambda_1^{\tilde\delta})(\{t^{\tilde\alpha}_{\tilde\gamma},C_{\tilde\delta}\}+t^{\tilde\beta}_{\tilde\gamma}t^{\tilde\alpha}_{\tilde\beta\tilde\delta})=\lambda_1^{\tilde\gamma}\lambda_2^{\tilde\delta}(t^{\tilde\beta}_{\tilde\gamma\tilde\delta}t^{\tilde\alpha}_{\tilde\beta}-\{t^{\tilde\alpha}_{\tilde\gamma\tilde\delta},\mathcal{H}\}), \quad (2.27)$$

which follow from the Jacobi identities for $\{\{C_{\tilde\sigma},\lambda_1^{\tilde\gamma}C_{\tilde\gamma}\},\lambda_2^{\tilde\sigma}C_{\tilde\sigma}\}$ and for $\{\{\mathcal{H},\lambda_1^{\tilde\gamma}C_{\tilde\gamma}\},\lambda_2^{\tilde\sigma}C_{\tilde\sigma}\}$.[5] Also we took into account that if the variables q,p are transformed to new variables $\tilde q=q+\triangle q$ and $\tilde p=p+\triangle p$, then the Poisson bracket of some quantities $A(\tilde q,\tilde p)$ and $B(\tilde q,\tilde p)$ with respect to $\tilde q,\tilde p$ are related with the Poisson brackets of $A(q,p)$ and $B(q,p)$ with respect to the old variables in first order in $\triangle q$, $\triangle p$ in the following manner

$$\begin{aligned}\{A(\tilde q,\tilde p),B(\tilde q,\tilde p)\}_{\tilde q,\tilde p} &= \{A(q,p),B(q,p)\}_{q,p}\\ +\frac{\delta}{\delta q_{\tilde i}}(\{A,B\})\triangle q_{\tilde i} &+ (q\to p)+O(\triangle q^2,\triangle p^2).\end{aligned} \quad (2.28)$$

We would like to stress that when we are performing the second transformation in (2.23,2.24) which follows the first one, then we must use the transformed variables. In particular, instead of $\lambda_2(\mathcal{N},x,t)$ we must take $\lambda_2(\hat I_{\lambda_1}\mathcal{N},x,t)$. This explains the appearence of the last terms in (2.25).

[5]With the exception of section 5 we consider for simplicity only the bosonic case from now on.

First let us consider the case, when the transformations are generated only by the constraints without extra compensating "trivial" transformations. In the particular case where the structure coefficents $t^{\tilde{\alpha}}_{\tilde{\beta}\tilde{\gamma}}$ do not depend on the canonical variables q, p the parameter $\bar{\lambda}$ also does not depend on them as can be seen from (2.25). Also, $\dot{t}^{\tilde{\alpha}}_{\tilde{\beta}\tilde{\gamma}} = 0$ in this case and thus the commutator of two transformations generated by the constraints only ($\xi = 0$) yields again a transformation generated by the constraint. Hence, *if the structure coefficients do not depend on the canonical variables the transformations generated by the constraints form a closed algebra off mass-shell.* On the other hand, if the structure coefficients do depend on the canonical variables that does not automatically imply that the algebra of transformations will not close even in the absence of trivial transformations. Actually, the q, p-dependence in the formula (2.25) for $\bar{\lambda}$ can, in principle, be cancelled. The price we pay for that is that the λ-parameters may become $\mathcal{N}$-dependent. As we shall see in section 7 this happens for gravity where some of the structure coefficients depend on q, if we consider transformations of functions which depend on the canonical variables (2.23). The algebra of transformations (2.14) can also be closed in all relevant cases when $\xi \neq 0$ if the $\lambda^{\tilde{\beta}}$ and ξ are related in a certain way. The corresponding transformations can be interpreted as Lagrangean symmetries when some of the constraints are nonlinear in the momenta. We shall discuss the cases which are of particular interest for us later on.

An interesting question to which we have no general answer is the following: what are the *sufficient* conditions to exponentiate the infinitesimal transformations (2.14) to finite ones. In the theories we shall consider we know the finite Lagrangean symmetries which can be formulated in the Hamiltonian formalism and this way one can find the finite transformation in the first order formalism. But in general it seems unlikely that the closing of the algebra of infinitesimal transformations is sufficient to exponentiate them since already for a free *nonrelativisic particle*, which very probably does not admit any finite local symmetry, the transformations (2.12) form a closed algebra. This difficult and very important question (i.e. for the functional integral) what are the conditions such that the transformations (2.14) can be made finite needs further investigation.

Constraints and the equations of motion. There is a very interesting and non-trivial connection between the equations of motion and the constraints. As it is wellknown, if we demand that the constraints are fulfilled on some initial hypersurface $t = t_0$, then due to the equations of motion they will be satisfied at later times. Actually, we have

$$\begin{aligned}\dot{C}_{\tilde{\alpha}} &= \frac{\delta C_{\tilde{\alpha}}}{\delta q_{\tilde{i}}}\dot{q}_{\tilde{i}} + \frac{\delta C_{\tilde{\alpha}}}{\delta p^{\tilde{i}}}\dot{p}^{\tilde{i}} \\ &= \mathcal{N}^{\tilde{\beta}} t^{\tilde{\gamma}}_{\tilde{\alpha}\tilde{\beta}} C_{\tilde{\gamma}} + t^{\tilde{\beta}}_{\tilde{\alpha}} C_{\tilde{\beta}} + \frac{\delta C_{\tilde{\alpha}}}{\delta q_{\tilde{i}}} EM(q_{\tilde{i}}) + \frac{\delta C_{\tilde{\alpha}}}{\delta p^{\tilde{i}}} EM(p^{\tilde{i}}),\end{aligned} \tag{2.29}$$

from which immediately follows that $\dot{C} \sim C$ if the equations of motion are satisfied. Thus we need to impose the constraints only on the initial hypersurface and then they will hold at any moment of time owing to the equations of motion.

Inversely, in some theories (e.g. gravity) we can get all of the equations of motions (or some of them as in string theory) if we only demand that the constraints are fulfilled for all t (i.e. everywhere) and that the symmetry transformations do not destroy this property. For example, in gravity and string theory this means that we demand that the constraints are valid everywhere and for any choice of spacelike hypersurfaces, because the symmetry transformations (diffeomorphism transformations) can be interpreted as a change of foliation of space-time. In general relativity this statement is known as *interconnection theorem* [11]. Usually, to prove this theorem it is assumed that the first four Einstein equation corresponding to the constraints are valid in any coordinate system (for any foliation) and then one immediately concludes that this can be true only if the remaining six Einstein equations are satisfied. Moreover, if one considers finite transformations then it suffices to demand that only the first Einstein equation must be fulfilled to conclude that the remaining equations must hold [11].

Note however, that to get the Einstein equations one needs to impose half of the Hamiltonian equations to express the momenta in term of the velocities. In the Hamiltonian formulation these equations are on the same footing as the other ones and thus the above arguments can hardly be seen as satisfactory in a Hamiltonian approach since we cannot state that the *whole* dynamics is encoded in the constraints. One does not deduce the Hamiltonian equations of motion only from the constraints and symmetries. Thus our purpose will be to fill this gap and derive the equations of motion using the constraints and the symmetry properties entirely in the Hamiltonian formalism, without postulating any of the Hamiltonian equations of motion.

For that let us consider how the constraints are changed under the symmetry transformations (2.14):

$$\begin{aligned} \delta_{\xi,\lambda} C_{\tilde\alpha} &= \frac{\delta C_{\tilde\alpha}}{\delta q^x_i}\delta_{\xi,\lambda} q^x_i + \frac{\delta C_{\tilde\alpha}}{\delta p^{ix}}\delta_{\xi,\lambda} p^{ix} \\ &= \frac{\delta C_{\tilde\alpha}}{\delta q^x_i} EM(q^x_i)\xi^x + \frac{\delta C_{\tilde\alpha}}{\delta p^{ix}} EM(p^{ix})\xi^x + \lambda^{\tilde\gamma} t^{\tilde\beta}_{\tilde\alpha\tilde\gamma} C_{\tilde\beta}. \end{aligned} \tag{2.30}$$

For the known theories the constraints are local functions of q and p and involve only spatial derivatives of q up to second and p up to first order. It follows then that the functional derivatives of the constraints have the form

$$\begin{aligned} \frac{\delta C_{\alpha y}}{\delta q^x_i} &= A^i_\alpha \delta(x,y) + B^{ia}_\alpha \frac{\partial}{\partial y^a}\delta(x,y) + D^{iab}_\alpha \frac{\partial^2}{\partial y^a \partial y^b}\delta(x,y) \\ \frac{\delta C_{\alpha y}}{\delta p^{ix}} &= E_{\alpha i}\delta(x,y) + F^a_{\alpha i}\frac{\partial}{\partial y^a}\delta(x,y), \end{aligned} \tag{2.31}$$

where $A, B, \ldots$ are functions of q^y and p^y. Substituting (2.30) into (2.31) a straightforward calculation yields

$$\begin{aligned} \delta_{\xi,\lambda} C_{\alpha y} &= (\dot C_{\alpha y} + \mathcal{N}^{\tilde\beta} t^{\tilde\gamma}_{\tilde\beta,\alpha y} C_{\tilde\gamma} + t^{\tilde\gamma}_{\alpha y} C_{\tilde\gamma})\xi_y \\ &+ \lambda^{\tilde\gamma} t^{\tilde\beta}_{\alpha y,\tilde\gamma} C_{\tilde\beta} + \Big(B^{ia}_\alpha EM(q^y_i) + F^a_{\alpha i} EM(p^{iy})\Big)\frac{\partial \xi^y}{\partial y^a} \\ &+ D^{iab}_\alpha \Big(2\frac{\partial EM(q^y_i)}{\partial y^a}\frac{\partial \xi^y}{\partial y^b} + EM(q^y_i)\frac{\partial^2 \xi^y}{\partial y^a \partial y^b}\Big). \end{aligned} \tag{2.32}$$

Here we used the explicit form for some of the indices $\tilde\alpha = \alpha, y$, $\tilde i = i, x$ etc; a, b run over the spatial indices and it is understood that there is no integration over y.

Now we can reformulate our question in the following manner: when can the equations of motion (or some of them) be the consequence of the equations

$$C_{\tilde\alpha} = 0 \quad \text{and} \quad \delta_{\xi,\lambda} C_{\tilde\alpha} = 0. \tag{2.33}$$

The first condition just means that the constraints are fulfilled everywhere and the second one that this statement does not depend on the chosen foliation.

From (2.30) we can immediately conclude that the equations of motion can be derived from (2.33) only if the following necessary conditions are satisfied:

- Some of the constraints should be nonlinear in the momenta, since, as we showed earlier, only in this case should we use the extra "trivial" transformations (and consequently $\xi \neq 0$).

- The system should have an infinite number of degrees of freedom. Otherwise there are no spatial derivatives of ξ and the pieces which are proportional to the equations of motion are absent.

- The constraints should involve spatial derivatives of the p and/or the q. Else all coefficients B, F, D in (2.31) vanish and the pieces proportional to the equations of motion are again absent.

If we demand that (2.33) holds for an arbitrary ξ, then from (2.30,2.31) we immediately get the following set of equations

$$\begin{aligned} D^{iab}_{\alpha} EM(q^y_i) &= 0 \\ B^{ia}_{\alpha} EM(q^y_i) + 2D^{iba}_{\alpha} \frac{\partial EM(q^y_i)}{\partial y^b} &= 0 \\ F^a_{\alpha i} EM(p^{iy}) &= 0 \end{aligned} \tag{2.34}$$

which can be solved to obtain the equations of motion. The equations of motion which we can get from (2.34) depends on the properties of the matrices D, B, F.

Now we will briefly review how the general results apply to particular systems:

Systems with a finite number of degrees of freedom: In this case no equations of motion follow from (2.33) even if $\xi \neq 0$ since there are no spatial derivatives of ξ.

Gauge theories All of the constraints are linear in the momenta and therefore the "trivial" transformations (2.11) are absent. Consequently, none of the equations of motion can be obtained from (2.33).

Bosonic string: One constraint is nonlinear in the momenta and hence $\xi \neq 0$. The matrices F, D are identically zero and $B \neq 0$. Then only some relations between the $EM(q)$ follow from (2.33) (see section 6).

Gravity: This is the most interesting case. One constraint is nonlinear and leads to $\xi \neq 0$ for the diffeomorphism transformations. The matrices F and D are non-singular. As is clear from (2.34) all Hamiltonian equations follow then from (2.33), that is the whole dynamics of general relativity in the Hamiltonian formulation is hidden in the requirement that the constraints are satisfied everywhere and for any foliation. Let us stress that in distinction to [11] we did not assume $EM(q) = 0$. These equations are also consequences of eqs. (2.33) and thus the interconnection theorem has been proven entirely within the Hamiltonian formalism (see section 7).

In the following we apply the general results of this section to concrete systems. First in sect. 3 to gauge theories which are trivial as regarding their symmetries, since all constraints are linear in the momenta and thus the gauge transformations are generated by the constraints themselves. Then we consider the relativistic particle where the constraint is quadratic in the momenta. We demonstrate the role played by the "trivial" transformations to recover the reparametization invariance. In sect. 5 we show how to proceed if several constraints are nonlinear in the momenta at the example of the locally supersymmetric relativistic particle [17]. The sections 6 and 7 are devoted to string theory and gravity. The different sections are selfcontained and the reader may skip those parts which are not of immediate interest for him/her.

Chapter 3

Yang Mills-theories

The action for the (non-abelian) gauge fields is

$$S = -\frac{1}{4}\int \mathrm{tr}\,[F_{\mu\nu}F^{\mu\nu}]d^3x dt \tag{3.1}$$

where [1]

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \\ A_\mu &= A_\mu^a T_a, \qquad [T_a, T_c] = i f_{ab}^c T_c, \end{aligned} \tag{3.2}$$

and it is invariant under local gauge transformations

$$A_\mu \longrightarrow e^{-i\theta}A_\mu e^{i\theta} + ie^{-i\theta}\partial_\mu e^{i\theta} \tag{3.3}$$

with $\theta = \theta^a T_a$. The functions $\theta^a = \theta^a(x,t)$ are arbitrary functions on space-time. The infinitesimal form of these gauge transformations is

$$\delta_\theta A_\mu^a = -\partial_\mu \theta^a - f_{bc}^a A_\mu^b \theta^c = -(D_\mu \theta)^a. \tag{3.4}$$

Now we will show that these infinitesimal gauge transformations are just the transformations generated by the constraints (see (2.8)).

In the usual way one can now transform the Lagrangean system into the corresponding Hamiltonian system and obtains the following first order action for Yang-Mills theories [16]

$$S = \int \Big[\vec{\pi}_{\tilde{a}} \cdot \dot{\vec{A}}^{\tilde{a}} - A^0_{\tilde{a}}(\vec{D}\cdot\vec{\pi})^{\tilde{a}} - \frac{1}{2}(\vec{\pi}_{\tilde{a}}\cdot\vec{\pi}^{\tilde{a}} + \vec{B}_{\tilde{a}}\cdot\vec{B}^{\tilde{a}})\Big]dt, \tag{3.5}$$

[1] $a, b, \ldots$ denote internal indices, $\mu, \nu \ldots$ space-time indices. The T_a are hermitean generators and the structure constants f_{ab}^c are totally antisymmetric.

where $\tilde{a} = (a, x)$, $\pi_i^{\tilde{a}}$ are the momenta conjugate to $A_i^{\tilde{a}}$, and

$$\begin{aligned} (\vec{D} \cdot \vec{\pi})^a &= \vec{\partial} \cdot \vec{\pi}^a + f^a_{bc} \vec{A}^b \cdot \vec{\pi}^c \\ \vec{B}^a &= -\vec{\partial} \times \vec{A}^a - \tfrac{1}{2} f^a_{bc} \vec{A}^b \vec{A}^c. \end{aligned} \tag{3.6}$$

Here we collected the spatial components into 3-vectors $\vec{A} = (A_1, A_2, A_3)$ (similarly for $\vec{\pi}$, $\vec{B}$) and assume the gauge group to be compact, so that $\vec{A}^a = \vec{A}_a$ etc.

The system (3.5) is a first class Hamiltonian system (2.1) for which the components $A^0_{\tilde{a}}$ play the role of Lagrangean multipliers, the constraints are just

$$C_{\tilde{a}} = (\vec{D} \cdot \vec{\pi})_{\tilde{a}}, \tag{3.7}$$

and the Hamiltonian

$$\mathcal{H} = \frac{1}{2} (\vec{\pi}^{\tilde{a}} \vec{\pi}_{\tilde{a}} + \vec{B}^{\tilde{a}} \vec{B}_{\tilde{a}}). \tag{3.8}$$

The constraints and Hamiltonian form a closed algebra with respect to the Poisson bracket

$$\{C_{ax}, C_{by}\} = f^c_{ab} \delta(x, y) C_{cx}, \qquad \{\mathcal{H}, C_{ax}\} = 0. \tag{3.9}$$

From that it follows that the structure coefficients are equal to

$$t^{cz}_{ax,by} = f^c_{ab}\, \delta(x - y) \delta(z - x), \qquad t^{ax}_{by} = 0. \tag{3.10}$$

Substituting (3.7) and (3.10) in formulae (2.8) we obtain the following symmetry transformations for the system (3.5)

$$\delta \vec{A}^{\tilde{a}} = \{\vec{A}^{\tilde{a}}, \lambda^{\tilde{b}} C_{\tilde{b}}\} = -(\vec{D}\lambda)^{\tilde{a}} \qquad \delta A^0_{\tilde{a}} = \delta \mathcal{N}^{\tilde{a}} = \dot{\lambda}^{\tilde{a}} - t^{\tilde{a}}_{\tilde{b}\tilde{c}} A^{0\tilde{b}} \lambda^{\tilde{c}} \tag{3.11}$$

and

$$\delta \vec{\pi}_{\tilde{a}} = \{\vec{\pi}_{\tilde{a}}, \lambda^{\tilde{b}} C_{\tilde{b}}\} = -f^a_{bc} \vec{\pi}^{bx} \lambda^{cx}. \tag{3.12}$$

These transformations correspond to symmetries of the corresponding Lagrangean system since the constraints (3.7) are linear in the momenta. The transformations (3.11) coincide with (3.4) if we identify $\lambda = \theta$. Hence it is clear that the whole group of gauge transformations (including time dependent ones) is generated by the constraints. It is easy to verify that the transformations for the momenta (3.12) follow from the first equation in (3.11) if we use the relation between velocities $\dot{\vec{A}}_{\tilde{a}}$ und momenta $\vec{\pi}_{\tilde{a}}$ (the first Hamiltonian equation) which defines the supspace $\mathcal{M}$ where the Lagrangean system lives. To compare the symmetries in the Lagrangean and Hamiltonian formulations we need to use these equations. However, the Lagrangean system lives in the subspace $\mathcal{M}$ (see (2.7)) while the transformations (3.11,3.12) can be viewed as symmetries in the whole phase space and hence the group of symmetries is richer in the Hamiltonian formalism since it acts also on trajectories which do not belong to $\mathcal{M}$.

The transformations (3.11,3.12) can be made finite in phase space off the hypersurface $\mathcal{M}$. Actually the action (3.5) is invariant under the transformation (3.3) if simultaneously the momenta are transformed as

$$\pi \longrightarrow e^{-i\theta}\pi e^{i\theta}. \tag{3.13}$$

To prove this we do not need to use any of the Hamiltonian equations. So this symmetry holds for all trajectories in phase space. This is why the global symmetry of Hamiltonian systems is richer as the usual gauge symmetry of the corresponding Lagrangean systems.

Chapter 4

Relativistic particle

It is convenient to describe the relativistic particle moving in 4-dimensional Minkowski spacetime by 4 scalar fields $\phi^\mu(t)$, $\mu = 0, 1, 2, 3$, in 1-dimensional 'spacetime' with coordinate t. The action has the form

$$S = -\frac{1}{2}\int \sqrt{-g}[g^{00}\dot{\phi}^\mu\dot{\phi}_\mu + m^2]dt \tag{4.1}$$

where the dot denotes differentiation with respect to time t and $\phi^\mu\phi_\mu = -(\phi^0)^2 + \sum_1^3(\phi^i)^2$. The m^2 term maybe viewed as 'cosmological constant' in 1-dimensional 'spacetime' with metric g_{00}.

The action (4.1) is manifestly invariant under general coordinate transformations in 1-dimensional 'spacetime' (reparametrization invariance). The infinitesimal form of these transformations reads

$$t \to t - \xi, \qquad g_{00} \to g_{00} + \mathcal{L}_\xi g_{00}, \qquad \phi^\mu \to \phi^\mu + \mathcal{L}_\xi\phi^\mu, \tag{4.2}$$

where $\mathcal{L}_\xi$ is the Lie-derivative in 1-dimensional 'spacetime'. Introducing the lapse funcion $\mathcal{N}$ according to

$$g_{00} = -\mathcal{N}^2 \tag{4.3}$$

and defining the momenta conjugated to ϕ_μ

$$\pi_\mu = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_\mu} = \frac{\dot{\phi}^\mu}{\mathcal{N}} \tag{4.4}$$

as a result of the Legendre transformation one finds the following first order action for the relativistic particle

$$S = \int [\pi^\mu\dot{\phi}_\mu - \mathcal{N}C]dt. \tag{4.5}$$

The lapse function $\mathcal{N}$ plays the role of a Lagrangean multiplier in (4.5) and the constraint is *quadratic in the momenta*

$$C = \frac{1}{2}(\pi^\mu \pi_\mu + m^2). \tag{4.6}$$

Of course, the structure coefficient vanishes.

The action (4.5) still should be invariant (at least in $\mathcal{M}$) under the infinitesimal diffeomorphisms (4.2), the explicit form of which is

$$\delta\phi^\mu = \dot{\phi}^\mu \xi \quad \text{and} \quad \delta\mathcal{N} = (\mathcal{N}\xi)^{\cdot} \ . \tag{4.7}$$

Since π and ϕ are independent variables in the first order formalism, we should add to (4.7) the transformation law for π to have the diffeomorphisms on the whole phase space. This transformation law for π, *which corresponds to the diffeomorphism group*, can be obtained at first in $\mathcal{M}$ (see (2.7)), where the Lagrangean system lives, from (4.4) as

$$\delta\pi^\mu = \frac{\delta\dot{\phi}^\mu}{\mathcal{N}} - \frac{\dot{\phi}^\mu}{\mathcal{N}^2}\delta\mathcal{N} = \dot{\pi}^\mu \xi \tag{4.8}$$

and then can be extended to the whole phase space and hence to trajectories for which (4.4) does not hold. Clearly the transformations (4.7,4.8) correspond to the reparametrisation (diffeomorphism) invariance of the relativistic particle in the Hamiltonian formalism. They coincide with (2.22)[1], which is a special combination of the 'trivial' and constraint-generated transformations.

The algebra of transformations (4.7,4.8) closes and forms a Lie algebra on the whole phase space. Their finite form reads

$$\phi^\mu(t) \to \phi^\mu(\tau(t)), \quad \pi_\mu(t) \to \pi_\mu(\tau(t)), \quad \mathcal{N}(t) \to \frac{d\tau}{dt}\mathcal{N}(\tau(t)). \tag{4.9}$$

It is easy to see (without using Hamilton's equations) that the action (4.5) is invariant under these finite tranformations completely off mass-shell.

The first order action (4.5) is also (off mass-shell) invariant under the transformations (2.8) generated by the constraints alone

$$\delta_\lambda\phi^\mu = \lambda\pi^\mu, \quad \delta_\lambda\pi_\mu = 0, \quad \delta\mathcal{N} = \dot{\lambda}. \tag{4.10}$$

[1]For systems with a finite number of degrees of freedom $\xi^y - \xi^x$ vanishes in (2.22)

It is clear that they are very different from the reparametrisation transformations (4.7,4.8) even in the subspace $\mathcal{M}$ and hence cannot correspond to any Lagrangean symmetry. Only *on mass-shell*,

$$\dot{\phi}^\mu = \mathcal{N}\pi^\mu \quad , \quad \dot{\pi}^\mu = 0 \tag{4.11}$$

do the transformations (4.10) coincide with the reparametrization transformations if we make the identification $\lambda = \mathcal{N}\xi$. However, as we argued earlier the comparison of infinitesimal transformations on mass-shell is meaningless.

If we demand that as a result of the transformation (4.10) the trajectory should stay in $\mathcal{M}$ then we immediately see that this can be true only for on-shell trajectories. Therefore (4.10) can be viewed as the dynamical equations in the subspace $\mathcal{M}$, where the Lagrangean system lives. Thus we conclude that the nonlinear constraint (4.6) generates the dynamics, rather than symmetries in $\mathcal{M}$. This explains the origin of the dynamics for superhamiltonian systems.

However, in the whole phase space, the infinitesimal transformations (4.10) can still be viewed as *off mass-shell* symmetries of the Hamiltonian system. Moreover they can be exponentiated to the finite ones

$$\phi^\mu(t) \to \phi^\mu(t) + \lambda(t)\pi^\mu(t), \quad \pi_\mu(t) \to \pi_\mu(t), \quad \mathcal{N}(t) \to \mathcal{N}(t) + \dot{\lambda}. \tag{4.12}$$

which are very different from (4.9). As we stressed already, the symmetry (4.10,4.12) do not correspond to diffeomorphisms of the Lagrangean system and it is not clear to us what is the relevance of this symmetry which exists only in the Hamiltonian version of the theory.

Chapter 5

The locally supersymmetric relativistic particle

The theory of the relativistic particle can be super-symmetrized and this leads to the simplest one-dimensional analog of supergravity, namely the theory for the locally supersymmetric relativisic particle [17]. For that we need to introduce in addition to the bosonic variables ϕ^μ fermionic variables ψ^μ which live in 1-dimensional 'spacetime' and they would describe spin-1/2 particles in 4-dimensional spacetime. To make the theory locally supersymmetric we also need the analog of the the spin-3/2 gravitino field in supergravity and which we denote by χ. Then the action for massless particles reads

$$S = -\frac{1}{2}\int dt \det(e_0^{\hat{0}})[g^{00}\dot{\phi}^2 - i\bar{\psi}\gamma^0\dot{\psi} - g^{00}\bar{\chi}_0\psi\dot{\phi}]. \tag{5.1}$$

To simplify the formulae we skipped all external indices. Here $e_0^{\hat{0}}$ is the einbein field in 1-dimensional 'spacetime' on which the bosonic fields $\phi, e_0^{\hat{0}}$ and fermionic ones ψ, χ_0 live. We denote by $\hat{0}$ the 'Lorentzian' index and by 0 the 'spacetime' index. The fermionic fields are assumed to be real Majorana fields and the 'spin 3/2' field χ is taken in the Rarita-Schwinger representation where it is considered as covariant vector of Majorana spinors. Of course in one-dimensional 'spacetime' this covariant vector has only one component.

Introducing the lapse function as in (4.3) and taking into account that

$$e_0^{\hat{0}} = \mathcal{N}, \quad e_{\hat{0}}^0 = \frac{1}{\mathcal{N}}, \quad \gamma^0 = e_{\hat{0}}^0\gamma^{\hat{0}} = \frac{i}{\mathcal{N}} \tag{5.2}$$

and for Majorana spinors

$$\bar{\psi} = \psi^\dagger \gamma^{\hat{0}} = i\psi \quad , \quad \bar{\chi} = \chi^\dagger \gamma^{\hat{0}} = i\chi \tag{5.3}$$

the action (5.1) becomes

$$S = \frac{1}{2} \int dt [\frac{1}{\mathcal{N}} \dot{\phi}^2 - i\psi\dot{\psi} - \frac{i}{\mathcal{N}} \chi\psi\dot{\phi}]. \tag{5.4}$$

This action is manifestly invariant under (infinitesimal) diffeomorphism transformations

$$t \to t - \xi^0 \quad \text{and} \quad Q \to Q + \mathcal{L}_\xi Q \tag{5.5}$$

which now have the explicit form

$$\delta\phi = \dot{\phi}\xi^0, \quad \delta\psi = \dot{\psi}\xi^0, \quad \delta\mathcal{N} = (\mathcal{N}\xi^0)^{\cdot}, \quad \delta\chi = (\chi\xi^0)^{\cdot}, \tag{5.6}$$

since ϕ, ψ are spacetime scalars and χ is a covariant vector. In addition it is also invariant under (infinitesimal) local supersymmetry transformations

$$\delta\phi = i\theta\psi, \quad \delta\psi = \theta(\dot{\phi} - \frac{i}{2}\chi\psi)\mathcal{N}^{-1}, \quad \delta\mathcal{N} = i\theta\chi, \quad \delta\chi = 2\dot{\theta}, \tag{5.7}$$

where θ is the time-dependent Grassmannian parameter of the supersymmetry transformations. Clearly the action (5.4) is invariant under simultaneous *infinitesimal* diffeomorphisms (5.6) and supersymmetry transformations (5.7). Our aim is to recover the corresponding off mass-shell symmetries (diffeomorphisms and local supersymmetry) in the first order Hamiltonian formalism.

The standard procedure leads to the following first order action for the locally supersymmetric particle

$$S = \int [\pi_\phi \dot{\phi} - \frac{1}{2} i\psi\dot{\psi} - \mathcal{N}^\alpha C_\alpha] dt \tag{5.8}$$

with Lagrangean multiplier fields

$$\mathcal{N}^0 = \mathcal{N} \quad \text{and} \quad \mathcal{N}^1 = \frac{1}{2}\chi. \tag{5.9}$$

Thus $\mathcal{N}^0$ is the bosonic lapse function and $\mathcal{N}^1$ proportional to the fermionic 'gravitino' field. The constraints

$$C_0 = \frac{1}{2}\pi_\phi^2 \qquad \text{and} \qquad C_1 = i\pi_\phi\psi \tag{5.10}$$

form a closed algebra with respect to the Poisson bracket, which are generalized to graded algebras to include fermionic variables as follows:

$$\{\phi, \pi_\phi\} = 1 \quad , \quad \{\psi, \psi\} = i. \tag{5.11}$$

Actually we have

$$\{C_0, C_0\} = \{C_0, C_1\} = 0 \quad \text{and} \quad \{C_1, C_1\} = -2iC_0. \tag{5.12}$$

As it follows from (5.11) the only nonvanishing structure coefficient is

$$t_{11}^0 = -2i. \tag{5.13}$$

The infinitesimal transformations (2.8) generated by the constraints (5.10) read

$$\begin{array}{c} \delta_\lambda \phi = \lambda^0 \pi_\phi + i\lambda^1 \psi, \quad \delta_\lambda \pi_\phi \;\; = \;\; 0, \quad \delta_\lambda \psi = \lambda^1 \pi_\phi \\ \delta_\lambda \mathcal{N} = \dot{\lambda}^0 + i\lambda^1 \chi \qquad\qquad\qquad \delta_\lambda \chi = 2\dot{\lambda}^1. \end{array} \tag{5.14}$$

where λ^0 and λ^1 are bosonic and Grassmannian variables, respectively. Actually they are nilpotent, i.e. $\{\{.,C\}C\} = 0$, and thus can easily be exponentiated to finite ones. One obtains the finite transformations $F(t) \to F(t) + \delta_{\lambda(t)} F(t)$, where $F(t)$ denotes any of the fields or Lagrangean multipliers appearing in (5.14). However, the transformations (5.14) are not really the symmetries of the Lagrangean system we are looking for.

To see that more clearly we first write the equations of motion which are gotten by varying the action (5.11) with respect to the dynamical variables ϕ, π_ϕ and ψ

$$\begin{aligned} EM(\phi) &= \dot{\phi} - \mathcal{N}\pi_\phi - \frac{i}{2}\chi\psi = 0, \\ EM(\pi_\phi) &= \dot{\pi}_\phi = 0, \\ EM(\psi) &= \dot{\psi} - \frac{1}{2}\pi_\phi \chi = 0. \end{aligned} \tag{5.15}$$

The subspace $\mathcal{M}$ in which the Lagrangean system lives is defined by the eq. $EM(\phi) = 0$. In this subspace we can read that the momentum π_ϕ under the transformations (5.6) and (5.7) should be tansformed as

$$\delta\pi_\phi = \dot{\pi}_\phi \xi^0 + \frac{i}{\mathcal{N}}\theta(\dot{\psi} - \frac{1}{2}\pi_\phi\chi). \tag{5.16}$$

Comparing (5.6,5.7) and (5.16) with the transformations (5.14) we immediately conclude that they coincide only if *all* equations of motion are satisfied, that is on mass-shell.

This agrees with our general considerations in sec.2 since for the supersymmetric particle both constraints in the action (5.8) are quadratic in the momenta π_ϕ and $\pi_\psi = -\frac{i}{2}\psi$ and hence the off-shell symmetries which correspond to the symmetries of the Lagrangean system cannot be generated by the constraints alone. Both of them take a off-shell trajectory which belongs to $\mathcal{M}$ away from this subspace. To return the trajectory back to $\mathcal{M}$ we need compensating transformations from the set of trivial transformations (2.11), one per nonlinear constraint. In particular the trivial transformations (2.12) in combination with the transformations generated by the constraint C_0 lead to the familiar reparametrization invariance (2.22).

Because the constraint C_1 is also quadratic in the momenta we take an extra compensating transformation from the set (2.11) and add it to (2.14) to obtain all Lagrangean symmetries in the Hamiltonian formalism. The resulting transformations read

$$\begin{aligned}
\delta\phi &= EM(\phi)\xi + \{\phi, \lambda^\alpha C_\alpha\} = (\dot{\phi} - \mathcal{N}\pi_\phi - \frac{i}{2}\chi\psi)\xi + \delta_\lambda\pi \\
\delta\pi_\phi &= EM(\pi_\phi)\xi + EM(\pi_\psi)\zeta + \{\pi_\phi, \lambda^\alpha C_\alpha\} = \dot{\pi}_\phi - i(\dot{\psi} - \frac{1}{2}\pi_\phi\chi)\zeta, \\
\delta\psi &= EM(\psi)\xi + EM(\phi)\zeta + \{\psi, \lambda^\alpha C_\alpha\} \\
&= (\dot{\psi} - \frac{1}{2}\pi_\phi\chi)\xi + (\dot{\phi} - \mathcal{N}\pi_\phi - \frac{i}{2}\chi\psi)\zeta + \delta_\lambda\psi, \\
\delta\mathcal{N} &= \delta_\lambda\mathcal{N} \\
\delta\chi &= \delta_\lambda\chi,
\end{aligned} \tag{5.17}$$

where δ_λ is given in (5.14) and $\pi_\psi = -\frac{i}{2}\psi$. Here ζ is the Grassmann parameter of the 'extra' transformation from the set of transformations (2.11) which we need to correct the gauge transformation generated by the nonlinear constraint C_1. Of course the parameters are not independent and are related by the requirement that the Hamiltonian symmetry is also a Lagrangean one. The corresponding condition (2.15), properly generalized to include fermionic variables, is satisfied by

$$\xi = \xi^0, \quad \zeta = \frac{\theta}{\mathcal{N}}, \quad \lambda^0 = \mathcal{N}\xi^0 \quad \text{and} \quad \lambda^1 = \theta + \frac{1}{2}\chi\xi^0, \tag{5.18}$$

expressing the 4 parameters $\xi, \zeta, \lambda^0, \lambda^1$ in terms of 2 independent parameters ξ^0 and θ. With this identification the symmetry transformations (5.17) of the Hamiltonian system are reduced exactly to the original diffeomorphism and supersymmetry transformations (5.6,5.7) and (5.16) for the Lagrangean system *without using any of the Hamiltonian equations.* However, the transformations (5.17) with parameters (5.18) are symmetries even off the subspace $\mathcal{M}$. They also form a closed algebra on the whole phase space [1]. Actually, denoting the transformations (5.17) by $\hat{I}(\xi^0, \theta)$ we find the following commutator of two subsequent transformations

$$[\hat{I}(\xi_2^0, \theta_2), \hat{I}(\xi_1^0, \theta_1)] = \hat{I}(\xi_3^0, \theta_3) - \hat{1}, \tag{5.19}$$

where

$$\begin{aligned} \xi_3^0 &= \dot{\xi}_1^0 \xi_2^0 - \xi_1^0 \dot{\xi}_2^0 + \frac{2i}{\mathcal{N}} \theta_2 \theta_1 \\ \theta_3 &= \dot{\theta}_1 \xi_2^0 - \dot{\theta}_2 \xi_1^0 + \frac{1}{\mathcal{N}} \theta_1 \theta_2 \chi \end{aligned} \tag{5.20}$$

Again this closure holds completely off mass shell. Hence we expect that the transformations (5.17,5.18) can be 'exponentiated' to finite symmetry transformations on the whole phase space and thus extend the original group of Lagrangean symmetries.

The same properties we expect to hold for supergravity theories in more dimensions. But because the computations are quiet involved we have so far refrained from repeating the above calculations for these more realistic theories.

We conclude this section by stressing that in the considered supersymmetric model neither of the constraints generates a symmetry transformation corresponding to a Lagrangean symmetry. They rather generate the dynamics of the Lagrangean system in the Hamiltonian formalism, similar as for the relativistic particle.

[1] It is worth noting that the transformation (5.17) without the extra ζ term form a closed algebra only on mass shell

Chapter 6

The bosonic string

The bosonic string propagating in a D-dimensional flat target space can be viewed as the theory for D massless scalar fields ϕ^μ, $\mu = 0, \ldots, D-1$ on a 2-dimensional world-sheet spacetime with metric $g_{\alpha\beta}$. The action for this theory can be written in a manifestly invariant form with respect to diffeomorphism transformations as [18]

$$S = -\frac{1}{2} \int \sqrt{-g} g^{\alpha\beta} \frac{\partial \phi^\mu}{\partial x^\alpha} \frac{\partial \phi_\mu}{\partial x^\beta} \, d^2 x, \tag{6.1}$$

where $x^\alpha \equiv (t, x)$ are the coordinates in the 2-dimensional spacetime. To simplify the formulae we shall skip the target-space index μ since it always appears in a trivial way and can easily be reinserted.

The diffeomorphism transformations which are off mass-shell symmetries of the action (6.1) are

$$x^\alpha \to x^\alpha - \xi^\alpha, \qquad g_{\alpha\beta} \to g_{\alpha\beta} + \mathcal{L}_\xi g_{\alpha\beta}, \qquad \phi \to \phi + \mathcal{L}_\xi \phi, \tag{6.2}$$

where ξ^α is the infinitesimal parameter. In addition the action is invariant with respect to Weyl transformations

$$g_{\alpha\beta} \to \Omega^2(x) g_{\alpha\beta} \quad \text{and} \quad \phi \to \phi. \tag{6.3}$$

To arrive at the first order formulation it is convenient to use the 1 + 1-decomposition for the world-sheet metric as [19]

$$g_{\alpha\beta} = -(\mathcal{N}^2 - \mathcal{N}^1 \mathcal{N}_1) dt^2 + 2\mathcal{N}_1 dx dt + \gamma_{11} dx^2, \tag{6.4}$$

where $\mathcal{N}$ and $\mathcal{N}_1$ are the lapse and shift functions, respectively. We rise and lower the spacial index '1' using the metric $\gamma_{11} \equiv \gamma$ of the 1-dimensional hypersurface t=constant in 2-dimensional spacetime. Correspondingly we have

$$\gamma^{11} = \frac{1}{\gamma}, \quad \mathcal{N}^1 = \frac{1}{\gamma}\mathcal{N}_1, \quad \sqrt{-g} = \mathcal{N}\sqrt{\gamma}. \tag{6.5}$$

Using (6.2) an easy calculation yields the following explicit transformation laws for

$$\mathcal{N}^0 = \frac{\mathcal{N}}{\sqrt{\gamma}}, \tag{6.6}$$

$\mathcal{N}^1$ and ϕ under diffeomorphism transformations $x^\alpha \to x^\alpha - \xi^\alpha$, $\xi^\alpha = (\xi^0, \xi^1)$:

$$\begin{aligned}
\delta\mathcal{N}^0 &= \delta(\frac{\mathcal{N}}{\sqrt{\gamma}}) = (\mathcal{N}^0\xi^0)^{\cdot} + \mathcal{N}^{1\prime}(\mathcal{N}^0\xi^0) - \mathcal{N}^1(\mathcal{N}^0\xi^0)' \\
&\qquad + \mathcal{N}^{0\prime}(\xi^1 + \mathcal{N}^1\xi^0) - \mathcal{N}^0(\xi^1 + \mathcal{N}^1\xi^0)', \\
\delta\mathcal{N}^1 &= (\xi^1 + \mathcal{N}^1\xi^0)^{\cdot} + \mathcal{N}^{1\prime}(\xi^1 + \mathcal{N}^1\xi^0) - \mathcal{N}^1(\xi^1 + \mathcal{N}^1\xi^0)' \\
&\qquad + \mathcal{N}^{0\prime}(\xi^1 + \mathcal{N}^1\xi^0) - \mathcal{N}^0(\xi^1 + \mathcal{N}^1\xi^0)', \\
\delta\phi &= \dot{\phi}\xi^0 + \phi'\xi^1.
\end{aligned} \tag{6.7}$$

Here dot and prime mean the differentiations with respect to the time and space coordinates $x^0 = t$ and $x^1 = x$, respectively. The transformation law for the momentum π conjugated to ϕ,

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\sqrt{\gamma}}{\mathcal{N}}(\dot{\phi} - \mathcal{N}^1\phi') \tag{6.8}$$

follows immediately from (6.7) and (6.8):

$$\delta\pi = \dot{\pi}\xi^0 + (\pi\xi^1)' + (\mathcal{N}^1\pi + \mathcal{N}^0\phi')\xi^{0\prime}. \tag{6.9}$$

In the first order Hamiltonian formulation the action (6.1) takes the form

$$S = \int (\pi\dot{\phi} - \mathcal{N}^\alpha C_\alpha)dxdt, \tag{6.10}$$

where the Lagrangean multipliers $\mathcal{N}^\alpha$ are just the functions defined in (6.5,6.6) (that is they are the lapse and shift functions up to $\sqrt{\gamma}$). The constraints

$$C_0 = \frac{1}{2}(\pi^2 + \phi'^2), \qquad \text{and} \qquad C_1 = \pi\phi' \tag{6.11}$$

form a closed algebra, i.e. are first class constraints, with respect to the standard Poisson brackets $\{\phi(x), \pi(y)\} = \delta(x, y)$:

$$\begin{aligned}\{C_i(x),C_i(y)\} &= C_1(x)\frac{\partial}{\partial x}\delta(x,y)-C_1(y)\frac{\partial}{\partial y}\delta(x,y)\\ \{C_0(x),C_1(y)\} &= C_0(x)\frac{\partial}{\partial x}\delta(x,y)-C_0(y)\frac{\partial}{\partial y}\delta(x,y),\end{aligned}\tag{6.12}$$

where $i=1,2$. Rewriting these relations in terms of the light-cone constraints $C_0 \pm C_1$ we immediately recognize them as Virasoro algebra [20].

Concerning the symmetries we first note that the Weyl symmetry (6.3) takes the trivial form in the Hamiltonian formalism

$$\mathcal{N}^0=\frac{\mathcal{N}}{\sqrt{\gamma}}\to\frac{\Omega\mathcal{N}}{\Omega\sqrt{\gamma}}=\mathcal{N}^0,\qquad \mathcal{N}^1=\frac{\mathcal{N}_1}{\sqrt{\gamma}}\to\mathcal{N}^1,\tag{6.13}$$

so that all variables in the first order action are Weyl invariant.

Because one of the constraints, namely C_0, is quadratic in the momentum, we need to combine gauge and reparametrization transformations as in (2.14) to recover the diffeomorphism invariance (6.7,6.8) in the Hamiltonian formalism. For the bosonic string the explicit transformation (2.14) reads

$$\begin{aligned}\delta\mathcal{N}^0 &= \dot{\lambda}^0+\mathcal{N}^{1\prime}\lambda^0-\mathcal{N}^1\lambda^{0\prime}+\mathcal{N}^{0\prime}\lambda^1-\mathcal{N}^0\lambda^{1\prime}\\ \delta\mathcal{N}^1 &= \dot{\lambda}^1+\mathcal{N}^{1\prime}\lambda^1-\mathcal{N}^1\lambda^{1\prime}+\mathcal{N}^{0\prime}\lambda^0-\mathcal{N}^0\lambda^{0\prime}\\ \delta\phi &= EM(\phi)\xi+\{\phi,\lambda^{\tilde{\alpha}}C_{\tilde{\alpha}}\}=(\dot{\phi}-\mathcal{N}^0\pi-\mathcal{N}^1\phi')\xi+\pi\lambda^0+\phi'\lambda^1,\\ \delta\pi &= EM(\pi)\xi+\{\pi,\lambda^{\tilde{\alpha}}C_{\tilde{\alpha}}\}=(\dot{\pi}-(\mathcal{N}^0\phi'+\mathcal{N}^1\pi)')\xi+(\phi'\lambda^0)'+(\pi\lambda^1)',\end{aligned}\tag{6.14}$$

where we need to assume that the parameters are related by the condition (2.18). This condition is solved if we express the parameters ξ,λ^0,λ^1 in terms of two independent parameters as

$$\xi=\xi^0,\qquad \lambda^0=\mathcal{N}^0\xi^0=\frac{\mathcal{N}}{\sqrt{\gamma}}\xi^0,\qquad \lambda^1=\xi^1+\mathcal{N}^1\xi^0,\tag{6.15}$$

and then we immediately recognize the transformations (6.14) as diffeomorphism transformations (6.7,6.8) without using the Hamiltonian equations. Once again we emphasize that the transformations (6.14) are infinitesimal symmetry transformations on the whole phase space whereas the transformations (6.7,6.8) are applicable only to trajectories on the hypersurface $\mathcal{M}$.

As a first step towards exponentiating the infinitesimal transformations (6.14), i.e. make them finite, we should check their algebra. Using the formulae for the particular choice (6.15) of parameters it easy to find that the commutator of two subsequent transformations $\hat{I}_{\xi,\lambda} \equiv \hat{I}(\vec{\xi})$, where $\vec{\xi} = (\xi^0, \xi^1)$ becomes

$$[\hat{I}(\vec{\eta}), \hat{I}(\vec{\xi})] = \hat{I}(\mathcal{L}_{\vec{\eta}}\vec{\xi}) - \hat{1}, \tag{6.16}$$

completely off mass shell. Hence the algebra of transformations (6.14) forms a (infinite dimensional) Lie-algebra even off the subspace $\mathcal{M}$.

Let us stress once more that the infinitesimal gauge transformations generated by the constraints only (that is the transformations (6.14) with ξ set to zero) are not symmetry transformations which could correspond to the diffeomorpisms of the Lagrangean system. The nonlinear constraint C_0 is reponsible for the dynamics.

The last remark concerns the connection between the constraints and the equations of motion for the string theory. Calculating the first functional derivative of the constraints with respect to the canonical variables we see that the B and E coefficients in (2.31) are

$$B_0 = E_1 = \phi'_y \quad , \quad B_1 = E_0 = \pi_y, \tag{6.17}$$

while the D and F coefficients vanish. Then the eqs. (2.34) reduce to

$$\phi'^{\mu} EM(\phi_\mu) = 0 \quad \text{and} \quad \pi^{\mu} EM(\phi_\mu) = 0 \tag{6.18}$$

where μ is the target-space index. From these equations we cannot conclude that all eqs. of motion should be satisfied. However, they put certain restrictions on the allowed $EM(\phi)$. Since the coefficients F are equal zero (the constraints do not involve any spatial derivatives of the momenta) the requirement that the constraints are satisfied everywhere and for any foliation does not tell us anything about the eqs. of motion $EM(\pi) = 0$. We will see in the next section that the interconnection theorem, which we just discussed, has much more interesting content in gravity.

Chapter 7

Gravity

General relativity without matter has the action

$$S = \int R\sqrt{-g}d^4x \tag{7.1}$$

(we adapt the sign and units conventions in [21]) and is invariant with respect to coordinate (or diffeomorphism) transformations, the infinitesimal form of which read

$$x^\alpha \to x^\alpha - \xi^\alpha, \qquad g_{\alpha\beta} \to g_{\alpha\beta} + \mathcal{L}_\xi g_{\alpha\beta}. \tag{7.2}$$

Rewriting the metric $g_{\alpha\beta}$ in the $3+1$-split form [19]

$$ds^2 = -(\mathcal{N}^2 - \mathcal{N}_i\mathcal{N}^i)dt^2 + 2\mathcal{N}_i dx^i dt + \gamma_{ij}dx^i dx^j, \tag{7.3}$$

where $\mathcal{N}$ is the lapse function, $\mathcal{N}_i$ are the shift functions, $\mathcal{N}_i = \gamma_{ij}\mathcal{N}^j$, and γ_{ij} is the metric of the 3-dimensional hypersurface Σ_t of constant time t, we derive from (7.2) the following explicit transformations for $\mathcal{N}$, $\mathcal{N}^i$, and γ_{ij}:

$$\begin{aligned}
\delta\mathcal{N} &= (\mathcal{N}\xi^0)^{\cdot} - \mathcal{N}^i(\mathcal{N}\xi^0)_{,i} + \mathcal{N}_{,m}(\xi^m + \mathcal{N}^m\xi^0)\ , \\
\delta\mathcal{N}^i &= (\xi^i + \mathcal{N}^i\xi^0)^{\cdot} - (\xi^i + \mathcal{N}^i\xi^0)_{,m}\mathcal{N}^m + \mathcal{N}^i{}_{,k}(\xi^k + \mathcal{N}^k\xi^0) \\
&\qquad - \mathcal{N}\gamma^{ij}(\mathcal{N}\xi^0)_{,j} + \gamma^{ij}\mathcal{N}_{,j}(\mathcal{N}\xi^0)\ , \\
\delta\gamma_{ij} &= (\dot{\gamma}_{ij} - \mathcal{N}_{i|j} - \mathcal{N}_{j|i})\xi^0 + {}^{(3)}\mathcal{L}_{\xi+\mathcal{N}\xi^0}\gamma_{ij}.
\end{aligned} \tag{7.4}$$

Here the comma denotes ordinary differentiation with respect to the corresponding space coordinate, the bar denotes covariant derivative in the 3 dimensional space Σ_t with metric γ_{ij}, γ^{ij} is the inverse 3-dimensional metric on Σ_t and ${}^{(3)}\mathcal{L}$ is the Lie derivative in Σ_t. This Lie derivative is to be taken in the direction $\xi + \mathcal{N}\xi^0 \equiv \{\xi^i + \mathcal{N}^i\xi^0\}$.

In the first order Hamiltonian formalism the ADM action for pure gravity takes the form

$$S = \int (\pi^{ij}\dot{\gamma}_{ij} - \mathcal{N}^{\alpha}\mathcal{H}_{\alpha})d^3x dt, \tag{7.5}$$

where π^{ij} are the momenta conjugated to γ_{ij} and the four Lagrangean multipiers are

$$\mathcal{N}^0 = \mathcal{N}, \qquad \text{and} \qquad \mathcal{N}^i = \gamma^{ij}\mathcal{N}_j, \tag{7.6}$$

that is the lapse and shift functions. Correspondingly the constraints $\mathcal{H}_\alpha$ are[1] [19, 21]

$$\mathcal{H}_0 = G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{\gamma}\,{}^{(3)}R, \qquad \mathcal{H}_i = -2\gamma_{ij}\pi^{jl}_{|l}, \tag{7.7}$$

where

$$G_{ijkl} = \frac{1}{2\sqrt{\gamma}}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}), \qquad \gamma = \det(\gamma_{ij}) \tag{7.8}$$

is the metric in superspace [21] and ${}^{(3)}R$ the instrinsic curvature of the hypersurface Σ_t of constant time t. With the help of the fundamental Poisson brackets

$$\{\gamma_{ij}(x), \pi^{kl}(y)\} = \delta_i^{(k}\delta_j^{l)}\,\delta(x,y) = \frac{1}{2}(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)\delta(x,y) \tag{7.9}$$

one checks that the constraints (7.7) are first class [21]

$$\begin{aligned}
\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} &= \gamma^{ij}(x)\mathcal{H}_j(x)\frac{\partial}{\partial x^i}\delta(x,y) - \gamma^{ij}(y)\mathcal{H}_j(y)\frac{\partial}{\partial y^i}\delta(x,y) \\
\{\mathcal{H}_i(x), \mathcal{H}_0(y)\} &= \mathcal{H}_0(x)\frac{\partial}{\partial x^i}\delta(x,y) \qquad (7.10) \\
\{\mathcal{H}_i(x), \mathcal{H}_j(y)\} &= \mathcal{H}_j(x)\frac{\partial}{\partial x^i}\delta(x,y) - \mathcal{H}_i(y)\frac{\partial}{\partial y^j}\delta(x,y).
\end{aligned} \tag{7.11}$$

Let us note that if we add matter (covariantly coupled to gravity) to (7.1) then the constraints contain extra pieces, but their algebra remains unchanged. Another interesing observation is the following: If we use $\sqrt{\gamma}\mathcal{H}_0$ instead of $\mathcal{H}_0$ as a constraint then the algebra of constraints looks very much like a natural generalization of the Virasoro algebra (6.12) to four dimensions. It is a nontrivial problem where the diffeomorphism invariance of the

[1] in this section we denote the constraints by $\mathcal{H}_\alpha$, a notation which is widely used in gravity

original action (7.1) is hidden in the first order Hamiltonian reformulation of gravity. There have been various attempts to reveal this symmetry (see, for instance [1, 2, 3])

Three of the constraints, namely the $\mathcal{H}_i$, are linear in momenta, so they should generate transformations which coincide with diffeomorphism transformations. This has been realized for time independent transformations some time ago [22]. However, the fourth constraint, namely $\mathcal{H}_0$, is quadratic in the momenta and hence cannot generate a symmetry of the corresponding Lagrangean system according to our general results in section 2. Only combined with a compensating transformation does it generate the symmetry we are looking for. Since the Hamiltonian is zero, this symmetry is exactly the reparametrization invariance (2.22). Assuming that the parameters in (2.14) are related such that the condition (2.18) is satisfied, we can write this off shell symmetry transformation for gravity in the following explicit manner

$$\begin{aligned} \delta\mathcal{N} &= \dot{\lambda}^0 - \mathcal{N}^j\lambda^0{}_{,j} + \mathcal{N}_{,j}\,\lambda^j, \\ \delta\mathcal{N}^i &= \dot{\lambda}^i - \mathcal{N}^j\lambda^i{}_{,j} + \mathcal{N}^i{}_{,j}\,\lambda^j - \mathcal{N}\gamma^{ij}\lambda^0{}_{,j} + \gamma^{ij}\mathcal{N}_{,j}\,\lambda^0, \\ \delta\gamma_{ij} &= EM(\gamma_{ij})\xi + \{\gamma_{ij}, \lambda^{\tilde{\alpha}}\mathcal{H}_{\tilde{\alpha}}\} \\ &= EM(\gamma_{ij})\xi + \frac{1}{\sqrt{\gamma}}(2\pi_{ij} - \gamma_{ij}\pi)\lambda^0 + {}^{(3)}\mathcal{L}_\lambda\gamma_{ij} \end{aligned} \tag{7.12}$$

and

$$\delta\pi^{ij} = EM(\pi^{ij})\xi + \{\pi^{ij}, \lambda^{\tilde{\alpha}}\mathcal{H}_{\tilde{\alpha}}\}. \tag{7.13}$$

Here the 5 parameters ξ, λ^α are to be expressed in terms of the four independent parameters ξ^α as

$$\xi = \xi^0, \qquad \lambda^0 = \mathcal{N}\xi^0, \qquad \lambda^i = \xi^i + \mathcal{N}^i\xi^0 \tag{7.14}$$

to resolve (2.17) and then it becomes evident that (7.12) is identical to (7.4). Again we need not use any of the Hamiltonian equations. A rather lengthy calculation shows that the transformation law for the momenta one gets by using the definition of the momenta in terms of γ_{ij}, $\mathcal{N}_k$ and (7.4) coincides with (7.13) also off mass shell.

Thus we found that in gravity the three constraints which are linear in the momenta generate the diffeomorphism transformations while the forth constraint $\mathcal{H}_0$ does it only in a particular combination with the 'trivial' transformation (2.12). This nonlinear in momenta constraint itself is responsible for the origin of the dynamics in the subspace $\mathcal{M}$ in the superhamiltonian reformulation of gravity.

The important question is how to read off the Lie algebra structure of the diffeomorphism group in the Hamiltonian formulation. Because for gravity the structure coefficients depend on the canonical variables (in distinction from the previous cases) one might expect that the algebra of infinitesimal transformations (7.12-7.14) cannot close in this case. Actually naively the dependence on the canonical variables can enter in the parameter $\bar{\lambda}$ for the commutator of two infinitesimal transformations with parameters through the γ-dependence of the structure coefficients (see (2.25)). Fortunately, this expectation is not confirmed. In particular, in the formula (2.25) for the $\bar{\lambda}$-parameter this γ-dependence of the various terms on the right hand side cancels for the concrete choice (7.14) for the $\mathcal{N}$-dependence of the parameters λ. The price we pay for that is the explicit dependence of the parameters of transformations on the Lagrangean mulitpliers, but not on the canonical variables γ, π. Starting from the general formulae (2.23-2.25) a straightforward but rather lengthy calculation shows that the transformations (7.12-7.14) form a Lie algebra completely off mass shell:

$$[\hat{I}(\eta), \hat{I}(\xi)] = \hat{I}(\mathcal{L}_\xi \eta) - \hat{1}, \qquad \xi = (\xi^0, \ldots, \xi^3), \quad \eta = (\eta^0, \ldots, \eta^3), \quad (7.15)$$

where ξ^0, ξ^i and η^0, η^i are defined in (7.14), as it should be for diffeomorphisms. The formula (7.15) holds even off the hypersurface $\mathcal{M}$ where the Lagrangean system lives.

There is a deep connection between the constraints and equations of motion in gravity. Calculating the derivative of the constraints in this case we shall find that all of the coefficients $A, \cdots, F$ in (2.31) do not vanish.

In particular, taking into account that the index k in the formulae (2.31,2.32) in the case of gravity is a composite one,, $i \equiv (j,k)$; a, b run over the same spatial index l and calculating the derivatives of $\mathcal{H}_i$ with respect to π^{jk} and $\mathcal{H}_0$ with respect ot γ_{np} we find

$$F^l_{ijk} = -2\gamma_{i(j}\delta^l_{k)} \quad \text{and} \quad D_0^{nplk} = -G^{nplk}, \qquad (7.16)$$

where G^{nplk} is the inverse of the superspace-metric, $G^{nplk}G_{lkij} = \delta_i^{(n}\delta_j^{p)}$. Then the first and last equations in (2.34) take the form

$$G^{nplk}EM(\gamma_{np}) = 0 \quad \text{and} \quad 2\gamma_{ij}EM(\pi^{jl}) = 0. \qquad (7.17)$$

Since the determinants det G and det γ are not equal zero the eqs.(7.17) have the unique solution

$$EM(\gamma_{np}) = 0 \quad \text{and} \quad EM(\pi^{jl}) = 0. \qquad (7.18)$$

The remaining equations in (2.34) are then automatically fulfilled. Thus, we see that in general relativity the whole dynamics follows from the requirement that the constraints are satisfied everywhere and they are preserved under diffeomorphisms.

Chapter 8

Discussion

In the previous sections we revealed the relevant local symmetries of Lagrangean systems in the first order Hamiltonian systems. We have seen that all symmetries have a similar structure in the Hamiltonian approach although they may look quite differently in the Lagrangean one. All fundamental field theories in physics, and in particular the ones we considered, are systems with first class constraints. If the constraints are linear in the momenta (as in Yang-Mills theories) then they generate the well known gauge symmetry. If some of the constraints are nonlinear in the momenta then only very special combinations of the transformations generated by the constraints and simple compensating transformations proportional to the equations of motion correspond to the off mass shell symmetries of the corresponding Lagrangean system. If only one of the constraints is nonlinear in the momenta, as in string theory and gravity, then the symmetries of the system consist of the gauge transformations generated by the linear constraints plus an extra reparametrization transformation related to the nonlinear constraint, but not just generated by this constraint. This takes place only if the Hamiltonian is equal to zero, i.e. is a super-Hamiltonian. All wellknown theories with nonlinear constraints possess a super-Hamiltonian. However, presently we do not know if there is a deep connection between the non-linearity of some of the constraints and the super-Hamiltonian character of the system. If there are more then one nonlinear constraint then one has to use extra transformations from the huge set of transformations (2.15) in a combination with the transformations generated by the nonlinear constraints to recover the Lagrangean symmetries.

In any case, the wellknown symmetries of the Lagrangean systems are manifest in the Hamiltonian formalism and even more transparent there. Different symmetries may look quite differently in the Lagrangean formalism (for example, local supersymmetry and diffeomorphisms) but they have the same formal structure in the Hamiltonian approach. In addition, the symmetry transformations for the Hamiltonian systems are richer as the corresponding ones for the Lagrangean systems. This is so since in the Hamiltonian approach the transformations are acting on the whole phase space and are symmetries for all off mass shell trajectories. For such general trajectories the momenta need not be related to the velocities, as it should be for Lagrangean systems.

We considered mainly the infinitesimal form of the symmetry transformations and checked the algebra of two subsequent infinitesimal transformations. We found that for all theories we studied (gauge theories, point particle, bosonic string and gravity) the algebras are closed completely off mass shell in the whole phase space, even off the subspace $\mathcal{M}$ in which the Lagrangean system lives. In particular, for gravity, where the structure constants depend on the canonical variables, we revealed a closed Lie algebra in the Hamiltonian formalism.

Different theories which are invariant under diffeomorphism transformations (as for example string theory and gravity) have similar constraint algebras but the constraints look quite differently. However, for the phase space transformations belonging to diffeomorphisms to form a Lie algebra the constraints themselves should have some underlying common structure which we did not reveal. For example, we could ask what kind of general conditions the constraints in string theory, dilaton coupled 2-dimensional gravity, gravity or higher derivative gravity, the constraints of which are looking quite differently, should satisfy to close the algebra. These interesting questions deserve further investigations.

The other question concerns the role of the transformations generated by the nonlinear constraints alone. We showed that they are responsible for the dynamics of Lagrangean systems in the superhamiltonianin formalism.

Also we have seen that there is a deep connection between the structure of the constraints and the dynamics. For example, in string theory some of the Hamiltonion equations and in gravity all of them automatically follow if we demand that the constraints are satisfied everywhere for any foliation of space time. The presence of the spatial derivatives of q and/or p is responsible for that on the technical level.

One possible application of the developed approach to phase space symmetries is a way to construct new theories possessing local symmetries in the Hamiltonian formalism (see, for instance [15, 17]). Actually in many cases the constraints have a clear physical interpretation (as the Gauss constraints in electromagnetism). So one starts by introducing contraints in the theory to satisfy some physical requirements. Then one should commute the constraints (leading to secondary constraints) such that the systems of original constraints together with the secondary ones form a first class system. Note that only first class constraints generate local symmetries [1]. The number of constraints is equal to the number of parameters of the symmetry transformations of the corresponding Lagrangean system.

Another interesting application of the considered formalism one could find in the quantized theories, in which we are ultimately interested. For example, in the functional integral approach to quantum theories it is more natural to consider the Hamiltonian functional integral as compared to the Lagrangean one. This is true in particular for theories which are invariant under diffeomorphisms. In the phase space functional integral at least the q, p-piece of the measure is just the well-defined Liouville measure. After performing the integration over the momenta we arrive at the functional integral in the Lagrangean formulation. However, even in the simple case of a first order action (2.1) which is quadratic in the momenta, a q-dependent function multiplying p^2 appears in the measure for the Lagrangean functional integral. For systems where the action is not quadratic in the momenta or even for gravity the question about the correct measure becomes quite nontrivial. Also, in the Hamiltonian version of the BRST-quantization it is not clear which symmetries (the ones generated by the constraints alone or the symmetries of the Lagrangean system) should we use to construct the BRST charge for systems with nonlinear constraints and whether these different charges lead to the same final quantization. Only in the simple cases of the relativistic particle and supersymmetric particle it has been demonstrated that the results in both cases are the same [24]. For both systems the two kinds of transformations can be written down in finite form.

For field theories this question has not been investigated. For theories with nonlinear constraints, and in particular gravity, there are two different BRST charges. One belonging to diffeomorphisms and one to the transformations generated by the constraints. They coincide only if we impose the equations of motion and this may be the reason why the Batalin-Vilkovisky

[1] For example, a system with $2n$ second class constraints can locally be transformed into a system with n first class constraints and n gauge fixings by a canonical transformation. Thus the gauge transformations generated by the first class constraints are automaticlly fixed by the n gauge fixings and no symmetries survive.

theorem [23] might break down when the two compared gauges are not infinitesimally close to each other [25]. The transformations generated by the two BRST charges differ by trivial transformations. The relevance of these trivial transformations can already be seen on the perturbative level in theories with nonclosing algebras (for instance in supergravity [26]).

One would like to hope that the results obtained in this paper could help to fill the gap in the study of symmetries of constraint Hamiltonian systems which, from our point of view, still exist even on the classical level in the current literature.

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Bibliography

[1] P. Bergman and A. Komar, Int. J. Theor. Phys. **5**(1972) 15.

[2] C.J. Isham and K.V. Kuchar, Ann. Phys. **164** (1985) 288; **164** (1985) 316.

[3] J. Lee and R. Wald, J. Math. Phys. **31** (1990) 725.

[4] J. Halliwell and J. Hartle, Phys. Rev. **D43** (1991) 1170.

[5] C. Teitelboim, PHys. Rev. Lett. **38** (1977) 1106.

[6] E. Fradkin and G. Vilkovisky, CERN report No. TH-2332 (1977), unpublished.

[7] B. Voronov and I. Tyutin, Theor. Math. Phys. **50** (1982) 218.

[8] M. Henneaux, Lectures given at the XX GIFT International Seminar on Theor. Phys., Jaca, Spain, june 1989.

[9] M. Henneaux and C. Teitelboim, 'Quantization of Gauge Systems', Princeton Univ. Press (1992).

[10] M. Henneaux, C. Teitelboim and J. Zanelli, Nucl. Phys. **B332** (1990) 169.

[11] see for example: C. Isham, Lectures presented at the NATO Advanced Study Institute 'Recent Problems in Mathematical Physics', Salamanca, June 1992, Imperial College preprint TP/91-92/25.

[12] B. DeWitt, Dynamical Theory of Groups and Fields, Gordon and Breach, New York, (1965).

[13] P.A.M. Dirac, Lectures on Quantum Mechanics, Academic, New York, (1964).

[14] R. Casalbuoni, Il Nuovo Cimento **33** (1976) 115.

[15] L. Feher, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Phys. Rep. **222** (1992) 1.

[16] P. Ramond, 'Field Theory, a Modern Primer', Benjamin/Cummings, Reading, M.A. (1981)

[17] L. Brink, S. Deser, B. Zumino, P. DiVecchia and P. Howe, Phys. Lett. **64B** (1976) 435.

[18] A.M. Polyakov, Phys. Lett. **103B** (1981) 207.

[19] R. Arnowitt, S. Deser and C.W. Misner in 'Gravitation, An Introduction to Current Research', ed. by L. Witten, Wiley, New York (1962).

[20] See e.g. M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory, Cambridge University Press (1987).

[21] B.S. DeWitt, Phys. Rev. **160** (1967) 1113.

[22] P.W. Higgs, Phys. Rev. Lett. **1** (1958) 373; ibid **3** (1959) 66.

[23] I. Batalin and G. Vilkovisky, Phys. Lett **69** (1977) 309.

[24] D. Nemeschansky, C. Preitschopf and M. Weinstein, Ann. of Phys. **183** (1988) 226.

[25] J. Guven and M. Ryan, Phys. Rev. **D45** (1992) 3559.

[26] R. Kallosh, Nucl. Phys. **141** (1978) 141.